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DISCRETELY COMPACT IMBEDDINGS IN SPACES OF CELL-CENTERED GRID FUNCTIONS

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Abstract: Compactness of imbeddings in discrete counterparts of Sobolev spaces is considered. We study the imbeddings in spaces of cell-centered grid functions, in one and two dimensional domains. No restrictions are made on the mesh-ratios of the underlying meshes.

Keywords: compactness of imbeddings, cell-centered grid functions.

1. Introduction

Results in compactness of imbeddings in spaces of grid functions can play a main role in the study of stability and convergence of finite difference schemes. In particular, they are important technical tools in order to establish supraconvergence results for schemes in non-uniform meshes (see e.g. [3]-[6] and [8]). In this paper we consider spaces of cell-centered grid functions. We prove discrete compactness of imbeddings in discrete versions of the Sobolev spaces $W_0^{m,p}$, $m = 1, 2$, $1 \leq p \leq \infty$, in one-dimensional domains. In two-dimensional domains we prove a similar result for the particular case $m = 1$ and $p = 2$. Grigorieff gives, in [7], correspondent results for spaces of vertex-centered grid functions for the one-dimensional case. In the case of non-uniform grids, the normed spaces which we consider in this paper do not coincide to those defined in [7] and an different kinds of proofs are needed.

2. Discrete approximation of $W_0^{m,p}(0, R)$

In this section, we start to introduce the discrete Sobolev spaces $W_0^{m,p}(0, R)$ as vector spaces of grid functions. We define the partition in the domain

$$G_h := \{0 = x_0 < x_1 < \cdots < x_N = R\}.$$

The set of the cell-centers is given by

$$S_h := \{x_{1/2}, x_{3/2}, \ldots, x_{N-1/2}\},$$

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where

$$x_{j-1/2} := \frac{x_{j-1}+x_j}{2}, \quad j = 1,\ldots,N.$$

For the grid functions v_h and w_h defined on $\bar{S}_h := S_h \cup \{x_0, x_N\}$ and G_h, respectively, the centered difference quotients are given by

$$(\delta v_h)_j := \frac{v_{j+1/2}-v_{j-1/2}}{h_{j-1/2}}, \quad j = 0,\ldots,N,$$

and

$$(\delta w_h)_{j-1/2} := \frac{w_j - w_{j-1}}{h_{j-1}}, \quad j = 1,\ldots,N,$$

where $x_{-1/2} := x_0$, $x_{N+1/2} := x_N$ and

$$h_{j-1/2} := x_{j+1/2} - x_{j-1/2}, \quad j = 0,\ldots,N,$$
$$h_{j-1} := x_j - x_{j-1}, \quad j = 1,\ldots,N.$$

We also consider $x_{-1} := x_0$, $x_{N+1} := x_N$, $h_{-1} := h_N := 0$. Let Λ be a sequence of mesh sizes $h = (h_0,\ldots,h_{N-1})$ such that

$$h_{max} := \max\{h_{j-1}, j = 1,\ldots,N\}$$

converges to zero.

Let $\overset{\circ}{W}{}_h^{m,p}$, $m = 0,1,2$, $p \in [1,\infty[$, be the space of grid functions on $\bar{S}_h$, which are zero on 0 and R, equipped with the norm

$$\|v_h\|_{W_h^{m,p}} := \left(\sum_{\ell=0}^{m} |v_h|^p_{W_h^{\ell,p}}\right)^{1/p},$$

where

$$|v_h|^p_{W_h^{0,p}} := \sum_{j=1}^{N} h_{j-1}|v_{j-1/2}|^p,$$
$$|v_h|^p_{W_h^{1,p}} := \sum_{j=0}^{N} h_{j-1/2}|\left(\delta v_h\right)_j|^p,$$
$$|v_h|^p_{W_h^{2,p}} := \sum_{j=1}^{N} h_{j-1}|\left(\delta^2 v_h\right)_{j-1/2}|^p.$$

If $p = \infty$ we consider the norm

$$\|v_h\|_{W_h^{m,\infty}} := \max_{0\le\ell\le m} |v_h|_{W_h^{\ell,\infty}},$$

where

$$\begin{aligned}
|v_h|_{W_h^{0,\infty}} &:= \max_{1\le j\le N} |(v_h)_{j-1/2}|, \\
|v_h|_{W_h^{1,\infty}} &:= \max_{0\le j\le N} |\,(\delta v_h)_j\,|, \\
|v_h|_{W_h^{2,\infty}} &:= \max_{1\le j\le N} |\,\left(\delta^2 v_h\right)_{j-1/2}|
\end{aligned}$$

and we denote the corresponding space by $\overset{\circ}{W}{}_h^{m,\infty}$. In analogy to the usual notation for Sobolev spaces, we use $\overset{\circ}{L}{}_h^p$ for $\overset{\circ}{W}{}_h^{0,p}$ and $\|.\|_{L_h^p}$ for the respective norm.

Let R_h be the operator that defines a restriction to S_h.

The discrete spaces introduced above form discrete approximations to $W_0^{m,p}(0,R)$ and $C_0^m[0,R]$ which we denote by $(W_0^{m,p}(0,R), \Pi\,\overset{\circ}{W}{}_h^{m,p})$ and $(C_0^m[0,R], \Pi\,\overset{\circ}{W}{}_h^{m,\infty})$, respectively. These approximations are considered in the following sense ([9], [10]): a sequence $(v_h)_{h\in\Lambda}$ is said to converge discretely in $(W_0^{m,p}(0,R), \Pi\,\overset{\circ}{W}{}_h^{m,p})$ to an element $v\in W_0^{m,p}(0,R)$,

$$v_h \to v \text{ in } (W_0^{m,p}(0,R), \Pi\,\overset{\circ}{W}{}_h^{m,p}) \quad (h\in\Lambda),$$

if for each $\epsilon>0$ there exists $\varphi\in C_0^\infty[0,R]$ such that

$$\|v-\varphi\|_{W^{m,p}(0,R)} \le \epsilon, \qquad \limsup\{\|v_h - R_h\varphi\|_{W_h^{m,p}}, h\in\Lambda\} \le \epsilon;$$

it is said to converge discretely in $(C_0^m[0,R], \Pi\,\overset{\circ}{W}{}_h^{m,\infty})$ to an element $v\in C_0^m[0,R]$,

$$v_h \to v \text{ in } (C_0^m[0,R], \Pi\,\overset{\circ}{W}{}_h^{m,\infty}) \quad (h\in\Lambda),$$

if

$$\|v_h - R_h v\|_{W_h^{m,\infty}} \to 0 \quad (h\in\Lambda).$$

We consider compact imbeddings of $\overset{\circ}{W}{}_h^{m,p}$, which have corresponding Sobolev imbedding results given by the Rellich-Kondrachov theorem (see e.g. [1]).

Theorem 1. *Let $m\in\{1,2\}$. The sequence of imbeddings*

$$J_h : \overset{\circ}{W}{}_h^{m,p} \to \overset{\circ}{W}{}_h^{m-1,q}, \quad h\in\Lambda,$$

for $1\le p,q\le\infty$, with $q<\infty$ if $p=1$, is discretely compact.

Proof: We first consider the imbedding for the case $m = 1$

$$J_h : \overset{\circ}{W}{}_h^{1,p} \to \overset{\circ}{L}{}_h^q, \quad h \in \Lambda.$$

Let $(v_h)_\Lambda \in \Pi \overset{\circ}{W}{}_h^{1,p}$ be a bounded sequence. Then the sequence $(w_h)_\Lambda \in \Pi W_0^{1,p}(0,R)$, which is linear in each interval $[x_{j-1/2}, x_{j+1/2}]$, $j = 0, \ldots, N$, satisfying

$$w_h(x_{j-1/2}) := v_h(x_{j-1/2}), \quad j = 0, \ldots, N+1,$$

is bounded in $\Pi W_0^{1,p}(0,R)$. In fact, if $p = \infty$ then

$$\|w_h\|_{W^{1,p}(0,R)} = \|v_h\|_{W_h^{1,p}}.$$

When $1 \leq p < \infty$ then by Friedrich's inequality

$$\|w_h\|_{W^{1,p}(0,R)} \leq C|w_h|_{W^{1,p}(0,R)}$$

and

$$|w_h|_{W^{1,p}(0,R)} = \Big(\int_0^R |w'(x)|^p \, dx\Big)^{1/p} = \Big(\sum_{j=0}^N h_{j-1/2} |(\delta v_h)_j|^p\Big)^{1/p} = |v_h|_{W_h^{1,p}(0,R)}.$$

We note that the imbedding $W^{1,p}(0,R) \to C[0,R]$, $1 < p < \infty$, is compact and consequently we can find is a subsequence $\Lambda' \subseteq \Lambda$ and a function $w \in C_0[0,R]$ such that

$$\max_{x\in[0,R]} |w_h(x) - w(x)| \to 0 \quad (h \in \Lambda').$$

Hence,

$$v_h \to w \text{ in } (C_0[0,R], \Pi \overset{\circ}{L}{}_h^\infty) \quad (h \in \Lambda').$$

Since for $1 \leq q < \infty$

$$\|v_h - R_h w\|_{L_h^q} \leq C\|v_h - R_h w\|_{L_h^\infty},$$

with $C = R^{1/q}$, we conclude the convergence

$$v_h \to w \text{ in } (L^q(0,R), \Pi \overset{\circ}{L}{}_h^q) \quad (h \in \Lambda').$$

Let us now consider $p = 1$. The imbedding $W^{1,1}(0,R) \to L^q(0,R)$ is compact and subsequently

$$w_h \to w \text{ in } L^q(0,R) \quad (h \in \Lambda')$$

for some subsequence $\Lambda' \subseteq \Lambda$ and $w \in L^q(0,R)$. We are now going to prove that $v_h \to w$ in $(L^q(0,R), \Pi \overset{\circ}{L}{}_h^q)$ $(h \in \Lambda')$. For each $\epsilon > 0$ and $c > 0$ it is possible to find $\varphi \in C_0^\infty[0,R]$ such that

$$\|w - \varphi\|_{L^q(0,R)} \leq \frac{\epsilon}{c}.$$

Let us consider the function ψ_h, which is linear in each interval $[x_{j-1/2}, x_{j+1/2}]$, $j = 0, \ldots, N$, and satisfies

$$\psi_h(x) := \varphi(x_{j-1/2}), \quad j = 0, \ldots, N+1.$$

There exists $c_q > 0$ such that

$$c_q^q \|v_h - R_h\varphi\|_{L_h^q}^q \leq \int_0^R |w_h - \psi_h|^q \, dx.$$

Since $\psi_h \to \varphi$ in $L^q(0,R)$ then

$$\int_0^R |w_h - \psi_h|^q \, dx \to \int_0^R |w - \varphi|^q \, dx \quad (h \in \Lambda').$$

Consequently, taking $c = c_q$, holds

$$\limsup\{\|v_h - R_h\varphi\|_{L_h^q}, h \in \Lambda'\} \leq \epsilon.$$

This concludes the first part of the proof, i.e., the sequence $J_h : \overset{\circ}{W}{}_h^{1,p} \to \overset{\circ}{L}{}_h^q$, $h \in \Lambda$ is discretely compact.

We consider now the sequence of imbeddings $(J_h)_\Lambda$, $J_h : \overset{\circ}{W}{}_h^{2,p} \to \overset{\circ}{W}{}_h^{1,q}$. Let $(v_h)_\Lambda \in \Pi \overset{\circ}{W}{}_h^{2,p}$ be bounded. The sequence $(w_h)_\Lambda$, where w_h is defined by

$$w_h(x_j) := (\delta v_h)_j, \quad j = 0, \ldots, N,$$

linear in each interval $[x_j, x_{j+1}]$, $j = 0, \ldots, N-1$, is bounded in $\Pi W^{1,p}(0,R)$. For $p > 1$ then Rellich-Kondrachov Theorem gives the existence of $\Lambda' \subset \Lambda$ and $w_1 \in C[0,R]$ such that

$$\max_{0 \leq j \leq N} |(\delta v_h)_j - w_1(x_j)| \to 0 \quad (h \in \Lambda').$$

Let $w_0(x) := \int_0^x w_1(t)\,dt$. We have, taking $v_0 = w_0 = 0$ into account,

$$
\begin{aligned}
|v_{j-1/2} - w_0(x_{j-1/2})| &\leq \sum_{i=0}^{N} \big|(\delta v)_i - w_1(x_i)\big| + \sum_{i=0}^{N} |w_1(x_i) - (\delta w_0)_i| \\
&\leq R \max_{0\leq i\leq N} \big|(\delta v)_i - w_1(x_i)\big| + R \max_{0\leq i\leq N} |w_0'(x_i) - (\delta w_0)_i|,
\end{aligned}
$$

$j = 0, \ldots, N+1$. Hence the convergence

$$
\max_{0\leq j\leq N+1} |v_{j-1/2} - w_0(x_{j-1/2})| \to 0 \quad (h \in \Lambda''),
$$

follows and we conclude that

$$
v_h \to w_0 \quad (h \in \Lambda') \text{ in } (C_0^1[0,R], \Pi \overset{\circ}{W}{}_h^{1,\infty}).
$$

For the case $p = 1$ the proof is analogous.

■

The next lemma is helpful in the proof of the compactness imbedding theorem for the two-dimensional case.

Lemma 1. *Let $(v_h)_\Lambda \in \Pi \overset{\circ}{W}{}_h^{1,p}$ be a bounded sequence, with $1 \leq p < \infty$. For any $\tau \in \mathbb{R}$, the step function defined by*

$$
w_h(x) := v_h(x_{j-1/2}),\ x \in]x_{j-1}, x_j], \quad j = 1, \ldots, N,
$$

and zero outside of these intervals, satisfies

$$
\int_I |w_h(x+\tau) - w_h(x)|^p\, dx \leq 3(|\tau| + h_{max})^p |v_h|^p_{W_h^{1,p}}, \tag{1}
$$

where I is any interval containing (x_0, x_N).

Proof: Let $\tau > 0$. Then

$$
\begin{aligned}
\int_I |w_h(x+\tau) - w_h(x)|^p\, dx \ \leq\ & \int_0^{R-\tau} |w_h(x+\tau) - w_h(x)|^p\, dx \\
& + \int_{-\tau}^{0} |w_h(x+\tau)|^p\, dx + \int_{R-\tau}^{R} |w_h(x)|^p\, dx.
\end{aligned}
$$

For f defined by

$$
f(x) := j, \quad x \in]x_{j-1}, x_j],
$$

we have, using Hölder's inequality,

$$\begin{aligned}
&\int_0^{R-\tau} |w_h(x+\tau) - w_h(x)|^p \, dx \\
&\quad \leq \int_0^{R-\tau} \Big(\sum_{k=f(x)}^{f(x+\tau)-1} \big| w_h(x_{k+1/2}) - w_h(x_{k-1/2}) \big| \Big)^p \, dx \\
&\quad \leq \int_0^{R-\tau} \Big(\sum_{k=f(x)}^{f(x+\tau)-1} h_{k-1/2} \Big)^{p-1} \sum_{k=f(x)}^{f(x+\tau)-1} h_{k-1/2} \left| (\delta v_h)_k \right|^p \, dx.
\end{aligned}$$

Since $\displaystyle\sum_{k=f(x)}^{f(x+\tau)-1} h_{k-1/2} \leq \tau + h_{max}$ we obtain

$$\begin{aligned}
&\int_0^{R-\tau} |w_h(x+\tau) - w_h(x)|^p \, dx \\
&\quad \leq (\tau + h_{max})^{p-1} \sum_{j=0}^{N_{R-\tau}} \Big(h_{j-1} \sum_{k=j}^{f(x_j+\tau)-1} h_{k-1/2} \left| (\delta v_h)_k \right|^p \Big) \\
&\quad \leq (\tau + h_{max})^{p-1} \sum_{k=0}^{N} \Big(h_{k-1/2} \left| (\delta v_h)_k \right|^p \sum_{j=s(k)}^{k} h_{j-1} \Big),
\end{aligned}$$

where $N_{R-\tau}$ and $s(k)$, are the biggest integer and the smallest integers, respectively, such that $\displaystyle\sum_{i=1}^{N_{R-\tau}} h_{i-1} \leq R - \tau$ and $f(x_{s(k)} + \tau) - 1 \geq k$. From

$$x_k - x_{s(k)} < \tau \qquad \sum_{j=s(k)}^{k} h_{j-1} < \tau + h_{max},$$

we conclude that

$$\int_0^{R-\tau} |w_h(x+\tau) - w_h(x)|^p \, dx \leq (\tau + h_{max})^p |v_h|^p_{W_h^{1,p}}. \tag{2}$$

On the other hand,

$$\int_{-\tau}^{0} |w_h(x+\tau)|^p \, dx = \int_0^{\tau} |w_h(x)|^p \, dx \leq \sum_{j=1}^{N_\tau} h_{j-1} \left| w_h(x_{j-1/2}) \right|^p,$$

with N_τ the smallest integer such that $\sum_{i=1}^{N_\tau} h_{i-1} \geq \tau$, and then

$$\begin{aligned}\int_{-\tau}^{0} |w_h(x+\tau)|^p \, dx &\leq \sum_{j=1}^{N_\tau} h_{j-1}\Big(\sum_{k=0}^{j-1} |v_h(x_{k+1/2}) - v_h(x_{k-1/2})|\Big)^p \\ &= \sum_{j=1}^{N_\tau} h_{j-1}\Big(\sum_{k=0}^{j-1} h_{k-1/2}|\big(\delta v_h\big)_k|\Big)^p.\end{aligned}$$

Since $\sum_{j=1}^{N_\tau} h_{j-1} \leq \tau + h_{max}$ and for $j \leq N_\tau$, $\sum_{k=0}^{j-1} h_{k-1/2} \leq \tau + h_{max}$, it follows by an application of Hölder's inequality

$$\int_{-\tau}^{0} |w_h(x+\tau)|^p \, dx \leq (\tau + h_{max})^p |v_h|^p_{W_h^{1,p}}. \tag{3}$$

In the same way as before, we have

$$\int_{R-\tau}^{R} |w_h(x)|^p \, dx \leq \sum_{j=N_{R-\tau}}^{N} h_{j-1}\Big(\sum_{k=j}^{N} h_{k-1/2}|\big(\delta v_h\big)_k|\Big)^p$$

and consequently

$$\int_{R-\tau}^{R} |w_h(x)|^p \leq (\tau + h_{max})^p |v_h|^p_{W_h^{1,p}}. \tag{4}$$

From (2), (3) and (4) we obtain (1).

The case $\tau < 0$ can be proved analogously.

■

3. Discrete approximation of $L^2(\Omega)$ and $W_0^{1,2}(\Omega)$, $\Omega \subset \mathbb{R}^2$

We now need norms for functions on two-dimensional grids. To this end we introduce discrete versions of the Sobolev spaces $W_0^{m,2}(\Omega)$, $m = 0, 1$, where Ω is an union of rectangles.

Let us first introduce the nonuniform grid G_H. In a rectangle $R = (x_{-1}, x_{N+1}) \times (y_{-1}, y_{M+1})$ which contains Ω we define the subset $G_H := R_1 \times R_2$, where

$$R_1 := \{x_{-1} < x_0 < \ldots < x_N < x_{N+1}\}$$

and

$$R_2 := \{y_{-1} < y_0 < \ldots < y_M < y_{M+1}\}.$$

The grid G_H is assumed to satisfy the following condition: The grid G_H is assumed to satisfy the following condition: the vertices of Ω are in the centers of the rectangles formed by G_H. If the case of a rectangular domain we allow $x_{-1} = x_0$, $x_{N+1} = x_N$, $y_{-1} = y_0$ and $y_{M+1} = y_M$.

Let

$$S_H := \{(x_{j-1/2}, y_{\ell-1/2}) : j = 0, \ldots, N+1,\ \ell = 0, \ldots, M+1\},$$

where $x_{j-1/2} := (x_{j-1} + x_j)/2$, $y_{\ell-1/2} := (y_{\ell-1} + y_\ell)/2$, and $\Omega_H := S_H \cap \Omega$, $\partial\Omega_H := S_H \cap \partial\Omega$, $\bar{\Omega}_H := \Omega_H \cup \partial\Omega_H$.

In the definition of the discrete norms we use the following centered divided differences in x-direction

$$(\delta_x v_H)_{j,\ell+1/2} := \frac{v_{j+1/2,\ell+1/2} - v_{j-1/2,\ell+1/2}}{h_{j-1/2}},$$

$$(\delta_x w_H)_{j-1/2,\ell+1/2} := \frac{w_{j,\ell+1/2} - w_{j-1,\ell+1/2}}{h_{j-1}},$$

where $h_{j-1/2} := x_{j+1/2} - x_{j-1/2}$, $h_{j-1} := x_j - x_{j-1}$. Correspondingly, the finite central difference with respect to the variable y are defined, with the mesh size vector k in place of h.

We denote by $\overset{\circ}{W}{}_H^{m,2}(R)$, $m = 0, 1$, the space of grid functions defined in C_H, that are zero on the set

$$\{(x_{j-1/2}, y_{\ell-1/2}) : j = 0, N+1, \ell = 0, \ldots, M+1 \vee j = 1, \ldots, N,\ \ell = 0, M+1\},$$

and equipped with the norm

$$\|v_H\|_{W_H^{m,2}(R)} := \Big(\sum_{r=0}^{m} |v_H|_{r,H}^2\Big)^{1/2},$$

where

$$
\begin{aligned}
|v_H|^2_{0,H} &:= \sum_{j=1}^{N}\sum_{\ell=1}^{M} h_{j-1}k_{\ell-1}|v_{j-1/2,\ell-1/2}|^2,\\
|v_H|^2_{1,H} &:= \sum_{j=0}^{N}\sum_{\ell=1}^{M} h_{j-1/2}k_{\ell-1}|\left(\delta_x v_H\right)_{j,\ell-1/2}|^2\\
&\quad +\sum_{j=1}^{N}\sum_{\ell=0}^{M} h_{j-1}k_{\ell-1/2}|\left(\delta_y v_H\right)_{j-1/2,\ell}|^2.
\end{aligned}
$$

Let P_{S_H} be the following operator that extends a grid function v_H in $\bar{\Omega}_H$ to S_H,

$$
P_{S_H}v_H := v_H \quad \text{in} \quad \bar{\Omega}_H, \quad P_{S_H}v_H := 0 \quad \text{in} \quad S_H\backslash\bar{\Omega}_H.
$$

We denote by $\overset{\circ}{W}{}_H^{m,2}(\Omega)$, $m=0,1$, the space of functions defined in $\bar{\Omega}_H$, that are zero on $\partial\Omega_H$, equipped with the norm

$$
\|v_H\|_{m,H} := \left(\sum_{r=0}^{m}|P_{S_H}v_H|^2_{r,H}\right)^{1/2}, \quad m=0,1.
$$

The space $\overset{\circ}{W}{}_H^{0,2}(\Omega)$ (also denoted by $\overset{\circ}{L}{}_H^2(\Omega)$) is endowed by the inner product

$$
(v_H,w_H)_H := \sum_{j=1}^{N}\sum_{\ell=1}^{M} h_{j-1}k_{\ell-1}(P_{S_H}v_H)_{j-1/2,\ell-1/2}(P_{S_H}\bar{w}_H)_{j-1/2,\ell-1/2}.
$$

When it is clear from the context that we use the extended function, we omit the notation P_{S_H}.

Let R_H be the operator that define the restriction to $\bar{\Omega}_H$.

The discrete spaces introduced above form discrete approximations of their continuous counterparts in the sense that we explain in what follows. Let Λ be a sequence of positive vectors $H=(h,k)$ of step-sizes such that the maximum step-size H_{max} converges to zero. A sequence $(v_H)_\Lambda \in \Pi\,\overset{\circ}{L}{}_H^2(\Omega)$ converges discretely to $v \in L^2(\Omega)$ in $(L^2(\Omega),\Pi\,\overset{\circ}{L}{}_H^2(\Omega))$, $v_H \to v$ in $(L^2(\Omega),\Pi\,\overset{\circ}{L}{}_H^2(\Omega))$ $(H\in\Lambda)$, if for each $\epsilon>0$ there exists $\varphi \in C_0^\infty(\Omega)$ such that

$$
\|v-\varphi\|_{L^2(\Omega)} \le \epsilon, \quad \lim_{H_{max}\to 0}\sup\{\|v_H - R_H\varphi\|_{0,H}\} \le \epsilon.
$$

A sequence $(v_H)_\Lambda \in \Pi \overset{\circ}{W}{}_H^{1,2}(\Omega)$ converges discretely to $v \in W_0^{1,2}(\Omega)$ in $(W_0^{1,2}(\Omega), \Pi \overset{\circ}{W}{}_H^{1,2}(\Omega))$, $v_H \to v$ in $(W_0^{1,2}(\Omega), \Pi \overset{\circ}{W}{}_H^{1,2}(\Omega))$ $(H \in \Lambda)$, if for each $\epsilon > 0$ there exists $\varphi \in C_0^\infty(\Omega)$ such that

$$\|v - \varphi\|_{W^{1,2}(\Omega)} \leq \epsilon, \qquad \lim_{H_{max} \to 0} \sup\{\|v_H - R_H \varphi\|_{1,H}\} \leq \epsilon.$$

A sequence $(v_H)_\Lambda$ converges weakly to v in $(L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2(\Omega))$, $v_H \rightharpoonup v$ in $(L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2(\Omega))$ $(H \in \Lambda)$, if

$$(w_H, v_H)_H \to (w, v)_0 \quad (H \in \Lambda)$$

for all $w \in L^2(\Omega)$ and $(w_H)_\Lambda \in \Pi \overset{\circ}{L}{}_H^2(\Omega)$ such that $w_H \to w$ in $(L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2(\Omega))$.

The following theorem was proved by Stummel in [9].

Theorem 2. *Let $(v_H)_\Lambda$ be a bounded sequence in $\Pi \overset{\circ}{L}{}_H^2(\Omega)$. Then, there exists a subsequence Λ' of Λ and $v \in L^2(\Omega)$, such that*

$$v_H \rightharpoonup v \quad in \quad (L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2(\Omega)) \quad (H \in \Lambda').$$

The discrete compactness result in the one-dimensional case was obtained using a correspondent result in the continuous case. Functions defined in all the domain which coincide with grid functions in the grid points were considered. The proof is based in the fact that this continuous functions are bounded if the correspondent grid functions have that property. In the two-dimensional case we could not find such continuous prolongations of grid functions.

The prove of the discrete compactness result that we present in the following is based in the Kolmogorov compactness theorem ([1], [11]) and uses the next lemma.

Lemma 2. *Let* $(v_H)_\Lambda \in \Pi \overset{\circ}{W}{}_H^{1,2}(\Omega)$ *be a bounded sequence. Let us consider the step function* w_H *defined by*

$$w_H(x,y) := v_{j+1/2,\ell+1/2}, \quad (x,y) \in (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1}) \subset \Omega,$$

and zero on $\mathbb{R}^2 \backslash \Omega$. *Let* Q *be a set containing* Ω. *Then, for all* $\tau = (\tau_1, \tau_2) \in \mathbb{R}^2$ *the following estimate holds*

$$\int_Q |w_H(x+\tau_1, y+\tau_2) - w_H(x,y)|^2 \, dxdy \leq 6(|\tau_1| + |\tau_2| + h_{max} + k_{max})^2 |v_H|^2_{1,H}. \tag{5}$$

Proof: For $\tau = (\tau_1, \tau_2) \in \mathbb{R}^2$ holds

$$\begin{aligned}
&\int_Q |w_H(x+\tau_1, y+\tau_2) - w_H(x,y)|^2 \, dxdy \\
&\leq \; 2\int_Q |w_H(x+\tau_1, y+\tau_2) - w_H(x, y+\tau_2)|^2 \, dxdy \\
&\quad +2\int_Q |w_H(x, y+\tau_2) - w_H(x,y)|^2 \, dxdy.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_Q |w_H(x+\tau_1, y+\tau_2) - w_H(x, y+\tau_2)|^2 \, dxdy \\
&\leq \; \sum_{\ell=1}^{M} k_{\ell-1} \int_{x_0-\tau_1}^{x_{N+\tau_1}} |w_H(x+\tau_1, y_{\ell-1/2}) - w_H(x, y_{\ell-1/2})|^2 \, dx,
\end{aligned}$$

then from Lemma 1, we obtain

$$\begin{aligned}
&\int_Q |w_H(x+\tau_1, y+\tau_2) - w_H(x, y+\tau_2)|^2 \, dxdy \\
&\leq \; 3(|\tau_1| + h_{max})^2 \sum_{\ell=1}^{M} k_{\ell-1} \sum_{j=0}^{N} h_{j-1/2} \left| \big(\delta_x(P_{C_H} v_H)\big)_{j,\ell-1/2} \right|^2 \\
&\leq \; 3(|\tau_1| + h_{max})^2 |v_H|^2_{1,H}.
\end{aligned}$$

Analogously,

$$\int_Q |w_H(x, y+\tau_2) - w_H(x,y)|^2 \, dxdy \leq 3(|\tau_2| + k_{max})^2 |v_H|^2_{1,H}.$$

We conclude that

$$\int_Q |w_H(x+\tau_1, y+\tau_2) - w_H(x,y)|^2\, dxdy$$
$$\leq\; 6\Big[(|\tau_1| + h_{max})^2 + (|\tau_2| + k_{max})^2\Big]|v_H|^2_{1,H}.$$

■

Theorem 3. *The sequence of imbeddings* $(J_H)_\Lambda$*,*

$$J_H : \overset{\circ}{W}{}^{1,2}_H(\Omega) \to \overset{\circ}{L}{}^2_H(\Omega) \quad (H \in \Lambda), \tag{6}$$

is discretely compact.

Proof: Let $(v_H)_\Lambda \in \Pi\, \overset{\circ}{W}{}^{1,2}_H(\Omega)$ be a bounded sequence. There exists M independent of H such that

$$\|v_H\|_{1,H} \leq M.$$

For $(w_H)_\Lambda$ from the previous lemma holds

$$\int_\Omega |w_H(x+\eta_1, y+\eta_2) - w_H(x,y)|^2\, dxdy \leq 6(|\tau_1| + |\tau_2| + h_{max} + k_{max})^2 M^2.$$

Since

$$\|w_H\|_{L^2(\Omega)} = |v_H|_{0,H} \leq M,$$

then $(w_H)_\Lambda$ is uniformly bounded in $\Pi L^2(\Omega)$. Using the Kolmogorov compactness theorem, we conclude that the sequence $(w_H)_\Lambda$ is relatively compact in $L^2(\Omega)$. There exists a sequence $\Lambda' \subseteq \Lambda$ and $w \in L^2(\Omega)$ such that

$$w_H \to w \text{ in } L^2(\Omega) \quad (H \in \Lambda').$$

In order to conclude the proof we need to prove that

$$v_H \to w \text{ in } (L^2(\Omega), \Pi\, \overset{\circ}{L}{}^2_H) \quad (H \in \Lambda').$$

Let $\epsilon > 0$. There exists $\varphi \in C_0^\infty(\Omega)$ such that

$$\|w - \varphi\|_{L^2(\Omega)} \leq \epsilon.$$

For the step function ψ_H defined by

$$\psi_H(x,y) := \varphi(x_{j+1/2}, y_{\ell+1/2}), \quad (x,y) \in (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1}) \subset \Omega,$$

null otherwise, we have

$$\|\psi_H - \varphi\|_{L^2(\Omega)} \to 0 \quad (H \in \Lambda)$$

and then

$$\|v_H - R_H\varphi\|_{0,H} = \|w_H - \psi_H\|_{L^2(\Omega)} \to \|w - \varphi\|_{L^2(\Omega)}.$$

■

The next lemma gives some more information about the imbeddings considered in the Theorem 3. Correspondent results for spaces of continuous functions are well known (see e.g. [2, Theorem 3.12]).

Lemma 3. *If* $(v_H)_\Lambda \in \Pi \overset{\circ}{W}{}_H^{1,2}(\Omega)$ *is bounded and weakly convergent to* v *in* $(L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2(\Omega))$ *then* $v \in W_0^{1,2}(\Omega)$.

Proof: Let $(v_H)_\Lambda$ be a bounded sequence in $\Pi \overset{\circ}{W}{}_H^{1,2}(\Omega)$ such that

$$v_H \rightharpoonup v \text{ in } (L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2(\Omega)) \quad (H \in \Lambda). \tag{7}$$

We consider $(w_H)_\Lambda$ from Lemma 2. From the proof of the last theorem, we know that $(w_H)_\Lambda$ converges to $w \in L^2(\Omega)$, for some $\Lambda' \subseteq \Lambda$. Let us consider the sequence $(\tilde{w}_H)_{\Lambda'}$ defined by

$$\tilde{w}_H := w_H \quad \text{in} \quad \Omega, \quad \tilde{w}_H := 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \Omega,$$

and the prolongation to $\mathbb{R}^2$ of w

$$\tilde{w} := w \quad \text{in} \quad \Omega, \quad \tilde{w} := 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \Omega.$$

We note that $\tilde{w}_H \to \tilde{w}$ in $L^2(\mathbb{R}^2)$ $(H \in \Lambda')$. For $\varphi \in C_0^\infty(\mathbb{R}^2)$ and all $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$, $\eta \neq 0$, we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \left| \big(\tilde{w}_H(x+\eta_1, y+\eta_2) - \tilde{w}_H(x,y)\big)\varphi(x,y) \right| dxdy \\ &\leq \left(\int_{\mathbb{R}^2} \left| \tilde{w}_H(x+\eta_1, y+\eta_2) - \tilde{w}_H(x,y) \right|^2 dxdy \right)^{1/2} \|\varphi\|_{L^2(\mathbb{R}^2)} \end{aligned}$$

Since (5), then

$$\int_{\mathbb{R}^2} \left| \big(\tilde{w}_H(x+\eta_1, y+\eta_2) - \tilde{w}_H(x,y)\big)\varphi(x,y) \right| dxdy \leq C\big(|\eta| + H_{max}\big)\|\varphi\|_{L^2(\mathbb{R}^2)}.$$

Taking the limit when $H_{max} \to 0$, results

$$\int_{\mathbb{R}^2} \left| \big(\tilde{w}_H(x+\eta_1, y+\eta_2) - \tilde{w}_H(x,y)\big)\varphi(x,y) \right| dxdy \leq C|\eta|\|\varphi\|_{L^2(\mathbb{R}^2)},$$

and consequently

$$\int_{\mathbb{R}^2} \frac{|\varphi(x-\eta_1, y-\eta_2) - \varphi(x,y)|}{|\eta|} |\tilde{w}(x,y)|\, dxdy \leq C\|\varphi\|_{L^2(\mathbb{R}^2)}.$$

Considering $\eta = \varepsilon(1,0)$ and the limit $\varepsilon \to 0$,we conclude that

$$\int_{\mathbb{R}^2} |\varphi_x(x,y)\tilde{w}(x,y)|\, dxdy \leq C\|\varphi\|_{L^2(\mathbb{R}^2)},$$

for $\varphi \in C_0^\infty(\mathbb{R}^2)$. Analogously, taking $\eta = \varepsilon(0,1)$, we obtain

$$\int_{\mathbb{R}^2} |\varphi_y(x,y)\tilde{w}(x,y)|\, dxdy \leq C\|\varphi\|_{L^2(\mathbb{R}^2)},$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. Consequently, $\tilde{w} \in W^{1,2}(\mathbb{R}^2)$. Since w is a restriction of $\tilde{w}$ to Ω and $\tilde{w} = 0$ in $\mathbb{R}^2 \backslash \Omega$ then $w \in W_0^{1,2}(\Omega)$.

Let us finally prove that $v = w$. Let $r \in L^2(\Omega)$ and $(r_H)_\Lambda \in \Pi \overset{\circ}{L}{}_H^2(\Omega)$, such that $r_H \to r$ in $(L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2(\Omega))$ $(H \in \Lambda)$. For the step function defined by

$$s_H(x,y) := r_H(x_{j+1/2}, y_{\ell+1/2}), \quad (x,y) \in (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1}) \subset \Omega,$$

zero in $\mathbb{R}^2 \backslash \Omega$, we have

$$(v_H, r_H)_H = (w_H, s_H)_0 \to (w, r)_0 \quad (H \in \Lambda).$$

Finally,

$$v_H \rightharpoonup w \text{ in } (L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2(\Omega)) \quad (H \in \Lambda).$$

Considering (7) we conclude that $v = w$.

■

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