Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 07–27

COHERENT PAIRS OF LINEAR FUNCTIONALS ON THE UNIT CIRCLE

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Abstract: In this paper we extend the concept of coherent pairs of measures from the real line to Jordan arcs and curves. We present a characterization of pairs of coherent measures on the unit circle: it is established that if (μ_0, μ_1) is a coherent pair of measures on the unit circle, then μ_0 is a semiclassical measure. Moreover, we obtain that the linear functional associated to μ_1 is a specific rational transformation of the linear functional corresponding to μ_0. Some examples are given.

Keywords: Orthogonal polynomials, differential equations, three term recurrence relations, measures on the unit circle, hermitian functionals.
AMS Subject Classification (2000): Primary 42C05.

1. Introduction

Let μ be a nontrivial positive Borel measure supported on a subset E of the real line. There exists a unique sequence $\{P_n\}$ of monic polynomials, with $\deg P_n = n$, such that

$$\int_E P_n(x)P_m(x)d\mu(x) = d_n^2\delta_{n,m},\ d_n \neq 0.$$

In this case $\{P_n\}$ is said do be the *sequence of monic orthogonal polynomials associated with* μ.

It is well known that $\{P_n\}$ satisfies a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x),\ n \geq 0, \tag{1}$$

where $P_{-1}(x) = 0$ and

$$c_{n+1} = \frac{\int_E P_{n+1}^2(x)d\mu(x)}{\int_E P_n^2(x)d\mu(x)},\quad b_n = \frac{\int_E xP_n^2(x)d\mu(x)}{\int_E P_n^2(x)d\mu(x)},\ n \geq 0.$$

Received October 4, 2007.
The work of the first and fourth author was supported by *CMUC/FCT*. The second author would like to thank *Unidade de Investigação (Matemática e Aplicações) da Universidade de Aveiro.* The work of the third author was supported by *Dirección General de Investigación (Ministerio de Ciencia y Tecnologia) of Spain* under grant *BFM 2000-0206-C04-01.* The first three authors thanks *INTAS project* ref. *2000-272* for their support. The work of M.N. Rebocho was supported by *FCT, Fundação para a Ciência e Tecnologia*, under grant *SFRH/BD/25426/2005.*

On the other hand, if (1) holds with $c_n > 0$, there exists the sequence of monic polynomials defined by (1) orthogonal with respect to the measure μ.

Let (μ_0, μ_1) be a pair of nontrivial positive Borel measures supported on subsets E_0 and E_1 of the real line. We introduce an inner product in the linear space $\mathbb{P}$ of polynomials with real coefficients

$$(p, q) = \int_{E_0} p(x)q(x)d\mu_0(x) + \lambda \int_{E_1} p'(x)q'(x)d\mu_1(x) \tag{2}$$

where $p, q \in \mathbb{P}$ and $\lambda \geq 0$.

This kind of inner products define a sequence $\{Q_n(x, \lambda)\}$ of monic polynomials that is orthogonal with respect to (2). It can be constructed using the standard Gram-Schmidt process. But these polynomials do not satisfy a three-term recurrence relation as (1). If $\{P_n\}$ and $\{R_n\}$ denote, respectively, the sequences of monic polynomials orthogonal with respect to μ_0, μ_1, then Iserles et al. introduced the concept of coherent pairs of measures (cf. [5]).

A pair of nontrivial Borel measures (μ_0, μ_1) supported on subsets of the real line is said to be *coherent* if the corresponding sequences of of monic orthogonal polynomials satisfy

$$R_n(x) = \frac{P'_{n+1}}{n+1}(x) + \alpha_n \frac{P'_n}{n}(x), \ \alpha_n \neq 0, \ n = 1, 2, ... \tag{3}$$

From here, a relation between $\{P_n\}$ and $\{Q_n(\cdot, \lambda)\}$ follows:

$$P_n(x) + \frac{n}{n-1}\alpha_{n-1}P_{n-1}(x) = Q_n(x, \lambda) + \beta_{n-1}(\lambda)Q_{n-1}(x, \lambda)$$

where $\beta_{n-1}(\lambda) = \gamma_{n-2}(\lambda)/\gamma_{n-1}(\lambda)$, γ_n is a polynomial of degree n in the variable λ, and $\{\gamma_n\}$ satisfies a three term recurrence relation.

In [5] the authors ask about the description of all coherent pairs of measures. The answer was given by Meijer in [7], where he proves that at least one of the measures must be a classical one (Laguerre or Jacobi). In particular, when the support is a compact subset of the real axis, the following cases appear:

a) $d\mu_0 = (1-x)^\alpha (1+x)^\beta dx, \ \alpha, \beta > -1\,;$
$d\mu_1 = \dfrac{(1-x)^{\alpha+1}(1+x)^{\beta+1}}{|x-\xi|}dx + M\delta(x-\xi), \ |\xi| \geq 1, \ M \geq 0\,;$

b) $d\mu_0 = (1-x)^\alpha (1+x)^\beta |x-\xi| dx,$
$d\mu_1 = (1-x)^{\alpha+1}(1+x)^{\beta+1}dx, \ \alpha, \beta > -1\,;$

c) $d\mu_0 = (1-x)^\alpha dx + M\delta(x+1), \quad d\mu_1 = (1-x)^{\alpha+1}dx, \quad \alpha, \beta > -1\,;$

d) $d\mu_0 = (1+x)^{\beta}dx + M\delta(x-1), \quad d\mu_1 = (1+x)^{\beta+1}dx,\ \alpha, \beta > -1\,.$

The aim of this contribution is the analysis of the concept of coherent pairs of measures supported on compact subsets of the complex plane. In particular, we will focus our attention when the support is the unit circle.

The structure of the manuscript is as follows. In section 2 we define coherent pairs of measures supported on Jordan arcs or curves using the connection between the corresponding sequences of orthogonal polynomials as in (3). As a consequence, the relation between these sequences and the sequence of monic orthogonal polynomials orthogonal with respect to the Sobolev inner product associated with the pair of measures (μ_0, μ_1) is deduced. In section 3 we present the basic results concerning hermitian orthogonality on the unit circle which will be used in the forthcoming sections. We give a sufficient condition for a sequence of orthogonal polynomials on the unit circle satisfying a first order structure relation to be semi-classical (see Theorem 3). This result is an extension to the result deduced by Branquinho and Rebocho in [3]. In section 4 we present a characterization of pairs of coherent measures on the unit circle; we prove that if (μ_0, μ_1) is a coherent pair of measures on the unit circle (μ_0, μ_1) then μ_0 is a semi-classical measure and the linear functional associated with μ_1 is a specific rational transformation of the linear functional corresponding to μ_0 (see, for example, [2]). Finally, in section 5, we study the companion coherent measure associated with the Bernstein-Szegő measure supported on the unit circle.

2. Coherent pairs of measures supported on Jordan arcs and curves

Let μ_0, μ_1 be positive Borel measures on E_0, E_1, respectively, which are Jordan curves or arcs. For $\lambda \in \mathbb{R}^+$, consider the inner product

$$\langle f, g \rangle_S = \langle f, g \rangle_0 + \lambda \langle f', g' \rangle_1\,,$$

where $\quad \langle f, g \rangle_k = \displaystyle\int_{E_k} f(\xi)\overline{g(\xi)}d\mu_k(\xi),\ k = 0, 1\,.$

Let us denote by $\{Q_n(.;\lambda)\}$, $\{P_n\}$, $\{R_n\}$, the sequences of monic polynomials orthogonal with respect to $\langle .,. \rangle_S$, $\langle .,. \rangle_0$, $\langle .,. \rangle_1$, respectively.
We also denote

$$S_{m,n} := \langle z^m, z^n \rangle_S = c^0_{m,n} + \lambda mn c^1_{m-1,n-1}\,,\ m, n \in \mathbb{N}$$

where $\left\{c_{m,n}^{k}\right\}_{n\in\mathbb{N}}$ are the moments with respect to the measures μ_k for $k = 0, 1$, respectively.

Taking into account this expression, we obtain the following representation in a determinantal form for the polynomials Q_n:

$$Q_n(z;\lambda) = \frac{\begin{vmatrix} c_{0,0}^{0} & c_{1,0}^{0} & \dots & c_{n,0}^{0} \\ c_{0,1}^{0} & c_{1,1}^{0} + \lambda c_{0,0}^{1} & \dots & c_{n,1}^{0} + \lambda n c_{n-1,0}^{1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{0,n-1}^{0} & c_{1,n-1}^{0} + \lambda(n-1)c_{0,n-2}^{1} & \dots & c_{n,n-1}^{0} + \lambda n(n-1)c_{n-1,n-2}^{1} \\ 1 & z & \dots & z^n \end{vmatrix}}{\begin{vmatrix} c_{0,0}^{0} & c_{1,0}^{0} & \dots & c_{n-1,0}^{0} \\ c_{0,1}^{0} & c_{1,1}^{0} + \lambda c_{0,0}^{1} & \dots & c_{n-1,1}^{0} + \lambda(n-1)c_{n-2,0}^{1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{0,n-1}^{0} & c_{1,n-1}^{0} + \lambda(n-1)c_{0,n-2}^{1} & \dots & c_{n-1,n-1}^{0} + \lambda(n-1)^2 c_{n-2,n-2}^{1} \end{vmatrix}}$$

or, equivalently,

$$Q_n(z;\lambda) = \frac{\begin{vmatrix} c_{0,0}^{0} & c_{1,0}^{0} & \dots & c_{n,0}^{0} \\ \frac{c_{0,1}^{0}}{\lambda} & \frac{c_{1,1}^{0}}{\lambda} + c_{0,0}^{1} & \dots & \frac{c_{n,1}^{0}}{\lambda} + n c_{n-1,0}^{1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{c_{0,n-1}^{0}}{\lambda(n-1)} & \frac{c_{1,n-1}^{0}}{\lambda(n-1)} + c_{0,n-2}^{1} & \dots & \frac{c_{n,n-1}^{0}}{\lambda(n-1)} + n c_{n-1,n-2}^{1} \\ 1 & z & \dots & z^n \end{vmatrix}}{\begin{vmatrix} c_{0,0}^{0} & c_{1,0}^{0} & \dots & c_{n-1,0}^{0} \\ \frac{c_{0,1}^{0}}{\lambda} & \frac{c_{1,1}^{0}}{\lambda} + c_{0,0}^{1} & \dots & \frac{c_{n-1,1}^{0}}{\lambda} + (n-1)c_{n-2,0}^{1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{c_{0,n-1}^{0}}{\lambda(n-1)} & \frac{c_{1,n-1}^{0}}{\lambda(n-1)} + c_{0,n-2}^{1} & \dots & \frac{c_{n-1,n-1}^{0}}{\lambda(n-1)} + (n-1)c_{n-2,n-2}^{1} \end{vmatrix}}.$$

Since the coefficients of the above polynomial are rational functions in λ, when λ tends to infinity we get

$$S_n(z) = \frac{\begin{vmatrix} c^0_{0,0} & c^0_{1,0} & \cdots & c^0_{n,0} \\ 0 & c^1_{0,0} & \cdots & nc^1_{n-1,0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c^1_{0,n-2} & \cdots & nc^1_{n-1,n-2} \\ 1 & z & \cdots & z^n \end{vmatrix}}{\begin{vmatrix} c^0_{0,0} & c^0_{1,0} & \cdots & c^0_{n-1,0} \\ 0 & c^1_{0,0} & \cdots & (n-1)c^1_{n-2,0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c^1_{0,n-2} & \cdots & (n-1)c^1_{n-2,n-2} \end{vmatrix}} \tag{4}$$

which is a monic polynomial of degree n.

Proposition 1. The polynomial S_n satisfies

(i) $\langle S_n, 1\rangle_0 = 0, \quad n \geq 1,$

(ii) $\langle S_n', z^k\rangle_1 = 0, \quad 0 \leq k \leq n-2,\ n \geq 2.$

Proof: It is a straightforward consequence of (4). ∎

Notice that from condition (ii) we deduce that $S_n'(z) = nR_{n-1}(z)$. On the other hand, from $R_{n-1}(z) = \sum_{k=1}^{n} \alpha_{n-1,k}\frac{P_k'(z)}{k}$ we get

$$\frac{S_n'(z)}{n} = \sum_{k=1}^{n} \alpha_{n-1,k}\frac{P_k'(z)}{k}\,,$$

and, by integration,

$$\frac{S_n(z)}{n} = \sum_{k=1}^{n} \alpha_{n-1,k}\frac{P_k(z)}{k} + \alpha_{n-1,0}\,.$$

But, according to condition (i) of Proposition 1, $\alpha_{n-1,0} = 0$. Therefore

$$\frac{S_n(z)}{n} = \sum_{k=1}^{n} \alpha_{n-1,k}\frac{P_k(z)}{k}\,,$$

or, equivalently,

$$S_n(z) = \sum_{k=1}^{n} a_{n-1,k}P_k(z) \tag{5}$$

where $a_{n-1,k} = n\alpha_{n-1,k}/k$ are the connection coefficients for the polynomial sequences $\{S_n\}$ and $\{P_n\}$.

On the other hand, from the Fourier expansion of S_n with respect to the polynomials $\{Q_n\}$ we get

$$S_n(z) = Q_n(z;\lambda) + \sum_{j=0}^{n-1} \beta_{n,j}(\lambda) Q_j(z;\lambda)$$

where, for $0 \le j \le n-1$,

$$\beta_{n,j}(\lambda) = \frac{\langle S_n(z), Q_j(z;\lambda)\rangle_S}{\langle Q_j(z;\lambda), Q_j(z;\lambda)\rangle_S} = \frac{\langle S_n(z), Q_j(z;\lambda)\rangle_0}{\langle Q_j(z;\lambda), Q_j(z;\lambda)\rangle_S}\,.$$

From this we do not get more information, but nevertheless if in (5) we assume that $a_{n-1,k} = 0$ for $k < n-s$ (with s a fixed nonnegative integer number), it follows that $\beta_{n,j}(\lambda) = 0$ for $j < n-s$. Thus, for $n \ge s$,

$$\sum_{k=n-s}^{n} a_{n-1,k} P_k(z) = \sum_{j=n-s}^{n} \beta_{n,j}(\lambda) Q_j(z;\lambda)\,. \tag{6}$$

Conversely, notice that if (6) holds, and $a_{n-1,n-s} \neq 0, \beta_{n,n-s}(\lambda) \neq 0$, then, from

$$\langle Q_n(z;\lambda), p(z)\rangle_S = \int_{E_0} Q_n(z;\lambda)p(z)d\mu_0 + \lambda \int_{E_1} Q_n'(z;\lambda)p'(z)d\mu_1,$$

we get

$$\langle \sum_{j=n-s}^{n} \beta_{n,j}(\lambda) Q_j(z;\lambda), p(z)\rangle_S = 0,\ p \in \mathbb{P}_{n-s-1},$$

i.e.,

$$\int_{E_1} \sum_{j=n-s}^{n} a_{n-1,j} P_j'(z) p'(z) d\mu_1 = 0,\ p \in \mathbb{P}_{n-s-1}.$$

From this the following relation holds

$$\sum_{j=n-s}^{n} a_{n-1,j} P_j'(z) = \sum_{j=n-s-1}^{n-1} b_{n,j} R_j(z).$$

Therefore, the following problem arises: To describe the measures μ_0, μ_1 such that the corresponding sequences of monic orthogonal polynomials $\{P_n\}$ and $\{R_n\}$ are related by

$$R_{n-1}(z) = \frac{P_n'(z)}{n} + \alpha_{n-1}\frac{P_{n-1}'(z)}{n-1}\,, \quad \alpha_{n-1} \neq 0\,, \quad n = 2,3,\ldots\,. \tag{7}$$

where for a sake of simplicity we write α_n instead of $\alpha_{n,n}$, as well as a_n instead of $a_{n,n}$.

For a coherent pair of measures we get some extra information about the sequence $(\beta_n(\lambda))$. Indeed,

$$P_n(z) + a_{n-1}P_{n-1}(z) = Q_n(z;\lambda) + \beta_{n-1}(\lambda)Q_{n-1}(z;\lambda)\,, \tag{8}$$

where

$$a_{n-1} = \frac{n}{n-1}\alpha_{n-1},$$

$$\begin{aligned}
\beta_{n-1}(\lambda) &= a_{n-1}\frac{\langle P_{n-1}, Q_{n-1}(.;\lambda)\rangle_0}{\langle Q_{n-1}(.;\lambda), Q_{n-1}(.;\lambda)\rangle_S} \\
&= a_{n-1}\frac{\|P_{n-1}\|_0^2}{\|Q_{n-1}(.;\lambda)\|_S^2}\,,\ n = 2, 3, \ldots\,.
\end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned}
&\|Q_{n-1}(.;\lambda)\|_S^2 \\
&\quad= \langle Q_{n-1}(.;\lambda), P_{n-1}\rangle_S \\
&\quad= \|P_{n-1}\|_0^2 + \lambda\langle Q'_{n-1}(.;\lambda), P'_{n-1}\rangle_1 \\
&\quad= \|P_{n-1}\|_0^2 + \lambda\langle Q'_{n-1}(.;\lambda), (n-1)R_{n-2} - a_{n-2}P'_{n-2}\rangle_1 \\
&\quad= \|P_{n-1}\|_0^2 + \lambda(n-1)^2\|R_{n-2}\|_1^2 - \lambda\bar{a}_{n-2}\langle Q'_{n-1}(.;\lambda), P'_{n-2}\rangle_1 \\
&\quad= \|P_{n-1}\|_0^2 + \lambda(n-1)^2\|R_{n-2}\|_1^2 + \bar{a}_{n-2}\langle Q_{n-1}(.;\lambda), P_{n-2}\rangle_0 \\
&\quad= \|P_{n-1}\|_0^2 + \lambda(n-1)^2\|R_{n-2}\|_1^2 \\
&\qquad\qquad + \bar{a}_{n-2}\langle P_{n-1} + a_{n-2}P_{n-2} - \beta_{n-2}(\lambda)Q_{n-2}(.,\lambda), P_{n-2}\rangle_0 \\
&\quad= \|P_{n-1}\|_0^2 + \lambda(n-1)^2\|R_{n-2}\|_1^2 + \bar{a}_{n-2}\left[a_{n-2} - \beta_{n-2}(\lambda)\right]\|P_{n-2}\|_0^2\,.
\end{aligned}$$

Now, substituting in (9), and using the preceding notation we have for $n = 3, 4, \ldots$

$$\beta_{n-1}(\lambda) = \frac{A_n}{B_n - \beta_{n-2}(\lambda)}\,, \tag{10}$$

where

$$\begin{aligned}
A_n &= \frac{a_{n-1}}{\bar{a}_{n-2}}\frac{\|P_{n-1}\|_0^2}{\|P_{n-2}\|_0^2} \\
B_n &= a_{n-2} + \frac{\|P_{n-1}\|_0^2 + \lambda(n-1)^2\|R_{n-2}\|_1^2}{\bar{a}_{n-2}\|P_{n-2}\|_0^2}\,,
\end{aligned}$$

with $\beta_1(\lambda) = \dfrac{\|P_1\|_0^2 a_1}{\lambda\|R_0\|_1^2 + \|P_1\|_0^2}$.

Notice that B_n is a polynomial of degree one in λ. In this way, once we obtain the coherent pairs we can deduce a representation for $\beta_{n-1}(\lambda)$, which are rational functions of λ and, eventually, from (8) we get an explicit expression for $Q_n(.;\lambda)$ in terms of $\{P_n\}$.

Theorem 1. *The sequence $(\beta_n(\lambda))$ is given by*

$$\beta_{n-1}(\lambda) = \frac{\gamma_{n-2}(\lambda)}{\gamma_{n-1}(\lambda)}\,,\ n = 2, 3, \dots \tag{11}$$

where $\{\gamma_n\}$ is a sequence of orthogonal polynomials associated with a positive Borel measure supported on $\mathbb{R}$.

Proof: Taking into account β_1 is a rational function in λ such that the degree of the numerator is zero and the degree of the denominator is one, by induction we get (11) where γ_n is a polynomial of degree n. Moreover, from (10),

$$\frac{\gamma_{n-1}(\lambda)}{\gamma_n(\lambda)} = \frac{A_{n+1}}{B_{n+1} - \gamma_{n-2}(\lambda)/\gamma_{n-1}(\lambda)}\,,\ n = 2, 3, \dots$$

i.e.

$$\gamma_n(\lambda) = \frac{B_{n+1}}{A_{n+1}}\gamma_{n-1}(\lambda) - \frac{1}{A_{n+1}}\gamma_{n-2}(\lambda)\,. \tag{12}$$

Taking into account that B_n is a polynomial of degree one in λ, we get that $\{\gamma_n\}$ is a sequence of polynomials orthogonal with respect to a linear functional. This is a straightforward consequence of the Favard Theorem, see [4], since they satisfy a three-term recurrence relation.

Indeed, if $\gamma_n(\lambda) = s_n\lambda^n +$ lower degree terms, then (12) becomes

$$s_n\tilde{\gamma}_n(\lambda) = \frac{B_{n+1}}{A_{n+1}}s_{n-1}\tilde{\gamma}_{n-1}(\lambda) - \frac{s_{n-2}}{A_{n+1}}\tilde{\gamma}_{n-2}(\lambda)\,,$$

or, equivalently, for $n = 2, 3, \dots$

$$\tilde{\gamma}_n(\lambda) = (\lambda + c_{n-1})\tilde{\gamma}_{n-1}(\lambda) - d_{n-1}\tilde{\gamma}_{n-2}(\lambda)$$

where

$$\begin{aligned} c_{n-1} &= \frac{|a_{n-1}|^2\|P_{n-1}\|_0^2 + \|P_n\|_0^2}{n^2\|R_{n-1}\|_1^2}\,, \\ d_{n-1} &= \frac{\|P_{n-1}\|_0^4|a_{n-1}|^2}{n^2(n-1)^2\|R_{n-1}\|_1^2\|R_{n-2}\|_1^2} > 0\,, \end{aligned}$$

and initial conditions $\tilde{\gamma}_0(\lambda) = 1$, $\tilde{\gamma}_1(\lambda) = \lambda + \|P_1\|_0^2/\|R_0\|_0^2$. Notice that, according to the Favard Theorem, $\{\tilde{\gamma}_n\}$ is a sequence of monic polynomials orthogonal with respect to a finite positive Borel measure supported on $\mathbb{R}$. ■

3. Quasi-Orthogonality on the Unit Circle

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and $\Lambda = \operatorname{span}\{z^k : k \in \mathbb{Z}\}$ be the linear space of Laurent polynomials with complex coefficients. Given a linear functional $u : \Lambda \to \mathbb{C}$, and the sequence of moments $(c_n)_{n\in\mathbb{Z}}$ of u, $c_n = \langle u, \xi^n \rangle$, $n \in \mathbb{Z}$, $c_0 = 1$, define the minors of the Toeplitz matrix $\Delta = (c_n)_{n\in\mathbb{N}}$, by

$$\Delta_k = \begin{vmatrix} c_0 & c_1 & \dots & c_k \\ c_{-1} & c_0 & \dots & c_{k-1} \\ \dots & \dots & \dots & \dots \\ c_{-k} & c_{-k+1} & \dots & c_0 \end{vmatrix}, \ \Delta_0 = c_0, \ \Delta_{-1} = 1, \ k \in \mathbb{N}.$$

u is said to be *hermitian* if $c_{-n} = \overline{c_n}, \forall n \in \mathbb{N}$, and *quasi-definite* (respectively, *positive definite*) if $\Delta_n \neq 0$ $(respectively, \Delta_n > 0)$, $\forall n \in \mathbb{N}$. We will denote by $\mathcal{H}$ the set of hermitian linear functionals defined on Λ.

In the positive-definite case, u has an integral representation given in terms of a nontrivial probability measure μ with infinite support on the unit circle $\mathbb{T}$,

$$\left\langle u, e^{in\theta} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} d\mu(\theta), \ n \in \mathbb{Z}.$$

The corresponding sequence of orthogonal polynomials, called orthogonal polynomials on the unit circle, OPUC in short, is then defined by

$$\frac{1}{2\pi} \int_0^{2\pi} P_n(e^{i\theta}) \overline{P}_m(e^{-i\theta}) d\mu(\theta) = e_n \delta_{n,m}, \ e_n > 0, \ n, m = 0, 1, \ldots.$$

If $P_n(z) = z^n$+lower degree terms, $\{P_n\}$ will be called a *sequence of monic orthogonal polynomials*, and we will denote it by MOPS. It is well known that MOPS on the unit circle satisfy the following recurrence relations, known as *Szegő recurrence relations*, for $n \geq 1$:

$$P_n(z) = zP_{n-1}(z) + a_n P_{n-1}^*(z), \quad P_n^*(z) = P_{n-1}^*(z) + \overline{a_n} z P_{n-1}(z)$$

with $a_n = P_n(0)$, $P_0(z) = 1$, and $P_n^*(z) = z^n \overline{P}_n(1/z)$.

$\{P_n^*\}$ satisfies, for $n \in \mathbb{N}$,

$$\langle u, P_n^*(z)\, z^{-k} \rangle = 0, \ k = 1, ..., n, \quad \langle u, P_n^*(z) \rangle = e_n. \tag{13}$$

The following relation holds (see [6])

$$(P_n')^{*_{n-1}}(z) = nP_n^*(z) - z(P_n^*)'(z),\ n \geq 1\,. \tag{14}$$

For $u \in \mathcal{H}$ and $A \in \mathbb{P}$, we define

$$\begin{aligned}\langle Au, f\rangle &= \langle u, A(z)f(z)\rangle, \quad f \in \Lambda\\ \langle (A+\overline{A})u, f\rangle &= \langle u, \left(A(z)+\overline{A}(1/z)\right) f(z)\rangle, \quad f \in \Lambda\,.\end{aligned}$$

Notice that $(A+\overline{A})u$ is a hermitian linear functional. We will use the notation $u^A = (A(z)+\overline{A}(1/z))u$.

Definition 1 (cf. [1])**.** Let $v \in \mathcal{H},\ p \in \mathbb{N}$, and let $\{P_n\}$ be a sequence of monic polynomials. $\{P_n\}$ is said to be $\mathbb{T}$-*quasi-orthogonal* of order p with respect to v if

i) $\langle v, P_n(z)\ z^{-k}\rangle = 0$, for every k with $p \leq k \leq n-p-1$ and for every $n \geq 2p+1$;

ii) There exists $n_0 \geq 2p$ such that $\langle v, P_{n_0}(z)\ z^{-n_0+p}\rangle \neq 0$.

Theorem 2 (cf. [1])**.** *Let $u \in \mathcal{H}$ be quasi-definite and let $\{P_n\}$ be the MOPS with respect to u. Then $\{P_n\}$ is $\mathbb{T}$-quasi-orthogonal of order p with respect to $v \in \mathcal{H} - \{0\}$ if and only if there exists only one polynomial B ($B \neq 0$) with* $\deg(B) = p$*, such that $v = u^B$.*

Taking into account Theorem 4.1 of [1] we give the following definition.

Definition 2. Let $u \in \mathcal{H}$ be quasi-definite and let $\{P_n\}$ be the MOPS associated with u. u is said to be *semiclassical* if there exists $\hat{u} \in \mathcal{H} - \{0\}$ such that the sequence $\{\widetilde{P}_n\}$ given by $\widetilde{P}_n(z) = \frac{1}{n} zP_n'(z), n \geq 1,\ \widetilde{P}_0(z) = 1$, is $\mathbb{T}$-quasi-orthogonal with respect to $\hat{u}$. In such a situation $\{P_n\}$ is said to be a *semiclassical sequence* of orthogonal polynomials.

In the sequel we define $f_n(z) = P_n(z)/P_n^*(z),\ \forall n \in \mathbb{N}$, and we study the conditions in order to $\{f_n\}$ satisfies a Riccati differential equation. This result will be useful to the following theorem. Using the Szegő recurrence relations we get

$$zf_n(z) = \frac{f_{n+1}(z) - a_{n+1}}{1 - \overline{a}_{n+1} f_{n+1}(z)},\ n = 1, ... \tag{15}$$

Lemma 1. *Let $\{P_n\}$ be a sequence of monic orthogonal polynomials on the unit circle and $\{P_n^*\}$ the sequence of reversed polynomials. If $\{f_n\}$ satisfies a*

Riccati differential equation with bounded degree polynomial coefficients, i.e.,

$$A_n f_n'(z) = B_n(z) f_n^2(z) + C_n(z) f_n(z) + E_n(z)\,, \ \forall n \in \mathbb{N} \tag{16}$$

then, for every $n \in \mathbb{N}$, the following relations hold,

$$A_{n+1} = A_n\,, \tag{17}$$

$$zB_{n+1} = \lambda_n^{-1}\left\{B_n - \overline{a}_{n+1}(zC_n + A_n) + \overline{a}_{n+1}^2 z^2 E_n\right\}\,, \tag{18}$$

$$zC_{n+1} = \lambda_n^{-1}\{(-2a_{n+1}B_n + (zC_n + A_n)(1 + |a_{n+1}|^2) - 2\overline{a}_{n+1}z^2E_n\}, \tag{19}$$

$$zE_{n+1} = \lambda_n^{-1}\left\{a_{n+1}^2 B_n - a_{n+1}(zC_n + A_n) + z^2 E_n\right\}\,, \tag{20}$$

with $\lambda_n = (1 - |a_{n+1}|^2)$.

Proof: If f_n satisfies (16), then

$$zA_n(zf_n)' = B_n(zf_n)^2 + (zC_n + A_n)zf_n + z^2E_n.$$

Using (15) in previous equation we get

$$zA_n\left(\frac{f_{n+1} - a_{n+1}}{1 - \overline{a}_{n+1}f_{n+1}}\right)' = B_n\left(\frac{f_{n+1} - a_{n+1}}{1 - \overline{a}_{n+1}f_{n+1}}\right)^2 + (zC_n + A_n)\left(\frac{f_{n+1} - a_{n+1}}{1 - \overline{a}_{n+1}f_{n+1}}\right) + z^2E_n\,.$$

Since

$$\left(\frac{f_{n+1} - a_{n+1}}{1 - \overline{a}_{n+1}f_{n+1}}\right)' = \frac{\lambda_n f_{n+1}'}{(1 - \overline{a}_{n+1}f_{n+1})^2} \quad \text{with} \quad \lambda_n = 1 - |a_{n+1}|^2,$$

from the previous equations we get

$$zA_n\frac{\lambda_n f_{n+1}'}{(1 - \overline{a}_{n+1}f_{n+1})^2} = B_n\left(\frac{f_{n+1}^2 + a_{n+1}^2 - 2a_{n+1}f_{n+1}}{(1 - \overline{a}_{n+1}f_{n+1})^2}\right) + (zC_n + A_n)\left(\frac{f_{n+1} - a_{n+1}}{1 - \overline{a}_{n+1}f_{n+1}}\right) + z^2E_n$$

as well as

$$\begin{aligned}\lambda_n\, zA_n f_{n+1}' &= \left\{B_n - \overline{a}_{n+1}(zC_n + A_n) + \overline{a}_{n+1}^2 z^2 E_n\right\} f_{n+1}^2 \\ &\quad + \left\{(-2a_{n+1}B_n + (zC_n + A_n)(1 + |a_{n+1}|^2) - 2\overline{a}_{n+1}z^2E_n\right\} f_{n+1} \\ &\quad + a_{n+1}^2 B_n - a_{n+1}(zC_n + A_n) + z^2E_n\end{aligned}$$

If we divide by $\lambda_n = (1-|a_n|^2)$ then

$$\begin{aligned} zA_n f'_{n+1} &= \lambda_n^{-1}\left\{B_n - \overline{a}_{n+1}(zC_n + A_n) + \overline{a}_{n+1}^2 z^2 E_n\right\} f_{n+1}^2 \\ &\quad + \lambda_n^{-1}\left\{(-2a_{n+1}B_n + (zC_n + A_n)(1+|a_{n+1}|^2) - 2\overline{a}_{n+1} z^2 E_n\right\} f_{n+1} \\ &\quad + \lambda_n^{-1}\left\{a_{n+1}^2 B_n - a_{n+1}(zC_n + A_n) + z^2 E_n\right\} \end{aligned}$$

Now, comparing the previous equation with (16) to $n+1$ and multiplied by z, i.e., with

$$zA_{n+1} f'_{n+1} = zB_{n+1} f_{n+1}^2 + zC_{n+1} f_{n+1} + zE_{n+1},$$

we get (17)-(20). ■

Theorem 3. *Let $\{P_n\}$ be a MOPS and $\{P_n^*\}$ be the sequence of reversed polynomials. If $\{P_n\}$ satisfies a structure relation with bounded degree polynomials, $n \geq 1$,*

$$z\Pi_n(z)P'_n(z) = G_n(z)P_n(z) + H_n(z)P_n^*(z) \tag{21}$$

$$z\Pi_n(z)(P_n^*)'(z) = S_n(z)P_n(z) + T_n(z)P_n^*(z) \tag{22}$$

then Π_n doesn't depend on n.

Let $p = \max\{\deg(G_n), \deg(H_n)+1, \deg(S_n), \deg(\Pi_1 - T_n)\},\ \forall n \in \mathbb{N}$. If there exists $n_0 \geq 2p$ such that $\deg(\Pi_1 - T_{n_0}) = p$, then $\{P_n\}$ is semi-classical.

Proof: If we multiply (21) by P_n^*, (22) by P_n, and divide the resulting equations by $(P_n^*)^2$, we get, after subtracting the corresponding equations,

$$\begin{aligned} & z\Pi_n\left(\frac{P'_nP_n^* - P_n(P_n^*)'}{(P_n^*)^2}\right) = \frac{(G_n - T_n)P_nP_n^* + H_n(P_n^*)^2 - S_n(P_n)^2}{(P_n^*)^2} \\ \Leftrightarrow\ & z\Pi_n\left(\frac{P_n}{P_n^*}\right)' = -S_n\left(\frac{P_n}{P_n^*}\right)^2 + (G_n - T_n)\frac{P_n}{P_n^*} + H_n \end{aligned}$$

Thus,

$$z\Pi_n f'_n = -S_n f_n^2 + (G_n - T_n) f_n + H_n.$$

From the previous lemma, $\Pi_n = \Pi_{n-1},\ \forall n \in \mathbb{N}$. Thus, $\Pi_n = \Pi_1,\ \forall n \in \mathbb{N}$.

Let us write (21) and (22) in the form

$$A\frac{zP'_n}{n} = \tilde{G}_nP_n + \tilde{H}_nP_n^* \tag{23}$$

$$A\frac{z(P_n^*)'}{n} = \tilde{S}_nP_n + \tilde{T}_nP_n^*,\ n \geq 1\,, \tag{24}$$

with $A=\Pi_1, \tilde{G}_n=G_n/n, , \tilde{H}_n=H_n/n, \tilde{S}_n=S_n/n, \tilde{T}_n=T_n/n$. Furthermore, if we use (14) in (24) then

$$A\left(\frac{zP_n'}{n}\right)^*=-\tilde{S}_nP_n+(A-\tilde{T}_n)P_n^* \tag{25}$$

On the other hand, from the hermitian character of u

$$\langle u^A,\frac{zP_n'}{n}\ z^{-k}\rangle ngle=\langle u,A\frac{zP_n'}{n}\ z^{-k}\rangle+\overline{\langle u,A\left(\frac{zP_n'}{n}\right)^*\ z^{k-n}\rangle}$$

Using (23) and (25) in previous equation we get

$$\begin{aligned}\langle u^A,\frac{zP_n'}{n}\ z^{-k}\rangle&=\langle u,\tilde{G}_nP_n\ z^{-k}\rangle\\&\quad+\langle u,\tilde{H}_nP_n^*\ z^{-k}\rangle-\overline{\langle u,\tilde{S}_nP_n\ z^{k-n}\rangle}+\overline{\langle u,(A-\tilde{T}_n)P_n^*\ z^{k-n}\rangle}\,.\end{aligned}\tag{26}$$

Since

$$\begin{aligned}&\langle u,\tilde{G}_nP_n\ z^{-k}\rangle=0,\quad k=\deg(\tilde{G}_n),...,n-1\\&\langle u,\tilde{H}_nP_n^*\ z^{-k}\rangle=0,\quad k=\deg(\tilde{H}_n)+1,...,n\\&\langle u,\tilde{S}_nP_n\ z^{k-n}\rangle=0,\quad k=1,...,n-\deg(\tilde{S}_n)\\&\langle u,(A-\tilde{T}_n)P_n^*\ z^{k-n}\rangle=0,\quad k=0,...,n-\deg(A-\tilde{T}_n)-1\end{aligned}$$

then, with $p=\max\{\deg(\tilde{G}_n),\deg(\tilde{H}_n)+1,\deg(\tilde{S}_n),\deg(A-\tilde{T}_n)\},\ \forall n\in\mathbb{N}$, it follows that

$$\langle u^A,\frac{zP_n'}{n}\ z^{-k}\rangle=0 \text{ for every } p\le k\le n-p-1 \text{ and for every } n\ge 2p+1\,.$$

Next we show that condition ii) of Definition 2,

$$\exists n_0\ge 2p:\langle u^A,\frac{zP_{n_0}'}{n_0}z^{-n_0+p}\rangle\neq 0,$$

holds for $n_0\ge 2p$ if and only if $\deg(A-\tilde{T}_{n_0})=p$.

From (26)

$$\begin{aligned}\langle u^A,\frac{zP_{n_0}'}{n_0}\ z^{-n_0+p}\rangle&=\langle u,\tilde{G}_{n_0}P_{n_0}\ z^{-n_0+p}\rangle+\langle u,\tilde{H}_{n_0}P_{n_0}^*\ z^{-n_0+p}\rangle\\&\quad-\overline{\langle u,\tilde{S}_{n_0}P_{n_0}\ z^{-p}\rangle}+\overline{\langle u,(A-\tilde{T}_{n_0})P_{n_0}^*\ z^{-p}\rangle}.\end{aligned}\tag{27}$$

Since $\deg(\tilde{G}_n) \le p,\ \deg(\tilde{H}_n) \le p-1,\ \deg(\tilde{S}_n) \le p,\ \forall n \in \mathbb{N}$, and $n_0 - p \ge p$, then

$$\langle u, \tilde{G}_{n_0} P_{n_0}\ z^{-n_0+p}\rangle = \langle u, \tilde{H}_{n_0} P^*_{n_0}\ z^{-n_0+p}\rangle = \overline{\langle u, \tilde{S}_{n_0} P_{n_0}\ z^{-p}\rangle} = 0.$$

Therefore, (27) is equivalent to

$$\langle u^A, \frac{zP'_{n_0}}{n_0}\ z^{-n_0+p}\rangle = \overline{\langle u, (A-\tilde{T}_{n_0})P^*_{n_0}\ z^{-p}\rangle}.$$

Taking into account the orthogonality relations (13) and $\deg(A - T_n) \le p$, we get

$$\langle u, (A-\tilde{T}_{n_0})P^*_{n_0}\ z^{-p}\rangle \ne 0 \Leftrightarrow \deg(A-\tilde{T}_{n_0}) = p\,.$$

Thus,

$$\langle u^A, \frac{zP'_{n_0}}{n_0}\ z^{-n_0+p}\rangle \ne 0 \Leftrightarrow \deg(A-\tilde{T}_{n_0}) = p.$$

Therefore, if there exists $n_0 \ge 2p$ such that $\deg(A - \tilde{T}_{n_0}) = p$, then the sequence $\{\frac{1}{n} zP'_n\}$ is $\mathbb{T}-$quasi-orthogonal of order p with respect to the hermitian functional u^A and we conclude that $\{P_n\}$ is semi-classical. ∎

4. Characterization Theorem

In the sequel we will use the vectors defined by

$$\psi_n(z) = [P_n(z)\ \ P^*_n(z)]^T,\ \ \vartheta_n(z) = [R_n(z)\ \ R^*_n(z)]^T\,,\ n \in \mathbb{N}\,.$$

We will use the Szegő recurrence relations in the matrix form for $\{\psi_n\}$,

$$\psi_n(z) = \mathcal{A}_n(z)\psi_{n-1}(z),\ \ \mathcal{A}_n(z) = \begin{bmatrix} z & a_n \\ \overline{a}_n z & 1 \end{bmatrix},\ n \in \mathbb{N}\,,\ \ a_n = P_n(0)\,, \tag{28}$$

and for $\{\vartheta_n\}$,

$$\vartheta_n(z) = \mathcal{B}_n(z)\vartheta_{n-1}(z),\ \ \mathcal{B}_n(z) = \begin{bmatrix} z & b_n \\ \overline{b}_n z & 1 \end{bmatrix},\ n \in \mathbb{N}\,,\ \ b_n = R_n(0)\,. \tag{29}$$

We will write $X^{(i,j)}$ to denote the entry (i,j) of a matrix X, $i,j = 1,2$.

Theorem 4. *Let (u,v) be a coherent pair of hermitian linear functionals on the unit circle and $\{P_n\},\{R_n\}$ the corresponding MOPS. Then, there exist $A \in \mathbb{P}$ and matrices $\mathcal{K}_n, \mathcal{M}_n$ of order two whose elements are bounded degree polynomials such that, for $n \ge 1$,*

$$zA(z)\psi'_n(z) = \mathcal{K}_n(z)\psi_n(z) \tag{30}$$

and

$$zA(z)\vartheta_n(z) = \mathcal{M}_n(z)\psi_n(z) \tag{31}$$

Moreover,
a) $\{P_n\}$ *is semi-classical;*
b) $\{R_n\}$ *is quasi-orthogonal of order* p *(* $p \leq 6$*) with respect to the functional* u^{zA}*. Thus, there exists a unique polynomial* B *of degree* p *such that* $u^{zA} = v^B$.

Proof: From

$$R_n = \frac{P'_{n+1}}{n+1} + \alpha_n \frac{P'_n}{n} \tag{32}$$

we get

$$R_n^* = \frac{(P'_{n+1})^{*_n}}{n+1} + \overline{\alpha}_n z \frac{(P'_n)^{*_{n-1}}}{n}.$$

Using (14) the last equation is equivalent to

$$R_n^* = P_{n+1}^* + \overline{\alpha}_n z P_n^* - z\frac{(P_{n+1}^*)'}{n+1} - \overline{\alpha}_n z^2 \frac{(P_n^*)'}{n}\,. \tag{33}$$

If we write (32) and (33) in a matrix form and use (28), we obtain

$$\vartheta_n = \mathcal{S}_n \psi_n + \mathcal{T}_n \psi'_n,\ n \geq 1, \tag{34}$$

with

$$\mathcal{S}_n = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathcal{A}_{n+1} + \begin{bmatrix} 0 & 0 \\ 0 & \overline{\alpha}_n z \end{bmatrix} + \begin{bmatrix} 1/(n+1) & 0 \\ 0 & -z/(n+1) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \overline{a}_{n+1} & 0 \end{bmatrix}$$
$$\mathcal{T}_n = \begin{bmatrix} 1/(n+1) & 0 \\ 0 & -z/(n+1) \end{bmatrix} \mathcal{A}_{n+1} + \begin{bmatrix} \alpha_n/n & 0 \\ 0 & -\overline{\alpha}_n z^2/n \end{bmatrix}.$$

Using (34) for $n+1$ and the recurrence relations (28) and (29), we get

$$\mathcal{H}_n \psi'_n = \tilde{\mathcal{M}}_n \psi_n \tag{35}$$

where the matrices $\mathcal{H}_n$ and $\tilde{\mathcal{M}}_n$ are given by

$$\mathcal{H}_n = \mathcal{B}_{n+1}\mathcal{T}_n - \mathcal{T}_{n+1}\mathcal{A}_{n+1},\ \ \tilde{\mathcal{M}}_n = \mathcal{S}_{n+1}\mathcal{A}_{n+1} + \mathcal{T}_{n+1}\begin{bmatrix} 1 & 0 \\ \overline{a}_{n+1} & 0 \end{bmatrix} - \mathcal{B}_{n+1}\mathcal{S}_n.$$

Now, if we multiply (35) by the adjoint matrix of $\mathcal{H}_n$, $\operatorname{adj}\mathcal{H}_n$, we get

$$h_n \psi'_n = \mathcal{K}_n \psi_n$$

where $h_n = \det(\mathcal{H}_n)$ is a non-zero polynomial and $\mathcal{K}_n = \operatorname{adj}(\mathcal{H}_n)\tilde{\mathcal{M}}_n$. Moreover, $h_n(0) = 0$, $\forall n \in \mathbb{N}$, and $\deg(h_n) \leq 5$, $\forall n \geq 1$. From Theorem 3 it follows that h_n is independent of n. Thus, we obtain (30) with $zA = h_1$ and $\mathcal{K}_n$ defined as above.

To obtain (31) we multiply (34) by zA and use (30). Thus, we obtain (31) with $\mathcal{M}_n = zA\mathcal{S}_n + \mathcal{T}_n\mathcal{K}_n$.

To prove assertion a) we remind that equations (30) can be written as equations of the same type as (21) and (22) of Theorem 3. Moreover, if

$$p = \max\{\deg(\mathcal{K}_n^{(1,1)}), \deg(\mathcal{K}_n^{(1,2)}) + 1, \deg(\mathcal{K}_n^{(2,1)}), \deg(A - \mathcal{K}_n^{(2,2)})\}, \ \forall n \in \mathbb{N},$$

then one can see that $p \leq 4$ and $\deg(A - \mathcal{K}_n^{(2,2)}) = p, \ n \geq 1$. Thus, from Theorem 3 we conclude that $\{P_n\}$ is semi-classical.

To prove assertion b) we use an analogue argument as in the proof of Theorem 3. We write (31) in the form

$$zAR_n = G_nP_n + H_nP_n^* \tag{36}$$

$$zAR_n^* = S_nP_n + T_nP_n^*, \ n \geq 1\,, \tag{37}$$

with $G_n, H_n, S_n, T_n \in \mathbb{P}$. From the definition of u^{zA} and the hermitian character of u, we have

$$\langle u^{zA}, R_n\, z^{-k}\rangle = \langle u, zAR_n\, z^{-k}\rangle + \overline{\langle u, zAR_n^*\, z^{k-n}\rangle} \tag{38}$$

On the other hand, using (36) and (37) in (38) we get, for $n, k \geq 0$,

$$\begin{aligned}\langle u^{zA}, R_n\, z^{-k}\rangle \\ = \langle u, G_nP_n\, z^{-k}\rangle + \langle u, H_nP_n^*\, z^{-k}\rangle + \overline{\langle u, S_nP_n\, z^{k-n}\rangle} + \overline{\langle u, T_nP_n^*\, z^{k-n}\rangle}.\end{aligned} \tag{39}$$

Using a similar reasoning as in the proof of Theorem 3, we obtain for

$$p = \max\{\deg(G_n), \deg(H_n) + 1, \deg(S_n), \deg(T_n)\}, \ \forall n \in \mathbb{N},$$

that

$\langle u^A, R_n\, z^{-k}\rangle = 0$ for every $p \leq k \leq n-p-1$ as well as for every $n \geq 2p+1$.

Thus the condition i) of Definition 2 is satisfied.

Then, we can also establish that condition ii) of Definition 2,

$$\exists n_0 \geq 2p : \langle u^A, R_{n_0} z^{-n_0+p}\rangle \neq 0,$$

holds for $n_0 \geq 2p$ if and only if $\deg(T_{n_0}) = p$. Moreover, we get that $p \leq 6$ and $\deg(T_n) = p, \ \forall n \geq 1$.

Thus $\{R_n\}$ is quasi-orthogonal of order p with respect to the functional u^{zA}. In this case, from Theorem 2, we conclude that there exists a polynomial B with $\deg(B) = p$ such that $u^{zA} = v^B$. ∎

5. Examples of Coherent Pairs on the Unit Circle

In this section we present the examples of coherent pairs corresponding to the Bernstein-Szegő class.

Theorem 5. *Let (μ_0, μ_1) be a coherent pair of measures supported on the unit circle. If μ_0 is the Lebesgue measure, then μ_1 belongs to the Bernstein-Szegő class, and the corresponding MOPS, $\{R_n\}$, is given by, $R_n(z) = z^{n-1}(z + c)$, $n \geq 1$, with c a constant, $|c| < 1$.*
Furthermore, $d\mu_1 = d\theta/(2\pi\,|z+c|^2)$.

Proof: If in (7) we assume the *sequence* $\{P_n\}$ *is a classical Hahn MOPS* in the sense that $\{P'_n/n\}$ is a sequence of monic polynomials orthogonal with respect to a measure supported on the unit circle, we know that $P_n(z) = z^n$ (see [6]). Therefore,

$$R_{n-1}(z) = z^{n-1} + \alpha_{n-1} z^{n-2}\,.$$

If we want that $\{R_n\}$ is a monic orthogonal polynomial sequence on the unit circle, then it will satisfy a forward recurrence relation

$$zR_{n-1}(z) + R_n(0)R^*_{n-1}(z) = R_n(z)\,, \tag{40}$$

and so

$$z^n + \alpha_n z^{n-1} = z^n + \alpha_{n-1} z^{n-1}\,,$$

that is, $\alpha_n = \alpha_{n-1} = \cdots = \alpha_2 = c$. As a consequence,

$$R_n(z) = z^{n-1}(z + c)\,.$$

Thus the MOPS $\{R_n\}$ belongs to the Bernstein-Szegő class and μ_1 is defined as stated (see [2], for example). ■

Theorem 6. *The only Bernstein-Szegő measure, μ_0, that admits a companion measure μ_1 supported on the unit circle such that it yields a coherent pair, is the Lebesgue measure.*

Proof: Let (μ_0, μ_1) be a coherent pair of measures supported on the unit circle and $\{P_n\}, \{R_n\}$ the corresponding MOPS. We will prove that if P_n belongs to the Bernstein-Szegő class, then $P_n(z) = z^n$.

Let us suppose that the monic orthogonal polynomial sequence $\{P_n\}$ is defined by $P_n(z) = z^{n-k}P_k(z)$ for $n \geq k$ (for a fixed nonnegative integer number k), where P_k is a monic polynomial of degree k such that $P_k(0) \neq 0$. Thus

$$P_n'(z) = (n-k)z^{n-k-1}P_k(z) + z^{n-k}P_k'(z) .$$

From (7) it follows that

$$R_n(z) = \frac{(n-k+1)z^{n-k}P_k(z) + z^{n+1-k}P_k'(z)}{n+1} + \alpha_n \frac{(n-k)z^{n-k-1}P_k(z) + z^{n-k}P_k'(z)}{n} ,$$

or, equivalently,

$$R_n(z) = z^{n-k-1}P_k(z)\left[\frac{n-k+1}{n+1}z + \alpha_n \frac{n-k}{n}\right] + z^{n-k}P_k'(z)\left[\frac{z}{n+1} + \frac{\alpha_n}{n}\right] .$$

Since $R_n(0) = 0$ for $n \geq k+2$ and taking into account (40), we have

$$R_n(z) = zR_{n-1}(z) , \quad n \geq k+2 .$$

Thus,

$$\begin{aligned} & z^{n-k-1}P_k(z)\left[\frac{n-k+1}{n+1}z + \alpha_n \frac{n-k}{n}\right] + z^{n-k}P_k'(z)\left[\frac{z}{n+1} + \frac{\alpha_n}{n}\right] \\ &= z^{n-k-1}P_k(z)\left[\frac{n-k}{n}z + \alpha_{n-1} \frac{n-k-1}{n-1}\right] + z^{n-k}P_k'(z)\left[\frac{z}{n} + \frac{\alpha_{n-1}}{n-1}\right] , \end{aligned}$$

$$\begin{aligned} z^{n-k-1}P_k(z) & \left[\left(\frac{n-k+1}{n+1} - \frac{n-k}{n}\right) z + \frac{n-k}{n}\alpha_n - \frac{n-k-1}{n-1}\alpha_{n-1}\right] \\ & + z^{n-k}P_k'(z)\left[\left(\frac{1}{n+1} - \frac{1}{n}\right) z + \frac{\alpha_n}{n} - \frac{\alpha_{n-1}}{n-1}\right] = 0 , \end{aligned}$$

$$\begin{aligned} P_k(z) & \left[\left(\frac{n-k+1}{n+1} - \frac{n-k}{n}\right) z + \frac{n-k}{n}\alpha_n - \frac{n-k-1}{n-1}\alpha_{n-1}\right] \\ & + zP_k'(z)\left[\left(\frac{1}{n+1} - \frac{1}{n}\right) z + \frac{\alpha_n}{n} - \frac{\alpha_{n-1}}{n-1}\right] = 0 . \end{aligned} \tag{41}$$

Since

$$P_k(0)\left[\frac{n-k}{n}\alpha_n - \frac{n-1-k}{n-1}\alpha_{n-1}\right] = 0 , \quad n \geq k+2,$$

and taking into account that $P_k(0) \neq 0$, we get for $n \geq k+2$,

$$\frac{n-k}{n}\alpha_n - \frac{n-1-k}{n-1}\alpha_{n-1} = 0\,.$$

Thus

$$\frac{2}{k+2}\alpha_{k+2} - \frac{1}{k+1}\alpha_{k+1} = 0\,, \text{ i.e.}$$

$$\alpha_{k+2} = \frac{k+2}{2(k+1)}\alpha_{k+1}\,,$$

and, as a consequence,

$$\alpha_{k+3} = \frac{k+3}{k+2}\frac{1}{3}\alpha_{k+2} = \frac{k+3}{k+1}\frac{1}{3!}\alpha_{k+1}\,.$$

In general, for $n \geq k+2$

$$\alpha_n = \frac{n}{k+1}\frac{1}{(n-k)!}\alpha_{k+1}\,.$$

Substituting this expression in (41),

$$\begin{aligned} 0 = P_k(z)&\left[\left(\frac{n-k+1}{n+1} - \frac{n-k}{n}\right)z\right] \\ &+ zP_k'(z)\left[-\frac{z}{n(n+1)} + \left(\frac{1}{(k+1)}\frac{1}{(n-k)!} - \frac{1}{(k+1)}\frac{1}{(n-k-1)!}\right)\alpha_{k+1}\right] \end{aligned}$$

$$\begin{aligned} 0 = P_k(z)&\left[\left(\frac{n-k+1}{n+1} - \frac{n-k}{n}\right)z\right] \\ &\qquad + zP_k'(z)\left[-\frac{z}{n(n+1)} + \frac{1}{(k+1)}\frac{1-n+k}{(n-k)!}\alpha_{k+1}\right]\,, \end{aligned}$$

$$\frac{k}{n(n+1)}P_k(z) - P_k'(z)\left[\frac{z}{n(n+1)} + \frac{1}{(k+1)}\frac{n-k-1}{(n-k)!}\alpha_{k+1}\right] = 0\,,$$

$$kP_k(z) - P_k'(z)\left[z + \frac{n(n+1)}{(k+1)}\frac{n-k-1}{(n-k)!}\alpha_{k+1}\right] = 0\,.$$

Since this equation is satisfied for all $n \geq k+2$, then $\alpha_{k+1} = 0$, as well as $P_k(z) = z^k$. But this contradicts the fact $P_k(0) \neq 0$, up to $k = 0$. In such a case we are in the previous situation. So we obtain that $P_n(z) = z^n$, $n \in \mathbb{N}$. ∎

Lemma 2. *If a sequence of monic polynomials $\{P_n\}$ orthogonal with respect to a linear functional v on the unit circle satisfies*

$$\frac{z^n}{n} + u_{n-1} = \frac{P_n(z)}{n} + \alpha_{n-1}\frac{P_{n-1}(z)}{n-1}\,, \; n = 2, 3, \ldots . \tag{42}$$

then $u_n = 0$, $n = 1, 2, \ldots .$

Furthermore, the corresponding moments c_n are zero for $n = 2, 3, \ldots .$

Proof: We will use induction arguments. For $n = 2$, (42) becomes $\frac{z^2}{2} + u_1 = \frac{P_2(z)}{2} + \alpha_1 P_1(z)$. If we apply the linear functional v in the above expression

$$\frac{c_2}{2} + u_1 c_0 = 0. \tag{43}$$

If we multiply by $1/z$ and using the linear functional v then we get

$$\frac{c_1}{2} + u_1 \bar{c}_1 = \alpha_1 \langle v, P_1(z)\overline{P_1}(1/z)\rangle.$$

From the last expression and taking into account v is quasi–definite then $c_1 \neq 0$.

For $n = 3$, (42) becomes $\frac{z^3}{3} + u_2 = \frac{P_3(z)}{3} + \alpha_2 P_2(z)$. If we multiply in the above expression by $1, 1/z, 1/z^2$, respectively and using the linear functional v we get

$$\frac{c_3}{3} + u_2 c_0 = 0\,, \tag{44}$$

$$\frac{c_2}{3} + u_2 \bar{c}_1 = 0\,, \tag{45}$$

$$\frac{c_1}{3} + u_2 \bar{c}_2 = \frac{\alpha_2}{2}\langle u, P_2(z)\overline{P_2}(1/z)\rangle\,.$$

Thus

$$\frac{c_1}{3} - 3|u_2|^2 c_1 = \frac{\alpha_2}{2}\langle u, P_2(z)\overline{P_2}(1/z)\rangle\,,$$

and, as a consequence, $c_1 \neq 0$ as well as $1 - 9|u_2|^2 \neq 0$.

For $n = 4$, (42) becomes $\frac{z^4}{4} + u_3 = \frac{P_4(z)}{4} + \alpha_3 \frac{P_3(z)}{3}$. Again, if we multiply the above expression by $1, 1/z, 1/z^2$, and $1/z^3$ respectively, then we get

$$\frac{c_4}{4} + u_3 c_0 = 0\,, \tag{46}$$

$$\frac{c_3}{4} + u_3 \bar{c}_1 = 0\,, \tag{47}$$

$$\frac{c_2}{4} + u_3 \bar{c}_2 = 0\,, \tag{48}$$

$$\frac{c_1}{4} + u_3 \bar{c}_3 = \frac{\alpha_3}{3}\langle u, P_3(z)\overline{P_3}(1/z)\rangle\,. \tag{49}$$

From (47) and (49) $c_1(1-16|u_3|^2) = \dfrac{\alpha_3}{3}\langle u, P_3(z)\overline{P_3}(1/z)\rangle$ i. e. $|u_3| \neq \dfrac{1}{4}$. Taking into account (48) we deduce that $c_2 = 0$, and, as a consequence of (45), $u_2 = 0$. On the other hand, from (44), $c_3 = 0$. Thus, taking into account (47), $u_3 = 0$ and, as a consequence, $c_2 = c_4 = 0$. Notice that $u_1 = 0$ from (43).
Finally,

$$\alpha_1\langle v, P_1(z)\overline{P_1}(1/z)\rangle = \frac{c_1}{2}\,,$$
$$\alpha_2\langle v, P_2(z)\overline{P_2}(1/z)\rangle = \frac{2}{3}c_1\,,$$
$$\alpha_3\langle v, P_3(z)\overline{P_3}(1/z)\rangle = \frac{3}{4}c_1\,.$$

If we assume $u_{n-2} = 0$ as well as $c_k = 0$ for $k = 2, 3, \ldots, n-1$, then we can multiply in (42) by $1, 1/z, \ldots, 1/z^{n-1}$, respectively. Using the linear functional v we get

$$\frac{c_n}{n} + u_{n-1} c_0 = 0 \tag{50}$$

$$\frac{c_{n-1}}{n} + u_{n-1} \bar{c}_1 = 0 \tag{51}$$

$$\frac{c_{n-2}}{n} + u_{n-1} \bar{c}_2 = 0$$
$$\vdots$$
$$\frac{c_2}{n} + u_{n-1} \bar{c}_{n-2} = 0$$

$$\frac{c_1}{n} + u_{n-1} \bar{c}_{n-1} = \frac{\alpha_{n-1}}{n-1}\langle u, P_{n-1}(z)\overline{P_{n-1}}(1/z)\rangle\,. \tag{52}$$

From (51) we get $u_{n-1} = 0$ and thus, from (50), $c_n = 0$. Taking into account (52)

$$\alpha_{n-1}\langle v, P_{n-1}(z)\overline{P_{n-1}(1/z)}\rangle = \frac{n-1}{n}c_1 \,.$$

Moreover, if v is a positive definite linear functional, then we get an integral representation of such a functional taking into account their moments c_0 and c_1. Indeed,

$$c_0 = \frac{A}{2\pi}\int_0^{2\pi} |z-\alpha|^2 \, d\theta \,, \quad c_1 = \frac{A}{2\pi}\int_0^{2\pi} z|z-\alpha|^2 \, d\theta$$

with $z = e^{i\theta}$.

Thus, $c_0 = (1+|\alpha|^2)A$, $c_1 = -\alpha A$. In other words, $\dfrac{\alpha}{1+|\alpha|^2} = -\dfrac{c_1}{c_0}$. ■

Theorem 7. *Let (μ_0, μ_1) be a coherent pair of measures supported on the unit circle. If μ_1 is the Lebesgue measure then μ_0 must be an absolutely continuous measure*

$$d\mu_0 = |z-\alpha|^2 \frac{d\theta}{2\pi} \,, \ z = e^{i\theta} \,.$$

Proof: If we assume μ_1 is the Lebesgue measure supported on the unit circle, i.e., $R_n(z) = z^n$, then (7) becomes

$$z^{n-1} = \frac{P_n'(z)}{n} + \alpha_{n-1}\frac{P_{n-1}'(z)}{n-1} \,, \ n = 2, 3, \ldots \,.$$

Integrating the above expression, there exists a sequence of complex numbers (u_n) such that

$$\frac{z^n}{n} + u_{n-1} = \frac{P_n(z)}{n} + \alpha_{n-1}\frac{P_{n-1}(z)}{n-1} \,, \ n = 2, 3, \ldots \,.$$

Using the previous lemma, the assertion follows. ■

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