CRITICAL OPERATORS FOR THE DEGREE OF THE MINIMAL POLYNOMIAL OF DERIVATIONS RESTRICTED TO GRASSMANN SPACES

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Abstract: Let $V$ be a finite dimension vector space. For a linear operator on $V$, $f$, $D(f)$ denotes the restriction of the derivation associated with $f$ to the $m$th Grassmann space of $V$. In [Cyclic Spaces for Grassmann Derivatives and Additive Theory, Bull. London Math. Soc. 26(1994) 140-146] Dias da Silva and Hamidoune obtained a lower bound for the degree of the minimal polynomial of $D(f)$, over an arbitrary field. Over a field of zero characteristic that lower bound is given by

$$\deg(P_{D(f)}) \geq m(\deg(P_f) - m) + 1.$$  

Using additive number theory results, results on the elementary divisors of $D(f)$ and methods presented by Marcus and Ali in [Minimal Polynomials of Additive Commutators and Jordan Products, J. Algebra 22(1972) 12-33] we obtain a characterization of equality cases in the former inequality, over a field of zero characteristic, whenever $m$ does not exceed the number of distinct eigenvalues of $f$.

Keywords: Grassmann space, derivation, minimal polynomial.

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1. Introduction

Let $\mathbb{F}$ be a field of zero characteristic and let $V$ be a finite dimension vector space over $\mathbb{F}$ such that $\dim V \geq m \geq 2$, where $m$ is an integer. Let $S_m$ be the symmetric group of degree $m$. For $\sigma \in S_m$, $P(\sigma)$ denotes the unique linear operator on the $m$th tensor power product of $V$, $\otimes^m V$, such that

$$P(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(m)},$$

for all $v_1, v_2, \ldots, v_m \in V$.

Let $\varepsilon$ be the alternating character on $S_m$ and consider the symmetrizer defined on $\otimes^m V$ by

$$T_\varepsilon = \frac{1}{m!} \sum_{\sigma \in S_m} \varepsilon(\sigma) P(\sigma).$$

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The $m$th Grassmann space of $V$ is $\wedge^m V = T_\epsilon(\otimes^m V)$. For $v_1, v_2, \ldots, v_m \in V$, $v_1 \wedge v_2 \wedge \cdots \wedge v_m$ denotes $T_\epsilon(v_1 \otimes v_2 \otimes \cdots \otimes v_m)$.

For a linear operator, $g$, on a vector space over $\mathbb{F}$, $P_g$ denotes the minimal polynomial of $g$ and $\deg(P_g)$ denotes its degree. The spectrum of $g$, i.e., the set of all eigenvalues of $g$ in the algebraic closure of $\mathbb{F}$, is denoted by $\sigma(g)$.

We are going to use the well known fact that, for a simple structure linear operator, the degree of its minimal polynomial is equal to the cardinality of its spectrum.

Let $f$ be a linear operator on $V$. The derivation associated with $f$ is the linear operator on $\otimes^m V$,
\[ f \otimes I_V \otimes \cdots \otimes I_V + I_V \otimes f \otimes \cdots \otimes I_V + \cdots + I_V \otimes I_V \otimes \cdots \otimes f. \]

The derivation associated with $f$ commutes with $T_\epsilon$ [2, section 3.2]. Hence, $\wedge^m V$ is an invariant subspace of the derivation associated with $f$. Let $D(f)$ denote the restriction of the derivation associated with $f$ to $\wedge^m V$. In [1] Dias da Silva and Hamidoune obtained a lower bound for the degree of the minimal polynomial of $D(f)$, over an arbitrary field. Over a field of zero characteristic that lower bound is given by
\[ \deg(P_{D(f)}) \geq m(\deg(P_f) - m) + 1. \] (1)

Using additive number theory results, results on the elementary divisors of $D(f)$ and methods presented in [3] we shall obtain a characterization of equality cases in (1) (for zero characteristic), whenever $m$ does not exceed the number of distinct eigenvalues of $f$.

2. Additive number theory results

Let $r$ and $n$ be positive integers. By $Q_{r,n}$ we denote the set of all strictly increasing maps from $\{1, \ldots, r\}$ into $\{1, \ldots, n\}$. If $\alpha \in Q_{r,n}$ we use the $r$-tuple notation for $\alpha$, that is, $\alpha = (\alpha(1), \ldots, \alpha(r))$.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite non-empty subset of $\mathbb{F}$, such that $|A| = n \geq m$, where $|A|$ denotes the cardinality of $A$.

By $\wedge^m A$ we denote the set of sums of $m$ distinct elements in $A$, that is,
\[ \wedge^m A = \left\{ \sum_{i=1}^{m} a_{\alpha(i)} : \alpha \in Q_{m,n} \right\}. \]

In [1] Dias da Silva and Hamidoune obtained a lower bound for the cardinality of $\wedge^m A$, for $A$ subset of an arbitrary field. In zero characteristic that
lower bound is given by
\[
| \wedge^m A | \geq m(|A| - m) + 1 .
\] (2)

For subsets of \( \mathbb{Q} \) it is well known a characterization of equality cases in (2).

**Lemma 1.** [6, Theorem 1.10] Let \( A \) be a finite subset of \( \mathbb{Q} \) such that \( |A| \geq m \geq 2 \). Then
\[
| \wedge^m A | = m(|A| - m) + 1
\]
if and only if one of the following cases holds:

1. \( |A| \in \{m, m+1\} \);
2. \( A \) is an arithmetic progression;
3. \( m = 2, |A| = 4 \) and there exist \( a \in \mathbb{Q}, q, q' \in \mathbb{Q} \setminus \{0\} \) such that \( q \neq q', q + q' \neq 0 \) and
\[
A = a + \{0, q, q', q + q'\} .
\]

The next two lemmas will be used to prove that Lemma 1 holds in any field of zero characteristic. The first one plays a similar role to the one played by Lemma 2.3 in [3].

**Lemma 2.** Let \( k, m \in \mathbb{N} \) be such that \( k \geq m \geq 2 \) and let \( \varphi \in S_k \). Then
\[
\left| \left\{ \left( \sum_{i=1}^{m} \alpha(i), \sum_{i=1}^{m} \varphi(\alpha(i)) \right) : \alpha \in Q_{m,k} \right\} \right| = m(k - m) + 1
\]
if and only if one of the following cases holds:

1. \( k \in \{m, m+1\} \);
2. \( \varphi \in \left\{ \text{id}, \begin{pmatrix} 1 & 2 & k - 1 & k \\ k & k - 1 & 2 & 1 \end{pmatrix} \right\} \);
3. \( m = 2, k = 4 \) and \( \varphi \in \{\text{id}, (2, 3), (1, 4), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3), (1, 2, 4, 3), (1, 3, 4, 2)\} \).

**Proof**
Let \( \varphi \in S_k \) and \( S = \left\{ \left( \sum_{i=1}^{m} \alpha(i), \sum_{i=1}^{m} \varphi(\alpha(i)) \right) : \alpha \in Q_{m,k} \right\} \). Note that
\[
|S| \geq \left| \left\{ \sum_{i=1}^{m} \alpha(i) : \alpha \in Q_{m,k} \right\} \right| = \left| \left[ \frac{m(m + 1)}{2}, \frac{m(2k - m + 1)}{2} \right] \cap \mathbb{N} \right| = m(k - m) + 1 .
\] (3)
Sufficient condition

(1): If \( k \in \{m, m+1\} \) then \(|Q_{m,k}| = m(k - m) + 1\). Since \(|S| \leq |Q_{m,k}|\), from (3), we have \(|S| = m(k - m) + 1\).

(2): If \( \varphi = \text{id} \) then \(|S| = |\{\sum_{i=1}^{m} \alpha(i) : \alpha \in Q_{m,k}\}| = m(k - m) + 1\).

Suppose \( \varphi = \begin{pmatrix} 1 & 2 & \cdots & k - 1 & k \\ k & k - 1 & \cdots & 2 & 1 \end{pmatrix} \).

Then
\[
\left| \left\{ \left( \sum_{i=1}^{m} \alpha(i), \sum_{i=1}^{m} \varphi(\alpha(i)) \right) : \alpha \in Q_{m,k} \right\} \right|
\]
\[
= \left| \left\{ \left( \sum_{i=1}^{m} \alpha(i), m(k + 1) - \sum_{i=1}^{m} \alpha(i) \right) : \alpha \in Q_{m,k} \right\} \right|
\]
\[
= m(k - m) + 1.
\]

(3): Suppose \( m = 2, k = 4 \) and \( \varphi \in \{\text{id}, (2,3), (1,4), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1243), (1342)\}\).

Since \( Q_{2,4} = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}\), \( S = \{(3, \varphi(1) + \varphi(2)), (4, \varphi(1) + \varphi(3)), (5, \varphi(1) + \varphi(4)), (5, \varphi(2) + \varphi(3)), (6, \varphi(2) + \varphi(4)), (7, \varphi(3) + \varphi(4))\}\).

For any possible choice of \( \varphi \) we have \( \varphi(1) + \varphi(4) = \varphi(2) + \varphi(3) \).
Therefore \(|S| = 5\).

Necessary condition

Let \( \varphi \in S_k \) and suppose \(|S| = m(k - m) + 1\), where \( k \geq m + 2 \).
Consider the sets of positive integers, \( B_1, \ldots, B_m \), given by

\[
B_1 = \left\{ 1 + \cdots + (m - 2) + (m - 1) + i : i = m, \ldots, k \right\}
\]
\[
= \frac{m(m+1)}{2} + \{0, 1, \ldots, k - m\};
\]

\[
B_j = \left\{ \underbrace{1 + \cdots + (m - j)}_{m-j} + i + \underbrace{\cdots + k}_{j-1} : i = m - j + 2, \ldots, k - j + 1 \right\}
\]
\[
= \frac{m(m+1)}{2} + (k - m)(j - 1) + \{1, 2, \ldots, k - m\}, \quad j = 2, \ldots, m.
\]

Then \( \sum_{i=1}^{m} \alpha(i) : \alpha \in Q_{m,k} = \left[ \frac{m(m+1)}{2}, \frac{m(2k-m+1)}{2} \right] \cap \mathbb{N} \) is the disjoint union of the sets \( B_1, B_2, \ldots, B_m \).

For \( j = 1, \ldots, m \), let \( S_j = \{(a, b) \in S : a \in B_j\} \). Then \(|S_j| \geq |B_j|\), for all \( j \), and \( S \) is the disjoint union of \( S_1, S_2, \ldots, S_m \).
From
\[ m(k - m) + 1 = |S| = \sum_{j=1}^{m} |S_j| \geq \sum_{j=1}^{m} |B_j| = m(k - m) + 1 , \]
it follows that \(|S_j| = |B_j|, j = 1, \ldots, m.\)

Let \( j \in \{1, \ldots, m-1\} \) and \( \ell \in \{m-j+2, \ldots, k-j\}. \) Define \( \alpha_{\ell,j}, \beta_{\ell,j} \in Q_{m,k} \) by
\[
\alpha_{\ell,j} = (1, \ldots, m-j-1, m-j+1, \ell, k-j+2, \ldots, k) \\
\beta_{\ell,j} = (1, \ldots, m-j-1, m-j, \ell+1, k-j+2, \ldots, k).
\]

Since \( \sum_{i=1}^{m} \alpha_{\ell,j}(i) = \sum_{i=1}^{m} \beta_{\ell,j}(i) \in B_j, \)
\[
\left( \sum_{i=1}^{m} \alpha_{\ell,j}(i), \sum_{i=1}^{m} \varphi(\alpha_{\ell,j}(i)) \right), \left( \sum_{i=1}^{m} \beta_{\ell,j}(i), \sum_{i=1}^{m} \varphi(\beta_{\ell,j}(i)) \right) \in S_j.
\]

From \(|S_j| = |B_j|\) it follows that \( \sum_{i=1}^{m} \varphi(\alpha_{\ell,j}(i)) = \sum_{i=1}^{m} \varphi(\beta_{\ell,j}(i)), \) that is,
\[
\varphi(m-j+1) + \varphi(\ell) = \varphi(m-j) + \varphi(\ell+1).
\]
Hence we have proved that
\[
\varphi(m-j+1) - \varphi(m-j) = \varphi(\ell+1) - \varphi(\ell), \quad (4)
\]
for \( j = 1, \ldots, m-1 \) and \( \ell = m-j+2, \ldots, k-j.\)

(I): \( k \geq m + 3 \)

First suppose \( m = 2. \) From (4) we have \( \varphi(\ell+1) - \varphi(\ell) = \varphi(2) - \varphi(1), \)
for \( \ell = 3, \ldots, k-1. \) Since \( k \geq 5, \alpha = (3, k-1) \) and \( \beta = (2, k) \) are in \( Q_{2,k}. \) From \( \alpha(1) + \alpha(2) = \beta(1) + \beta(2) \in B_2 \) and \( |S_2| = |B_2|, \) it follows that \( \varphi(3) + \varphi(k-1) = \varphi(2) + \varphi(k) \) and so, \( \varphi(3) - \varphi(2) = \varphi(k) - \varphi(k-1) = \varphi(2) - \varphi(1). \) Hence, for \( m = 2 \) we have
\[
\varphi(i+1) - \varphi(i) = \varphi(2) - \varphi(1), \quad i = 1, 2, \ldots, k-1.
\]

Next we prove that this is also true for \( m \geq 3. \) Suppose \( m \geq 3 \) and let \( i \in \{1, \ldots, m-2\}. \)

Taking \( j = i \) and \( \ell = m-i+2 \) in (4) we obtain \( \varphi(m-i+1) - \varphi(m-i) = \varphi(m-i+3) - \varphi(m-i+2). \)
Taking \( j = i + 1 \) and \( \ell = m - (i + 1) + 3 \leq k - (i + 1) \) in (4) we obtain \( \varphi(m - i) - \varphi(m - i - 1) = \varphi(m - i + 3) - \varphi(m - i + 2) \).

Then \( \varphi(m - i + 1) - \varphi(m - i) = \varphi(m - i) - \varphi(m - i - 1) \), for \( i = 1, \ldots, m - 2 \).

Hence
\[
\varphi(i + 1) - \varphi(i) = \varphi(2) - \varphi(1), \quad i = 1, \ldots, m - 1.
\]

Taking \( j = 2 \) and \( \ell = m \) in (4) we get \( \varphi(m + 1) - \varphi(m) = \varphi(m - 1) - \varphi(m - 2) = \varphi(2) - \varphi(1) \).

For \( i = m + 1, \ldots, k - 1 \), taking \( j = 1 \) and \( \ell = i \) in (4) we have
\[
\varphi(i + 1) - \varphi(i) = \varphi(m) - \varphi(m - 1) = \varphi(2) - \varphi(1).
\]

Thus we have proved that
\[
\varphi(i + 1) - \varphi(i) = \varphi(2) - \varphi(1), \quad i = 1, \ldots, k - 1.
\]

Let \( r = \varphi(2) - \varphi(1) \neq 0 \). Then \( \varphi(i) = \varphi(1) + (i - 1)r \), for \( i = 1, \ldots, k \).

If \( r > 0 \) then \( \varphi(1) < \varphi(2) < \cdots < \varphi(k) \) and \( \varphi = \text{id} \).

If \( r < 0 \) then \( \varphi(1) > \varphi(2) > \cdots > \varphi(k) \) and
\[
\varphi = \begin{pmatrix}
1 & 2 & \cdots & k - 1 & k \\
k & k - 1 & \cdots & 2 & 1
\end{pmatrix}.
\]

(II): \( k = m + 2 \)

In this case, from (4), we have
\[
\varphi(k - j + 1) - \varphi(k - j) = \varphi(k - j - 1) - \varphi(k - j - 2), \quad j = 1, \ldots, k - 3.
\]

That is,
\[
\varphi(k) - \varphi(k - 1) = \varphi(k - 2) - \varphi(k - 3) = \cdots = \begin{cases}
\varphi(2) - \varphi(1) & \text{if } k \text{ is even} \\
\varphi(3) - \varphi(2) & \text{if } k \text{ is odd}
\end{cases}
\]

and
\[
\varphi(k - 1) - \varphi(k - 2) = \varphi(k - 3) - \varphi(k - 4) = \cdots = \begin{cases}
\varphi(3) - \varphi(2) & \text{if } k \text{ is even} \\
\varphi(2) - \varphi(1) & \text{if } k \text{ is odd}
\end{cases}.
\]

Let
\[
r = \begin{cases}
\varphi(2) - \varphi(1) & \text{if } k \text{ is even} \\
\varphi(3) - \varphi(2) & \text{if } k \text{ is odd}
\end{cases}
\quad \text{and} \quad
r' = \begin{cases}
\varphi(3) - \varphi(2) & \text{if } k \text{ is even} \\
\varphi(2) - \varphi(1) & \text{if } k \text{ is odd}
\end{cases}.
\]

Suppose \( k \geq 5 \). Since \( m = k - 2 \), \( B_1 = \frac{(k - 2)(k - 1)}{2} + \{0, 1, 2\} \) and \( B_2 = \frac{(k - 2)(k - 1)}{2} + \{3, 4\} \). Let
\[
\alpha = (1, 2, \ldots, k - 4, k - 2, k), \quad \beta = (1, 2, \ldots, k - 5, k - 3, k - 2, k - 1) \in Q_{k - 2, k}.
\]
Since \( \sum_{i=1}^{k-2} \alpha(i) = \sum_{i=1}^{k-2} \beta(i) = \frac{(k-2)(k-1)}{2} + 3 \in B_2 \) and \( |S_2| = |B_2| \),
we have \( \sum_{i=1}^{k-2} \varphi(\alpha(i)) = \sum_{i=1}^{k-2} \varphi(\beta(i)) \), that is, \( \varphi(k-4) + \varphi(k) = \varphi(k-3) + \varphi(k-1) \).

Then \( r = r' \) and \( \varphi(i+1) - \varphi(i) = \varphi(2) - \varphi(1) \), for \( i = 1, \ldots, k-1 \).
As we have seen in (1),
\[
\varphi \in \left\{ \text{id}, \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ k & k-1 & \cdots & 2 & 1 \end{pmatrix} \right\}.
\]

For \( k = 4 \) and \( m = 2 \), from (5), we have \( \varphi(1) + \varphi(4) = \varphi(2) + \varphi(3) \).
Then \( \{\varphi(1), \varphi(4)\} \in \{\{1, 4\}, \{2, 3\}\} \) and case (3) holds.

Next lemma is a straightforward generalization of Lemma 2.1 from [3].

Lemma 3. Let \( m \geq 2 \) and let \( V \) be an \( n \)-dimensional vector space over a field of zero characteristic, \( \mathbb{F} \). Let \( k \in \mathbb{N} \) and let \( u_1, \ldots, u_k \in V \) be distinct. Then there exists a basis \( \{f_1, \ldots, f_n\} \) of \( V^* \), such that, for each \( j \in \{1, \ldots, n\} \), \( f_j(u_1), \ldots, f_j(u_k) \) are \( k \) distinct elements in \( \mathbb{F} \) and
\[
\left| \left\{ \sum_{i=1}^{m} u_{\alpha(i)} : \alpha \in Q_{m,k} \right\} \right| \geq |\wedge^m \{f_j(u_1), \ldots, f_j(u_k)\}| \geq m(k - m) + 1.
\]

Proposition 1. Let \( \mathbb{F} \) be a field of zero characteristic and let \( A \) be a finite subset of \( \mathbb{F} \) such that \( |A| \geq m \geq 2 \). Then
\[
|\wedge^m A| = m(|A| - m) + 1
\]
if and only if one of the following cases holds:

1. \( |A| \in \{m, m+1\} \);

2. \( A \) is an arithmetic progression;

3. \( m = 2, |A| = 4 \) and there exist \( a \in \mathbb{F}, q, q' \in \mathbb{F} \setminus \{0\} \) such that \( q \neq q', q + q' \neq 0 \) and
\[
A = a + \{0, q, q', q + q'\}.
\]

Proof The sufficient condition’s proof is obvious, so we include only the necessary condition’s proof. Suppose \( A = \{a_1, \ldots, a_k\} \), where \( k = |A| \geq m + 2 \geq 4 \), and \( |\wedge^m A| = m(k - m) + 1 \).
Consider the vector space over \( \mathbb{Q} \),

\[ W = \left\{ \sum_{i=1}^{k} q_i a_i : q_i \in \mathbb{Q} \right\} \]

and let \( n = \dim_{\mathbb{Q}} W \leq k \). From Lemma 3 there exists a basis of \( W^* \), \( \{f_1, \ldots, f_n\} \), such that, for \( t = 1, \ldots, n \),

\[ |\{f_t(a_1), \ldots, f_t(a_k)\}| = k \]

and

\[ |\wedge^m A| \geq |\wedge^m \{f_t(a_1), \ldots, f_t(a_k)\}| \geq m(k - m) + 1. \]

Since \( |\wedge^m A| = m(k - m) + 1 \), it follows that

\[ |\wedge^m \{f_t(a_1), \ldots, f_t(a_k)\}| = m(k - m) + 1, \quad t = 1, \ldots, n. \quad (6) \]

From (6) and Lemma 1, for each \( t \in \{1, \ldots, n\} \), one of the following cases holds.

(i): \( \{f_t(a_1), \ldots, f_t(a_k)\} \) is an arithmetic progression;

(ii): \( m = 2, k = 4 \) and there exist \( b_t' \in \mathbb{Q}, q_t, q_t' \in \mathbb{Q} \setminus \{0\} \) such that

\[ q_t \neq q_t', q_t + q_t' \neq 0 \quad \text{and} \quad \{f_t(a_1), f_t(a_2), f_t(a_3), f_t(a_4)\} = b_t' + \{0, q_t, q_t', q_t + q_t'\}. \]

First we assume that \( k \geq 5 \) or \( m \geq 3 \). Then, for each \( t \in \{1, \ldots, n\} \), \( \{f_t(a_1), \ldots, f_t(a_k)\} \) is an arithmetic progression. For some \( \varphi_t \in S_k, r_t \in \mathbb{Q} \setminus \{0\} \) and \( b_t \in \mathbb{Q} \) we have

\[ f_t(a_i) = b_t + r_t \varphi_t(i), \quad i = 1, \ldots, k. \]

Considering the basis \( \left\{ \frac{1}{r_1} f_1, \frac{1}{r_1} f_2, \ldots, \frac{1}{r_1} f_n \right\} \) of \( W^* \), for which Lemma 3 is also true, we can assume that \( r_1 = 1 \).

Let \( \{e_1, \ldots, e_n\} \) be the basis of \( W \) having \( \{f_1, f_2, \ldots, f_n\} \) as its dual. Then

\[ a_i = \sum_{j=1}^{n} f_j(a_i) e_j = \sum_{j=1}^{n} (b_j + r_j \varphi_j(i)) e_j, \quad i = 1, \ldots, k. \quad (7) \]

Reordering \( a_1, \ldots, a_k \) in such way that

\[ f_1(a_i) = b_1 + i, \quad i = 1, \ldots, k \]

we may suppose that \( \varphi_1 = \text{id} \).
From (7) it follows that

\[ a_i = (b_1 + i)e_1 + \sum_{j=2}^{n} (b_j + r_j \varphi_j(i)) e_j, \quad i = 1, \ldots, k. \]

Let \( b = \sum_{j=1}^{n} b_j e_j \in W \) and consider the elements in \( W \) given by \( d_i = a_i - b \), for \( i = 1, \ldots, k \). Since \( |A| = |B| \), and \( A \) is an arithmetic progression if and only if \( \{d_1, \ldots, d_k\} \) is an arithmetic progression, we may assume that \( b = 0 \), that is, we may assume that \( b_j = 0 \), for \( j = 1, \ldots, n \).

Then

\[ a_i = ie_1 + \sum_{j=2}^{n} r_j \varphi_j(i) e_j, \quad i = 1, \ldots, k. \]

If \( n = 1 \) then \( a_i = i e_1 \) for \( i = 1, \ldots, k \) and \( A \) is an arithmetic progression. Suppose \( n > 1 \) and let \( t \in \{2, \ldots, n\} \). For each \( \alpha \in Q_{m,k} \) we have

\[
\sum_{\ell=1}^{m} a_{\alpha(\ell)} = \sum_{\ell=1}^{m} \alpha(\ell) e_1 + \sum_{\ell=1}^{m} \sum_{j=2}^{n} r_j \varphi_j(\alpha(\ell)) e_j = \sum_{\ell=1}^{m} \alpha(\ell) e_1 + r_t \left( \sum_{\ell=1}^{m} \varphi_t(\alpha(\ell)) \right) e_t + \sum_{j \neq t} \sum_{\ell=1}^{m} \varphi_j(\alpha(\ell)) r_j e_j
\]

and, since \( e_1, \ldots, e_n \) are linearly independent,

\[
m(k-m) + 1 = |A| \geq \left| \left\{ \left( \sum_{\ell=1}^{m} \alpha(\ell), \sum_{\ell=1}^{m} \varphi_t(\alpha(\ell)) \right) : \alpha \in Q_{m,k} \right\} \right| \geq m(k-m) + 1.
\]

Then

\[
\left| \left\{ \left( \sum_{\ell=1}^{m} \alpha(\ell), \sum_{\ell=1}^{m} \varphi_t(\alpha(\ell)) \right) : \alpha \in Q_{m,k} \right\} \right| = m(k-m) + 1
\]

and, from Lemma 2,

\[ \varphi_t \in \left\{ \text{id}, \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ k & k-1 & \cdots & 2 & 1 \end{pmatrix} \right\}, \quad t = 2, \ldots, n. \]
Suppose $\varphi_{t_1} = \varphi_{t_2} = \cdots = \varphi_{t_p} = \text{id}$ and

$$\varphi_{s_1} = \varphi_{s_2} = \cdots = \varphi_{s_q} = \begin{pmatrix} 1 & 2 & \cdots & k - 1 & k \\ k & k - 1 & \cdots & 2 & 1 \end{pmatrix},$$

where $p + q = n$, $t_1 = 1$ and $\{t_2, \ldots, t_p, s_1, s_2, \ldots, s_q\} = \{2, \ldots, n\}$. Then, for $i = 1, \ldots, k,$

$$a_i = ie_1 + \sum_{j=2}^{n} r_j \varphi_j(i)e_j$$

$$= ie_1 + \sum_{j=1}^{p} r_{t_j} ie_{t_j} + \sum_{j=1}^{q} r_{s_j} (k - i + 1)e_{s_j}$$

$$= i \left( e_1 + \sum_{j=1}^{p} r_{t_j} e_{t_j} - \sum_{j=1}^{q} r_{s_j} e_{s_j} \right) + (k + 1) \sum_{j=1}^{q} r_{s_j} e_{s_j}$$

and $A$ is an arithmetic progression.

Next suppose $k = 4$ and $m = 2$. For $t = 1, \ldots, n \leq 4$ there exist $b'_t \in \mathbb{Q}$, $q_t, q'_t \in \mathbb{Q} \setminus \{0\}$ such that $q_t \neq q'_t$, $q_t + q'_t \neq 0$ and

$$\{f_t(a_1), f_t(a_2), f_t(a_3), f_t(a_4)\} = b'_t + \{0, q_t, q'_t, q_t + q'_t\}.$$  

(This includes both cases (i) and (ii)).

For $j \in \{1, \ldots, n\}$ there exists a permutation $\varphi_j \in S_4$ such that

$$f_j(a_i) = b'_j + \left( \left[ \frac{\varphi_j(i)}{2} \right] - \left[ \frac{\varphi_j(i) - 1}{2} \right] \right) q_j + \left[ \frac{\varphi_j(i) - 1}{2} \right] q'_j, \quad i = 1, 2, 3, 4,$$

where, for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.

As in case $k \geq 5$ or $m \geq 3$ we can assume that $q_1 = 1$, $\varphi_1 = \text{id}$ and $b'_j = 0$, for each $j$. Then, for $i = 1, 2, 3, 4$,

$$a_i = \sum_{j=1}^{n} f_j(a_i)e_j = \left( \left\lfloor \frac{i}{2} \right\rfloor - \left\lfloor \frac{i - 1}{2} \right\rfloor \right) e_1 + \sum_{j=2}^{n} f_j(a_i)e_j$$

and
\[ a_1 + a_2 = e_1 + \sum_{j=2}^{n}(f_j(a_1) + f_j(a_2))e_j; \]
\[ a_1 + a_3 = q_1e_1 + \sum_{j=2}^{n}(f_j(a_1) + f_j(a_3))e_j; \]
\[ a_1 + a_4 = (1 + q_1')e_1 + \sum_{j=2}^{n}(f_j(a_1) + f_j(a_4))e_j; \]
\[ a_2 + a_3 = (1 + q_1')e_1 + \sum_{j=2}^{n}(f_j(a_2) + f_j(a_3))e_j; \]
\[ a_2 + a_4 = (2 + q_1')e_1 + \sum_{j=2}^{n}(f_j(a_2) + f_j(a_4))e_j; \]
\[ a_3 + a_4 = (1 + 2q_1')e_1 + \sum_{j=2}^{n}(f_j(a_3) + f_j(a_4))e_j. \]

Since \( q_1' \neq 0, q_1' \neq 1 = q_1, 1 + q_1' = q_1 + q_1' \neq 0 \) and \(| \wedge^2 A | = 5\), it follows that \( a_2 + a_3 = a_1 + a_4 \) and (3) holds.

3. Elementary divisors

Let \( m \geq 2 \), let \( \mathbb{F} \) be a field of zero characteristic and let \( V \) be a finite dimensional vector space over \( \mathbb{F} \) such that \( \dim V \geq m \). Let \( f \) be a linear operator on \( V \). The following characterization of the elementary divisors of \( D(f) \) is well known ([4, 5]).

Let

\[(X - \mu_i)^{n_i}, \quad i = 1, 2, \ldots, \ell \]

be the elementary divisors of \( f \), where \( \mu_1, \ldots, \mu_\ell \in \mathbb{F} \) are not necessarily distinct. Let \( k_1, k_2, \ldots, k_\ell \) be nonnegative integers such that

\[ k_1 + k_2 + \cdots + k_\ell = m \quad \text{and} \quad k_i \leq n_i, \quad i = 1, 2, \ldots, \ell. \quad (8) \]

Let \( r_1, r_2, \ldots, r_\ell \) be nonnegative integers such that

\[ 2r_i \leq k_i(n_i - k_i), \quad i = 1, 2, \ldots, \ell. \quad (9) \]

For \( s \in \{1, 2, \ldots, \ell\} \) define

\[ E_s = k_s(n_s - k_s) - 2r_s + 1 \quad \text{and} \quad \mathcal{E}_s = \sum_{i=1}^{s} E_i. \]

For \( t_1, t_2, \ldots, t_{\ell-1} \) integers such that

\[ 1 \leq t_s \leq \min\{\mathcal{E}_s - 2(t_1 + \cdots + t_{s-1}) + s - 1, E_{s+1}\}, \quad s = 1, \ldots, \ell - 1, \quad (10) \]

define

\[ \eta(r_1, \ldots, r_\ell, t_1, \ldots, t_{\ell-1}) = \mathcal{E}_\ell - 2(t_1 + t_2 + \cdots + t_{\ell-1}) + \ell - 1. \]

Let \( s \in \{1, 2, \ldots, \ell\} \). For each positive integer \( j \) we denote by \( p_{s,j} \) the number of partitions of \( j \) into not more than \( k_s \) parts, each part at most \( n_s - k_s \) and define \( p_{s,0} = 1 \).
For each $s \in \{1, 2, \ldots, \ell\}$ let
\[
    c_s = \begin{cases} 
        1 & \text{if } r_s = 0 \\
        p_{s,r_s} - p_{s,r_s-1} & \text{if } r_s > 0 
    \end{cases}.
\]

**Theorem 1.** [4, 5] The elementary divisors of $D(f)$ are
\[
    \left( X - \sum_{s=1}^{\ell} k_s \mu_s \right)^{\eta(r_1, \ldots, r_{\ell-1})}, \; c_1 c_2 \cdots c_\ell \text{ times},
\]
when $k_1, \ldots, k_\ell, r_1, \ldots, r_\ell, t_1, \ldots, t_{\ell-1}$ run over the sets of nonnegative integers satisfying (8), (9) and (10).

**Remark 1.** For $k_1, \ldots, k_\ell, r_1, \ldots, r_\ell, t_1, \ldots, t_{\ell-1}$ satisfying (8), (9) and (10), we have
\[
    \eta(r_1, \ldots, r_{\ell-1}) \leq \mathcal{E}_{\ell} - \ell + 1 \leq \sum_{s=1}^{\ell} k_s (n_s - k_s) + 1.
\]

**Remark 2.** If we consider $r_1 = \cdots = r_\ell = 0$ and $t_1 = \cdots = t_{\ell-1} = 1$, we obtain $c_1 = \cdots = c_\ell = 1$ and
\[
    \eta(0, \ldots, 1, \ldots, 1) = \sum_{s=1}^{\ell} k_s (n_s - k_s) + 1.
\]
It follows that, if $k_1 + \cdots + k_\ell = m$ and $0 \leq k_i \leq n_i$, $i = 1, \ldots, \ell$, then
\[
    \left( X - \sum_{s=1}^{\ell} k_s \mu_s \right)^{\sum_{s=1}^{\ell} k_s (n_s - k_s) + 1}
\]
is an elementary divisor of $D(f)$.

The following well know results can be obtained as corollaries from Theorem 1.

**Corollary 1.** If $a_1, \ldots, a_r \in \mathbb{F}$ are the distinct eigenvalues of $f$ and
\[
    (X - a_j)^{m_{i,j}}, \; j = 1, 2, \ldots, s_i, \; i = 1, \ldots, r
\]
are the elementary divisors of $f$ then
\[
    \sigma(D(f)) = \left\{ \sum_{i=1}^{r} m_i a_i : \sum_{i=1}^{r} m_i = m, \; m_i \in \mathbb{N}_0 \; \text{and} \; m_i \leq \sum_{j=1}^{s_i} n_{i,j}, \; i = 1, \ldots, r \right\}.
\]
Corollary 2. If $f$ is of simple structure then also $D(f)$ is of simple structure.

Corollary 3.

(1) $\wedge^m \sigma(f) \subseteq \sigma(D(f))$;
(2) If $\dim V = |\sigma(f)|$ then $\wedge^m \sigma(f) = \sigma(D(f))$.

For $m = 2$ there is a considerably simpler characterization for the elementary divisors of $D(f)$.

Theorem 2. [2, Chapter 7, Theorem 2.6] Let $$ (X - \mu_i)^{n_i}, \quad i = 1, 2, \ldots, \ell, $$ be the elementary divisors of $f$, where $\mu_1, \ldots, \mu_\ell \in \overline{\mathbb{F}}$ are not necessarily distinct. The elementary divisors of the restriction of the derivation associated with $f$ to $\wedge^2 V$ are:

$$ (X - 2\mu_i)^k, \quad k = 2n_i - 3, 2n_i - 7, \ldots, \left\{ \begin{array}{ll} 1 & \text{if } n_i \text{ is even} \\ 3 & \text{if } n_i \text{ is odd} \end{array} \right., \quad 1 \leq i \leq \ell $$

and

$$ (X - \mu_i - \mu_j)^{n_i + n_j - 2t + 1}, \quad 1 \leq t \leq \min\{n_i, n_j\}, \quad 1 \leq i < j \leq \ell. $$

4. Main Result

Theorem 3. Let $\mathbb{F}$ be a field of zero characteristic, let $V$ be a vector space over $\mathbb{F}$ with finite dimension $n \geq m$ and let $f$ be a linear operator on $V$. Suppose $r := |\sigma(f)| \geq m$. Let $D(f)$ be the restriction of the derivation associated with $f$ to $\wedge^m V$. Then

$$ \deg(P_{D(f)}) = m(\deg(P_f) - m) + 1 $$

if and only if one of the following cases holds:

(1): $r = m = n$;
(2): $r = m + 1 = n$;
(3): The elementary divisors of $f$ are $$ X - b_1, \ldots, X - b_{m-1}, (X - b_m)^2, $$
where $b_1, \ldots, b_m \in \overline{\mathbb{F}}$ are distinct;
(4): $r \geq m + 1$ and the elementary divisors of $f$ are $$ X - b_i, \quad s_i \text{ times}, \quad i = 1, \ldots, r, $$
where $b_1, \ldots, b_r$ is an arithmetic progression with first term $b_1$, $s_1 = \cdots = s_{m-1} = 1$ and $s_{r-m+2} = \cdots = s_r = 1$;
(5): \( m = 2 \) and the elementary divisors of \( f \) are
\[ X - b, \ (X - b - q)^2, \ X - b - 2q, \]
where \( b, q \in \mathbb{F} \) and \( q \neq 0 \);

(6): \( m = 2 \) and the elementary divisors of \( f \) are
\[ X - b, \ X - b - q, \ X - b - q', \ X - b - q - q', \]
where \( b \in \mathbb{F}, q, q' \in \mathbb{F} \setminus \{0\}, q \neq q' \) and \( q + q' \neq 0 \);

(7): \( m = 2 \) and the elementary divisors of \( f \) are
\[ (X - b_1)^2, (X - b_2)^2, \]
where \( b_1, b_2 \in \mathbb{F} \) and \( b_1 \neq b_2 \).

**Proof**

**Sufficient condition**

(1), (2) and (6): In any of these cases \( f \) is of simple structure and \( \dim V = |\sigma(f)| \). Then (Corollaries 2, 3 and Proposition 1)
\[
\deg(P_{D(f)}) = |\sigma(D(f))| = |\wedge^m \sigma(f)| = m(r - m) + 1 = m(\deg(P_f) - m) + 1.
\]

(3): By Corollary 1, the eigenvalues of \( D(f) \) are the \( m \) elements
\[
z_i = b_m + \sum_{j=1}^{m} b_j, \quad i = 1, \ldots, m
\]
and (Remark 2) \( X - z_1, X - z_2, \ldots, X - z_{m-1}, (X - z_m)^2 \) are elementary divisors of \( D(f) \). Since \( \dim \wedge^m V = \binom{m+1}{m} = m + 1 \), it follows that
\[
P_{D(f)} = (X - z_m)^2 \prod_{i=1}^{m-1} (X - z_i)
\]
and \( \deg(P_{D(f)}) = m + 1 = m(\deg(P_f) - m) + 1 \).

(4): Suppose \( b_i = b_1 + (i - 1)q \), where \( q \in \mathbb{F} \setminus \{0\} \). From Corollary 1, \( \sigma(D(f)) \) is the set
\[
\left\{ mb_1 + q \sum_{i=1}^{r} m_i(i - 1) : \sum_{i=1}^{r} m_i = m \text{ and } 0 \leq m_i \leq s_i, i = 1, \ldots, r \right\}.
\]
Since \( s_1 = \cdots = s_{m-1} = 1 \) and \( s_{r-m+2} = \cdots = s_r = 1 \),
Without loss of generality we may assume that 

\[ \sum_{i=1}^{r} m_i(i - 1) : m_1 + \cdots + m_r = m \text{ and } 0 \leq m_i \leq s_i, i = 1, \ldots, r \]

\[ = \left[ \frac{m(m-1)}{2}, mr - \frac{m(m+1)}{2} \right] \cap \mathbb{N}. \]

Then

\[ \sigma(D(f)) = \left\{ mb_1 + qz : z \in \left[ \frac{m(m-1)}{2}, mr - \frac{m(m+1)}{2} \right] \cap \mathbb{N} \right\} = \wedge^m \sigma(f). \]

Since \( f \) is of simple structure, also \( D(f) \) is of simple structure and \( \deg(P_{D(f)}) = |\sigma(D(f))| = rm - m^2 + 1 = m \deg(P_f) - m^2 + 1. \)

(5): From Theorem 2 the elementary divisors of \( D(f) \) are

\[ (X - 2b - q)^2, \quad X - 2b - 2q, \quad X - 2b - 2q, \quad (X - 2b - 3q)^2. \]

Then \( P_{D(f)} = (X - 2b - 2q)(X - 2b - q)^2(X - 2b - 3q)^2 \) and \( \deg(P_{D(f)}) = 5 = 2 \deg(P_f) - 3. \)

(7): In this case \( P_{D(f)} = (X - 2b_1)(X - 2b_2)(X - b_1 - b_2)^3 \) and \( \deg(P_{D(f)}) = 5 = 2 \deg(P_f) - 3. \)

Necessary condition

Suppose \( \deg(P_{D(f)}) = m \deg(P_f) - m^2 + 1. \) Let \( a_1, \ldots, a_r \in \overline{\mathbb{F}} \) (where \( r \geq m \)) be the distinct eigenvalues of \( f \) and let

\[ (X - a_i)^{n_{i,j}}, \quad j = 1, 2, \ldots, t_i, \quad i = 1, \ldots, r \]

be the elementary divisors of \( f \), where, for each \( i \), \( n_i := n_{i,1} \geq n_{i,2} \geq \cdots \geq n_{i,t_i} \). Then \( P_f = (X - a_1)^{n_1} \cdots (X - a_r)^{n_r}. \)

Consider the \( \mathbb{Q} \)-vector space, \( W = \{ \sum_{i=1}^{r} q_i a_i : q_i \in \mathbb{Q} \} \). Let \( d \) be its dimension and let \( \{ f_1, \ldots, f_d \} \) be a basis of \( W^* \) satisfying the conditions in Lemma 3, for the distinct elements in \( W, a_1, a_2, \ldots, a_r. \)

From Lemma 3, \( f_1(a_1), f_1(a_2), \ldots, f_1(a_r) \) are distinct rational numbers. Without loss of generality we may assume that \( a_1, a_2, \ldots, a_r \) are ordered in such way that

\[ f_1(a_1) < f_1(a_2) < \cdots < f_1(a_r). \] (11)

We consider two cases: \( r = m \) and \( r \geq m + 1. \)

(1): \( r = m. \)

Suppose that \( n_i \geq 2 \) for some \( i \in \{1, \ldots, r\} \). Let \( \ell \) be the biggest element in \( \{1, \ldots, r\} \) such that

\[ n_\ell = \max\{n_i : i = 1, \ldots, r\} \geq 2 \]
and let $k$ be the biggest element in \( \{1, \ldots, r\} \setminus \{\ell\} \) such that 
\[
n_k = \max\{n_i : i = 1, \ldots, \ell - 1, \ell + 1 \ldots, r\}.
\]
Let 
\[
z_i = a_\ell + \sum_{j=1}^{m} a_j, \quad i = 1, \ldots, m.
\]

\( z_1, \ldots, z_m \) are distinct eigenvalues of \( D(f) \) and, since \( n_\ell \geq 2 \), 
\[
(X - z_\ell)\sum_{j=1}^{m}(n_j-1)+1 \quad \text{and} \quad (X - z_i)^{\sum_{j=1}^{m}(n_j-1)+2(n_\ell-2)+1} 
\]
are elementary divisors of \( D(f) \).

Then, for some monic polynomial \( q(X) \in \mathbb{F}[X] \setminus \{0\} \), 
\[
P_{D(f)} = q(X)(X - z_\ell)^{\sum_{j=1}^{m}(n_j-1)+1} \prod_{i=1}^{m} (X - z_i)^{\sum_{j=1}^{m}(n_j-1)+2(n_\ell-2)+1}.
\]

\[
\deg(q(X)) = m \deg(P_f) - m^2 + 1 - \deg(P_f) + m - 1 - \sum_{i=1}^{m} \sum_{j=1}^{m} (n_j - 1) - 
\]
\[
(m - 1)(2n_\ell - 3)
\]
\[
= (m - 1) \deg(P_f) - m^2 + m - \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{i \neq \ell} n_j + (m - 1)(m - 2) - 
\]
\[
(m - 1)(2n_\ell - 3)
\]
\[
= (m - 1) \deg(P_f) - m^2 - \sum_{i=1}^{m} (\deg(P_f) - n_i - n_\ell) - 2(m - 1)n_\ell + 
\]
\[
m^2 + m - 1
\]
\[
= -(m - 1)n_\ell + \sum_{i=1}^{m} n_i + m - 1
\]
\[
= \deg(P_f) - mn_\ell + m - 1.
\]
We consider two subcases:

(i): \( n_k \geq 2 \):

From the definition of \( k \), \( n_\ell \geq n_k \) and \( \deg(P_f) \leq n_\ell + (m - 1)n_k \).

Then \( 0 \leq \deg(q(X)) \leq (m - 1)(n_k - n_\ell + 1) \) and so \( n_k \in \{ n_\ell, n_\ell - 1 \} \). Suppose \( n_k = n_\ell - 1 \). Then \( \deg(q(X)) = 0 \) and

\[
\sigma(D(f)) = \{ z_1, \ldots, z_m \}. \tag{14}
\]

If \( k < \ell \),

\[
w_1 = a_k + a_1 + \cdots + a_{m-1} = 2a_k + \sum_{j=1, j \neq k}^{m-1} a_j
\]

is an eigenvalue of \( D(f) \) such that \( f_1(w_1) < f_1(z_m) < \cdots < f_1(z_1), \) which contradicts (14).

If \( k > \ell \),

\[
w_2 = a_k + a_2 + \cdots + a_m = 2a_k + \sum_{j=2, j \neq k}^{m} a_j
\]

is an eigenvalue of \( D(f) \) such that \( f_1(w_2) > f_1(z_1) > \cdots > f_1(z_m), \) which contradicts (14).

Then \( n_k = n_\ell \geq 2 \) and, from the definitions of \( k \) and \( \ell \), we have \( k < \ell \). Also in this case

\[
w_1 = a_k + a_1 + \cdots + a_{m-1} = 2a_k + \sum_{j=1, j \neq k}^{m-1} a_j
\]

is an eigenvalue of \( D(f) \) not in \( \{ z_1, \ldots, z_m \} \). Therefore

\[
(X - w_1)^{\sum_{j=1}^{m-1} (n_j - 1) + 2(n_k - 2) + 1}
\]

divides \( q(X) \) and, from (13), it follows that

\[
\sum_{j=1, j \neq k}^{m-1} (n_j - 1) + 2(n_k - 2) + 1 \leq \deg(P_f) - mn_\ell + m - 1,
\]

that is,

\[
\deg(P_f) - n_k - n_m - m + 2 + 2n_k - 3 \leq \deg(P_f) - mn_\ell + m - 1.
\]
Since \( n_k = n_\ell \geq n_m \), we obtain \( mn_\ell \leq 2m \) and \( n_k = n_\ell = 2 \). Then \( m + 2 = r + 2 \leq \deg(P_f) \leq 2m \).

If \( m = 2 \) then \( P_f = (X - a_1)^2(X - a_2)^2 \), \( \sigma(D(f)) = \{2a_1, 2a_2, a_1 + a_2\} \) and (Theorem 2) \( (X - a_1 - a_2)^3 \) is an elementary divisor of \( D(f) \). Since \( \deg(P_{D(f)}) = 5 \) we have \( P_{D(f)} = (X - 2a_1)(X - 2a_2)(X - a_1 - a_2)^3 \). Suppose \( t_1 \geq 2 \). Then \( (X - a_1)^{n_{1,2}} \) is another elementary divisor of \( f \) associated with \( a_1 \). Hence \( (X - 2a_1)^{2+n_{12}-1} \) is an elementary divisor of \( D(f) \), which contradicts \( n_{1,2} \geq 1 \). Then \( t_1 = 1 \) and, similarly, \( t_2 = 1 \). Condition (7) holds.

Assume now that \( r = m \geq 3 \). Suppose \( n_q = 2 \) for some \( q \in \{1, \ldots, m\} \setminus \{\ell, k\} \). Then \( \deg(P_f) \geq r + 3 = m + 3 \). From the definitions of \( \ell \) and \( k \) we have \( q < k < \ell \). Then

\[
\begin{align*}
w_1 &= a_k + a_1 + \cdots + a_{m-1} = 2a_k + \sum_{j=1}^{m-1} a_j \\
w_3 &= a_q + a_1 + \cdots + a_{m-1} = 2a_q + \sum_{j=1}^{m-1} a_j
\end{align*}
\]

are eigenvalues of \( D(f) \) such that \( f_1(w_3) < f_1(w_1) < f_1(z_m) < \cdots < f_1(z_1) \).

Therefore,

\[
(X - w_1)^{\sum_{j=1}^{m-1} (n_j - 1) + 2(n_k - 2) + 1} (X - w_3)^{\sum_{j=q}^{m-1} (n_j - 1) + 2(n_q - 2) + 1}
\]

has degree, at most, equal to the degree of \( q(X) \), that is,

\[
2 \deg(P_f) - 2m - 2n_m + n_k + n_q - 2 \leq \deg(P_f) - m - 1,
\]

which contradicts \( \deg(P_f) \geq m+3 \), since \( n_k = n_q = 2 \) and \( n_m \leq 2 \). So, for \( r = m \geq 3 \) and \( n_k \geq 2 \) it must be \( n_k = n_\ell = 2 \) and \( n_i = 1 \) for \( i \in \{1, \ldots, m\} \setminus \{\ell, k\} \). Then \( \deg(P_f) = m + 2 \) and \( \deg(q(X)) = 1 \). Since \( w_1 \in \sigma(D(f)) \setminus \{z_1, \ldots, z_m\} \), it follows that \( q(X) = X - w_1 \). Since

\[
(X - w_1)^{\sum_{j=1}^{m-1} (n_j - 1) + 2(n_k - 2) + 1}
\]
is an elementary divisor of $D(f)$ it follows that $\ell = m$ and, from (12), we have

$$P_{D(f)} = (X - z_m)^3 \left( \prod_{i=1 \atop i \neq k}^{m-1} (X - z_i)^2 \right) (X - z_k)(X - w_1).$$

If $k \leq m - 2$, then

$$w_4 = a_k + a_1 + \cdots + a_{m-2} + a_m = 2a_k + a_m + \sum_{j=1 \atop j \neq k}^{m-2} a_j$$

is also an eigenvalue of $D(f)$, and again we have a contradiction, since $f_1(w_1) < f_1(w_4) < f_1(z_m) < \cdots < f_1(z_1)$.

Then $k = m - 1$. If $m \geq 4$ then $w_5 = a_3 + \cdots + a_{m-2} + 2a_{m-1} + 2a_m$ is also an eigenvalue of $D(f)$ and, from $f_1(w_1) < f_1(z_m) < \cdots < f_1(z_1) < f_1(w_5)$, we have a contradiction.

Then $m = 3$, $\ell = 3$, $k = 2$, $P_f = (X - a_1)(X - a_2)^2(X - a_3)^2$ and

$$P_{D(f)} = (X - z_3)^3(X - z_1)^2(X - z_2)(X - w_1).$$

Since $(X - 2a_2 - a_3)^2$ is an elementary divisor of $D(f)$, $2a_2 + a_3 \in \{z_1, z_3\} = \{a_2 + 2a_3, a_1 + a_2 + a_3\}$, and, once more, we obtain a contradiction.

(ii): $n_k = 1$.

In this case $\deg(P_f) = n_\ell + m - 1$ and, from (13), we obtain $0 \leq \deg(q(X)) = (n_\ell - 2)(1 - m)$. Then $n_\ell = 2$, $\deg(q(X)) = 0$ and

$$P_{D(f)} = (X - z_\ell)^2 \prod_{i=1 \atop i \neq \ell}^{m} (X - z_i).$$

Suppose $t_q \geq 2$ for some $q \in \{1, \ldots, m\} \setminus \{\ell\}$. Then, for $i = 1, \ldots, m$, $y_i = a_q + \sum_{j=1 \atop j \neq i}^{m} a_j$ is an eigenvalue of $D(f)$ and $f_1(y_1) > f_1(y_2) > \cdots > f_1(y_m)$. Since $\sigma(D(f)) = \{z_1, \ldots, z_m\}$ and $f_1(z_1) > f_1(z_2) > \cdots > f_1(z_m)$, it has to be $z_i = y_i$, for all $i$, which contradicts $a_q \neq a_\ell$. Then $t_q = 1$, for all $q \in \{1, \ldots, m\} \setminus \{\ell\}$. 
Now suppose $t_{\ell} \geq 2$. Then $(X - a_{\ell})^2$ and $(X - a_{\ell})^{m,2}$ are elementary divisors of $f$. If $\ell \geq 2$ then
\[
(X - z_1) \sum_{j=1}^{m} (n_j-1)(n_{\ell,1}+n_{\ell,2}+n_{1,2})+1
\]
is an elementary divisor of $D(f)$, with degree $n_{\ell,1}+n_{\ell,2}+n_{1,2} \geq 2$ and we obtain a contradiction. Then $\ell = 1$ and
\[
(X - z_2) \sum_{j=3}^{m} (n_j-1)(n_{1,1}+n_{1,2})+1
\]
is an elementary divisor of $D(f)$ with degree $n_{1,1}+n_{1,2}+1 \geq 2$. Once more, we obtain a contradiction. Condition (3) holds.

For $r = m$ it remains to consider the case $n_1 = \cdots = n_m = 1$, that is, the case $f$ is of simple structure.

Suppose $t_{\ell} \geq 2$ for some $\ell \in \{1, \ldots, m\}$. Then $z_1, \ldots, z_m$, defined as before, are $m$ distinct eigenvalues of $D(f)$, to which $X - z_i, \quad i = 1, \ldots, m$ are associated elementary divisors. Then $m \deg(P_f) - m^2 + 1 \geq m$ and this contradicts $\deg(P_f) = r = m$. It follows that $t_1 = \cdots = t_m = 1$ and condition (1) holds.

\[ (\text{II}): r \geq m + 1: \]

Consider the $m$ subsets of $W$ given by
\[
L_1 = \{a_1 + \cdots + a_{m-2} + a_i : i = m, \ldots, r\}, \\
L_j = \{a_1 + \cdots + a_{m-2} + a_i + a_{r-j+2} + \cdots + a_r : i = m - j + 2, \ldots, r - j + 1\}, \quad j = 2, \ldots, m,
\]
and the $m$ subsets of $Q$ given by
\[
L'_1 = f_1(L_1) = \{f_1(a_1) + \cdots + f_1(a_{m-2}) + f_1(a_i) : i = m, \ldots, r\}, \\
L'_j = f_1(L_j) = \{f_1(a_1) + \cdots + f_1(a_{m-2}) + f_1(a_i) + f_1(a_{r-j+2}) + \cdots + f_1(a_r) : i = m - j + 2, \ldots, r - j + 1\}, \quad j = 2, \ldots, m.
\]

For $j = 1, \ldots, m$ let $m'_j$ and $M'_j$ be, respectively, the minimum and the maximum of $L'_j$. Since $f_1(a_1) < f_1(a_2) < \cdots < f_1(a_r)$, we have $M'_j < m'_{j+1}$, for $j = 1, \ldots, m - 1$. Then, for $1 \leq j < k \leq m$, $L'_j \cap L'_k = \emptyset$. Hence, $L_j \cap L_k = \emptyset$, for $1 \leq j < k \leq m$, and the elements in the disjoint union $\bigcup_{j=1}^{m} L_j$ are $m(r - m) + 1$ distinct eigenvalues of
$D(f)$, with associated elementary divisors

\[
\left( X - a_i - \sum_{k=1}^{m-1} a_k \right)^{\sum_{k=1}^{m-1} (n_k - 1) + (n_i - 1) + 1}, \quad i = m, \ldots, r;
\]

\[
\left( X - a_i - \sum_{k=1}^{m-j} a_k - \sum_{k=r-j+2}^{r} a_k \right)^{\sum_{k=1}^{m-j} (n_k - 1) + \sum_{k=r-j+2}^{r} (n_k - 1) + (n_i - 1) + 1},
\]

\[i = m - j + 2, \ldots, r - j + 1, \quad j = 2, \ldots, m.\]

Let $t(X)$ be the product of these elementary divisors. Then

\[
\text{deg}(t(X)) = (r - m + 1) \left( \sum_{k=1}^{m-1} n_k - m + 1 \right) + \sum_{i=m}^{r} n_i + \\
\sum_{j=2}^{m} \sum_{i=m-j+2}^{r-j+1} \left( \sum_{k=1}^{m-j} n_k + \sum_{k=r-j+2}^{r} n_k + n_i - m + 1 \right)
\]

\[= (r - m) \sum_{k=1}^{m-1} n_k + \text{deg}(P_f) + (-m + 1)(rm - m^2 + 1) +
\]

\[+ (r - m - 1) \sum_{j=2}^{m} \left( \sum_{k=1}^{m-j} n_k + \sum_{k=r-j+2}^{r} n_k \right) + \sum_{j=2}^{m} (\text{deg}(P_f) - n_{m-j+1})
\]

\[= (r - m - 1) \sum_{k=1}^{m-1} n_k + m \text{deg}(P_f) + (-m + 1)(rm - m^2 + 1) +
\]

\[+ (r - m - 1) \sum_{j=2}^{m} \left( \sum_{k=1}^{m-j} n_k + \sum_{k=r-j+2}^{r} n_k \right)
\]
Let \( q(X) \in \mathbb{F}[X] \) be such that \( P_{D(f)} = q(X)t(X) \). Since \( n_i \geq 1 \), for all \( i \), we have

\[
\deg(q(X)) = m \deg(P_D) - m^2 + 1 - \deg(t(X)) = -m^2 + 1 - (r - m - 1) \sum_{k=1}^{m-1} n_k + (m - 1)(rm - m^2 + 1) - (r - m - 1)(m - 1)^2 - \sum_{j=2}^{r} \left( \sum_{k=1}^{m-j} n_k + \sum_{k=r-j+2}^{r} n_k \right) \\
\leq -m^2 + 1 - (r - m - 1)(m - 1) + (m - 1)(rm - m^2 + 1) - (r - m - 1)(m - 1)^2 \\
= 0.
\]

Then \( q(X) = 1 \), \( P_{D(f)} = t(X) \) and

\[
\sigma(D(f)) = \bigcup_{j=1}^{m} L_j. 
\]

Then also

\[
f_1(\sigma(D(f))) = \bigcup_{j=1}^{m} L'_j. 
\]

Suppose that \( n_\ell \geq 2 \) or \( t_\ell \geq 2 \) for some \( \ell \in \{1, \ldots, m - 1\} \). Then

\[
c = 2a_\ell + \sum_{j=1}^{m-1} a_j \in \sigma(D(f)).
\]

Since

\[
f_1(c) = f_1(a_\ell) + \sum_{i=1}^{m-1} f_1(a_i) < \sum_{i=1}^{m} f_1(a_i) = m'_1,
\]

we obtain a contradiction with (16).

Suppose that \( n_\ell \geq 2 \) or \( t_\ell \geq 2 \), for some \( \ell \in \{r - m + 2, \ldots, r\} \). Then

\[
d = 2a_\ell + \sum_{j=r-m+2}^{r} a_j \in \sigma(D(f))
\]
and, from
\[
f_1(d) = f_1(a_\ell) + \sum_{j=r-m+2}^{r} f_1(a_j) > \sum_{j=r-m+1}^{r} f_1(a_j) = M'_m,
\]
we obtain a contradiction with (16).

Hence
\[
n_i = t_i = 1, \quad \text{for } i \in \{1, \ldots, m-1\} \cup \{r-m+2, \ldots, r\}. \quad (17)
\]

Suppose \( f \) is not of simple structure. Then \( r-m+1 \geq m \) and \( n_\ell \geq 2 \) for some \( \ell \in \{m, \ldots, r-m+1\} \). Let \( \ell \) be the smallest element in \( \{m, \ldots, r-m+1\} \) such that \( n_\ell \geq 2 \). Notice that \( \ell \leq r-m+1 \leq r-1 \).

Suppose \( \ell \leq r-2 \) and consider
\[
\begin{align*}
x_i &= \sum_{j=1}^{m-1} a_j + a_i, \quad i = m, \ldots, \ell; \\
y_i &= \sum_{j=1}^{m-2} a_j + a_i + a_r, \quad i = \ell + 1, \ldots, r-1; \\
v_i &= \sum_{j=1}^{m-2} a_j + a_\ell + a_i, \quad i = m, \ldots, r.
\end{align*}
\]

Since \( f_1(x_m) < f_1(x_{m+1}) < \cdots < f_1(x_\ell) < f_1(v_m) < f_1(v_{m+1}) < \cdots < f_1(v_r) < f_1(y_{\ell+1}) < \cdots < f_1(y_{r-1}) < m'_3 \), the elements in (18) are \( 2r - 2m + 1 \) distinct eigenvalues of \( D(f) \), not in \( \bigcup_{j=3}^{m} L_j \).

From (17) and \( n_1 = \cdots = n_{\ell-1} = 1 \), we conclude that
\[
\begin{align*}
(X - x_i), \quad i = m, \ldots, \ell - 1; \\
(X - x_\ell)^{n_\ell}; \\
(X - y_i)^{n_i}, \quad i = \ell + 1, \ldots, r - 1; \\
(X - v_i)^{n_\ell+n_i-1}, \quad i = m, \ldots, r, \quad i \neq \ell; \\
(X - v_\ell)^{2n_\ell-3}
\end{align*}
\]
are elementary divisors of $D(f)$. Then

$$m \deg(P_f) - m^2 + 1 \geq \ell - m + n_\ell + \sum_{i=\ell+1}^{r-1} n_i + \sum_{i=m}^{r} (n_\ell + n_i - 1) +$$

$$2n_\ell - 3 + \sum_{j=3}^{m} \sum_{i=m-j+2}^{r-j+1} n_i$$

$$\Rightarrow m \deg(P_f) - m^2 + 1 \geq \ell - m - 3 + \sum_{i=\ell}^{r-1} n_i + \sum_{i=m}^{r} n_i + (r - m)(n_\ell - 1) +$$

$$n_\ell + \sum_{j=3}^{m} \sum_{i=m-j+2}^{r-j+1} n_i$$

$$\Rightarrow m \deg(P_f) - m^2 + 1 \geq \ell - m - 3 + (\deg(P_f) - \ell) + (\deg(P_f) - m + 1) +$$

$$(r - m + 1)n_\ell - r + m + \sum_{j=3}^{m} \sum_{i=m-j+2}^{r-j+1} n_i.$$

For $3 \leq j \leq m$ we have $m - j \leq m - 3$ and $r - j \geq r - m$. So, if $i \leq m - j + 1$ or $i \geq r - j + 2$ then $n_i = 1$. Hence

$$\sum_{i=m-j+2}^{r-j+1} n_i = \deg(P_f) - m$$

and

$$m \deg(P_f) - m^2 + 1 \geq 2\deg(P_f) - m - 2 - r + (r - m + 1)n_\ell +$$

$$(m - 2)(\deg(P_f) - m)$$

$$\Rightarrow -m^2 + 1 \geq (r - m + 1)n_\ell - r - m - m^2 + 2m - 2$$

$$\Rightarrow (r - m + 1)n_\ell \leq r - m + 3.$$

From the last inequality, since we are assuming that $n_\ell \geq 2$, we have $r = m + 1$ and, from $\ell \leq r - 2 = m - 1$, we obtain a contradiction with (17). Then $\ell = r - 1$ and, from $\ell \leq r - m + 1$, it follows that $m = 2$.

So, if $f$ is not of simple structure then $m = 2$, $n_{r-1} \geq 2$ and

$$n_i = t_i = 1, \text{ for } i \in \{1, \ldots, r\} \setminus \{r - 1\}.$$
In this case, 

\[ x_i = a_1 + a_i, \quad i = 2, \ldots , r - 1; \]
\[ v_i = a_{r-1} + a_i, \quad i = 2, \ldots , r \]

are 2r − 3 distinct eigenvalues of \( D(f) \). Since \( n_i = 1 \) for \( i \neq r - 1 \) we obtain

\[ 2 \deg(P_f) - 3 \geq \sum_{i=2}^{r-1} n_i + \sum_{i \neq r-1}^{r-1} (n_{r-1} + n_i - 1) + 2n_{r-1} - 3 \]

\[ \Rightarrow 2 \deg(P_f) - 3 \geq \deg(P_f) - 2 + (r - 2)(n_{r-1} - 1) + \deg(P_f) - n_{r-1} - 1 + 2n_{r-1} - 3 \]

\[ \Rightarrow \quad 2 \geq (r - 1)(n_{r-1} - 1). \]

Since \( r \geq m + 1 = 3 \), it follows that \( r = 3 \) and \( n_2 = 2 \). From (17) we obtain \( t_1 = t_3 = 1 \). Suppose \( t_2 \geq 2 \). Then \((X - a_2)^2\) and \((X - a_2)^{n_{2,2}}\) are elementary divisors of \( f \) and

\[ (X - a_1 - a_2)^2, \quad (X - a_2 - a_3)^2, \quad (X - 2a_2)^{n_2,2+1} \]

are elementary divisors of \( D(f) \) associated to distinct eigenvalues. Hence \( 5 = \deg(P_{D(f)}) \geq 5 + n_{2,2} \), which leads to a contradiction. Then \( t_2 = 1 \) and

\[ P_{D(f)} = (X - a_1 - a_2)^2(X - 2a_2)(X - a_2 - a_3)^2. \]

Since \( a_1 + a_3 \in \sigma(D(f)) \) it follows that \( a_1 + a_3 = 2a_2 \) and \( a_1, a_2 = a_1 + (a_3 - a_2), a_3 = a_1 + 2(a_3 - a_2) \) is an arithmetic progression. Condition (5) holds.

Next suppose \( f \) is of simple structure. From (15) and (2) it follows that

\[ m(r - m) + 1 = |\sigma(D(f))| \geq |\land^m \sigma(f)| \geq m(r - m) + 1. \]

Hence

\[ \sigma(D(f)) = \land^m \sigma(f) \]

(19)

and

\[ |\land^m \sigma(f)| = m(r - m) + 1. \]

Then (Proposition 1) one of the following conditions holds:
(a): $\sigma(f)$ is an arithmetic progression:

Let $b$ and $q$ be, respectively, the first term and the difference of that arithmetic progression. Since $b, b + q \in W$, then also $q \in W$ and $f_1(b), f_1(b+q), \ldots, f_1(b+(r-1)q)$ is an arithmetic progression in $\mathbb{Q}$ with difference $f_1(q) \neq 0$ (from (11)).

If $f_1(q) > 0$ then $f_1(b) < f_1(b+q) < \cdots < f_1(b+(r-1)q)$ and so, from (11), we have $a_i = b + (i - 1)q$, for $i = 1, \ldots, r$.

From (17) we have $t_i = 1$ for all $i \in \{1, \ldots, m-1\} \cup \{r - m + 2, \ldots, m\}$ and condition (4) holds.

(b): $r = m + 1$:

If $m \geq 3$ then $r \geq 4$ and, from (17), we have $t_i = 1$, $i = 1, \ldots, r$. Condition (2) holds.

If $m = 2$ then $r = 3$ and, from (17), $t_1 = t_3 = 1$. Suppose $t_2 \geq 2$.

Then, from (19) and Corollary 1, we have

$$2a_2 \in \sigma(D(f)) = \{a_1 + a_2, a_1 + a_3, a_2 + a_3\}.$$  

Therefore $2a_2 = a_1 + a_3$ and $\sigma(f)$ is an arithmetic progression with first term $a_1$ and difference $a_2 - a_1$. Condition (4) holds.

(c): $m = 2$, $r = 4$ and

$$\sigma(f) = a + \{0, d, d', d + d'\}$$

for some $a \in \mathbb{F}$, $d, d' \in \mathbb{F} \setminus \{0\}$ such that $d \neq d'$ and $d + d' \neq 0$.

Then

$$\sigma(D(f)) = \wedge^2 \sigma(f) = 2a + \{d, d', d + d', 2d + d', d + 2d'\}. $$

Let

$$X - a, \quad s_1 \text{ times}$$

$$X - a - d, \quad s_2 \text{ times}$$

$$X - a - d', \quad s_3 \text{ times}$$

$$X - a - d - d', \quad s_4 \text{ times}$$

be the elementary divisors of $f$. From (17) we know that, at least, two of the numbers $s_1, s_2, s_3, s_4$ are equal to 1.

If $s_1 = s_2 = s_3 = s_4 = 1$ then condition (6) holds.

Suppose $s_1 \geq 2$. Then $2a \in \sigma(D(f))$. Hence $2d + d' = 0$ or $d + 2d' = 0$. Then $\sigma(f)$ is an arithmetic progression. Similarly, if $s_i \geq$
for some $i \in \{2, 3, 4\}$, then $\sigma(f)$ is an arithmetic progression. As we have seen in (a), condition (4) holds.

References


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