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ON THE STABILITY OF A CLASS OF SPLITTING METHODS FOR INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT: The classical convection-diffusion-reaction equation has the unphysical property that if a sudden change in the dependent variable is made at any point, it will be felt instantly everywhere. This phenomena violate the principle of casuality.

Over the years, several authors have proposed modifications in an effort to overcome the propagation speed defect. The purpose of this paper is to study, from analytical and numerical point of view a modification to the classical model that take in to account the memory effects. Besides the finite speed of propagation, we establish an energy estimate to the exact solution. We also present a numerical method which has the same qualitative property of the exact solution. Finally we illustrate the theoretical results with some numerical simulations.

KEYWORDS: Integro-differential equations, Splitting methods, Stability, Convergence.

1. Introduction

The classical heat equation for the temperature u

$$\frac{\partial u}{\partial t}(x,t) = \frac{k}{\gamma}\Delta u(x,t),$$

on a bar, where γ represents the heat capacity, is obtained combining the Fourier's law for the heat flux q

$$q(x,t) = -k\nabla u(x,t),$$

where k denotes the thermal conductivity, with the Mass Conservation law

$$\frac{\partial u}{\partial t}(x,t) + \nabla q(x,t) = 0. \tag{1}$$

The classical heat equation has the unphysical property that if a sudden change in the temperature is made at a point of the bar, it will be felt instantly everywhere. This property, known as a infinite speed of propagation, is not present in heat conduction phenomena and is consequence of the violation of principle of casuality by the Fourier law for the flux. In fact, this

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law establish that the observed heat flux at some point at some time is consequence of the space temperature variation at the same point and at the same time.

In order to overcome the limitation of the traditional heat equation, over the years, several authors have proposed modifications to Fourier's flux including in its definition a certain memory term as an effort to avoid the infinite propagation speed ([6], [15], [21]).

Attending that the heat flux q at point x and at time t should be consequence of the temperature variation at point x but at some passed time, Cattaneo, in [6], proposed the following heat flux definition

$$q(x,t+\tau) = -k\nabla u(x,t),$$

where τ is a relaxation time. Considering a first order approximation to the flux and integrating the first order differential equation

$$\frac{\partial q}{\partial t}(x,t) + \frac{1}{\tau}q(x,t) = -\frac{k}{\tau}\nabla u(x,t),$$

we obtain the so-called Cattaneo's flux

$$q(x,t) = -\frac{k}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\nabla u(x,s)\,ds. \tag{2}$$

Note that, when $\tau \to 0$, the Cattaneo's flux tends to the classical Fourier's flux. Combining (2) with (1) we obtain, for the temperature, the Cattaneo's equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{k}{\tau\gamma}\int_0^t e^{-\frac{t-s}{\tau}}\Delta u(x,s)\,ds. \tag{3}$$

The simplest initial boundary value problem (IBVP) – (3) with $u(x,0) = u_0(x)$ – that gives rise to finite speed of propagation is defined using (3) ([6], [21]). In fact, if we impose some regularity on the initial condition u_0, we may prove that this IBVP is equivalent to a hyperbolic IBVP defined by telegraph equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{1}{\tau}\frac{\partial u}{\partial t}(x,t) = \frac{k}{\gamma\tau}\Delta u(x,t),$$

that transmits waves with finite velocity $c = \sqrt{\dfrac{k}{\gamma\tau}}$.

In [15], Joseph and Preziosi argue that there is no real conductor where the heat conduction phenomenon can be modeled by the Cattaneo's equation. So, they propose the use of a modified flux defined by

$$q(x,t+\tau) = -k\nabla u(x,t+\tau^*), \qquad \tau > \tau^*,$$

with two relaxation parameters. Considering the first order approximation to the flux and to the gradient of the concentration we obtain

$$\frac{\partial q}{\partial t}(x,t) + \frac{1}{\tau}q(x,t) = -\frac{k}{\tau}\nabla u(x,t) - k\frac{\tau^*}{\tau}\frac{\partial}{\partial t}\nabla u(x,t), \tag{4}$$

which allows us to obtain the following heat flux

$$q(x,t) = -k\frac{\tau^*}{\tau}\nabla u(x,t) - \frac{k}{\tau}\left(1-\frac{\tau^*}{\tau}\right)\int_0^t e^{-\frac{t-s}{\tau}}\nabla u(x,s)\,ds.$$

If we take $k = k_1 + k_2$, where k_1 represent the effective thermal conductivity and k_2 the elastic conductivity, and $\frac{\tau^*}{\tau} = \frac{k_1}{k}$, we obtain the so-called Jeffrey's heat flux ([15])

$$q(x,t) = -k_1\nabla u(x,t) - \frac{k_2}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\nabla u(x,s)\,ds.$$

From the Mass Conservation law (1) is easy to show that, in this case, the temperature u satisfies the Jeffrey's equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{k_1}{\gamma}\Delta u(x,t) + \frac{k_2}{\tau\gamma}\int_0^t e^{-\frac{t-s}{\tau}}\Delta u(x,s)\,ds.$$

In the last years several analytical and numerical studies on the solution of IBVP defined by using integro-differential equation as the Jeffrey's equation, arise in the literature. For instance, in [2], the authors considered the Jeffrey's equation

$$\frac{\partial u}{\partial t}(x,t) = \alpha\Delta u(x,t) + \frac{D}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\Delta u(x,s)\,ds,\ x\in(a,b), t>0, \tag{5}$$

with homogeneous Dirichlet boundary conditions. From an analytical point of view they establish an energy estimate which was fundamental to prove the stability of the IBVP with respect to perturbations of the initial condition. From a numerical viewpoint they propose a splitting method which simulates the heat transport as the superposition of two phenomena: diffusion and memory in time, being the memory treated by using the telegraph equation.

Reaction-diffusion integro-differential equations have been also considered in the literature in order to overcome some unphysical behavior present by the solution of the classical Fisher equation

$$\frac{\partial u}{\partial t}(x,t) = D\Delta u(x,t) + f(u(x,t)). \tag{6}$$

In fact, if the reaction term f is defined by $f(u(x,t)) = Uu(x,t)(1-u(x,t))$, then the traveling wave solution $u(x,t) = \phi(x-ct)$ connecting the stationary states $u = 0$ (unstable) and $u = 1$ (stable) satisfies $c \geq \sqrt{4DU}$. Then when the reaction parameter U goes to infinity, the propagation speed c goes also to infinity and this behavior is unphysical ([9], [10]). For instance in [4] the Fisher-Kolmogorov-Petrovskii-Piskunov equation

$$\frac{\partial u}{\partial t}(x,t) = f(u(x,t)) + \frac{D}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\Delta u(x,s)\,ds,\ x \in (a,b), t > 0, \tag{7}$$

with homogeneous Dirichlet boundary conditions, was studied from analytical and numerical point of view. The stability of the model was established and some numerical methods were proposed.

In [12] and [13], reaction-transport systems with memory and long range interaction were modeled by the following integro-differential equation

$$\frac{\partial u}{\partial t}(x,t) = \int_0^t \alpha(t-s)\Big(\int_{\mathbb{R}} u(x+\mu,s)\phi(\mu)\,d\mu - u(x,s)\Big)\,ds + f(u(x,t)),\ x \in \mathbb{R}, \tag{8}$$

where $\alpha(s)$ and $\phi(\mu)$ represent kernel functions. The initial value problem defined by (8) was studied in [14] from analytical and numerical point of view where estimates for the L^2 norm of the solution and the L^2 norm of its past were established. These estimates were deduced for the continuous model and for the discrete models proposed in that paper.

The use of memory terms in the definition of the flux in some biological applications leads to new models. These new models enables us to study quantities that the classical models do not give any information. For instance, in [3], it was consider a new model for percutaneous absorption of a drug which consists in integro-differential equation

$$\frac{\partial u}{\partial t}(x,t) = \mu - \gamma u(x,t) + \beta\nabla u(x,t) + \frac{D}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\Delta u(x,s)\,ds,\ x \in (a,b), t > 0, \tag{9}$$

with appropriate initial and boundary conditions. The authors studied the qualitative properties of the model and its numerical approximation and they compare their model with the classical one based on the classical Fick's law for the flux.

In this paper we will consider the IBVP

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= f(u(x,t)) + \beta\nabla u(x,t)\\ &+\alpha\Delta u(x,t) + \frac{D}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\Delta u(x,s)\,ds,\ \ x\in(a,b), t>0, \quad (10)\end{aligned}$$

where $\alpha, D \geq 0$, $\tau > 0$, $\beta \in \mathbb{R}$, with initial and boundary conditions

$$\begin{aligned} &u(x,0) = u_0(x),\, x\in(a,b),\\ &u(a,t) = u(b,t) = 0,\, t>0. \end{aligned} \quad (11)$$

Our aim is to study the qualitative properties of the solution of (10)–(11) and the stability of the model with respect to the L^2 norm and also with respect of the L^2 norm of the past in time of the gradient. From the numerical point of view we propose a splitting methods which allows us to compute a numerical approximation presenting the qualitative behavior of the solution of (10)–(11). The paper is organized as follows. In Section 2 the qualitative behavior of model (10)–(11) is studied and its stability is concluded. In Section 3 we study a family of θ numerical methods in terms of its stability and accuracy. Finally, in Section 4 some numerical simulations are included illustrating the theoretical results obtained in Section 3.

2. Energy estimates

Let us consider the IBVP (10)–(11). We establish, in the following result, an estimate for the energy functional

$$E(u)(t) = \|u(t)\|^2 + \frac{D}{\tau}\|\int_0^t e^{-\frac{t-s}{\tau}}\nabla u(s)\,ds\|^2, \quad (12)$$

for $t > 0$, where $\|.\|$ represents the usual L^2 norm.

Theorem 1. *Let u be a solution of (10)–(11) satisfying, for each $t\in[0,T]$, $|u(x,t)| \leq L$ ($L\in\mathbb{R}^+$), for $x\in[a,b]$, and $\frac{\partial u}{\partial t}(t)$, $\nabla u(t)$, $\Delta u(t)$, $\int_0^t e^{-\frac{t-s}{\tau}}\Delta u(s)\,ds$*

$\in L^2[a,b]$. If f is continuously differentiable and $f(0)=0$, then

$$E(u)(t) \leq e^{2\max\{-\frac{1}{\tau}, f'_{max}-\frac{\alpha}{(b-a)^2},\}t}\|u_0\|^2, \tag{13}$$

for each $t \in (0,T]$, where $f'_{max} = \max_{|u|\leq L} f'(u)$.

Proof: Multiplying (10) by u, with respect to the L^2 inner product $(.,.)$ and integrating by parts, we easily get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 \leq f'_{max}\|u(t)\|^2 - \alpha\|\nabla u(t)\|^2 - \frac{D}{\tau}(\int_0^t e^{-\frac{t-s}{\tau}}\nabla u(s)\,ds, \nabla u(t)). \tag{14}$$

As

$$\begin{aligned}(\int_0^t e^{-\frac{t-s}{\tau}}\nabla u(s)\,ds, \nabla u(t)) = &\frac{1}{2}\frac{d}{dt}\|\int_0^t e^{-\frac{t-s}{\tau}}\nabla u(s)\,ds\|^2 \\ &+\frac{1}{\tau}\|\int_0^t e^{-\frac{t-s}{\tau}}\nabla u(s)\,ds\|^2,\end{aligned}$$

we deduce from (14) the differential inequality

$$\frac{d}{dt}E(u)(t) \leq 2\max\{-\frac{1}{\tau}, f'_{max} - \frac{\alpha}{(b-a)^2}\}E(u)(t), \tag{15}$$

which allows us to obtain (13).

□

According to the previous theorem, the solution u satisfies

$$\|u(t)\| \leq e^{\max\{-\frac{1}{\tau}, f'_{max}-\frac{\alpha}{(b-a)^2}\}t}\|u_0\|$$

and the "average in time" of its gradient

$$\|\int_0^t e^{-\frac{t-s}{\tau}}\nabla u(s)\,ds\| \leq e^{\max\{-\frac{1}{\tau}, f'_{max}-\frac{\alpha}{(b-a)^2}\}t}\|u_0\|.$$

If $f'_{max} < 0$ then

$$\|u(t)\| \to 0 \quad \text{and} \quad \|\int_0^t e^{-\frac{t-s}{\tau}}\nabla u(s)\,ds\| \to 0, \qquad \text{as} \quad t \to \infty.$$

Remark 1. *We remark that, as particular cases, we conclude the following:*

(1) *For the IBVP defined by the Fisher-Kolmogorov-Petrovskii-Piskunov equation (7) we have, as in [4],*

$$E(u)(t) \leq e^{2\max\{-\frac{1}{\tau}, f'_{max}\}t}\|u_0\|^2,\ t \geq 0.$$

(2) *For the IBVP defined by the Fisher equation (6) it can be shown that*

$$\|u(t)\| \leq e^{f'_{max}t}\|u_0\|.$$

(3) *For the Jefferey's IBVP defined by (5) holds, as in* [1],

$$E(u)(t) \leq e^{-2\min\{\frac{1}{\tau},\frac{\alpha}{(b-a)^2}\}t}\|u_0\| \to 0, \qquad as \quad t \to \infty.$$

(4) *For Cattaneo's IBVP we may conclude that*

$$E(u)(t) \leq \|u_0\|^2.$$

(5) *For the classical heat IBVP it is known that*

$$\|u(t)\| \leq e^{-\frac{\alpha}{(b-a)^2}t}\|u_0\| \to 0, \qquad as \quad t \to \infty.$$

Let us now consider the stability behavior of the solution u under perturbations in the initial condition u_0.

Let u and u_ϵ be solutions of (10) satisfying the same boundary conditions (not necessarily homogeneous) and initial conditions u_0 and $u_0 + \epsilon$, respectively. The influence of ϵ on the solution of is estimated in the following result.

Theorem 2. *Let u and u_ϵ be solutions of (10) satisfying the same boundary conditions and initial conditions u_0 and $u_0 + \epsilon$, respectively. If, for these solutions, the hypothesis of Theorem 1 are satisfied then*

$$E(u - u_\epsilon)(t) \leq e^{2\max\{-\frac{1}{\tau},f'_{max}-\frac{\alpha}{(b-a)^2},\}t}\|\epsilon\|^2,$$

for each $t \in (0, T]$, where $f'_{max} = \max_{|u|\leq L} f'(u)$.

Proof: Let us first note that $v_\epsilon = u - u_\epsilon$ satisfies

$$\frac{\partial v_\epsilon}{\partial t}(x,t) = f(u(x,t)) - f(u_\epsilon(x,t)) + \beta\nabla v_\epsilon(x,t) \tag{16}$$

$$+\alpha\Delta v_\epsilon(x,t) + \frac{D}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\nabla v_\epsilon(x,s)\,ds,\ x \in (a,b), t > 0, \tag{17}$$

and the conditions

$$u_\epsilon(x,0) = -\epsilon(x),\ x \in (a,b),$$

$$u_\epsilon(a,t) = u_\epsilon(b,t) = 0,\ t > 0.$$

Multiplying equation (16) with respect to the L^2 inner product $(.,.)$ we obtain

$$
\begin{aligned}
(\frac{\partial v_\epsilon}{\partial t}(t), v_\epsilon(t)) &= (f(u(t)) - f(u_\epsilon(t)), v_\epsilon(t)) + \beta(\nabla v_\epsilon(t), v_\epsilon(t)) \\
&\quad + \alpha(\Delta v_\epsilon(t), v_\epsilon(t)) + \frac{D}{\tau}(\int_0^t e^{-\frac{t-s}{\tau}} \Delta v_\epsilon(s)\, ds, v_\epsilon(t)).
\end{aligned}
$$

As $(f(u(t)) - f(u_\epsilon(t)), v_\epsilon(t)) \leq f'_{max}\|v_\epsilon\|^2$, the proof is concluded following the same steps of the proof of the last theorem.

□

3. Numerical methods

Let us consider in $[a,b]$ a grid $G_h = \{x_i : i = 0, \dots, N\}$ with $x_0 = a$, $x_N = b$ and $x_i - x_{i-1} = h$, $i = 1, \dots, N$. In what follows, we will consider the second-order centered finite difference operator Δ_h and $\nabla_{h,s(\beta)}$ which corresponds to $\nabla_{h,-}$, the first-order backward finite difference operator when $\beta < 0$, or to $\nabla_{h,+}$, the first-order forward finite difference operator when $\beta > 0$, defined by the usual way. Let us also consider the time grid $\{t_n, n = 0, \dots, M\}$ such that $t_0 = 0$, $t_M = T$ and $t_{n+1} - t_n = \Delta t$.

The class of splitting methods that we study are based on the following functional splitting

I. Reaction:

$$
\begin{cases}
\frac{\partial u_1}{\partial t}(x,t) = f(u_1(x,t)), & x \in (a,b), t \in (t, t+\Delta t] \\
u_1(x,t) = u(x,t), & x \in (a,b)
\end{cases}
$$

II. Advection and Diffusion:

$$
\begin{cases}
\frac{\partial u_2}{\partial t}(x,t) = \beta\nabla u_2(x,t) + \alpha\Delta u_2(x,t), & x \in (a,b), t \in (t, t+\Delta t] \\
u_2(x,t) = u_1(x, t+\Delta t), & x \in (a,b)
\end{cases}
$$

III. Diffusion Memory:

$$
\begin{cases}
\frac{\partial u_3}{\partial t}(x,t) = \frac{D}{\tau}\int_0^t e^{-\frac{t-s}{\tau}} \Delta u_3(x,s)\, ds, & x \in (a,b), t \in (t, t+\Delta t] \\
u_3(x,t) = u_2(x, t+\Delta t), & x \in (a,b)
\end{cases}.
$$

By SM we denote the splitting method obtained combining I_h, II_h and III_h defined by

I_h. Reaction:

$$\begin{cases} u_{1,h}^{n+1} &= u_{1,h}^{n} + \Delta t(\theta f(u_{1,h}^{n}) + (1-\theta) f(u_{1,h}^{n+1})), \quad \theta \in [0,1] \\ u_{1,h}^{n} &= u_h^n \end{cases}$$

II_h. Advection and Diffusion:

$$\begin{cases} u_{2,h}^{n+1} &= u_{2,h}^{n} + \Delta t \beta \nabla_{h,\mathrm{s}(\beta)} u_{2,h}^{n+1} + \Delta t \alpha \Delta_h u_{2,h}^{n+1} \\ u_{2,h}^{n} &= u_{1,h}^{n+1} \end{cases}$$

III_h. Diffusion Memory:

$$\begin{cases} u_{3,h}^{n+1} &= u_{3,h}^{n} + \Delta t^2 \dfrac{D}{\tau} \displaystyle\sum_{j=1}^{n} e^{-\frac{t_{n+1}-t_j}{\tau}} \Delta_h u_h^j + \Delta t^2 \dfrac{D}{\tau} \Delta_h u_{3,h}^{n+1} \\ u_{3,h}^{n} &= u_{2,h}^{n+1} \end{cases}$$

where

$$u_h^j(x_0) = u_h^j(x_N) = 0,\ j = 1, \dots, M-1,\ u_h^0(x_i) = u_0(x_i),\ i = 1, \dots, N. \tag{18}$$

Finally we will consider $u(x_i, t_{n+1}) \approx u_h^{n+1}(x_i) = u_{3,h}^{n+1}(x_i)$, $i = 1, \dots, N$.

The stability and accuracy properties of the described splitting method will be compared with the correspondent properties of the non-splitting scheme (NSM)

$$\begin{aligned} u_h^{n+1} &= u_h^n + \Delta t(\theta f(u_h^n) + (1-\theta) f(u_h^{n+1})) + \Delta t \beta D_{\mathrm{s}(\beta)x} u_h^{n+1} \\ &\quad + \Delta t \alpha \Delta_h u_h^{n+1} + \frac{D \Delta t^2}{\tau} \sum_{j=1}^{n+1} e^{-(t_{n+1}-t_j)/\tau} \Delta_h u_h^j, \quad \theta \in [0,1]. \end{aligned} \tag{19}$$

3.1. Stability. In order to study the stability of the numerical methods, let us introduce some notation. We denote by $L^2(G_h)$ the space of grid functions v_h defined in G_h such that $v_h(x_0) = v_h(x_N) = 0$. In this space, we will consider the discrete inner product

$$(v_h, w_h)_h = h \sum_{i=1}^{N-1} v_h(x_i) w_h(x_i), \qquad v_h, w_h \in L^2(G_h). \tag{20}$$

We denote by $\|\cdot\|_h$ the norm induced by this inner product. We will introduce other notations:

$$(v_h, w_h)_{h+} = h\sum_{i=1}^{N} v_h(x_i)w_h(x_i), \tag{21}$$

$$\|v_h\|_{h+} = \left(h\sum_{i=1}^{N} v_h(x_i)^2\right)^{1/2}, \tag{22}$$

for grid functions defined on $G_h \cup \{x_N\}$.

We remark that holds the following discrete Friedrichs-Poincaré inequality

$$\|v_h\|_h^2 \leq (b-a)^2\|\nabla_{h,-}v_h\|_{h,+}^2.$$

Our goal is to obtain an estimate for the fully discrete version of the energy (12) given by

$$E_h(u_h^{n+1}) = \|u_h^{n+1}\|_h^2 + \frac{D}{\tau}\|\Delta t\sum_{j=1}^{n} e^{-(t_{n+1}-t_j)/\tau}\nabla_{h,-}u_h^j\|_{h+}^2. \tag{23}$$

We will prove the following result for the SM defined by I_h–III_h and (18).

Theorem 3. *Let u_h^j be a solution of the SM defined by I_h–III_h and (18), such that $|u_h^j(x_i)| \leq L$ ($L \in \mathbb{R}^+$), for $i = 1,\ldots,N$ and $j = 1,\ldots,M$. If f is continuously differentiable and $f(0) = 0$, then*

$$E_h(u_h^{n+1}) \leq (S(\Delta t,\theta))^{n+1}\|u_0\|_h^2, \tag{24}$$

for $n = 0,\ldots,M-1$, where the stability factor $S(\Delta t,\theta)$ is defined by

$$S(\Delta t,\theta) = \frac{1}{(1+2\alpha\Delta t(b-a)^{-2})(1-\Delta t(\theta {f'_{max}}^2 + 2(1-\theta)f'_{max}))} \tag{25}$$

for $\theta \in [0,1]$, and $f'_{max} = \max_{|u|\leq L} f'(u)$, provided that

$$1-\Delta t(\theta {f'_{max}}^2 + 2(1-\theta)f'_{max}) > 0. \tag{26}$$

Proof: Let us first consider I_h

$$u_{1,h}^{n+1} = u_{1,h}^n + \Delta t(\theta f(u_{1,h}^n) + (1-\theta)f(u_{1,h}^{n+1})).$$

Multiplying this equation by $u_{1,h}^{n+1}$, with respect to the L^2 inner product $(.,.)_h$, we get

$$(u_{1,h}^{n+1}, u_{1,h}^{n+1})_h = (u_{1,h}^n, u_{1,h}^{n+1})_h + \Delta t(\theta(f(u_{1,h}^n), u_{1,h}^{n+1}) + (1-\theta)(f(u_{1,h}^{n+1}), u_{1,h}^{n+1})_h).$$

Due to the fact that $f(0)=0$ we obtain

$$(1-\Delta t(\theta {f'_{max}}^2+2(1-\theta)f'_{max}))\|u_{1,h}^{n+1}\|_h^2 \leq (1+\Delta t\theta)\|u_{1,h}^n\|_h^2$$

which implies

$$\|u_{1,h}^{n+1}\|_h^2 \leq \frac{1+\Delta t\theta}{1-\Delta t(\theta {f'_{max}}^2+2(1-\theta)f'_{max})}\|u_{1,h}^n\|_h^2, \tag{27}$$

provided that Δt satisfies (26).

Let us now consider II_h

$$u_{2,h}^{n+1} = u_{2,h}^n + \Delta t\beta\nabla_{h,\mathrm{s}(\beta)}u_{2,h}^{n+1} + \Delta t\alpha\Delta_h u_{2,h}^{n+1}.$$

Proceeding as before and using summation by parts we get

$$\begin{aligned}(u_{2,h}^{n+1},u_{2,h}^{n+1})_h = (u_{2,h}^n,u_{2,h}^{n+1})_h - \Delta t\beta(\nabla_{h,\mathrm{s}(\beta)}u_{2,h}^{n+1},u_{2,h}^{n+1})_h \\ -\Delta t\alpha\|\nabla_{h,-}u_{2,h}^{n+1}\|_{h+}^2.\end{aligned} \tag{28}$$

We remark that

$$\beta(\nabla_{h,\mathrm{s}(\beta)}u_{2,h}^{n+1},u_{2,h}^{n+1})_h \leq 0. \tag{29}$$

In fact, for instance for $\beta>0$, taking $v_h := u_{2,h}^{n+1}$ we have

$$\begin{aligned}\beta(\nabla_{h,+}v_h,v_h)_h &= \beta\left(\sum_{i=2}^N v_iv_{i-1} - \sum_{i=1}^{N-1}v_i^2\right)\\ &\leq \beta\left(\frac{1}{2}\sum_{i=1}^N\left(v_i^2+v_{i-1}^2\right) - \sum_{i=1}^{N-1}v_i^2\right)\\ &\leq 0.\end{aligned}$$

Taking (29) in (28) and using the discrete Friedrichs-Poincaré inequality we obtain

$$\|u_{2,h}^{n+1}\|_h^2 \leq \frac{1}{1+2\alpha\Delta t(b-a)^{-2}}\|u_{2,h}^n\|_h^2. \tag{30}$$

Finally let us consider III_h

$$u_{3,h}^{n+1} = u_{3,h}^n + \Delta t^2\frac{D}{\tau}\sum_{j=1}^n e^{-\frac{t_{n+1}-t_j}{\tau}}\Delta_h u_h^j + \Delta t^2\frac{D}{\tau}\Delta_h u_{3,h}^{n+1}.$$

As in the previous cases we get

$$\|u_{3,h}^{n+1}\|_h^2 = (u_{3,h}^n, u_{3,h}^{n+1})_h - \Delta t^2 \frac{D}{\tau}(\sum_{j=1}^{n} e^{-\frac{t_{n+1}-t_j}{\tau}} \nabla_{h,-} u_h^j, \nabla_{h,-} u_{3,h}^{n+1})_{h+} \\ - \Delta t^2 \frac{D}{\tau} \|\nabla_{h,-} u_{3,h}^{n+1}\|_{h+}^2 .$$

Using the same arguments as before and due to the fact that

$$2(\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} \Delta_{h,-} u_h^j, \nabla_{h,-} u_h^{n+1})_{h+} = \\ \|\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} \nabla_{h,-} u_h^j\|_{h+}^2 - e^{-2\Delta t/\tau} \|\sum_{j=1}^{n} e^{-\frac{t_n-t_j}{\tau}} \nabla_{h,-} u_h^j\|_{h+}^2 + \|\nabla_{h,-} u_h^{n+1}\|_{h+}^2$$

we obtain

$$\|u_{3,h}^{n+1}\|_h^2 + \frac{D}{\tau}\|\Delta t(\sum_{j=1}^{n} e^{-\frac{t_{n+1}-t_j}{\tau}} \nabla_{h,-} u_h^j + \nabla_{h,-} u_{3,h}^{n+1})\|_{h+}^2 \\ \leq \|u_{3,h}^n\|_h^2 + \frac{D}{\tau}\|\Delta t \sum_{j=1}^{n} e^{-\frac{t_{n+1}-t_j}{\tau}} \nabla_{h,-} u_h^j\|_{h+}^2. \tag{31}$$

Attending that $u_{3,h}^n = u_{2,h}^{n+1}$ and using in (31) inequality (30) we obtain

$$\|u_{3,h}^{n+1}\|_h^2 + \frac{D}{\tau}\|\Delta t(\sum_{j=1}^{n} e^{-\frac{t_{n+1}-t_j}{\tau}} \nabla_{h,-} u_h^j + \nabla_{h,-} u_{3,h}^{n+1})\|_{h+}^2 \\ \leq \frac{1}{1 + 2\alpha\Delta t(b-a)^{-2}}\|u_{2,h}^n\|_h^2 + \frac{D}{\tau}\|\Delta t \sum_{j=1}^{n} e^{-\frac{t_{n+1}-t_j}{\tau}} \nabla_{h,-} u_h^j\|_{h+}^2. \tag{32}$$

Finally, as we have $u_{2,h}^n = u_{1,h}^{n+1}$, from inequalities (32) and (27) we conclude the proof.

□

Following the same arguments, we may obtain an estimate for the discrete energy (23) of the solution of the NSM defined by (19) and (18).

Theorem 4. *Let u_h^j be a solution of the NSM defined by (19) and (18), such that $|u_h^j(x_i)| \leq L$ ($L \in \mathbb{R}^+$), for $i = 1, \ldots, N$ and $j = 1, \ldots, M$. If f is* *continuously differentiable and $f(0) = 0$, then*

$$E_h(u_h^{n+1}) \leq (S(\Delta t, \theta))^{n+1} \|u_0\|_h^2, \tag{33}$$

for $n = 0, \ldots, M-1$, where the stability factor $S(\Delta t, \theta)$ is defined by

$$S(\Delta t, \theta) = \frac{1 + \Delta t \theta}{\min\{1, 1 - \Delta t(\theta {f'_{max}}^2 + 2(1-\theta) f'_{max} - (2\alpha + D/\tau \Delta t)(b-a)^{-2})\}} \tag{34}$$

for $\theta \in [0, 1]$ and $f'_{max} = \max_{|u| \leq L} f'(u)$, provided that

$$1 - \Delta t(\theta {f'_{max}}^2 + 2(1 - \theta f'_{max}) - (2\alpha + D/\tau \Delta t)(b-a)^{-2}) > 0.$$

□

We denote by SI the splitting method I_h–III_h with $\theta = 0$ (implicit reaction), SE the splitting method I_h–III_h with $\theta = 0$ (explicit reaction), by FI the non-splitting scheme (19) with $\theta = 0$ (implicit reaction) and by IMEX the non-splitting scheme (19) with $\theta = 1$ (explicit reaction) and by S_i, S_e, S_{fi} and S_{imex} we represent the corresponding stability factors.

In Figures 1 – 2 we plot the defined stability factors as functions of the time step. As we expected, in what concerns the stability, these figures confirm the advantage of the implicit schemes. If we compare the splitting schemes with the non-splitting ones we may see, specially for $f'_{max} < 0$ (Figure 1), that the stability factor for the splitting method with implicit reaction is less or equal than the stability factor of the non-splitting scheme with implicit reaction.

The behavior of the stability conditions to the time step is considered in Figure 3. Let $SC(\Delta t, \theta)$ be defined by

$$SC(\Delta t, \theta) = \max\{0, 1 - \Delta t(\theta {f'_{max}}^2 + 2(1-\theta) f'_{max})\},$$

for splitting schemes and by

$$SC(\Delta t, \theta) = \max\{0, 1 - \Delta t(\theta {f'_{max}}^2 + 2(1-\theta) f'_{max} - (2\alpha + D/\tau \Delta t)(b-a)^{-2})\},$$

for non-splitting ones. By SC_i, SC_e, SC_{fi} and SC_{imex} we denote the previous functions. Figure 3 illustrates the fact that the restrictions to the stability imposed by the explicit schemes are more restrictive.

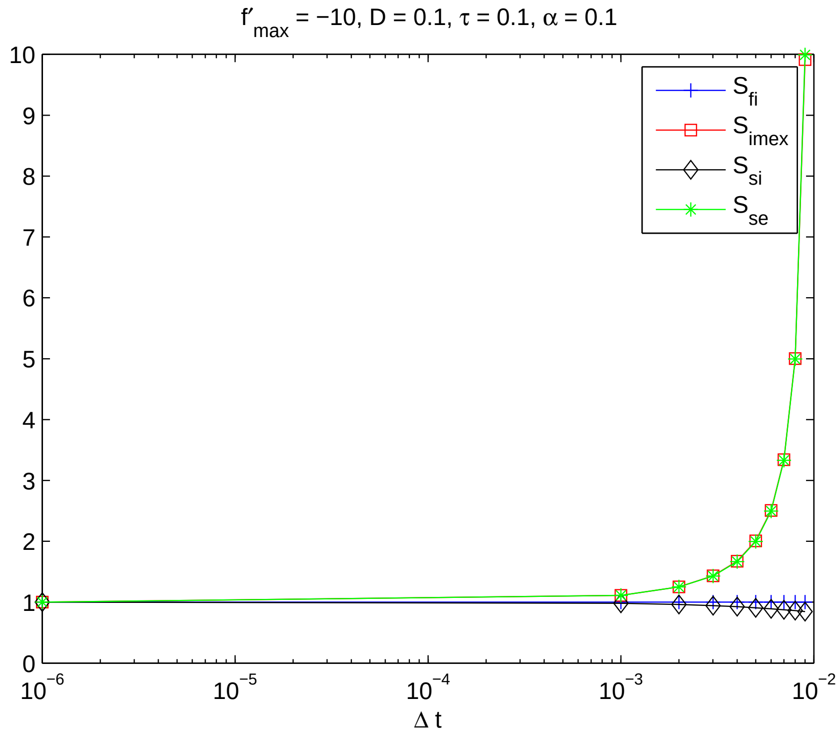

FIGURE 1. Stability factor: $f'_{max} < 0$.

3.2. Error estimates. In this section we will study the convergence of the numerical schemes proposed in the previous section. Let $e_h^j(x_i) = u_h^j(x_i) - u(x_i, t_j)$ be the global error of the approximation $u_h^j(x_i)$ obtained by the numerical method I_h–III_h with boundary conditions (18), and let $T_h^j(x_i)$ be the corresponding truncation error. Following the proof of Theorem 3 we may prove the next result.

Theorem 5. *Let u_h^j, $j = 1, \ldots, M$, be the numerical solution of (10)–(11) obtained with I_h–III_h with boundary conditions (18). If f is continuously differentiable and $f(0) = 0$, then*

$$E_h(e_h^{n+1}) \leq \sum_{j=0}^{n} \overline{S}^{j+1}(\Delta t, \theta)\Delta t \|T_h^{n+1-j}\|_h^2,$$

with $\|T_h^\ell\|_h^2 = \max_{k=1,2,3} \|T_{k,h}^\ell\|_h^2$, where $T_{k,h}^\ell$ denotes the truncation error corresponding to problem k for $k = 1, 2, 3$, and

$$\overline{S}(\Delta t, \theta) = \frac{M(\Delta t, \theta)}{m(\Delta t, \theta)},$$

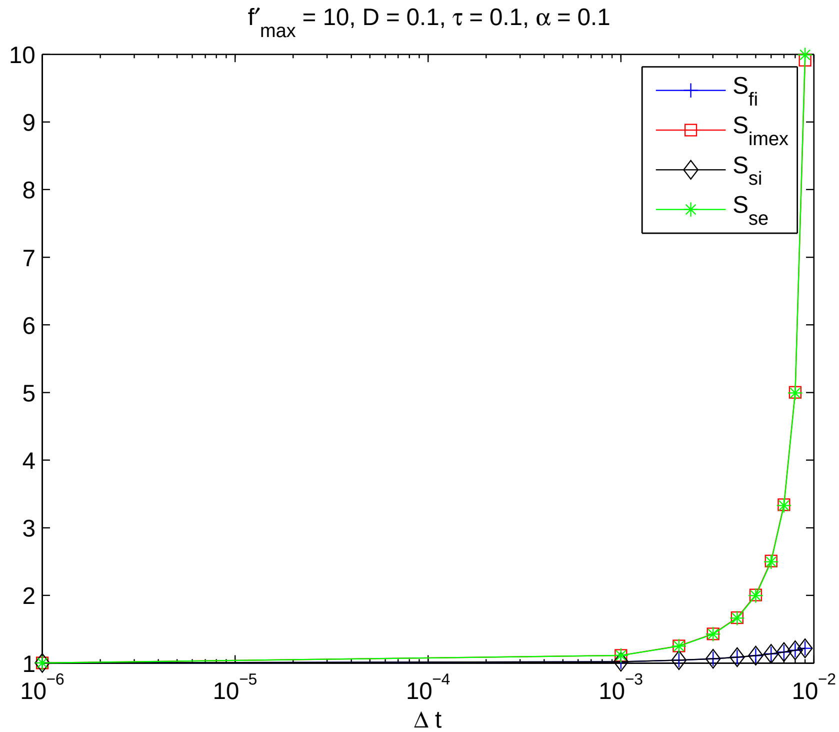

FIGURE 2. Stability factor: $f'_{max} > 0$.

where

$$M(\Delta t, \theta) = \max\{e^{-2\Delta t/\tau}, \frac{1 + \Delta t\theta}{1 - \Delta t(\theta {f'_{max}}^2 + 2(1-\theta) f'_{max} + 1)}\}$$

and

$$m(\Delta t, \theta) = \min\{1, 1 - \Delta t(1 - \frac{D}{\tau}\Delta t(b-a)^{-2})\},$$

provided that

$$1 - \Delta t(\theta {f'_{max}}^2 + 2(1-\theta) f'_{max} + 1) > 0 \tag{35}$$

and

$$1 - \Delta t(1 - \frac{D}{\tau}\Delta t(b-a)^{-2}) > 0. \tag{36}$$

Proof: Let

$$e^j_{k,h}(x_i) = u^j_{k,h}(x_i) - u_k(x_i, t_j), \quad k = 1, 2, 3,$$

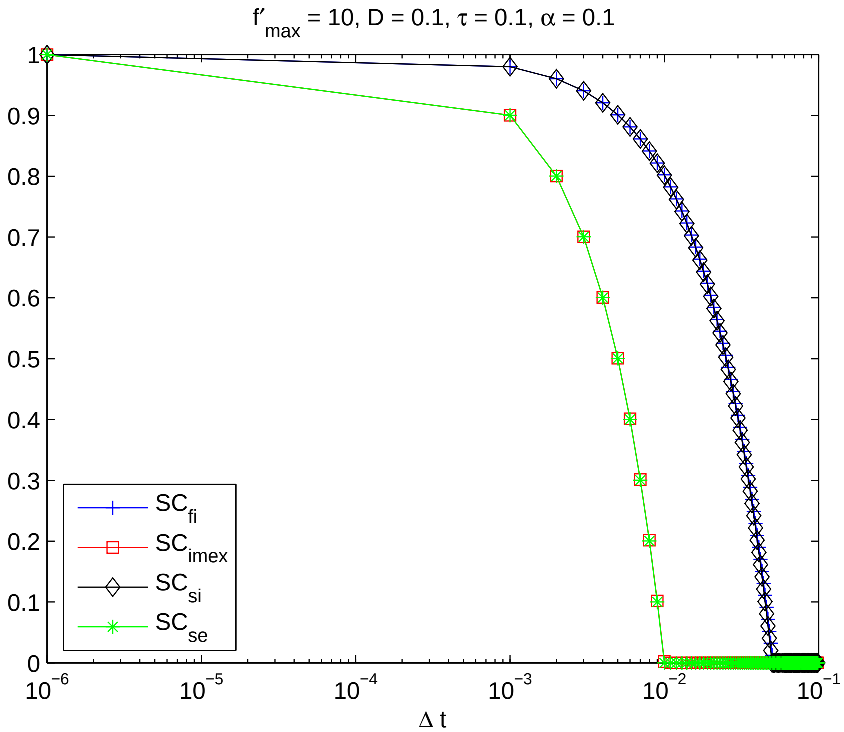

FIGURE 3. Stability conditions.

be the error for the different subproblems I_h–III_h. Let us first consider I_h. Considering the error equation for $e^{n+1}_{1,h}$ it can be shown that

$$\|e^{n+1}_{1,h}\|^2_h \leq \frac{(1+\Delta t)\theta\|e^n_{1,h}\|^2_h + \Delta t\|T^{n+1}_{1,h}\|^2_h}{1-\Delta t(\theta {f'_{max}}^2 + 2(1-\theta)f'_{max}+1)}, \tag{37}$$

provided that (35) holds.

Let us now consider II_h. Using on the error equation for $e^{n+1}_{2,h}$ the arguments considered on the proof of inequality (30), it can be shown that

$$\|e^{n+1}_{2,h}\|^2_h = \|e^n_{2,h}\|^2_h - 2\alpha(b-a)^{-2}\|\nabla_{h,-}e^{n+1}_{2,h}\|^2_{h+} + 2\Delta t(T^{n+1}_{2,h}, e^{n+1}_{2,h})_h.$$

Attending that

$$(T^{n+1}_{2,h}, e^{n+1}_{2,h})_h \leq \frac{1}{\epsilon^2}\|T^{n+1}_{2,h}\|^2_h + \epsilon^2\|e^{n+1}_{2,h}\|^2_h$$

holds for $\epsilon \neq 0$, considering $\epsilon^2 = 2\alpha(b-a)^{-2}$, we obtain

$$\|e^{n+1}_{2,h}\|^2_h \leq \|e^n_{2,h}\|^2_h + \frac{\Delta t}{2\alpha\Delta t(b-a)^{-2}}\|T^{n+1}_{2,h}\|^2_h. \tag{38}$$

Finally let us consider III_h. From the error equation for $e_{3,h}^{n+1}$, considering the procedures used on the proof of inequality (31), it can be shown

$$\begin{aligned} m(\Delta t,\theta)(\|e_{3,h}^{n+1}\|_h^2+\frac{D}{\tau}\|\Delta t(\sum_{j=1}^{n}e^{-\frac{t_{n+1}-t_j}{\tau}}\nabla_{h,-}e_h^j+\nabla_{h,-}e_{3,h}^{n+1})\|_{h+}^2) \\ \leq \|e_{3,h}^{n}\|_h^2+\frac{D}{\tau}e^{-2\Delta t/\tau}\|\Delta t\sum_{j=1}^{n}e^{-\frac{t_{n+1}-t_j}{\tau}}\nabla_{h,-}e_h^j\|_{h+}^2+\Delta t\|T_{3,h}^{n+1}\|_h^2. \end{aligned} \tag{39}$$

Combining (37)–(39) and attending that $e_{1,h}^n=e_h^n$, $e_{2,h}^n=e_{1,h}^{n+1}$ and $e_{3,h}^n=e_{2,h}^{n+1}$, we conclude that

$$E_h(e_h^{n+1})\leq \overline{S}(\Delta t,\theta)E_h(e_h^n).$$

As $E_h(e_h^0)=0$, we conclude the proof.

□

According to the Theorem 5, we conclude that, if

$$M(\Delta t,\theta)=e^{-2\Delta t/\tau},$$

we have

$$E_h(e_h^{n+1})\leq e^{CT}\|T_h\|_{h,\infty}^2,$$

where $C=-2/\tau$ if $m(\Delta t,\theta)=1$, and

$$C=\frac{1}{1-\Delta t(1-\frac{D}{\tau}\Delta t(b-a)^{-2})}$$

if

$$m(\Delta t,\theta)=1-\Delta t(1-\frac{D}{\tau}\Delta t(b-a)^{-2}).$$

For

$$M(\Delta t,\theta)=\frac{1+\Delta t\theta}{1-\Delta t(\theta {f'_{max}}^2+2(1-\theta)f'_{max}+1)}$$

we may obtain similar results.

The convergence of the θ family of methods is now consequence of the consistency, that is, $\|T_h\|_{h,\infty}=\mathcal{O}(h,\Delta t)$.

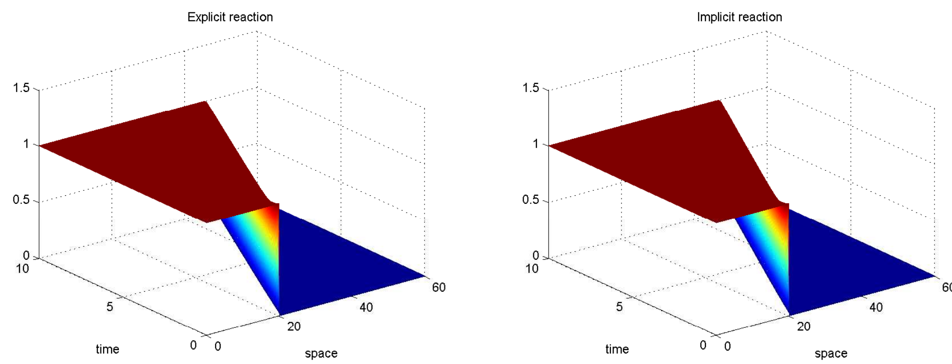

FIGURE 4. Splitting methods: $\Delta t = 10^{-2}$.

4. Numerical results

In this section we will consider the following IBVP

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= Uu(1-u) + \beta \nabla u(x,t) \\ &\quad + \alpha \Delta u(x,t) + \frac{D}{\tau}\int_0^t e^{-\frac{t-s}{\tau}} \Delta u(x,s)\, ds,\ \ x \in (0,60), t \in (0,T],\end{aligned}$$

where $U = 10$, $\alpha = 0.1$, $D = 0.1$, $\tau = 0.1$, $\beta = -0.3$, with initial and boundary conditions

$$u(x,0) = \begin{cases} 1, & x \in [0,20], \\ 0 & x \in (20,60], \end{cases}$$

$$u(0,t) = 1, \quad u(60,t) = 0,\ t \in (0,T].$$

To illustrate the theoretical results of the previous section, we will compare the the splitting methods defined by I_h–III_h, with $h = 0.1$ and different values of Δt. In Figures 4 – 5 we plot the numerical solutions obtained with method SI (implicit reaction) and with IMEX (explicit reaction) for $T = 10$.

As we expect, the implicit method perform better when we increase the time step.

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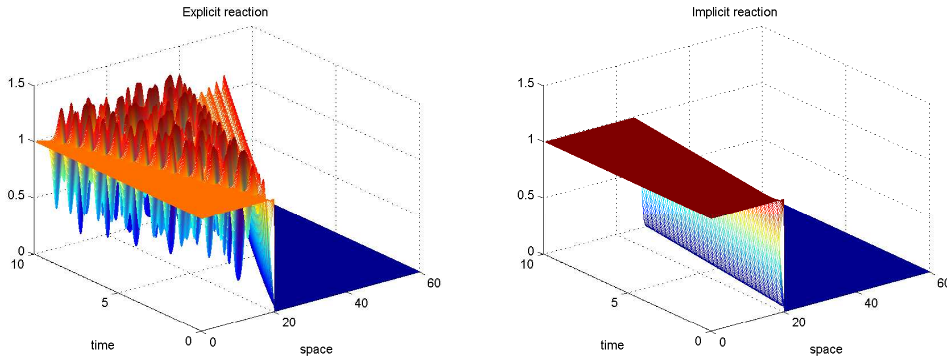

FIGURE 5. Splitting methods: $\Delta t = 0.3$.

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