Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 07–12

MONOTONE INSERTION AND MONOTONE EXTENSION OF FRAME HOMOMORPHISMS

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Abstract: The purpose of this paper is to introduce monotonization in the setting of pointfree topology. More specifically, monotonically normal locales are characterized in terms of monotone insertion and monotone extensions theorems.

Keywords: Frame, locale, sublocale, monotonically normal locale, diameter, metrizable locale, frame homomorphism, monotone insertion property, monotone extension property, upper semicontinuous, lower semicontinuous.
AMS Subject Classification (2000): 06D22, 54D15, 54C30, 54E35.

1. Introduction

The purpose of this paper is to introduce monotonization into pointfree topology. We first recall that pointfree topology deals with complete lattices in which finite meets distribute over arbitrary joins. These lattices are called frames or locales. A map between two frames is a frame homomorphism if it preserves arbitrary joins and finite meets. The resulting category is denoted by **Frm**. One source of frames is given by the lattice $\mathcal{O}X$ of all open subsets of a topological space X. The assignment $X \mapsto \mathcal{O}X$ gives rise to a contravariant functor $\mathcal{O} : \mathbf{Top} \to \mathbf{Frm}$ which makes a continuous map $f : X \to Y$ into the frame homomorphism $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ determined by $\mathcal{O}f(U) = f^{-1}(U)$ for all $U \in \mathcal{O}Y$.

What is then meant by a monotonization? Suppose we have a concept consisting of sets P, Q and a specific map $\Delta : P \to Q$. Suppose further that we can enrich the concept by claiming that both P and Q carry partial orderings $\leq_P$ and $\leq_Q$ and then require the map $\Delta : (P, \leq_P) \to (Q, \leq_Q)$ to be monotone, i.e., order-preserving. In this way we have arrived at a new concept which is just the monotonization of the former concept. Usually,

Received March 31, 2007.
The first and the second named authors acknowledge financial support from the Ministry of Education and Science of Spain and FEDER under grant MTM2006-14925-C02-02. The first named author also acknowledges financial support from the University of the Basque Country under grant UPV05/101. The third named author acknowledges financial support from the Centre of Mathematics of the University of Coimbra/FCT.

monotonization yields a specialization of the original concept. It should be remarked that a particular concept may have different monotonizations (cf. [5]).

To illustrate the monotonization procedure, let X be a topological space with topology $\mathcal{O}X$ (and $\mathcal{C}X$ being the family of all closed sets of X), let $P = \{(K,U) \in \mathcal{C}X \times \mathcal{O}X : K \subseteq U\}$ and $Q = \mathcal{O}X$. Then X is normal if and only if there exists a map $\Delta : P \to Q$ with $K \subseteq \Delta(K,U) \subseteq \overline{\Delta(K,U)} \subseteq U$ for all $(K,U) \in P$. Such a map is called a normality operator. Now observe that both P and Q carry natural orderings. Namely, P is ordered by componentwise inclusion $\leq_P$, i.e., $(K_1,U_1) \leq_P (K_2,U_2)$ if and only if $K_1 \subseteq K_2$ and $U_1 \subseteq U_2$, while $\leq_Q$ is the usual inclusion. One may ask what happens if one requires $\Delta : (P,\leq_P) \to (Q,\leq_Q)$ to be monotone. A space X for which there exists a monotone normality operator is called monotonically normal. We refer to [8] for a survey of those spaces.

The paper is organized as follows. Section 2 sets up the basic terminology of pointfree topology. In Section 3 we characterize normal locales in terms of certain operators. We introduce hereditarily normal locales as those locales in which every sublocale is normal and prove that this is equivalent to the requirement that each open sublocale be normal. Section 4 deals with monotonically normal locales. These are characterized in several ways in terms of monotone normality-type operators. We show that every metrizable locale is monotonically normal. Section 5 provides the monotone localic Katětov-Tong insertion theorem. On one hand it is a monotonization of the localic insertion theorem of [14], while on the other it is a pointfree variant of the monotone insertion theorem of [11]. In Section 6, the monotone localic insertion theorem is used to characterize monotonically normal locales in terms of monotone extenders. When applied to topological spaces it gives the result proved directly in [20]. We point out that, in contrast to [20], our argument is free of the T_1-axiom.

2. Background in locales

Here we gather together some basic frame-theoretic terminology that we shall need in what follows. Some other specific concepts will be defined when actually needed. Our main references for frames and locales are [10] and [16].

A *frame* L is a complete lattice satisfying the frame distributive law

$$a \wedge \bigvee B = \bigvee\{a \wedge b : b \in B\}$$

for all $a \in L$ and $B \subseteq L$. A *frame homomorphism* is a map $f : L \to M$ which preserves finite meets (including the top 1) and arbitrary joins (including the bottom 0). The bounds of L may occasionally be denoted by 1_L and 0_L. The resulting category is denoted **Frm**. The set of all frame homomorphisms from L to M is denoted by $\mathbf{Frm}(L, M)$. The category of locales is the dual category $\mathbf{Loc} = \mathbf{Frm}^{op}$ of **Frm**.

Due to the frame distributive law, all the maps $a \wedge (\cdot) : L \to L$ preserve arbitrary joins and, thus, have right (Galois) adjoints $a \to (\cdot) : L \to L$, which means that $a \wedge c \leq b$ iff $c \leq a \to b$. Thus, in a frame L we have

$$a \to b = \bigvee \{c \in L : a \wedge c \leq b\},$$

and for $a, b \in L$ and $A \subseteq L$ the following hold:

(H1) $a \leq b \to a$,
(H2) $a \wedge (a \to b) = a \wedge b$,
(H3) $a \to b = 1$ iff $a \leq b$,
(H4) $(\bigvee A) \to b = \bigwedge_{a \in A}(a \to b)$.

The *pseudo-complement* of $a \in L$ is $a^* = a \to 0$. Then $a \wedge a^* = 0$, $a \leq a^{**}$ and $(\bigvee A)^* = \bigwedge_{a \in A} a^*$. Also, $a \leq b$ iff $b^* \leq a^*$.

The subobjects in **Loc** (equivalently, the quotients in **Frm**) have been described in several equivalent ways in the literature. The definition of a sublocale that we adopt here is taken from [10, Exercise II.2.3]. It follows the lines of [15].

A subset $S \subseteq L$ is a *sublocale* of L if it satisfies the following:

(S1) For every $A \subseteq S$, $\bigwedge A \in S$,
(S2) For every $a \in L$ and $s \in S$, $a \to s \in S$.

Partially ordered by inclusion, the set of all sublocales of L is a complete lattice (more precisely, a co-frame, i.e. a dual of a frame) in which $\{1\}$ is the bottom and L is the top. The sets

$$\mathfrak{o}(a) = \{a \to b : b \in L\} \quad \text{and} \quad \mathfrak{c}(a) = {\uparrow} a = \{b \in L : a \leq b\}$$

are sublocales of L for all $a \in L$. They will be referred to as the *open* and *closed* sublocales of L, respectively.

Each sublocale $S \subseteq L$ is a frame with the same meets as in L and with the same Heyting operation $\to$, since the latter merely depends on the meet operation. However, the joins in S may differ from those of L. Thus, $1_S = 1$, but in general $0_S \neq 0$. In particular, $0_{\mathfrak{o}(a)} = a^*$ and $0_{\mathfrak{c}(a)} = a$.

A sublocale $S \subseteq L$ determines the surjection $c_S \in \mathbf{Frm}(L, S)$ given by

$$c_S(a) = \bigwedge(S \cap \mathfrak{c}(a))$$

for all $a \in L$. In particular, $c_S(s) = s$ for all $s \in S$.

Let us also recall that $A \subseteq L$ *generates* L if each element of L is a join of a family of meets of finite subsets of A.

Following [1] and [2], the *locale of reals* is the locale $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q) where $p, q \in \mathbb{Q}$, subject to the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,
(R3) $(p, q) = \bigvee\{(r, s) : p < r < s < q\}$,
(R4) $1 = \bigvee_{p,q\in\mathbb{Q}}(p, q)$.

By (R3), $(p, q) = \bigvee \varnothing = 0$ if $p \geq q$. We write:

$$(p, -) = \bigvee_{q\in\mathbb{Q}} (p, q) \quad \text{and} \quad (-, q) = \bigvee_{p\in\mathbb{Q}} (p, q).$$

Then $(p, -) \wedge (-, q) = (p, q)$.

An obvious equivalent representation of the locale of reals is the following [12]: $\mathfrak{L}(\mathbb{R})$ is the locale generated by the elements $(p, -)$ and $(-, q)$ where $p, q \in \mathbb{Q}$, subject to the following relations:

(R1′) $(p, -) \vee (-, q) = 1$ whenever $p < q$,
(R2′) $(p, -) \wedge (-, q) = (p, q)$ whenever $q \leq p$,
(R3′) $(p, -) = \bigvee_{r>p}(r, -)$,
(R4′) $(-, q) = \bigvee_{r<q}(-, r)$,
(R5′) $1 = \bigvee_{p\in\mathbb{Q}}(p, -) = \bigvee_{q\in\mathbb{Q}}(-, q)$.

Following [7] (cf. also [12]), we denote by $\mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_l(\mathbb{R})$ the sublocales of $\mathfrak{L}(\mathbb{R})$ generated by all the elements $(p, -)$ and $(-, q)$ $(p, q \in \mathbb{Q})$, respectively. As in [7], we let

$$\mathrm{USC}(L) = \{f \in \mathbf{Frm}(\mathfrak{L}_l(\mathbb{R}), L) : \bigcap_{q\in\mathbb{Q}} \mathfrak{o}(f(-, q)) = \{1\}\}$$

and

$$\mathrm{LSC}(L) = \{g \in \mathbf{Frm}(\mathfrak{L}_u(\mathbb{R}), L) : \bigcap_{p\in\mathbb{Q}} \mathfrak{o}(g(p, -)) = \{1\}\}.$$

Members of USC(L) [resp., LSC(L)] are called *upper* [resp., *lower*] *semi-continuous real functions* on L.

Remark 2.1. The two extra algebraic conditions defining $\mathrm{USC}(L)$ and $\mathrm{LSC}(L)$ are just translations in terms of sublocales of the original conditions of [7] stated in terms of congruences. We recall that the reason of these extra conditions is that when $L = \mathcal{O}X$, there is a one-to-one correspondence between $\mathrm{USC}(\mathcal{O}X)$ and the set $\mathrm{USC}(X, \mathbb{R})$ of all upper semicontinuous real functions on X as well as between $\mathrm{LSC}(\mathcal{O}X)$ and the set $\mathrm{LSC}(X, \mathbb{R})$ of all lower semicontinuous real functions on X (see [7, Corollary 4.3]). With these conditions our pointfree insertion and extension theorems become true generalizations of their topological counterparts (see also [14] for a related discussion).

Partial orders: (1) $\mathrm{USC}(L)$ is partially ordered by:

$$f_1 \leq f_2 \;\Leftrightarrow\; f_2(-,q) \leq f_1(-,q) \text{ for all } q \in \mathbb{Q}.$$

(2) $\mathrm{LSC}(L)$ is partially ordered by:

$$g_1 \leq g_2 \;\Leftrightarrow\; g_1(p,-) \leq g_2(p,-) \text{ for all } p \in \mathbb{Q}.$$

(3) $\mathrm{C}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), L)$ is partially ordered by:

$$h_1 \leq h_2 \;\Leftrightarrow\; h_{1|\mathrm{LSC}(L)} \leq h_{2|\mathrm{LSC}(L)} \text{ and } h_{2|\mathrm{USC}(L)} \leq h_{1|\mathrm{USC}(L)}.$$

Members of $\mathrm{C}(L)$ are called *continuous real functions* [1] on L.

3. Normal and hereditarily normal locales

Recall that a locale L is called *normal* if, given $a, b \in L$ with $a \vee b = 1$, there exist $u, v \in L$ such that $a \vee u = 1 = b \vee v = 1$ and $u \wedge v = 0$. Clearly, one can select $v = u^*$. Thus, L is normal if and only if, whenever $a \vee b = 1$, there exists a $u \in L$ satisfying $a \vee u = 1 = b \vee u^*$. For a future monotonization (in Section 4), it will be convenient to restate the definition of normality in the following terms. Let

$$\mathcal{D}_L = \{(a, b) \in L \times L : a \vee b = 1\}.$$

A locale L is normal if and only if there exists a function $\Delta : \mathcal{D}_L \to L$ such that

$$a \vee \Delta(a, b) = 1 = b \vee \Delta(a, b)^*$$

for all $(a, b) \in \mathcal{D}_L$. The function Δ is called a *normality operator*.

Notation and terminology. For an arbitrary function $\Delta : \mathcal{D}_L \to L$ we let $\Delta^{op}(a, b) = \Delta(b, a)$. The function Δ will be called *self-disjoint* whenever the pointwise meet $\Delta \wedge \Delta^{op}$ is equal to 0.

Remarks 3.1. (1) If $\Delta : \mathcal{D}_L \to L$ is a normality operator, then so is $\Delta^{\circledast}$ defined by

$$\Delta^{\circledast}(a,b) = \Delta(b,a)^*.$$

(2) If Δ_1 and Δ_2 are normality operators, then so is $\Delta_1 \wedge \Delta_2$ (the pointwise meet).
(3) In particular, each normal locale L admits a self-disjoint normality operator Θ. Indeed, if Δ is a normality operator, then $\Theta = \Delta \wedge \Delta^{\circledast}$ has the required property.

Proposition 3.2. *Let L be a locale. The following are equivalent:*

(1) *L is normal.*
(2) *There exists a self-disjoint $\Theta : \mathcal{D}_L \to L$ such that $a \vee \Theta(a,b) = 1$ for all $(a,b) \in \mathcal{D}_L$.*

Proof: (1) implies (2) by Remark 3.1(3). If (2) holds true, then $b \vee \Theta(a,b)^* \geq b \vee \Theta(b,a) = 1$, hence Θ is a normality operator. ■

A locale will be called *hereditarily normal* if every its sublocale is normal.

Proposition 3.3. *A locale L is hereditarily normal if and only if every open sublocale of L is normal.*

Proof: The “only if” part is obvious. To prove the “if” part, let L be a locale whose open sublocales are normal. Let $S \subseteq L$ be an arbitrary sublocale of L. In order to prove that S is normal, let $a, b \in S$ be such that $a \vee_S b = 1$ and consider the open sublocale

$$T = \mathfrak{o}(a \vee b) = \{(a \vee b) \to x : x \in L\} = \{c_T(x) : x \in L\}.$$

By (H4), $c_T(a) = (a \vee b) \to a = (a \to a) \wedge (b \to a) = b \to a$. Likewise, $c_T(b) = a \to b$. So, we have

$$(b \to a) \vee_T (a \to b) = c_T(a) \vee_T c_T(b) = c_T(a \vee b) = 1.$$

Since T is normal, by Remark 3.1(3), there exists a normality operator $\Theta : \mathcal{D}_T \to T$ such that $\Theta \wedge \Theta^{op} = 0_T = (a \vee b)^*$. We have

$$\begin{aligned}(a \vee b) \to (a \vee \Theta(b \to a,\, a \to b)) &= c_T(a \vee \Theta(b \to a,\, a \to b)) \\ &= c_T(a) \vee_T \Theta(b \to a,\, a \to b) \\ &= (b \to a) \vee_T \Theta(b \to a,\, a \to b) = 1,\end{aligned}$$

which, by (H3), yields $a \vee b \leq a \vee \Theta(b \to a,\, a \to b)$. Thus

$$a \vee b = (a \vee b) \wedge (a \vee \Theta(b \to a,\, a \to b)) = a \vee (b \wedge \Theta(b \to a,\, a \to b)). \quad (1)$$

We show that $\Sigma : \mathcal{D}_S \to S$ defined by $\Sigma(x, y) = c_S(y \wedge \Theta(y \to x,\, x \to y))$ satisfies condition (2) of Proposition 3.2. Indeed, by (1) we get

$$a \vee_S \Sigma(a, b) = c_S(a \vee (b \wedge \Theta(b \to a,\, a \to b))) = c_S(a \vee b) = a \vee_S b = 1$$

and

$$\begin{aligned}(\Sigma \wedge \Sigma^{op})(a, b) &= c_S(a \wedge b \wedge (\Theta \wedge \Theta^{op})(b \to a,\, a \to b)) \\ &= c_S((a \wedge b) \wedge (a \vee b)^*) = c_S(0) = 0_S.\end{aligned}$$

We have shown that S is normal. ■

4. Monotonically normal locales

Convention 4.1. (1) For $(P, \leq)$ a partially ordered set, any subset

$$\mathcal{P} \subseteq P \times P$$

will always be assumed to have the componentwise order inherited from $P^{op} \times P$, i.e.,

$$(a, b) \leq_{\mathcal{P}} (c, d) \;\Leftrightarrow\; c \leq_P a \text{ and } b \leq_P d.$$

In what follows, P will either be L, $L^{\mathbb{N}}$ or $L^{\mathbb{Q}}$, where the latter two sets are ordered componentwise.

(2) Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two partially ordered sets. A map

$$\phi : (P, \leq_P) \to (Q, \leq_Q)$$

is called *monotone* [resp, *antitone*] iff:

$$x \leq_P y \Rightarrow \phi(x) \leq_Q \phi(y) \text{ [resp., } \phi(y) \leq_Q \phi(x)] \text{ for all } x, y \in P.$$

By monotonizing the concept of a normal locale one arrives at the concept of a monotonically normal locale. Specifically, a locale L is called *monotonically normal* if there exists a monotone function $\Delta : \mathcal{D}_L \to L$ such that

$$a \vee \Delta(a, b) = 1 = b \vee \Delta(a, b)^*$$

for all $(a, b) \in \mathcal{D}_L$. We call Δ a *monotone normality operator.*

We start with a monotone variant of Remarks 3.1.

Remarks 4.2. (1) If $\Delta : \mathcal{D}_L \to L$ is a monotone normality operator, then so is $\Delta^{\circledast} : \mathcal{D}_L \to L$ defined by $\Delta^{\circledast}(a,b) = \Delta(b,a)^*$. Indeed, $\Delta^{\circledast}$ is a normality operator (Remark 3.1(1)) and is monotone, because $(a,b) \le (c,d)$ in $\mathcal{D}_L$ if and only if $(d,c) \le (b,a)$ in $\mathcal{D}_L$ and $(\cdot)^*$ is antitone.

(2) If Δ_1 and Δ_2 are monotone normality operators, then so is the pointwise meet $\Delta_1 \wedge \Delta_2$.

(3) Each monotonically normal locale L admits a self-disjoint monotone normality operator Θ (cf. [9, Lemma 2.2]). Indeed, if Δ is a monotone normality operator, then with $\Theta = \Delta \wedge \Delta^{\circledast}$ one has $\Theta(a,b) \wedge \Theta^{op}(a,b) \le \Delta(a,b) \wedge \Delta(a,b)^* = 0$. Notice that if Θ is self-disjoint, then $\Theta \le \Theta^{\circledast}$.

(4) A self-disjoint and monotone function $\Delta : \mathcal{D}_L \to L$ such that $a \vee \Delta(a,b) = 1$ for all $(a,b) \in \mathcal{D}_L$ is clearly a monotone normality operator (cf. Proposition 3.2).

Before moving to a canonical example of a monotonically normal locale, viz. metrizable locales, we give a number of obvious examples.

Examples 4.3. (1) A topological space X is monotonically normal if and only if $\mathcal{O}X$ is monotonically normal (note that here, as in [11] and [6], we do not assume the T_1-axiom to be a part of the definition of monotone normality). When T_1-separation is a part of the definition of monotone normality, then any monotonically normal space is hereditarily monotonically normal. It has already been pointed out in [6] that this need not be the case without T_1 (cf. Example 4.8).

(2) If 1 is coprime (i.e., $a \vee b = 1$ implies $a = 1$ or $b = 1$), then $\mathcal{D}_L = (\{1\} \times L) \cup (L \times \{1\})$ and L is monotonically normal. In fact, $\Delta : \mathcal{D}_L \to L$ defined by

$$\Delta(a,b) = \begin{cases} 0 & \text{if} \quad a = 1 \text{ and } b \neq 1, \\ 1 & \text{if} \quad b = 1. \end{cases}$$

is a monotone normality operator (see also [14, Example 4.2]).

(3) Both $\mathfrak{L}(\mathbb{R})$ and $\mathfrak{L}[p,q] = \uparrow ((-,p) \vee (q,-))$ (where $p < q$) are monotonically normal.

We shall now show that metrizable locales [17] (further developed, among others, in [18], [19] and [3]) are monotonically normal. Before doing this, some preparatory material is needed which is taken from the just cited papers. Given $A \subseteq L$ and $b \in L$, we put

$$A(b) = \bigvee \{a \in A : a \wedge b \neq 0\}.$$

Clearly, $A(\cdot) : L \to L$ preserves arbitrary joins and — as such — admits a right (Galois) adjoint $\alpha_A : L \to L$ given by

$$\alpha_A(a) = \bigvee\{b \in L : A(b) \le a\}.$$

Consequently, $A(b) \le a$ iff $b \le \alpha_A(a)$. In particular, $A(\alpha_A(a)) \le a$.

A *metric diameter* on a locale L is a map $d : L \to [0, \infty]$ satisfying the following conditions:

(D1) $d(0) = 0$,
(D2) d is monotone,
(D3) $d(a \vee b) \le d(a) + d(b)$ whenever $a \wedge b \neq 0$,
(D4) for each $\varepsilon > 0$ the set $B_\varepsilon = \{a \in L : d(a) < \varepsilon\}$ is a cover of L (i.e. $\bigvee B_\varepsilon = 1$),
(D5) $a = \bigvee\{b \in L : B_\varepsilon(b) \le a$ for some $\varepsilon > 0\}$ for all $a \in L$,
(D6) $d(a) = \bigvee\{d(b \vee c) : b, c \le a$ and $d(b) \vee d(c) < \varepsilon\}$ for all $a \in L$ and $\varepsilon > 0$.

A *metric locale* is a pair (L, d) where d is a metric diameter on L. A locale that admits a metric diameter is called *metrizable*. We note that a topological space X is metrizable if and only if $\mathcal{O}X$ is metrizable as a locale. The passage from a metric ρ to a metric diameter d is provided by the usual diameter $d(U) = \sup\{\rho(x, y) : x, y \in U\}$ for all $U \in \mathcal{O}X$.

In what follows, we write α_ε instead of α_{B_ε}.

Lemma 4.4. [18, Lemma 1.10]. *For each $A \subseteq L$ and $a \in L$ the following hold:*

(1) $a \vee \alpha_\varepsilon(a)^* = 1$ *for all* $\varepsilon > 0$.
(2) $a = \bigvee_{\varepsilon>0} \alpha_\varepsilon(a)$.

Proof: (1) Let $c \in B_\varepsilon$. If $c \wedge \alpha_\varepsilon(a) = 0$, then $c \le \alpha_\varepsilon(a)^*$. Otherwise, $c \le B_\varepsilon(\alpha_\varepsilon(a)) \le a$. Hence $a \vee \alpha_\varepsilon(a)^* \ge \bigvee B_\varepsilon = 1$ by (D4).

(2) Since $B_\varepsilon(b) \le a$ if and only if $b \le \alpha_\varepsilon(a)$, we have by (D5):

$$a = \bigvee_{\varepsilon>0} \bigvee\{b \in L : b \le \alpha_\varepsilon(a)\} = \bigvee_{\varepsilon>0} \alpha_\varepsilon(a).$$ ∎

Proposition 4.5. *Each metrizable locale is monotonically normal.*

Proof: Let L be metrizable. Define $\Delta : \mathcal{D}_L \to L$ by

$$\Delta(a,b) = \bigvee_{\varepsilon>0} (\alpha_\varepsilon(a)^* \wedge \alpha_\varepsilon(b))$$

for all $(a,b) \in \mathcal{D}_L$. Since α_ε is monotone and $(\cdot)^*$ is antitone, Δ is easily seen to be monotone. By Lemma 4.4, we have

$$\begin{aligned} a \vee \Delta(a,b) &= \bigvee_{\varepsilon>0} ((a \vee \alpha_\varepsilon(a)^*) \wedge (a \vee \alpha_\varepsilon(b))) \\ &= \bigvee_{\varepsilon>0} (a \vee \alpha_\varepsilon(b)) \\ &= a \vee \bigvee_{\varepsilon>0} \alpha_\varepsilon(b) \\ &= a \vee b = 1. \end{aligned}$$

Also,

$$\begin{aligned} (\Delta \wedge \Delta^{op})(a,b) &= \bigvee_{\varepsilon_1,\varepsilon_2>0} (\alpha_{\varepsilon_1}(a)^* \wedge \alpha_{\varepsilon_2}(a) \wedge \alpha_{\varepsilon_2}(b)^* \wedge \alpha_{\varepsilon_1}(b)) \\ &\leq \bigvee_{\varepsilon_1\geq\varepsilon_2} (\alpha_{\varepsilon_1}(a)^* \wedge \alpha_{\varepsilon_2}(a) \vee \bigvee_{\varepsilon_1<\varepsilon_2} (\alpha_{\varepsilon_2}(b)^* \wedge \alpha_{\varepsilon_1}(b)) = 0. \end{aligned}$$

Hence, by Remark 4.2(4), Δ is a monotone normality operator. ■

Remark 4.6. We note that the proof of Proposition 4.5 merely uses (D4), (D5) and the fact that $\{B_\varepsilon : \varepsilon > 0\}$ is a chain under inclusion (the latter property is used in proving that Δ is self-disjoint). According to [17] and [18], a system $\mathcal{C}$ of covers of a locale L is called *admissible* if it satisfies condition (D5), i.e.,

$$a = \bigvee\{b \in L : \underset{C\in\mathcal{C}}{\exists} C(b) \leq a\} = \bigvee_{C\in\mathcal{C}} \alpha_C(x),$$

i.e., we have (2) of Lemma 4.4. Also, we have $a \vee \alpha_C(a)^* = 1$ for all $a \in L$. Consequently:

> *Each locale that admits a chain of admissible covers is monotonically normal.*

We now have the following equivalent formulation of monotone normality (cf. [13, Proposition 3], and a part of Theorem 2.4 in [4]; see also [6, Proposition 3.1]).

Proposition 4.7. *For a locale L, the following are equivalent:*

(1) *L is monotonically normal.*
(2) *There exists a self-disjoint $\Sigma : \mathcal{D}_L \to L$ such that $\Sigma(a,b) \leq b$, $a \vee \Sigma(a,b) = 1$, and $\Sigma(a,\cdot)$ is monotone for all $(a,b) \in \mathcal{D}_L$.*

Proof: (1) $\Rightarrow$ (2): By Remark 4.2(3), let Θ be a self-disjoint monotone normality operator of L. Further, for each $(a,b) \in \mathcal{D}_L$, let

$$\Sigma(a,b) = b \wedge \bigvee_{(a,c)\leq(a,b)} \Theta(a,c).$$

Clearly, $\Sigma(a,b) \leq b \wedge \Theta(a,b)$, so that $\Sigma(a,b) \leq b$ and $\Sigma \leq \Theta$. Consequently $\Sigma \wedge \Sigma^{op} \leq \Theta \wedge \Theta^{op} = 0$. Also,

$$a \vee \Sigma(a,b) = (a \vee b) \wedge \Big(\bigvee_{(a,c)\leq(a,b)} (a \vee \Theta(a,c))\Big) = 1.$$

Finally, if $(a,b) \leq (a,d)$ in $\mathcal{D}_L$, then

$$\Sigma(a,b) = b \wedge \bigvee_{(a,c)\leq(a,b)} \Theta(a,c) \leq d \wedge \bigvee_{(a,c)\leq(a,d)} \Theta(a,c) = \Sigma(a,d).$$

(2) $\Rightarrow$ (1): Let Σ be the operator of (2). Define $\Delta : \mathcal{D}_L \to L$ by

$$\Delta(a,b) = \bigvee_{(c,b)\leq(a,b)} \Sigma(c,b)$$

for all $(a,b) \in \mathcal{D}_L$. Clearly, $a \vee \Delta(a,b) \geq a \vee \Sigma(a,b) = 1$. Since $\Sigma \wedge \Sigma^{op} = 0$, we have $\Sigma \leq \Sigma^{\circledast}$. Thus

$$\begin{aligned} \Delta(a,b)^* &= \bigwedge_{(c,b)\leq(a,b)} \Sigma(c,b)^* \\ &= \bigwedge_{(b,a)\leq(b,c)} \Sigma^{\circledast}(b,c) \\ &\geq \bigwedge_{(b,a)\leq(b,c)} \Sigma(b,c) \\ &\geq \Sigma(b,a). \end{aligned}$$

Therefore

$$b \vee \Delta(a,b)^* \geq b \vee \Sigma(b,a) = 1.$$

Finally, if $(a,b) \leq (a_1,b_1)$, then $\Delta(a,b) \leq \bigvee_{(c,b)\leq(a_1,b_1)} \Sigma(c,b) = \Delta(a_1,b_1)$. ■

It is easy to show that any closed sublocale of a (monotonically) normal locale is (monotonically) normal. However, in contrast to the topological situation (with the T_1-separation axiom), the localic monotone normality fails to be an hereditary property. The following example shows that a monotonically normal locale may have an open sublocale which fails to be normal.

Example 4.8. Let L be a non-normal locale. Add a new element $\infty \notin L$ to L and consider $M = L \cup \{\infty\}$ with its natural ordering, i.e., $a \le \infty$ for all $a \in L$. Then ∞ is coprime in M and so $\mathcal{D}_M = (\{\infty\} \times M) \cup (M \times \{\infty\})$ and M becomes a monotonically normal locale (see Example 4.3). Finally, the open sublocale

$$\mathfrak{o}(1) = \{a \in M : 1 \to a = a\} = (L \backslash \{1\}) \cup \{\infty\}$$

is clearly isomorphic to L, hence fails to be normal.

It is well known that a topological space is normal if and only if every two separated F_σ-sets have disjoint open neighborhoods. A monotone variant of that statement is in [11, Proposition 3.1], while its localic variant is in [14, Lemma 3.2]. The following provides its monotone localic counterpart.

The statement involves the set

$$\mathcal{U}_L = \Big\{(a,b) \in L^{\mathbb{N}} \times L^{\mathbb{N}} : (a(n), \bigwedge b(\mathbb{N})),\ (b(n), \bigwedge a(\mathbb{N})) \in \mathcal{D}_L\Big\}$$

partially ordered according to Convention 4.1. For a sequence $a \in L^{\mathbb{N}}$, we define $a^{\downarrow}(n) = \bigwedge_{i \le n} a(n)$ (clearly, $a^{\downarrow}(\cdot)$ is antitone).

Lemma 4.9. *For a locale L, the following are equivalent:*

(1) *L is monotonically normal.*
(2) *There exists a monotone function $\Upsilon : \mathcal{U}_L \to L$ such that $a(n) \vee \Upsilon(a,b) = 1$ and $b(n) \vee \Upsilon(a,b)^* = 1$ for all $(a,b) \in \mathcal{U}_L$ and $n \in \mathbb{N}$.*

Proof: (1) $\Rightarrow$ (2): Let $\Delta : \mathcal{D}_L \to L$ be a monotone normality operator. Recall that so is $\Delta^{\circledast}$. Define $\Upsilon : \mathcal{U}_L \to L$ by

$$\Upsilon(a,b) = \bigvee_{n \in \mathbb{N}} \big(\Delta(a^{\downarrow}(n), \bigwedge b(\mathbb{N})) \wedge \Delta^{\circledast}(\bigwedge a(\mathbb{N}), b^{\downarrow}(n))\big).$$

It is easy to see that Υ is monotone. Let $(a,b) \in \mathcal{U}_L$. Then with $u_n = \Delta(a^{\downarrow}(n), \bigwedge b(\mathbb{N}))$ and $v_n = \Delta^{\circledast}(\bigwedge a(\mathbb{N}), b^{\downarrow}(n))$ one has

$$a^{\downarrow}(n) \vee u_n = \bigwedge b(\mathbb{N}) \vee u_n^* = 1 = \bigwedge a(\mathbb{N}) \vee v_n = b^{\downarrow}(n) \vee v_n^*$$

for all n. Thus

$$a(n) \vee \Upsilon(a,b) \geq a^{\downarrow}(n) \vee (u_n \wedge v_n) \geq (a^{\downarrow}(n) \vee u_n) \wedge (\bigwedge a(\mathbb{N}) \vee v_n) = 1.$$

Since both (u_n) and (v_n^*) are monotone, it follows that for all n and m one has

$$u_n \wedge v_n \wedge u_m^* \wedge v_m^* \leq \begin{cases} u_n \wedge u_n^* & \text{if } n \leq m \\ v_n \wedge v_n^* & \text{if } n > m \end{cases} = 0,$$

i.e., $u_m^* \wedge v_m^* \leq \bigwedge_n (u_n \wedge v_n)^* = \Upsilon(a,b)^*$ for all m. So, we get

$$b(n) \vee \Upsilon(a,b)^* \geq b^{\downarrow}(n) \vee (u_n^* \wedge v_n^*) \geq (\bigwedge b(\mathbb{N}) \vee u_n^*) \wedge (b^{\downarrow}(n) \vee v_n^*) = 1.$$

(2) $\Rightarrow$ (1): This is obvious. In fact, if $(a,b) \in \mathcal{D}_L$, then for $a(n) = a$ and $b(n) = b$ one has $(a,b) \in \mathcal{U}_L$ and $\Theta(a,b) = \Upsilon(a,b)$ defines a monotone normality operator for L. ■

We still need yet a more specific normality-type operator. For this purpose, for each $\alpha \in L^{\mathbb{Q}}$ and $r \in \mathbb{Q}$, define $\alpha_r = \alpha(r)$ and let $\mathcal{S}_L$ denote the collection of all ordered pairs $(\alpha, \beta) \in L^{\mathbb{Q}} \times L^{\mathbb{Q}}$ where α is monotone, β is antitone and

$$\alpha_s \vee \beta_r = 1 \text{ whenever } r < s.$$

The set $\mathcal{S}_L$ is partially ordered according to Convention 4.1.

Proposition 4.10. *For a locale L, the following are equivalent:*

(1) *L is monotonically normal.*
(2) *There exists a monotone function $\Gamma : \mathcal{S}_L \to L^{\mathbb{Q}}$ such that for all $(\alpha, \beta) \in \mathcal{S}_L$ and $r < s$ the following holds:*

$$\Gamma(\alpha,\beta)_r \vee \alpha_s = \Gamma(\alpha,\beta)_r \vee \Gamma(\alpha,\beta)_s^* = \beta_r \vee \Gamma(\alpha,\beta)_s^* = 1.$$

Proof: (1) $\Rightarrow$ (2): Let $\{r_n : n \in \mathbb{N}\}$ be an indexation of $\mathbb{Q}$. For each $(\alpha, \beta) \in \mathcal{S}_L$ we will inductively define a family $\{\gamma_{r_i} = \Gamma(\alpha,\beta)_{r_i} : i \in \mathbb{N}\}$ such that

$$\begin{cases} \gamma_{r_j} \vee \alpha_s = \gamma_{r_i} \vee \gamma_{r_j}^* = \beta_r \vee \gamma_{r_i}^* = 1 & \text{if } r < r_i < r_j < s \ \ (i, j < n), \\ \Gamma(\alpha,\beta) \leq \Gamma(\bar{\alpha}, \bar{\beta}) & \text{if } (\alpha,\beta) \leq (\bar{\alpha}, \bar{\beta}). \end{cases} \qquad (I_n)$$

In doing so, we shall use the following sets:

$$A_n = \{\alpha_r : r > r_n\}, \quad B_n = \{\beta_r : r < r_n\},$$
$$C_n = \{\gamma_{r_i} : r_i < r_n,\, i < n\}, \quad D_n = \{\gamma_{r_i}^* : r_i > r_n,\, i < n\}.$$

Now we proceed inductively. For $n = 2$, if $r < r_1 < s$, then, since α is monotone and β is antitone, one has

$$\alpha_s \vee \bigwedge B_1 \geq \alpha_s \vee \beta_{r_1} = 1 \quad \text{and} \quad \beta_r \vee \bigwedge A_1 \geq \beta_r \vee \alpha_{r_1} \geq 1.$$

Hence we have $(a, b) \in \mathcal{U}_L$ with a and b being monotone enumerations of A_1 and B_1, respectively. Using Lemma 4.9, we put $\gamma_{r_1} = \Gamma(\alpha, \beta)_{r_1} = \Upsilon(a, b)$.

Assume we have constructed $\{\gamma_{r_i} : i < n\}$ satisfying (I_n). Let c and d be monotone enumerations of $A_n \cup D_n$ and $B_n \cup C_n$, respectively. As above, we check that $(c, d) \in \mathcal{U}_L$. Indeed, if $r < r_i < r_j < s (i, j < n)$, then

$$\begin{aligned} \alpha_s \vee \bigwedge (B_n \cup C_n) = & \ (\alpha_s \vee \bigwedge B_n) \wedge (\alpha_s \vee \bigwedge C_n) \\ \geq & \ (\alpha_s \vee \beta_{r_n}) \wedge \bigwedge_{r_j < r_n} (\alpha_s \vee \gamma_{r_j}) = 1 \end{aligned}$$

and, similarly,

$$\gamma_{r_i}^* \vee \bigwedge (B_n \cup C_n) \geq (\gamma_{r_i}^* \vee \beta_{r_n}) \wedge \bigwedge_{r_i < r_j} (\gamma_{r_i}^* \vee \gamma_{r_j}) = 1.$$

Analogously,

$$\beta_r \vee \bigwedge (A_n \cup D_n) = 1 = \gamma_{r_i} \vee \bigwedge (A_n \cup D_n).$$

Thus, using Lemma 4.9 again, we define $\gamma_{r_n} = \Upsilon(c, d)$ and (I_{n+1}) holds true.

(2) $\Rightarrow$ (1): As in Proposition 4.9. ∎

5. Monotone localic Katětov-Tong insertion theorem

Given $f \in \mathrm{USC}(L)$ and $g \in \mathrm{LSC}(L)$, we define

$$f \lhd_{u,l} g \ \Leftrightarrow \ f(-, q) \vee g(p, -) = 1 \text{ for all } p < q \text{ in } \mathbb{Q},$$

and

$$g \lhd_{l,u} f \ \Leftrightarrow \ f(-, p) \wedge g(p, -) = 0 \text{ for all } p \in \mathbb{Q}.$$

Remark 5.1. It is easy to see that there is a one-to-one correspondence between $\mathrm{C}(L)$ and the set

$$\mathcal{A}_L = \{(f, g) \in \mathrm{USC}(L) \times \mathrm{LSC}(L) : f \lhd_{u,l} g \text{ and } g \lhd_{l,u} f\}.$$

Indeed, given an $h \in \mathrm{C}(L)$ and restricting it to $\mathfrak{L}_l(\mathbb{R})$ and $\mathfrak{L}_u(\mathbb{R})$ yields the pair $(h_{|\mathfrak{L}_l(\mathbb{R})}, h_{|\mathfrak{L}_u(\mathbb{R})}) \in \mathcal{A}_L$. Conversely, any $(f, g) \in \mathcal{A}_L$ gives rise to an

$h \in \mathrm{C}(L)$ defined by $h(p,q) = f(-,q) \wedge g(p,-)$. In such a case we shall write $h = \langle f, g\rangle$.

Let

$$\mathrm{UL}(L) = \{(f,g) \in \mathrm{USC}(L) \times \mathrm{LSC}(L) : f \lhd_{u,l} g\}.$$

This set has the componentwise order inherited from $\mathrm{USC}(L)^{op} \times \mathrm{LSC}(L)$, i.e.,

$$(f_1, g_1) \leq (f_2, g_2) \;\Leftrightarrow\; f_2 \leq f_1 \text{ and } g_1 \leq g_2.$$

A *scale* (*descending trail* in [1]) in L is a map $\tau : \mathbb{Q} \to L$ such that $\tau(r) \vee \tau^*(s) = 1$ whenever $r < s$, and $\bigvee \tau(\mathbb{Q}) = 1 = \bigvee \tau^*(\mathbb{Q})$ where $\tau^* = (\cdot)^* \circ \tau$.

Lemma 5.2. [1, Lemma 2]. *Each scale τ in L generates an $h \in \mathrm{C}(L)$ defined by $h(p,q) = \bigvee\{\tau(r) \wedge \tau^*(s) : p < r < s < q\}$.* ■

Remark 5.3. If τ_1 and τ_2 are scales with $\tau_1 \leq \tau_2$ in $L^{\mathbb{Q}}$ and h_1 and h_2 are the corresponding frame homomorphisms, then $h_1 \leq h_2$.

We shall need the *characteristic maps* $\chi_a^u \in \mathrm{USC}(L)$ and $\chi_a^l \in \mathrm{LSC}(L)$, for any $a \in L$, defined by

$$\chi_a^u(-,q) = \begin{cases} 0, & \text{if } q \leq 0, \\ a, & \text{if } 0 < q \leq 1, \\ 1, & \text{if } q > 1, \end{cases} \quad \text{and} \quad \chi_a^l(p,-) = \begin{cases} 1, & \text{if } p < 0, \\ a, & \text{if } 0 \leq p < 1, \\ 0, & \text{if } p \geq 1. \end{cases}$$

We are eventually in a position to give a monotone version of the localic Katětov-Tong theorem of [14] (the reader should consult [14] for a criticism of the localic insertion theorem of [12] which has not been a true generalization of the Katětov-Tong insertion theorem; see also Remark 2.1 and [7]). When applied to $L = \mathcal{O}X$ it yields the monotone insertion theorem of [11]. When $(f,g) \in \mathrm{UL}(L)$ and $h \in \mathrm{C}(L)$ we shall simply write $f \leq h \leq g$ whenever $f \leq h_{|\mathfrak{L}_l(\mathbb{R})}$ in $\mathrm{USC}(L)$ and $h_{|\mathfrak{L}_u(\mathbb{R})} \leq g$ in $\mathrm{LSC}(L)$.

Theorem 5.4. *For a locale L, the following are equivalent:*

1. *L is monotonically normal.*
2. *There exists a monotone function $\Lambda : \mathrm{UL}(L) \to \mathrm{C}(L)$ such that $f \leq \Lambda(f,g) \leq g$ for all $(f,g) \in \mathrm{UL}(L)$.*

Proof: (1) $\Rightarrow$ (2): Let $(f,g) \in \mathrm{UL}(L)$ and let $\varphi, \gamma \in L^{\mathbb{Q}}$ be defined by $\varphi(r) = f(-,r)$ and $\gamma(r) = g(r,-)$. Then $(\varphi,\gamma) \in \mathcal{S}_L$. Let $\Gamma : \mathcal{S}_L \to L^{\mathbb{Q}}$ be the monotone function given by Proposition 4.10. Then the map $\tau : \mathbb{Q} \to L$ defined by $\tau(r) = \Gamma(\varphi, \gamma)(r)$ becomes a scale that generates the required $\Lambda(f, g) \in \mathrm{C}(L)$.

We first observe that Λ is monotone. Indeed, $(f, g) \leq (f_1, g_1)$ in $\mathrm{UL}(L)$ if and only if $(\varphi, \gamma) \leq (\varphi_1, \gamma_1)$ in $\mathcal{S}_L$. Thus, $\tau \leq \tau_1$ in $L^{\mathbb{Q}}$ and, consequently (Remark 5.3), $\Lambda(f, g) \leq \Lambda(f_1, g_1)$.

It remains to show that $f \leq \Lambda(f, g)_{|\mathfrak{L}_l(\mathbb{R})}$ and $\Lambda(f, g)_{|\mathfrak{L}_u(\mathbb{R})} \leq g$. By Proposition 4.10, if $r < s$ we have

$$1 = \varphi(s) \vee \Gamma(\varphi, \gamma)(r) \leq f(-, s) \vee \tau^{**}(r).$$

Hence $\bigvee_{r<s} \tau^*(r) \leq f(-, s)$ and this just says that $\Lambda(f, g)_{|\mathfrak{L}_u(L)}(-, s) \leq f(-, s)$. The second inequality follows similarly.

(2) $\Rightarrow$ (1): If $(a, b) \in \mathcal{D}_L$, then $(\chi_a^u, \chi_b^l) \in \mathrm{UL}(L)$. Now it suffices to observe that $\Delta : \mathcal{D}_L \to L$ defined by $\Delta(a, b) = \Lambda(\chi_a^u, \chi_b^l)(\frac{1}{2}, -)$ is a monotone normality operator. ■

6. Monotone extension property

An $h \in \mathrm{C}(L)$ is said to be *bounded* if $h(p, q) = 1$ for some $p < q$. In the sequel, $\mathrm{C_b}(L)$ stands for all the bounded members of $\mathrm{C}(L)$.

Remark 6.1. Let $a \in L$ and $h = \langle f, g \rangle \in \mathrm{C_b}(\uparrow a)$ (recall Remark 5.1). If $h(p, q) = 1$, we define $\hat{f} \in \mathbf{Frm}(\mathfrak{L}_l(\mathbb{R}), L)$ and $\hat{g} \in \mathbf{Frm}(\mathfrak{L}_u(\mathbb{R}), L)$ by

$$\hat{f}(-, s) = \begin{cases} 0, & \text{if } s \leq p, \\ f(-, s), & \text{if } p < s < q, \\ 1, & \text{if } s \geq q, \end{cases} \quad \text{and} \quad \hat{g}(r, -) = \begin{cases} 1, & \text{if } r < p, \\ g(r, -), & \text{if } p < r < q, \\ 0, & \text{if } r \geq q. \end{cases}$$

Moreover, $\hat{f} \in \mathrm{USC}(L)$ and $\hat{g} \in \mathrm{LSC}(L)$ since the extra condition defining upper (resp., lower) semicontinuity follows from $\hat{f}(-, p) = 0$ (resp., $\hat{g}(q, -) = 0$). Finally, it is easy to check that $\hat{f} \lhd_{u,l} \hat{g}$ i.e., $(\hat{f}, \hat{g}) \in \mathrm{UL}(L)$.

Note that the construction above is only possible when the continuous function h is bounded.

We shall say that L has the *monotone bounded extension property* if for each $a \in L$ there exists a monotone function $\Phi_a : \mathrm{C_b}(\uparrow a) \to \mathrm{C_b}(L)$ such that $c_{\uparrow a} \circ \Phi_a(h) = h$ for all $h \in \mathrm{C_b}(\uparrow a)$. Thus, $\Phi_a(h)$ extends h whenever

$$\Phi_a(h)(p, q) \vee a = h(p, q).$$

Proposition 6.2. *Every monotonically normal locale has the monotone bounded extension property.*

Proof: Let L be monotonically normal and let $h = \langle f, g\rangle \in \mathrm{C_b}(\uparrow a)$ with $(f, g) \in \mathcal{A}_{\uparrow a}$ (see Remark 5.1). By Remark 6.1 and Theorem 5.4, Φ_a : $\mathrm{C_b}(\uparrow a) \to \mathrm{C_b}(L)$ defined by $\Phi_a(h) = \Lambda(\hat{f}, \hat{g})$ is as required. ■

We would like to emphasize that the previous result could also be stated in terms of unbounded real functions. However, its proof is quite technical and so we omit it.

The particular case $L = \mathcal{O}X$ of the next theorem was proved in [20, Theorem 2.3]. Our proof, when applied to the case $L = \mathcal{O}X$, provides another proof of Theorem 2.3 of [20]. It is worth mentioning that the proof in [20] depends upon the T_1-axiom, while our argument is free of it.

Theorem 6.3. *For a locale L, the following are equivalent:*

(1) *L is monotonically normal.*
(2) *For every $a \in L$ there exists a monotone extender*

$$\Phi_a : \mathrm{C_b}(\uparrow a) \to \mathrm{C_b}(L)$$

satisfying the following conditions: if $a_2 \leq a_1$ and $h_i \in \mathrm{C_b}(\uparrow a_i)$ $(i = 1, 2)$ are such that $h_i \leq h_j$ in $\mathrm{UL}(\uparrow a_2)$ $(i \neq j)$, then $\Phi_{a_i}(h_i) \leq \Phi_{a_j}(h_j)$ $(i \neq j)$.

Proof: (1) ⇒ (2): For each $a \in L$ let $\Phi_a : \mathrm{C_b}(\uparrow a) \to \mathrm{C_b}(L)$ be the monotone extender given by Proposition 6.2, i.e., $\Phi_a(h) = \Lambda(f, g)$ where $(f, g) \in \mathcal{A}_{\uparrow a}$ generates h (see Remark 5.1). If $a_2 \leq a_1$, $h_i = \langle f_i, g_i\rangle$ $(i = 1, 2)$, and $h_1 \leq h_2$, then $(f_1, g_1) \leq (f_2, g_2)$ and, consequently, $\Phi_{a_1}(h_1) \leq \Phi_{a_2}(h_2)$. This is the case with $i = 1$ and $j = 2$. Similarly, one verifies the second case.

(2) ⇒ (1): We shall exhibit a function $\Sigma : \mathcal{D}_L \to L$ satisfying condition (2) of Proposition 4.7. For each $(a, b) \in \mathcal{D}_L$ consider the characteristic maps of a and b that take values in the locale $\uparrow(a \wedge b)$ (rather than in L), i.e., $\chi^u_a \in \mathrm{USC}(\uparrow(a \wedge b))$ and $\chi^l_b \in \mathrm{LSC}(\uparrow(a \wedge b))$.

Since $a \vee b = 1$, it follows that $\chi^u_a \lhd_{u,l} \chi^l_b$, while $\chi^l_b \lhd_{l,u} \chi^u_a$ follows from the fact that $0_{\uparrow(a\wedge b)} = a \wedge b$. By Remark 5.1, $h_{ab} = \langle \chi^u_a, \chi^l_b\rangle \in \mathrm{C_b}(\uparrow(a \wedge b))$. By hypothesis, there exists a monotone extender

$$\Phi_{a\wedge b} : \mathrm{USC}(\uparrow(a \wedge b)) \to \mathrm{C_b}(L)$$

satisfying the additional conditions of (2). Define $\Sigma : \mathcal{D}_L \to L$ by

$$\Sigma(a,b) = \Phi_{a\wedge b}(h_{ab})(\tfrac{1}{2},-) \wedge \Phi_{a\wedge b}(h_{ba})(-,\tfrac{1}{2}).$$

Claim 1: $\Sigma \wedge \Sigma^{op} = 0$. Indeed,

$$\Sigma(a,b) \wedge \Sigma(b,a) \leq \Phi_{a\wedge b}(h_{ab})(\tfrac{1}{2},-) \wedge \Phi_{a\wedge b}(h_{ab})(-,\tfrac{1}{2}) = 0.$$

Claim 2: $\Sigma(a,b) \leq b$ and $a \vee \Sigma(a,b) = 1$. Indeed, since $\Phi_{a\wedge b}$ is an extender, one has $\Phi_{a\wedge b}(h) \vee (a \wedge b) = h$. Thus

$$(a\wedge b) \vee \Sigma(a,b) = h_{ab}(\tfrac{1}{2},-) \wedge h_{ba}(-,\tfrac{1}{2}) = \chi_b^l(\tfrac{1}{2},-) \wedge \chi_a^u(-,\tfrac{1}{2}) = b.$$

In particular, $\Sigma(a,b) \leq b$ and $a \vee \Sigma(a,b) \geq a \vee b = 1$

Claim 3: $\Sigma(a,b) \leq \Sigma(a,c)$ whenever $(a,b) \leq (a,c)$ in $\mathcal{D}_L$. Indeed, we have $\Phi_{a\wedge c}(h_{ca}) \leq \Phi_{a\wedge b}(h_{ba})$ and $\Phi_{a\wedge c}(h_{ac}) \geq \Phi_{a\wedge b}(h_{ab})$ and the assertion follows. ■

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