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REAL PALEY-WIENER THEOREMS FOR THE KOORNWINDER-SWARTTOUW q-HANKEL TRANSFORM

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ABSTRACT: We derive two real Paley-Wiener theorems in the setting of quantum calculus. The first uses techniques due to Tuan and Zayed [21] in order to describe the image of the space $L_q^2(0,R)$ under Koornwinder and Swarttouw q-Hankel transform [14] and contains as a special case a description of the domain of the q-sampling theorem associated with the q-Hankel transform [1]. The second characterizes the image of compactly supported q-smooth functions under a rescaled version of the q-Hankel transform and is a q-analogue of a recent result due to Andersen [6].

KEYWORDS: Paley-Wiener theorems, q-Hankel transform.
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1. Introduction

The original Paley-Wiener theorem asserts that the Paley-Wiener space

$$PW = \left\{ f \in L^2(\mathbf{R}) : f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ixt} u(t)\, dt, u \in L^2(-\pi,\pi) \right\}$$

is composed by functions allowing analytic continuation to the whole complex plane as entire functions of exponential type at most π. Since the proof of this theorem does not lend very naturally to other integral transformations, alternative approaches using real variable methods have been developed in order to give a description of the space PW and its generalizations. For instance, Bang [8] proved that

$$\lim_{n\to\infty} \left\| \frac{d^n}{dx^n} f \right\|_p^{\frac{1}{n}} = \sup\{|\lambda| : \lambda \in \mathrm{supp}\mathcal{F}f\},$$

and, as a consequence,

$$PW = \left\{ f \in L^2(\mathbf{R}) : \lim_{n\to\infty} \left\| \frac{d^n}{dx^n} f \right\|_2^{\frac{1}{n}} = \pi \right\}.$$

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Tuan proved a complementary statement using the primitive operator [18] and extended Bang´s result to other transforms, replacing the operator $\frac{d}{dx}$ by a second order operator possessing the kernel of the integral transformation as eigenfunction [16], [17]. A unified approach to obtain such propositions for a general Sturm-Liouville transform is due to Tuan and Zayed [21]. A problem that attracted many attention in recent years was the extension of Paley-Wiener theorems to the Dunkl transform on the real line [15], [7], [19].

A class of Paley-Wiener theorems sitting inside the Schwarz space was obtained by Andersen in [5], where it is shown that the Fourier transform is a bijection between smooth functions supported in $[-R,R]$ and the space of all Schwartz functions satisfying, for all $N \in \mathbf{N}_0$,

$$\sup_{x\in\mathbf{R},\ n\in\mathbf{N}_0} R^{-n}n^{-N}(1+|x|)^N \left|\frac{d^n}{dx^n}f\right| < \infty.$$

An analogous result for the Hankel transform was given in [6], where it is proved that the Hankel transform with kernel $(xy)^{-\nu}J_\nu(xy)$, in the space $L^1(\mathbf{R}_+, x^{2\nu+1}dx)$, is a bijection between the space of even smooth functions supported in $[-R,R]$ and the space of all even Schwartz functions satisfying, for all $N \in \mathbf{N}_0$,

$$\sup_{x\in\mathbf{R},\ n\in\mathbf{N}_0} R^{-n}n^{-N}(1+|x|)^N \left|\Delta_\nu^n f\right| < \infty,$$

where Δ_ν stands for the second order differential operator having $(xy)^{-\nu}J_\nu(xy)$ as eigenfunctions with eigenvalue y^2.

In many cases, the Paley-Wiener theorems give a description of the functions for which a sampling formula is valid. For instance, PW is the domain space for the celebrated Whittaker-Shannon-Koltenikov theorem. In [1], a sampling theorem valid for functions in the following q-Bessel version of the Paley-Wiener space has been derived:

$$PW_q^\nu = \left\{ f \in L_q^2\left(\mathbf{R}^+\right) : f\left(x\right) = \int_0^1 \left(tx\right)^{\frac{1}{2}} J_\nu^{(3)}\left(xt;q^2\right) u\left(t\right) d_qt, u \in L_q^2\left(0,1\right) \right\}, \tag{1}$$

where $J_\nu^{(3)}(z;q)$ is the third Jackson (or Hahn-Exton) q-Bessel function. The functions in PW_q^ν can be recovered from a very sparse grid of sampling points, located near the arithmetic progression $\{q^{-n}, n \in \mathbf{N}\}$. It is desirable to describe such functions in terms of growth conditions. Since the space PW_q^ν is the image under Koornwinder and Swarttouw´s q-Hankel transform

[14] of the space $L_q^2(0,1)$, it can be described using a Paley-Wiener type theorem.

In the present paper we provide two real Paley-Wiener theorems for the q-Hankel transform in terms of second order q-difference equations whose eigenfunctions are q-Bessel functions. In the third section we obtain, using some of Tuan and Zayed´s techniques from [21], a Paley-Wiener theorem for square q-integrable functions that includes a description of PW_q^ν as a special case. Then, section 4 uses a different normalization of the q-Hankel transform in order to obtain a q-analogue of Andersen´s Paley-Wiener theorem for the Hankel transform. To this end we will make use of the properties of the q-Bessel functions studied by Fitouhi, Hamza and Bouzeffour [10]. We should stress that Fitouhi and Dhaoudi [11] obtained a q-Paley-Wiener for the q-sine transform, but their result goes in a different direction of ours, characterizing growth by means of a certain q-hyperbolic cosine.

2. Preliminaries

Choose a number $0 < q < 1$. In what follows, the standard conventional notations from [12] will be used

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n}(1 - aq^{k-1}),$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n.$$

The q-difference operator D_q is

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}. \tag{2}$$

The set $\mathbf{R}_q$ is defined as

$$\mathbf{R}_q = \{q^k, k = 0, \pm 1, \pm 2, ...\}.$$

The third Jackson q-Bessel function is defined by the power series

$$J_\nu^{(3)}(z;q) = z^\nu \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{\frac{k(k+1)}{2}}}{(q^{\nu+1};q)_k (q;q)_k} z^{2k}, \tag{3}$$

In the preprint [3] it is shown how this function can be used to construct a theory of Fourier series on q-linear grids.

Jackson´s q-integral in the interval $(0,a)$ is defined as

$$\int_0^a f(t)\, d_qt = (1-q)\, a \sum_{n=0}^{\infty} f(aq^n)\, q^n, \tag{4}$$

and in the interval $(0,\infty)$ as

$$\int_0^\infty f(t)\, d_qt = (1-q) \sum_{n=-\infty}^{\infty} f(q^n)\, q^n. \tag{5}$$

The notation $L_q^p(X)$ will stand for the Banach space induced by the norm

$$\|f\|_{L_q^p(X)} = \left[\int_X |f(t)|^p\, d_qt\right]$$

and in the presence of a weight we will write

$$\|f\|_{L_q^p(X,w(t))} = \left[\int_X |f(t)|^p\, w(t) d_qt\right]$$

Define, after Koornwinder and Swarttouw [14], a q-Hankel transform for functions f in $L_q^1(0,\infty)$:

$$\left(H_q^\nu f\right)(x) = \int_0^\infty (xt)^{\frac{1}{2}}\, J_\nu^{(3)}\left(xt;q^2\right) f(t)\, d_qt. \tag{6}$$

It was shown in [14] that such a q-Hankel transform satisfies the inversion formula

$$f(t) = \int_0^\infty (xt)^{\frac{1}{2}}\, J_\nu^{(3)}\left(xt;q^2\right) \left(H_q^\nu f\right)(x)\, d_qx = \left(H_q^\nu \left(H_q^\nu f\right)\right)(t), \tag{7}$$

where t takes the values $q^k, k = 0, \pm 1, \pm 2, ...$ As a result, it satisfies Parseval identity

$$\|f\|_{L_q^2(0,1)} = \left\|H_q^\nu f\right\|_{L_q^2(0,1)} \tag{8}$$

and provides a Hilbert space isometry between $L_q^2(0,1)$ and the space PW_q^ν.

Setting $A = 1$, $B = 0$ and $M = 1$ in Lemma 1 of [2] we infer that $u(x) = x^{\frac{1}{2}} J_\nu^{(3)}\left(x;q^2\right)$ satisfies

$$\left[\frac{q^{\frac{3}{2}-\nu}}{(1-q^2)} - \frac{(1-q^{\nu-\frac{1}{2}})(1-q^{-\nu-\frac{1}{2}})}{(1-q^2)x^2}\right]^{-1} D_q^2 u(x) = -u(qx)$$

This justifies defining the operator $L_{q,\nu,x}$ by

$$L_{q,\nu,x}f(x) = -\left[\frac{q^{\frac{3}{2}-\nu}}{(1-q^2)} - \frac{(1-q^{\nu-\frac{1}{2}})(1-q^{-\nu-\frac{1}{2}})}{(1-q^2)x^2}\right]^{-1} D_q^2 f(q^{-1}x).$$

Clearly,

$$L_{q,\nu,x}u(xy) = y^2 u(xy).$$

We use x on the subscript to indicate that the q-differences are taken with respect to x. When there is no possible confusion we drop the subscript.

3. A real Paley-Wiener theorem for L^2 functions.

Let $R > 0$ and $L_{q,\nu}^n f$ denote n repeated applications of the operator $L_{q,\nu}$ to f. Define the Paley-Wiener space $PW_{q,R}^\nu$ as

$$PW_{q,R}^\nu = \{f \in C_q^\infty(\mathbf{R}^+) : L_{q,\nu}^n f \in L_q^2(\mathbf{R}^+), n = 0, 1, ... \text{ and } \lim_{n\to\infty} \left\|L_{q,\nu}^n f\right\|^{\frac{1}{2n}} = R\}$$

The main result in this section will depend on the following Lemma..

Lemma 1 Let $x^n F(x) \in L_q^2(\mathbf{0}, \infty)$ for all $n = 0, 1, 2, ...$ Then

$$\lim_{n\to\infty} \left[\int_0^\infty x^{4n} \left|F(x)\right|^2 d_q x\right]^{\frac{1}{4n}} = \sup_{x \in \sup pF} |x| \tag{9}$$

Proof. Proceed exactly as in the proof of Lemma 2 in [21], with $m = 1$, $\lambda = x^2$ and replacing the measure $\int_{-\infty}^{\infty} d\rho_j(\lambda)$ by $\int_0^\infty d_q x$. □

Theorem 1. *The q-Hankel transform is a bijection of $L_q^2(0, R)$ onto $PW_{q,R}^\nu$.*

Proof. Let $R > 0$ and assume that $H_q^\nu(f) \in L_q^2(0, R)$. Then $x^n H_q^\nu(f) \in L_q^2(0, \infty)$ for $n = 0, 1, ...$ A repeated application of the operator $L_{q,\nu,x}$ to the identity (7) gives, if $y \in \mathbf{R}_q$,

$$\begin{aligned} L_{q,\nu,y}^n f(y) &= \int_0^\infty L_{q,\nu,y}^n (xy)^{\frac{1}{2}} J_\nu^{(3)}(xy; q^2) H_q^\nu(f)(x) d_q x \\ &= (-1)^n \int_0^\infty x^{2n} (xy)^{\frac{1}{2}} J_\nu^{(3)}(xy; q^2) H_q^\nu(f)(x) d_q x \\ &= (-1)^n H_q^\nu(x^{2n} H_q^\nu(f)) \end{aligned}$$

using Parseval identity (8) we have

$$\left\|L_{q,\nu}^n f\right\|^2 = \int_0^\infty x^{4n} \left|H_q^\nu(f)(x)\right|^2 d_q x. \tag{10}$$

Applying (9) gives

$$\lim_{n\to\infty}\left\|L_{q,\nu}^{n}f\right\|^{\frac{1}{2n}}=\sup_{x\in\sup pF}|x|=R$$

and $f\in PW_{q,R}^{\nu}$.

Conversely, let $f\in PW_{q,R}^{\nu}$. The definition of $PW_{q,R}^{\nu}$ implies that $(L_{q,\nu})^{n}f\in L_{q}^{2}(0,R)$ and by (10) also $x^{n}H_{q}^{\nu}(f)\in L_{q}^{2}(0,R)$. Using (9) and again (10) gives

$$\sup_{x\in\sup \mathrm{p}H_{q}^{\nu}(f)(x)}|x|=\lim_{n\to\infty}\left[\int_{0}^{\infty}x^{4n}\left|H_{q}^{\nu}(f)(x)\right|^{2}d_{q}x\right]^{\frac{1}{4n}}=\lim_{n\to\infty}\left\|L_{q,\nu}^{n}f\right\|^{\frac{1}{2n}}=R$$

and (9) shows that $H_{q}^{\nu}(f)\in L_{q}^{2}(0,R)$. $\square$

Remark 1. *In particular, H_{q}^{ν} provides a bijection between $L_{q}^{2}(0,1)$ and the space $PW_{q,1}^{\nu}$. In face of (1), this is equivalent to the identity*

$$PW_{q,1}^{\nu}=PW_{q}^{\nu}$$

and we have reached our first goal of finding a description of the space PW_{q}^{ν}. In Theorem 2 of [1] *it is proved that $x^{\nu-u+\frac{1}{2}}J_{u}^{(3)}(x;q^{2})\in PW_{q}^{\nu}$.*

Remark 2. *The proof of the above theorem uses ideas from section 2 of* [21]*, where the authors dealt with general Sturm-Liouville problems and therefore had to deal with many assumptions that are verified automatically in the case of our q-Hankel transform. Many of these assumptions were later removed in* [20] *We remark that the paper* [4] *lays the foundations for a q-analogue Sturm-Liouville theory.*

Remark 3. *Theorem 1 is reminiscent of Theorem 5 in* [6] *and of Theorem 2 in* [16].

4. A real Paley-Wiener space contained in the q-Schwartz space

Denote by l_{q}^{R} the the sequence space on $\mathbf{R}_{q}\cap(0,R)$ (observe that this is the proper q-analogue of the space $C^{\infty}(0,R)$, since any sequence function can be extended to a C^{∞} one).

Denote by $S_{q}(\mathbf{R}_{q})$ the q-Schwartz space, the space of restrictions on $\mathbf{R}_{q}$ of functions such that

$$\sup_{x\in\mathbf{R}_{q};0\leq k\leq n}\left|(1+x^{2})^{m}D_{q}^{k}f(x)\right|<+\infty$$

In this section we will use a q-Bessel function which results after minor changes from $J_\nu^{(3)}$. We will follow exactly the normalization of [10] where the authors derived the basic properties that we are going to list. The preprint [9] also contains a detailed introduction to the concepts we are using. The only difference in our presentation is that we replace "even functions on $\mathbf{R}$" by "functions on $\mathbf{R}^+$", an equivalent class.

The q-Hankel transform h_q^ν is defined, for functions in $L_q^1((0,\infty), x^{2\nu+1})$, as

$$h_q^\nu(f)(y) = \int_0^\infty f(x) j_\nu(xy; q^2) x^{2\nu+1} d_q x$$

where

$$j_\nu(x; q^2) = (1-q^2)^\nu \frac{\Gamma_{q^2}(\alpha+1)}{((1-q)q^{-1}z)^\nu} J_\nu^{(3)}((1-q)q^{-1}z; q^2)$$

This is a q-analogue of the transform considered in [6]. It is shown in Theorem 3 of [9] that h_q^ν is an isomorphism of $S_q(\mathbf{R}_q)$ into itself.

Define the operator

$$\Delta_{q,\nu,x} f(x) = -\frac{D_q\left[x^{2\nu+1} D_q f\right](q^{-1}x)}{x^{2\nu+1}}$$

The functions $j_\nu(x; q^2)$ are eigenvalues of Δ_ν^q with eigenvalues y^2 [10, (43)]

$$\Delta_{q,\nu,x}\left[j_\nu(x; q^2)\right] = y^2 j_\nu(x; q^2)$$

We also have [9, (23)]

$$h_q^\nu(\Delta_{q,\nu,x} f) = \frac{y^2}{q^{2\nu+1}} h_q^\nu(f) \tag{11}$$

For all $x \in \mathbf{R}_q$, we have the growth estimate [10, (48)]

$$\left|j_\nu(x; q^2)\right| \leq \frac{1}{(q; q^2)_\infty^2}. \tag{12}$$

Remark 4. *Some emphasis should be put on the fact that estimate (12) is only valid on the set $\mathbf{R}_q$. Actually, the function $j_\nu(x; q^2)$ is unbounded on the real line, since it is an entire function of order zero. Nevertheless, remains bounded at the grid $\{q^k\}$. Luckily, this is all we are going to need, since the support points of the q-integral are located over $\mathbf{R}_q$.*

Define the real Paley-Wiener space $pw_{q,R}^{\nu}$ as

$$pw_{q,R}^{\nu} = \{f \in S_q(\mathbf{R}_q)\colon \sup_{x\in\mathbf{R}, n\in\mathbf{N}_0} \left(\frac{R}{q}\right)^{-2n} A_{n,N}, q(1+|x|)^{2N} \left|\Delta_{q,\nu,x}^{n} f(x)\right| < \infty\}, \tag{13}$$

where $A_{n,N,q} = \frac{(q^{2n};q^{-2})_N}{q^{2N}(1-q)^{2N}}$. The elements in $pw_{q,R}^{\nu}$ satisfy the growth condition on their q-differences:

$$\sup_{x\in\mathbf{R}, n\in\mathbf{N}_0} \left|\Delta_{q,\nu,x}^{n} f(x)\right| < C \left(\frac{R}{q}\right)^{2n} \frac{1}{A_{n,N,q}} \frac{1}{(1+|x|)^{2N}}.$$

The next theorem is a generalization of Theorem 3 in [6] and the proof follows making the necessary adaptations to deal with the q-setting.

Theorem 2. *The q-Hankel transform h_q^{ν} is a bijection of l_q^R onto $pw_{q,R}^{\nu}$.*

Proof. Let $f \in pw_{q,R}^{\nu}$ and consider $y \in R_q$ outside $[0, R]$. Iterating (11) n times we obtain

$$\begin{aligned} h_q^{\nu}(f)(y) &= \frac{q^{2n(2\nu+1)}}{y^{2n}} h_q^{\nu}([(\Delta_{q,\nu})^n f]) \\ &= \frac{q^{2n(2\nu+1)}}{y^{2n}} \int_0^{\infty} (\Delta_{q,\nu})^n f(x) j_{\nu}(xy;q^2) x^{2\nu+1} d_q x. \end{aligned}$$

Therefore, (if $2N \geq 2\nu+3$), for a positive constant C, we have, using (12) and (13),

$$\begin{aligned} \left|h_q^{\nu}(f)(y)\right| &\leq \frac{q^{2n(2\nu+1)}}{y^{2n}} \frac{1}{(q;q^2)_{\infty}^2} \int_0^{\infty} (\Delta_{q,\nu})^n f(x) x^{2\nu+1} d_q x \\ &\leq C \left(\frac{Rq^{2\nu+1}}{y}\right)^{2n} \frac{(1-q)^{2N}}{((q^{2n};q^{-2})_N (q;q^2)_{\infty}^2} \int_0^{\infty} (1+|x|)^{-2N+2\nu+1} d_q x. \end{aligned}$$

Since $\nu > -\frac{1}{2}$, $|q| < 1$ and $R < y$, this last quantity clearly approaches zero as $n \to \infty$. It follows that supp $h_q^{\nu}(f) \subset [0, R]$. Conversely let $f \in C_q^{\infty}(0, \infty)$.

Fix $N \in N_0$. Then, for $n \in N_0$,

$$
\begin{aligned}
y^{2N}\Delta^n_{q,\nu,y} h^\nu_q(f)(y) &= \int_0^\infty f(x) y^{2N}\Delta^n_{q,\nu,y} j_\nu(xy;q^2)x^{2\nu+1}d_qx \qquad (14)\\
&= \int_0^\infty x^{2n} f(x) y^{2N} j_\nu(xy;q^2)x^{2\nu+1}d_qx\\
&= \int_0^\infty x^{2n} f(x) \Delta^N_{q,\nu,x} j_\nu(xy;q^2)x^{2\nu+1}d_qx\\
&= \int_0^\infty \Delta^N_{q,\nu,x}\left[x^{2n} f(x)\right] j_\nu(xy;q^2)x^{2\nu+1}d_qx. \qquad (15)
\end{aligned}
$$

It remains to estimate $\Delta^N_{q,\nu,x}\left[x^{2n}f(x)\right]$. A calculation gives

$$
\begin{aligned}
\Delta_{q,\nu,x}\left[x^{2n}f(x)\right] &= \left(\frac{x}{q}\right)^{2n-2} \frac{(1-q^{2n})(1-q^{2n-1})}{(1-q)^2} \{ \left(\frac{1-q^{2\nu+1}}{1-q^{2n-1}} + q^{2\nu+1}\right) f(x)\\
&+ \left(\frac{1-q^{2\nu+1}}{1-q^{2n-1}}\frac{1-q}{1-q^{2n}}q^{2n-1} + \frac{1-q^2}{1-q^{2n-1}}q^{2\nu+2n-1}\right) xD_qf(x)\\
&+ \frac{(1-q)^2}{(1-q^{2n})(1-q^{2n-1})} q^{2\nu+4n-1}x^2D_q^2f(x)\}.
\end{aligned}
$$

Taking into account that for nonnegative n holds $\frac{1-q}{1-q^{2n}} < 1$, iteration of the above calculation gives, if $n > N$,

$$
\Delta^N_{q,\nu,x}\left[x^{2n}f(x)\right] = \left(\frac{x}{q}\right)^{2n-2N} \frac{(q^{2n};q^{-2})_N}{(1-q)^{2N}} f_N(x),
$$

where f_N is a function such that $\mathrm{supp} f_N \subset \mathrm{supp} f$, and

$$
\|f_N\|_\infty \leq C\sum_{k=0}^{2N} \left\|D^k_q f\right\|_\infty,
$$

with C a constant depending in ν and R but not on n. We thus get

$$
\left|\Delta^N_{q,\nu,x}\left[x^{2n}f(x)\right]\right| \leq C\left(\frac{x}{q}\right)^{2n-2N} \frac{(q^{2n};q^{-2})_N}{(1-q)^{2N}} \sum_{k=0}^{2N}\left\|\frac{d^k}{dx^k}f\right\|_\infty \qquad (16)
$$

Now, a short calculation using the definition of the q-integral (4) gives

$$
\int_0^R x^{2\nu+1}d_qx = \frac{1-q}{1-q^{2\nu+2}}R^{2\nu+2}. \qquad (17)
$$

Inserting estimate (16) on (14)-(15) gives, using (17) and (12),

$$\left|y^{2N}\Delta_{q,\nu,y}^{n}h_q^{\nu}(f)(y)\right| \leq \widetilde{C}\left(\frac{1}{q}\right)^{2n-2N} R^{2n-2N+2\nu+2}\frac{(q^{2n};q^{-2})_N}{(1-q)^{2N-1}}\sum_{k=0}^{2N}\left\|\frac{d^k}{dx^k}f\right\|_{\infty},$$

where $\widetilde{C}$ is another constant depending in ν and R but not on n. This shows that $h_q^{\nu}(f) \in pw_{q,R}^{\nu}$. $\square$

Remark 5. *In section 3.2 of* [10] *it is shown that*

$$j_{\nu+p}(x;q^2) = \int_0^1 t^{2\nu+1}W_{p-1}(t;q^2)j_{\nu}(xt;q^2)d_qt$$

where $W_{p-1}(t;q^2)$ is a smooth function. As a result, $j_{\nu+p}(x;q^2) \in pw_{q,1}^{\nu}$ and satisfies

$$\sup_{x\in\mathbf{R},n\in\mathbf{N}_0}\left|\Delta_{q,\nu,x}^{n}[j_{\nu+p}(x;q^2)]\right| < C\left(\frac{1}{q}\right)^{2n}\frac{1}{A_{n,N,q}}\frac{1}{(1+|x|)^{2N}}.$$

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