ORDERS AND ACTIONS OF BRANCHED COVERINGS OF HYPERBOLIC LINKS

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Abstract: Let \( M, M' \) be compact oriented 3-manifolds and \( L' \) a link in \( M' \) whose exterior has positive Gromov norm. We prove that the topological types of \( M \) and \((M', L')\) determine the degree of a strongly cyclic covering \( p : M \to M' \), branched over \( L' \). Moreover, if \( M' \) is an homology sphere then these topological types determine also the covering up to conjugacy.

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1. Introduction

Let \( M, M' \) be compact, oriented 3-manifolds and \( G \) a group that acts orientation-preservingly on \( M \), with \( M/G \cong M' \). Suppose that \( M \) and \( M' \) are hyperbolic. If \( G \) acts freely then the natural projection \( p : M \to M' \) is a covering and the order of \( G \) is given by \(|G| = \text{vol}(M)/\text{vol}(M')\), where \text{vol} is the hyperbolic volume. A similar reasoning applies to manifolds which are not necessarily hyperbolic but whose JSJ decompositions [11, 12] contain hyperbolic pieces; it suffices to take the ratio between the sums of volumes of hyperbolic pieces of \( M \) and \( M' \).

In this paper we consider the uniqueness problem of \(|G|\) and of the conjugacy class of \( G \) in \( \text{Diff}^+(M) \) when the action is not free. Let \( L' \subset M' \) be a nonempty link and \( G \) a group that acts orientation-preservingly on \( M \), such that the natural projection \( p : M \to M/G \cong M' \) is a branched covering, with branch set \( L' \). The pre-image of \( L' \) in \( M \) is a link \( L \). The link \( L' \subset M' \) is prime if every embedded sphere in \( M' \) that cuts \( L' \) transversally in two points bounds a submanifold that intersects \( L' \) in an unknotted arc.

If \( p : M \to M' \) is a branched covering over \( L' \) with covering group \( G \), and the stabiliser of each point \( x \in L = p^{-1}(L') \) equals \( G \) then \( p \) is called a covering of \( M' \) strongly branched over \( L' \). When \( p : M \to M' \) is a strongly branched covering, \( G \) is a cyclic group and the branching order of every component of \( L' \) is the same and equal to \( d = |G| \). We will call \( p \) an \((L', d)\)-covering and

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note $O = (M', L', d)$ its quotient orbifold, that is, the orbifold with underlying manifold $M'$, singular set $L'$ and branching order $d$ along every component of $L'$.

The main results of this paper are the following theorems.

**Theorem 1.** Let $M$ and $M'$ be compact, orientable, 3-manifolds, whose boundary is a (possibly empty) disjoint union of tori. Let $L' \subset M'$ be a prime link. If the exterior of $L'$ in $M'$ is irreducible and its JSJ decomposition contains an hyperbolic piece, then there exists at most one positive integer $d$ for which $M$ is a $d$-fold covering of $M'$, strongly branched over $L'$.

**Theorem 2.** Let $M'$ be an integral homology sphere and $L' \subset M'$ an hyperbolic link. Then, for each manifold $M$, any two cyclic coverings $p_1, p_2 : M \to M'$ branched over $L'$, of prime degrees, are conjugated.

For completeness sake, in Theorem 1 we allow $d$ to be 1, that is, we show that $M'$ is not a self-covering strongly branched over a link. In [19] we consider the analogous problem for links whose exterior is a graph manifold.

To prove Theorem 1 we define the volume of an orbifold in section 2, and show that, under certain conditions, this volume increases with the branching order of the orbifold, a result that is interesting on its own. In section 3 we deduce Theorem 1 and in section 4 we prove Theorem 2.

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### 2. Volumes of orbifolds

Since the exterior $E'$ of $L'$ is irreducible and $L'$ is a prime link, it follows that $O$ is irreducible. By [3], there is a finite family $T$ of essential toric 2-suborbifolds, disjoints, embedded in $O$ such that each connected component of the complement $M' - N(T)$ of a regular neighbourhood $N(T)$ of $T$ is either atoroidal or a Seifert orbifold. Moreover, such a minimal family is unique up to isotopy. We call it the JSJ family of $O$. By Thurston’s Orbifold Theorem [2], the interior of the atoroidal pieces is either hyperbolic, euclidean, or admits a Seifert fibration. If $O$ is not atoroidal, it may also admit a geometry modelled on the Lie group Sol. We define the volume of $O$ as the sum of the hyperbolic volumes of all the hyperbolic pieces of the JSJ decomposition of $O$, and note it $\text{vol}(O)$. By Mostow’s rigidity theorem [1, 16], these hyperbolic volumes are topological invariants, which implies that the volume of $O$ is well-defined.
In this section, we establish a result relating the volume of an orbifold with its branching order. We make use of the following result of Souto [22] to compare the volume of a manifold with the volume of a metric defined on it with sectional curvature bounded below.

**Proposition 3.** Let $M$ be a closed, orientable, irreducible, geometrisable 3-manifold. Let $g$ be a metric on $M$ with sectional curvature bounded below by $-1$. Then $\text{vol}(M) \leq \text{vol}(M, g)$.

In Theorem 6 we will consider cone-manifold structures on a manifold, so we derive from this proposition the following result.

**Proposition 4.** Let $M$ be a closed orientable, irreducible, geometrisable 3-manifold and $C$ an hyperbolic cone-manifold structure on $M$. Then $\text{vol}(M) \leq \text{vol}(C)$.

**Proof.** To apply the previous proposition we need to replace the singular metric $g_C$ defined on $M$ by a smooth metric. For each $\varepsilon > 0$, we will obtain a smooth metric $g_\varepsilon$ on $M$, with sectional curvature bounded below by $-1$, such that

$$\text{vol}(M, g_\varepsilon) < u(\varepsilon) \text{vol}(M, g_C),$$

where $u$ is a function such that $\lim_{\varepsilon \to 0} u(\varepsilon) = 1$.

Let $V$ be a tubular neighbourhood of the singular set $L$ of $C$, with radius $r_0$. The cone hyperbolic metric $g_C$ is given on $V$ by

$$ds^2 = dr^2 + f^2(r) d\theta^2 + g^2(r) dz^2$$

where $f(r) = (d_1/d_2) \sinh r$ and $g(r) = \cosh r$. For $\delta > 0$ sufficiently small, we replace the singular metric $g_C$ in $V$ by a metric $g_\varepsilon^*$ given by

$$ds^2 = dr^2 + \varphi^2(r) d\theta^2 + \gamma^2(r) dz^2$$

where $\varphi, \gamma : [0, r_0 - \delta] \to [0, +\infty)$ are smooth functions such that:

1. in a neighbourhood of 0, $\varphi(r) = r$ and $\gamma(r)$ is constant;
2. in a neighbourhood of $r_0 - \delta$, $\varphi(r) = f(r + \delta)$ and $\gamma(r) = g(r + \delta)$;
3. $\forall r \in [0, r_0 - \delta]$, $\frac{\varphi''(r)}{\varphi(r)} \leq 1 + \varepsilon$, $\frac{\gamma''(r)}{\gamma(r)} \leq 1 + \varepsilon$ and $\frac{\varphi'(r)\gamma'(r)}{\varphi(r)\gamma(r)} \leq 1 + \varepsilon$.

Note $f_\delta(r) = f(r + \delta)$ and $g_\delta(r) = g(r + \delta)$. Let $r_1$ be the smallest positive solution of the equation $f_\delta(r) = r$. Set $\varphi = f_\delta$ in $[2r_1, r_0 - \delta]$. To define $\varphi$ in $[0, 2r_1]$, set $\varphi = \text{id}$ in a small neighbourhood of 0 and extend smoothly with $\varphi'' < 0 < \varphi'$ until $r_1$, $\varphi(r_1) \approx f_\delta(r_1)$ and $\varphi'(r_1) \approx f'_\delta(r_1)$. Then extend
ϕ smoothly to \([r_1,2r_1]\) so that \(ϕ''\) becomes rapidly close to \(f''_δ\) and finally maintain \(ϕ, \varphi'\) and \(ϕ''\) close to \(f_δ, f'_δ, f''_δ\) (see Figure 1).

Now we construct \(γ\). Set \(γ = g_δ\) in a small neighbourhood of \(r_0 - δ\). Extend \(γ\) smoothly to \([2r_1, r_0 - δ]\) putting \(γ'' < g''_δ, γ' < g'_δ\) except in a small neighbourhood of \(2r_1\) and \(γ''(2r_1) = γ'(2r_1) = 0\). Finally extend \(γ\) to \([0, 2r_1]\) keeping it constant. Choosing a small \(δ\) (hence a small \(r_1\) gives \(γ \approx g_δ, γ' \approx g'_δ\) in \([2r_1, r_0 - δ]\).

\[\]

**Figure 1.** Graphs of \(ϕ\) and \(ψ\)

By construction, the quotients given in (3) (which are symmetrical to the sectional curvatures of \(g^*_ε\)), are all bounded above by \(1 + ε\): in \([0, 2r_1]\), the latter two quotients are zero and the bound on the first quotient comes from the closeness of \(ϕ''\) and \(f''_δ\); in \([2r_1, r_0 - δ]\), the bound comes from \(ϕ = f_δ\) and the closeness of \(γ\) and \(g_δ\) and of their derivatives.

Condition (2) insures that we may glue the metric \(g^*_ε\) in \(V\) to the metric \(g_C\) in \(C - V\) along \(∂V\). The smooth riemannian metric obtained in \(M'\), which we note also \(g^*_ε\), has sectional curvature bounded below by \(-1 - ε\).

When \(δ \to 0\), the functions \(ϕ\) and \(γ\) approach respectively \(f\) and \(g\) so that volume \(\text{vol}(V, g^*_ε)\) approaches \(\text{vol}(V, g_C)\). Then, for \(δ > 0\) sufficiently small, \(g^*_ε\) is a smooth metric on \(M'\), with sectional curvature bounded below by \(-1 - ε\), such that \(\text{vol}(V, g^*_ε) < (1 + ε) \text{vol}(V, g_C)\). Then \(g_ε = (1 + ε)^{1/2} g^*_ε\) is a metric on \(M'\) with sectional curvature bounded below by \(-1\) and

\[\text{vol}(M, g_ε) < (1 + ε)^{3/2} \text{vol}(M, g^*_ε) < (1 + ε)^{5/2} \text{vol}(M, g_C)\]

Since by Proposition 3, \(\text{vol}(M) \leq \text{vol}(M, g_ε)\), for every \(ε > 0\), it follows that \(\text{vol}(M) \leq \text{vol}(C)\).

The following proposition will be useful to classify the non-hyperbolic orbifolds that may appear.
**Proposition 5.** Let $O = (M', L', d)$ be a geometric 3-orbifold where $L'$ is nonempty and $d \geq 3$. If $O$ is not hyperbolic, then either $O - L'$ admits a Seifert fibration which induces a Seifert fibration on $O$, or $O = (S^3, L', 3)$ where $L'$ is the figure-eight knot.

**Proof.** Since $O$ is geometric but not hyperbolic, it is a spherical, euclidean, Seifert or Sol orbifold. We study each case separately.

Suppose that $O$ is a Seifert orbifold. Since $d > 2$, every component of $L'$ is a fibre of $O$ and $O - L'$ admits a Seifert fibration.

If $O$ is a Sol orbifold, there is a non free action of a group $G$ of isometries of Sol such that $\text{Sol}/G \cong O$. Let $x \in \text{Sol}$ be a singular point of this action. Since the stabiliser of $x$ is the dihedral group $D_4$ [21], then $d = 2$, which contradicts the hypothesis that $d \geq 3$.

Suppose now that $O$ is a spherical orbifold that doesn’t admit Seifert fibrations. By [7], the singular set of $O$ contains vertices, which contradicts the hypothesis that $L'$ is a link.

If $O$ is an euclidean orbifold that doesn’t admit Seifert fibrations and its singular set is a link, the classification of crystallographic groups [6] shows that the underlying space of $O$ is the sphere $S^3$ and its singular set is the figure-eight knot with branching order $d = 3$.

Now we use Propositions 4 and 5 to obtain the main result of this section.

**Theorem 6.** Let $M'$ be a compact, orientable 3-manifold whose boundary is a (possibly empty) disjoint union of tori and $L' \subset M'$ a nonempty link. For $i = 1, 2$, let $O_i = (M, L', d_i)$, with $2 \leq d_1 < d_2$. If $O_1$ and $O_2$ are irreducible then $\text{vol}(O_1) \leq \text{vol}(O_2)$.

**Proof.** For $i = 1, 2$, consider the minimal families $F_i$ of toric incompressible non parallel suborbifolds that decompose $O_i$ in geometric pieces. Since, by definition, vol is additive with respect to these families, we may suppose that $F_1$ and $F_2$ do not contain tori.

Since the branching degree of $O_i$ is constant, $F_i$ is either empty, a union of euclidean turnovers $S^2(3, 3, 3)$, or a union of pillows $S^2(2, 2, 2, 2)$. The last case is not possible for $F_2$, since $d_2 > 2$. Furthermore, if $F_2$ contained an euclidean turnover, then $O_1$ would contain a spherical turnover $S^2(2, 2, 2)$. Since $O_1$ is irreducible, this spherical turnover bounds a discal orbifold whose singular set is a graph, which contradicts the hypothesis that $L'$ is a link. Then $F_2$ is empty and $O_2$ is a geometric orbifold.

Suppose $O_2$ is not hyperbolic. By Proposition 5, either $O_2 - L'$ admits a
Seifert fibration which induces a Seifert fibration on $O_2$, or $O_2 = (S^3, L', 3)$ where $L'$ is the figure-eight knot. In the first case, both $O_1$ and $O_2$ are Seifert-fibred and $\text{vol}(O_1) = \text{vol}(O_2) = 0$. In the latter case, $O_1 = (S^3, L', 2)$ is a spherical orbifold since its double covering is a lens space [20]. We have again $\text{vol}(O_1) = \text{vol}(O_2) = 0$.

There remains the case where $O_2$ is an hyperbolic orbifold.
Suppose that $\partial M$ is a nonempty union of tori $T_1, \ldots, T_n$. By Thurston’s Hyperbolic Surgery Theorem [23, 8, 2], there exist pairs of integers $(a_i, b_i)_{i=1, \ldots, n}$ such that the closed orbifold $\overline{O}_2$ obtained by Dehn filling of ratio $(a_i, b_i)$ along the tori $T_i \subset \partial O_2$ verify

$$\text{vol}(O_2) - \varepsilon < \text{vol}(\overline{O}_2) < \text{vol}(O_2).$$

Since $O_1$ is not necessarily hyperbolic, Thurston’s Hyperbolic Surgery Theorem does not apply directly to it. Consider the JSJ decomposition of $O_1$

$$O_1 = \bigcup_j O_1^j.$$

Each torus $T_i$ belongs to the boundary of a piece $O^j_1$ of this decomposition. If $O^j_1$ is hyperbolic, we may apply Thurston’s Hyperbolic Surgery Theorem to obtain a pair $(a_i, b_i)$ such that the closed orbifold $\overline{O}^j_1$ obtained from $O^j_1$ by Dehn filling of ratio $(a_i, b_i)$ along $T_i$ is hyperbolic and its hyperbolic volume is close to $\text{vol}(O^j_1)$. If $O^j_1$ is a Seifert orbifold, we may choose the pair $(a_i, b_i)$ such that $\overline{O}^j_1$ is still a Seifert orbifold. For that, it suffices that the fibres of $O^j_1$ are not meridians of the glued solid torus. Then $\text{vol}(\overline{O}^j_1) = \text{vol}(O^j_1) = 0$, that is, we choose a Dehn filling on this Seifert piece that doesn’t change its volume.

Since the pieces of the JSJ decomposition of the closed orbifold $\overline{O}_1$ are the orbifolds $\overline{O}^j_1$, we have that $\text{vol}(\overline{O}_1)$ and $\text{vol}(O_1)$ are close.
Since $\overline{O}_1$ is a very good orbifold [14], there exists a manifold $M$ and a finite group $G \subset \text{Diff}^+(M)$ such that $M/G \cong \overline{O}_1$. Lift the hyperbolic metric of $\overline{O}_2$ by the projection $p : M \to \overline{O}_1$ induced by the action of $G$, to obtain an hyperbolic cone-manifold structure $C$ on $M$. Its singular set is the singular set $L$ of the $G$-action with cone-angle $2\pi(d_1/d_2) < 2\pi$. By Proposition 4, $\text{vol}(M) \leq \text{vol}(C)$. Since the volumes of $M$ and $C$ are $d_1$ times the volumes of $\overline{O}_1$ and $\overline{O}_2$, respectively, then $\text{vol}(\overline{O}_1) \leq \text{vol}(\overline{O}_2)$. It follows from the above
construction that \( \text{vol}(O_1) < \text{vol}(O_2) + \varepsilon \), for every \( \varepsilon > 0 \), which shows that \( \text{vol}(O_1) \leq \text{vol}(O_2) \).

We note that for \( d_2 \geq d_1 \geq 3 \) we could prove Theorem 6 without using the method of Besson-Courtois-Gallot involved in the proof of Proposition 3. If \( d \geq 4 \), we can assume that both \( O_1 \) and \( O_2 \) are closed hyperbolic orbifolds. We can deform the cone hyperbolic metric of \( O_1 \) to the cone hyperbolic metric of \( O_2 \). By [2], when we increase the cone-angles from 0, no degeneration occurs, and the Schl"afli formula [10, 13, 15] shows that the hyperbolic volume decreases.

In the case \( d_1 = 3 \), it still possible to use the same reasoning but we need to cut \( O_1 \) along the euclidean turnovers \( S^2(3,3,3) \) of the JSJ family of \( O_1 \) and cut \( O_2 \) along the corresponding family of hyperbolic turnovers \( S^2(d_2,d_2,d_2) \). This family is isotopic to a family of totally geodesic turnovers [2]. Now we must use the theory of deformations of hyperbolic manifolds with totally geodesic boundary [5] to conclude that the hyperbolic volume also decreases in this case.

3. Uniqueness of the degree

In this section we use Theorem 6 to prove Theorem 1.

Proof of Theorem 1. For \( i = 1,2 \), let \( p_i : M \to M' \) be an \((L,d_i)\)-covering, and suppose \( d_1 < d_2 \).

For the case \( d_1 = 1 \), we consider the Gromov simplicial volumes [9, 23] of the pairs \((M,\partial M)\) and \((M',\partial M')\). Since \( p_1 : (M,\partial M) \to (M',\partial M') \) is a diffeomorphism, we have \( \|(M,\partial M)\| = \|(M',\partial M')\| \). On the other hand \( p_2 : (M,\partial M) \to (M',\partial M') \) has degree \( d_2 > 1 \), which gives

\[
\|(M,\partial M)\| \geq d_2\|(M',\partial M')\| > \|(M',\partial M')\|,
\]

a contradiction.

We suppose now that \( 2 \leq d_1 < d_2 \). For \( i = 1,2 \), let \( O_i \) be the quotient orbifold of \( M \) by \( p_i \). By Theorem 6, \( \text{vol}(O_1) \leq \text{vol}(O_2) \). Lifting the geometric decomposition of \( O_i \) to \( M \), we get a \( G_i \)-invariant geometric decomposition of \( M \). Hence,

\[
0 \leq \frac{\text{vol}(M)}{d_1} = \text{vol}(O_1) \leq \text{vol}(O_2) = \frac{\text{vol}(M)}{d_2} < \infty.
\]

Since \( d_1 < d_2 \), this inequality shows that \( M \) has null volume. Therefore both orbifolds \( O_i \) contain no hyperbolic pieces.
Let $E'_0$ be an hyperbolic piece of the geometric decomposition of the exterior $E'$ of $L'$ (whose existence is granted by hypothesis) and $L'_0$ the union of the connected components of $L'$ touching $E'_0$. Since $\text{vol}(O_i) = 0$, the link $L'_0$ is nonempty. We denote $O^0_i$ the correspondent geometric suborbifold of $O_i$, $i = 1, 2$. Proposition 5 shows that $O^0_2 = (S^3, L', 3)$, where $L'$ is the figure-eight knot. Hence $O^0_2 = O_2$ and $M'$ is an euclidean manifold. Again, $O_1 = (S^3, L', 2)$ is a spherical orbifold. Therefore $M$ admits both an euclidean and a spherical metric, which is impossible [21].

We end this section with an easy corollary.

**Corollary 7.** Let $L \subset S^3$ be a prime link. If $L$ is not a an iterated cable then all cyclic coverings of $L$ are nonhomeomorphic.

**Proof.** This follows from the fact that the JSJ decomposition of the exterior of a link has no hyperbolic piece iff the link is an iterated cable. □

**4. Uniqueness of the action**

In this section we prove Theorem 2. Let $p_1, p_2 : M \to M'$ be two branched coverings over an hyperbolic link $L'$ with prime degrees. Then $p_1, p_2$ are strongly branched coverings. Theorem 1 allows us to suppose that the degrees of $p_1$ and $p_2$ are the same. Then Theorem 2 is a consequence of the following theorem.

**Theorem 8.** Let $M$ be a closed orientable manifold, $M'$ is a $\mathbb{Z}_d$-homology sphere where $d$ is prime, and $L' \subset M'$ an hyperbolic link. Then any two $(L', d)$–coverings $p_1, p_2 : M \to M'$ are conjugated.

To prove this, we consider first the easier cases ($d = 2$, $L'$ is a knot, $M$ is not hyperbolic) and in Proposition 13 we prove the remaining case.

For $i = 1, 2$, let $G_i$ be the covering group of the $(L', d)$–covering $p_i : M \to M'$. Denote $E'$, $E_i$ the exteriors of $L'$ and $L = p_i^{-1}(L')$, respectively. Since the branched covering $p_i : M \to M'$ induces a cyclic covering $p_i|_{E_i} : E_i \to E'$, we have the exact sequences

$$1 \longrightarrow \pi_1(E_i) \xrightarrow{p_i *} \pi_1(E') \xrightarrow{\rho_i} \mathbb{Z}_d \longrightarrow 1.$$ 

Since $\mathbb{Z}_d$ is abelian, the representation $\rho_i : \pi_1(E') \to \mathbb{Z}_d$ factors

$$\pi_1(E') \xrightarrow{\rho_i} \mathbb{Z}_d \xrightarrow{} H_1(E').$$
We call the homomorphism \( \rho_i : H_1(E') \to \mathbb{Z}_d \) the holonomy of the covering \( p_i \). The image \( \rho(\mu) \) of the meridian of each component of \( L' \) by the holonomy of the covering is nontrivial.

Since \( M' \) is a \( \mathbb{Z}_d \)-homology sphere, the Mayer-Vietoris sequence of the triple \( (M', E', V') \), where \( V' \) is a tubular neighbourhood of \( L' \), shows that \( H_1(E'; \mathbb{Z}_d) \cong \mathbb{Z}_d^n \), where \( n \) is the number of components of \( L' \).

**Proposition 9.** Let \( p_1, p_2 : M \to M' \) be two \( (L', d) \)-coverings and its holonomies \( \rho_1, \rho_2 : H_1(E') \to \mathbb{Z}_d \). If \( \ker(\rho_1) = \ker(\rho_2) \), then \( p_1 \) and \( p_2 \) are conjugated.

**Proof.** Since \( \ker(\rho_1) = \ker(\rho_2) \), the coverings \( p_1|_{E_1} \) et \( p_2|_{E_2} \) are equivalent, that is, the following diagram is commutative,

\[
\begin{array}{ccc}
E_1 & \overset{\rho_i}{\longrightarrow} & E_2 \\
\downarrow p_1 & & \downarrow p_2 \\
E' & \overset{=}{{\longrightarrow}} & E'
\end{array}
\]

where \( h : E_1 \to E_2 \) is a diffeomorphism and \( \overset{=}{{\longrightarrow}} \) represents the identity. Then, if we note \( G_i|_{E_i} \) the group of restrictions to \( E_i \) of the diffeomorphisms of \( G_i \), we have

\[ hG_2|_{E_2}h^{-1} = G_1|_{E_1}. \]

Now we want to extend \( h : E_1 \to E_2 \) to a diffeomorphism \( M \to M \). The inverse image by \( p_i \) of \( d \) times the meridian \( \mu \) of each component of \( L' \) is a meridian \( \mu_i \) of a component of \( L_i \). Then \( h(\mu_1) = \mu_2 \), that is, the meridian of each component of \( L_1 \) is sent by \( h \) over the meridian of a component of \( L_2 \). This shows that \( h \) can be extended to \( M \). \( \square \)

**Corollary 10.** Theorem 8 is true for \( d = 2 \).

**Proof.** There is a single homomorphism \( \rho : H_1(E') \to \mathbb{Z}_2 \) such that the image of the meridians of each connected component of \( L' \) is nontrivial. \( \square \)

Now we suppose that the singular set \( L' \) of \( M' \) is a knot and, with this condition, we prove that \( G_1 \) and \( G_2 \) are conjugated in \( \text{Diff}^+(M) \).

**Proposition 11.** If \( M' \) is a \( \mathbb{Z}_d \)-homology sphere and \( L' \) is a knot in \( M' \), then any two \( (L', d) \)-coverings \( p_1, p_2 : M \to M' \) are conjugated.

**Proof.** Since \( L' \) is a knot, \( H_1(E') \cong \mathbb{Z}_d \). Therefore, a nontrivial homomorphism \( \rho : H_1(E') \to \mathbb{Z}_d \) is unique, up to right compositions by an automorphism of \( G_i \), for \( d \) is prime. Then the kernels of the holonomies \( \rho_1 \) et \( \rho_2 \) are the same for the two actions. By Proposition 9, \( p_1 \) and \( p_2 \) are conjugated.
Corollary 12. Theorem 8 is true when \( d \geq 3 \) and \( M \) is not an hyperbolic manifold.

Proof. Since \( d \geq 3 \) and \( M' \) is a \( \mathbb{Z}_d \)-homology sphere, Thurston’s Orbifold Theorem shows that \( O \) is a geometric orbifold. Since \( O \) is not hyperbolic and the exterior \( E' \) of \( L' \) is hyperbolic, Proposition 5 shows that \( O = (S^3, L', 3) \), where \( L' \) is the figure-eight knot. The conclusion that the two \( \mathbb{Z}_3 \)-actions on \( M \) are conjugated follows from Proposition 11.

We now prove Theorem 8 when \( M \) is an hyperbolic manifold, \( L' \) is disconnected and \( d \geq 3 \).

Proposition 13. Let \( M \) be a closed hyperbolic manifold. Let \( G_1 \) and \( G_2 \) two nonfree actions of the cyclic group \( \mathbb{Z}_d \) on \( M \), with \( d \geq 3 \) prime. Then the actions of \( G_1 \) and \( G_2 \) are conjugated if and only if the quotient orbifolds are diffeomorphic.

The orbifold theorem shows that the actions of \( G_1 \) and \( G_2 \) are conjugated to isometric actions. We may then suppose that \( G_1 \) and \( G_2 \) are isometry groups of \( M \). Note
\[
L_i = \text{Fix}(G_i),
\]
the set of fixed points of \( G_i \). Since \( G_i \) is a cyclic group of prime order, the covering of \( M' \) by \( M \) is strongly branched. Therefore both links \( L_i \) contain the same number of connected components as \( L' \).

Lemma 14. The group \( G_i \) is the group of isometries of \( M \) that fix \( L_i \) pointwise.

Proof. Let \( K \) be a component of \( L_i \) and \( x \) an isometry of \( M \) such that \( K \subseteq \text{Fix}(x) \). Let \( \tilde{K} \) be a component of the covering of \( K \) in \( \mathbb{H}^3 \). Since \( x \) and the generator \( g_i \) of \( G_i \) fix \( K \) pointwise, there are isometries \( \tilde{x} \) and \( \tilde{g}_i \) of \( \mathbb{H}^3 \) that project respectively over \( x \) and \( g_i \) and fix \( \tilde{K} \) pointwise. Then \( \tilde{x} \) and \( \tilde{g}_i \) are rotations around the hyperbolic line \( \tilde{K} \), thus commuting in \( \text{PSL}_2(\mathbb{C}) \). Then \( x \) and \( g_i \) commute in \( \text{Isom}(M) \). This shows that \( x \) projects by \( p_i \) over an isometry \( x' \) of \( O \) that fixes \( L' \) pointwise. Since \( M' \) is a \( \mathbb{Z}_d \)-homology sphere and \( L' \) is disconnected, it follows that the isometry \( x' \) is trivial [4], and therefore \( x \in G_i \).

Since \( M \) is an hyperbolic manifold, the isometry group of \( M \) is finite. Since \( d \) is a prime number, \( \text{Isom}(M) \) contains a Sylow \( d \)-group \( S \). After conjugating by an isometry, we may suppose that \( G_1 \) and \( G_2 \) are in \( S \). We will prove that \( G_1 \) and \( G_2 \) are the same. Note \( N_i = N_S(G_i) \) the normaliser of \( G_i \) in \( S \).
**Lemma 15.** The group $N_i$ is the group of isometries of $M$ that fixes $L_i$ setwise.

**Proof.** Let $x \in \text{Isom}(M)$ be such that $x(L_i) = L_i$. Then $\text{Fix}(x^{-1}G_ix) = x(\text{Fix}G_i) = x(L_i) = L_i$. Lemma 14 shows then that $x^{-1}G_ix \subseteq G_i$, and therefore $x \in N_i$. The reciprocal inclusion is immediate. □

The following proposition was proved in [17] in a more general form, where the degree is not necessarily prime.

**Proposition 16.** If $G_1 \neq G_2$, then $L'$ has $d$ components.

**Proof.** By Sylow theory, either $N_1 = S$ or $N_1$ contains a subgroup $xG_1x^{-1} \neq G_1$, where $x \in S - N_1$.

In the first case, we have $G_2 \subset N_1$. Then $G'_2 = p_1(G_2)$ is a nontrivial subgroup of $\text{Isom}^+(O)$. Since $\text{Fix}(G'_2)$ is nonempty and $M'$ is a $Z_d$–homology sphere, then $\text{Fix}(G'_2)$ is a knot. Then the number of components of $L_2 = \text{Fix}(G_2)$ divides $|G_1| = d$. Since $d$ is prime, the link $L_2$ (and therefore $L'$) has $d$ components.

In the second case, let $y \in xG_1x^{-1} - G_1$. Then $\text{Fix}(y) = \text{Fix}(xg_1x^{-1}) = x(\text{Fix}(g_1)) = x(L_1)$, which is a link with the same number of components as $L_1$. Since $y \in N_1 - G_1$, then $y' = p_1(y)$ is a nontrivial isometry of $O$. Since $\text{Fix}(y') \neq \emptyset$ and $M'$ is a $Z_d$–homology sphere, then $\text{Fix}(y')$ is a knot. Then the number of components of $\text{Fix}(y)$ divides $|G_1| = d$ and as before $L'$ has $d$ components.

Since $d$ is prime, an element of $N_i$ either preserves each component of $L_i$, or it permutes cyclically the components of $L_i$. We consider both cases in Propositions 17 and 18.

**Proposition 17.** If every element of $N_1$ (or $N_2$) preserves each component of $L_1$ (respectively $L_2$), then $G_1 = G_2$.

**Proof.** By hypothesis, $N_1$ contains only hyperbolic transformations which keeps invariant a tubular neighbourhood of each component of $L_1$. They act on this tubular neighbourhood as rotations along and around its axis with order a power of $d$. Then $N_1$ is a subgroup of $Z_{d^r} \oplus Z_{d^s}$, where the first factor corresponds to rotations around the components of $L_1$ and the second corresponds to rotations along these components.

The group $N_1'$ induces a group of isometries $N_1'$ of the quotient orbifold $O$, which is a subgroup of $Z_{d^r-1} \oplus Z_{d^s}$. We will prove that $N_1'$ is cyclic. First notice that $N_1'$ cannot contain a nontrivial element $(a, 0)$, since this
isometry of $O$ would act on a tubular neighbourhood of $L'$ as a rotation around the components of $L'$. Since $M'$ is a $\mathbb{Z}_d$–homology sphere, this is impossible by Smith theory. Then $N'_1$ cannot contain two distinct elements $(a_1, b)$ and $(a_2, b)$, since it would contain also the nontrivial element $(a_1 - a_2, 0)$. Therefore $N'_1$ is generated by the unique element $\eta = (a, b)$ with minimal positive second coordinate.

Let $k$ be the smallest integer such that $A' = \text{Fix}(\eta^d)$ is nonempty. Since $M'$ is a $\mathbb{Z}_d$–homology sphere, $A'$ is a knot. For every element $x'$ of $N'_1$, we have either $\text{Fix}(x') = \emptyset$, or $\text{Fix}(x') = A'$, since $N'_1$ is cyclic. Then, for every element $x$ of $N_1 - G_1$, we have $\text{Fix}(x) \subseteq p_1^{-1}(A')$. Since $d$ is a prime number,

$$A = p_1^{-1}(A')$$

is either a knot, or a link with $d$ components. Now consider an element $x \in N_1$ such that $\text{Fix}(x)$ is a link with $d$ components. Then $\text{Fix}x = L_1$ or $\text{Fix}x = A$, according to if $x \in G_1$ or not.

Now we want to prove that $N_S(N_1) = N_1$. Choose any element $y \in N_S(N_1)$. The preceding argument shows that either $y(L_1) = L_1$, or $y(L_1) = A$. In the second case, we have $y^2(L_1) = L_1$. Then, the isometry $y$ has even order, which contradicts the hypothesis that $d$ is prime and greater than 2. It follows that $y(L_1) = L_1$. Lemma 15 shows that $y \in N_1$, for every $y \in N_S(N_1)$. Then $N_S(N_1) = N_1$. Sylow theory shows then that $N_1 = S$.

We have proven that $S$ is commutative. Then $G_1$ and $G_2$ commute and therefore $G'_1 = p_2(G_1)$ is a subgroup of $\text{Isom}^+(O)$. Since $M'$ is a $\mathbb{Z}_d$–homology sphere, if $G'_1$ was nontrivial, $\text{Fix}G'_1$ would be a knot. Then $G_2$ would permute the components of $L_1$, which contradicts the hypothesis, and therefore $G'_1$ is trivial, and $G_1 = G_2$. \hfill \Box

To conclude the proof of Theorem 13, there remains to prove the case where $N_1$ and $N_2$ contain both elements that permute cyclically the $d$ components of $L_1$, respectively $L_2$.

**Proposition 18.** If, for $i = 1, 2$, $N_i$ contains an element $x_i$ that permutes cyclically the $d$ components of $L_i$, then $G_1 = G_2$.

**Proof.** Let $x_i \in N_i$ be such an element. We have $x_i g_i x_i^{-1} \in G_i$, and therefore it exists an integer $k_i \in \{0, 1, \ldots, d - 1\}$ such that

$$x_i g_i x_i^{-1} = g_i^{k_i}.$$
Then $x_i^2 g_i x_i^{-2} = x_i g_i^k x_i^{-1} = (x_i g_i x_i^{-1})^k = g_i^k$ and, more generally,

$$x_i^l g_i = g_i^l x_i^l.$$  

Then, for $l = d$, we obtain $x_i^d g_i = g_i^{k_i} x_i^d$. Since $x_i^d$ and $g_i$ keep invariant each component of $L_i$, they commute. It follows that

$$g_i = g_i^{k_i}.$$  

Since $d$ is prime, we obtain from the Fermat’s little theorem the congruence $k_i^d \equiv k_i \pmod{d}$, and therefore $g_i = g_i^{k_i}$ and $k_i = 1$. Then $x_i$ commutes with $g_i$.

Then $g_i$ acts locally as a rotation around each component of $L_i$ with the same angle of rotation. Then, the arrow

$$H_1(M' - L'; \mathbb{Z}_d) \cong \mathbb{Z}_d \oplus \cdots \oplus \mathbb{Z}_d \rightarrow G_i$$

sends each meridian to the same power of $g_i$, up to automorphisms of $G_i$.

Then the kernels of the holonomies associated to $G_1$ and $G_2$ are the same. By Proposition 9, $G_1$ and $G_2$ are conjugated. □

References


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