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# NON-FICKIAN DELAY REACTION-DIFFUSION EQUATIONS : THEORETICAL AND NUMERICAL STUDY

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ABSTRACT: The Fisher's equation is established combining the Fick's law for the flux and the mass conservation law. Assuming that the reaction term depends on the solution at some past time, a delay parameter is introduced and the delay Fisher's equation is obtained. Modifying the Fick's law for the flux considering a temporal memory term, integro-differential equations of Volterra type were introduced in the literature.

In these paper we study reaction-diffusion equations obtained combining the two modifications: a temporal memory term in the flux and a delay in the reaction term. The delay integro-differential equations, also known as delay Volterra integro-differential equations, are studied in the theoretical view point: stability estimates are established. Numerical methods which mimic the theoretical models are studied. Numerical experiments illustrating the established results are also included.

**Key words:** Delay reaction-diffusion equation, integro-differential equation, retarded Volterra integro-differential equations, numerical method, stability, convergence.

*Mathematics Subject Classification (2000):* 65M06, 65M20, 65M15

## 1. Introduction

Nonlinear delay reaction-diffusion equations

$$\frac{\partial u}{\partial t}(x,t) = D_1 \frac{\partial^2 u}{\partial x^2}(x,t) + f(u(x,t), u(x,t-\tau)),\ x \in (a,b), t \in (0,T] \qquad (1)$$

where $\tau > 0$ is a delay parameter, $D_1 > 0$ is the diffusion coefficient, are largely used on the description of biological phenomena.

The independent $x$-version of the equation (1) was considered in the literature to model a wide range of phenomena in biosciences (see [7]).

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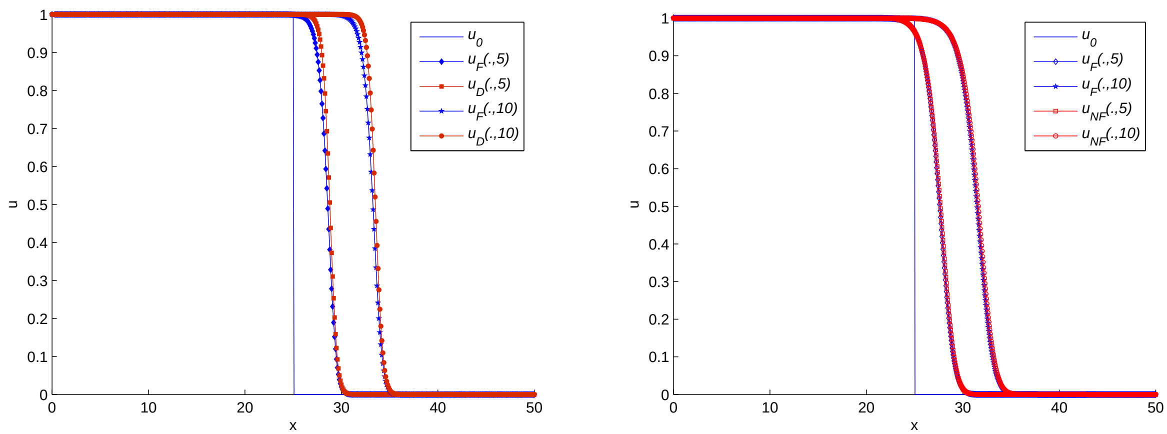


FIGURE 1. Behaviour of the solutions: $u_F$ of the equation (2) and $u_D$ of the equation (1) for different values of $\tau$ ($\tau = 0.2$ (left), $\tau = 0.05$ (right)).

Such equation arises naturally in the literature replacing the classical Fisher equation (also known as Fisher-Kolmogorov-Petrovski-Piskunov equation)

$$\frac{\partial u}{\partial t}(x,t) = D_1 \frac{\partial^2 u}{\partial x^2}(x,t) \, + f(u(x,t)), \; x \in (a,b), t \in (0,T], \tag{2}$$

when the dependence of the reaction term on the solution $u$ evaluated at the present time $t$ and at past time $t-\tau$ is assumed. The simplest model is the one obtained replacing the diffusion Verhulst's equation by the logistic delay equation (1) with the reaction term

$$f(u(x,t), u(x,t-\tau)) = Uu(x,t)\big(1 - u(x,t-\tau)\big).$$

Other reaction terms arise in the grow population phenomena (see for instance [12], [29]).

In Figure 1 we compare, for the logistic term $f(u) = Uu(1-u)$, the behaviour of the solution $u_F$ of the classical Fisher equation (2) with the solution $u_D$ of (1) for $U = 2, D_1 = 0.1, \tau = 0.05, 0.2$, in $[0, 50]$ and with the initial condition

$$u_0(x) = \begin{cases} 1, \; x \leq 25, \\ \\ 0, \; x > 25. \end{cases} \tag{3}$$

As it can be shown and as it is illustrated in Figure 1, when $\tau \to 0$, the solution of (1) converges for the solution of (2).

Delay Fisher's equation (1) is based on Fick's law for the flux $J(x,t)$ - the Fickian flux-

$$J(x,t) = -D_1\frac{\partial u}{\partial t}(x,t), \tag{4}$$

which is combined with the mass conservation law

$$\frac{\partial u}{\partial t}(x,t) = -\frac{\partial J}{\partial x}(x,t) + f(u(x,t), u(x,t-\tau)). \tag{5}$$

It should be stressed that in the context of heat conduction problems, the flux $J$ is known as Fourier flux.

For the logistic term $f(u) = U(1-u)u$, if the initial condition is, for instance, a step solution connecting the stationary states $u = 0$ (unstable state) and $u = 1$ (stable state), then equation (2) has a travel wave solution $u(x,t) = \phi(x-ct)$ such that $c \geq \sqrt{4D_1U}$. Consequently, if the reaction parameter goes to infinity, then the propagation speed also goes to infinity. This pathologic behaviour is not presented in the physical phenomena but it is introduced by the mathematical model. In order to avoid this limitation of the classical Fisher's equation in the context of reaction-diffusion phenomena, in [15] and [16], was introduced the following non Fickian flux

$$J(x,t) = -\frac{D_1}{\beta}\int_0^t e^{-\frac{t-s}{\beta}}\frac{\partial u}{\partial x}(x,s)\,ds. \tag{6}$$

Infinite propagation speed is also a characteristic of the solution of heat equation. In fact, if a sudden change in the temperature is made at a point it will be felt instantaneously everywhere. This property, known as a infinite propagation speed, is not present in heat conduction phenomena and is consequence of the violation, by the Fourier law, of the principle of casuality for the flux. Such behaviour induced Cattaneo to introduce in [11] the expression (6) for the heat flux including, in flux definition, a certain memory term as an effort to avoid the infinite propagation speed ([11],[24], [35]).

The time memory flux (6) can be obtained assuming that a flux observed at some time should be related with the gradient of the solution at some past time, that is

$$J(x,t+\beta) = -D_1\frac{\partial u}{\partial x}(x,t), \tag{7}$$

where $\beta$ is a delay parameter. Considering the first order approximation in (7), we obtain

$$\frac{\partial J}{\partial t}(x,t) + \frac{1}{\beta}J(x,t) = -\frac{D_1}{\beta}\frac{\partial u}{\partial x}(x,t),$$

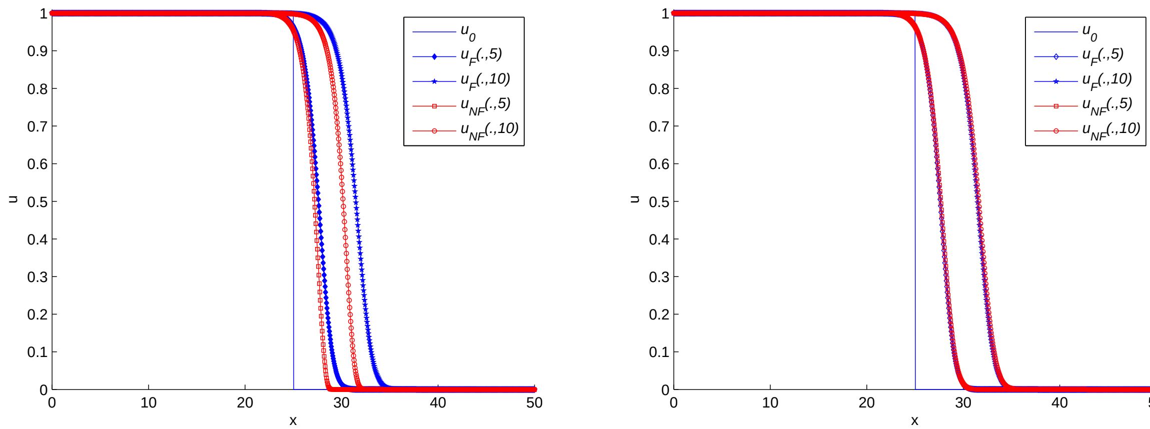


FIGURE 2. Behaviour of the solutions: $u_F$ of the equation (2) and $u_{NF}$ of the equation (9) for different values of $\beta$ ($\beta = 0.4$ (left), $\beta = 0.05$ (right)).

whose solution is given by (6).

Combining (6) with the mass conservation law

$$\frac{\partial u}{\partial t}(x,t) = -\frac{\partial J}{\partial x}(x,t) + f(u(x,t)), \tag{8}$$

the integro-differential equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{D_1}{\beta}\int_0^t e^{-\frac{t-s}{\beta}}\frac{\partial^2 u}{\partial x^2}(x,s)\,ds + f(u(x,t))\;, x \in (a,b), t > 0, \tag{9}$$

is obtained.

In Figure 2 we compare, for the logistic term $f(u) = Uu(1-u)$, the behaviour of the solution of the classical Fisher equation (2) with the solution of (9) for $U = 1, D_1 = 0.2, \beta = 0.05, 0.4$, in $[0, 50]$ and with the initial condition (3). As it can be shown and as it is illustrated in Figure 2, when $\beta \to 0$, the solution (9) converges to the solution of (2).

Let us now consider that in the reaction-diffusion system, the flux presents two different components: one depending only on the diffusion and the second one taking into account the time memory effect. This means that the flux $J(x,t)$ is splitted in two terms: the Fick's flux $J_1(x,t)$ and the Cattaneo's flux $J_2(x,t)$ being the last one defined by (6). If we consider the mass conservation law (8) then equation (9) is replaced by

$$\frac{\partial u}{\partial t}(x,t) = D_1\frac{\partial^2 u}{\partial x^2}(x,t) + \frac{D_2}{\beta}\int_0^t e^{-\frac{t-s}{\beta}}\frac{\partial^2 u}{\partial x^2}(x,s)\,ds + f(u(x,t)), \tag{10}$$

for $x \in (a,b), t > 0$. Otherwise, if we consider the mass conservation law (5), then equation (10) is replaced by the Volterra integro-differential equation

$$\frac{\partial u}{\partial t}(x,t) = D_1 \frac{\partial^2 u}{\partial x^2}(x,t) + \frac{D_2}{\beta}\int_0^t e^{-\frac{t-s}{\beta}} \frac{\partial^2 u}{\partial x^2}(x,s)\, ds + f(u(x,t), u(x,t-\tau)), \tag{11}$$

for $x \in (a,b), t > 0$.

In some biological applications where delay models induced an erratic behaviour, the $x$-independent version of (11) have been applied. Without be exhaustive we mention [7], [14], [19], [23],[32],[34], [36],[37]. Nevertheless, the equation (11) should be consider to model such biological phenomena if the spatial distribution is taken into account and the diffusion flux has two main contributions: Fickian and the non Fickian.

The theoretical and numerical analysis of the Volterra integro-differential equations (9) and (10) were consider for instance in [2]- [5], [8], [13],[15]-[17], [24], [30], [31], [33]. Existence results for the solution of IBVP defined with the retarded Volterra integro-differential equations (11) were established in [9], [10].

Our aim in this paper is to study, from analytical and numerical viewpoints, the solutions of retarded Volterra integro-differential equations (11) with the conditions

$$u(a,t) = u_a(t),\ u(b,t) = u_b(t),\ t \in (0,T], \tag{12}$$

$$u(x,t) = u_0(x,t),\ x \in (a,b),\ t \in [-\tau, 0]. \tag{13}$$

From analytical viewpoint we establish estimates to the $L^2$ norm of the solution and to the $L^2$-norm of the past in time of its gradient. Such estimates enable us to conclude the stability of the mathematical model with respect to perturbations of the initial condition. From numerical point of view we propose simple numerical methods that present the qualitative behaviour of the continuous counterparts. The results obtained in this paper can be seen as extension to the retarded Volterra integro-differential equations of the results previously obtained by the authors in [2]- [5], [8], for the solutions of integro-differential equations. We point out that these results can be seen also as extensions of the results obtained in [18] for delay reaction-diffusion equations.

The paper is organized as follows. In Section 2 we study the retarded Volterra integro-differential equations (11) with the conditions (12), (13). The stability of the stationary states of (11) is analysed in Section 3. A simple discretization of (11), (12), (13) is introduced in Section 4. The discrete version of the results proved for the continuous models in Section 2 are established in Section 5. Finally, in Section 6 are included some numerical simulations illustrating the theoretical results obtained in Section 5.

## 2. On the continuous retarded Volterra integro-differential problem

**2.1. Stability.** In this section we analyse the stability of the IBVP (11), (12), (13) with respect to perturbations of the initial condition (13).

We use the following notation: by $v(t)$ we denote the $x$-function if $v$ is defined in $[a,b]\times[0,T]$ and $t$ is fixed. We represent by $(.,.)$ the usual $L^2$ inner product and by $\|.\|$ the usual $L^2$ norm. By $H_0^1(a,b)$ we represent the usual Sobolev space $H^1(a,b)$ of functions $v$ null on the boundary points $a$ and $b$. Let $L^2(0,T,H_0^1(a,b))$ be the space of functions $v$ defined in $[a,b]\times[0,T]$ such that, for $t\in[0,T]$, $v(t)\in H_0^1(a,b)$ and

$$\int_0^T \|v(t)\|_1^2\,dt < \infty,$$

where $\|.\|_1$ denotes the usual norm in $H^1(a,b)$. Let $L^2(0,T,L^2(a,b))$ be defined as $L^2(0,T,H_0^1(a,b))$ replacing $H_0^1(a,b)$ by $L^2(a,b)$.

In what follows we assume the following assumptions:

$$M\tau = T, \tag{14}$$

$$u(x,t)\in[c,d], (x,t)\in[a,b]\times[-\tau,T], \tag{15}$$

for some $c,d$.

$$f \text{ is continuously differentiable in } [c,d]\times[c,d]\,, f(0,0)=0, \tag{16}$$

and we use the notations:

$$(\frac{\partial f}{\partial x})_{max} = \max_{(x,y)\in[c,d]\times[c,d]}\frac{\partial f}{\partial x}(x,y),\ (\frac{\partial f}{\partial y})_{max} = \max_{(x,y)\in[c,d]\times[c,d]}\frac{\partial f}{\partial y}(x,y)\,.$$

We establish, in the following result, an estimate for the energy functional

$$E_{g,u}(t) = \|u(t)\|^2 + 2D_1\int_0^t \|\frac{\partial u}{\partial x}(s)\|^2\,ds + \frac{D_2}{\beta}\|\int_0^t e^{-\frac{t-s}{\beta}}\frac{\partial u}{\partial x}(s)\,ds\|^2\,, t\in[0,T],$$

depending on the behaviour of the initial condition $u_0(x,t)$ for $x\in[a,b]$ and $t\in[-\tau,0]$.

**Theorem 1.** *Let $u$ be a solution of (11)-(13) with homogeneous boundary conditions and such that*

$$u \in L^2(0,T,H_0^1(a,b)),\ \frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial x^2} \in L^2(0,T,L^2(a,b)).$$

*Then, under the assumptions (14)-(16) and for $m=1,\ldots,M$, we have*

$$E_{g,u}(t) \leq e^{mC\tau}(1+2\eta^2\tau)^m \max_{s\in[-\tau,0]} \|u_0(s)\|^2, t \in [(m-1)\tau, m\tau], \tag{17}$$

*where*

$$C = \max\{1, \frac{1}{2\eta^2}\Big(\frac{\partial f}{\partial y}\Big)^2_{max} + 2\Big(\frac{\partial f}{\partial x}\Big)_{max}\} \tag{18}$$

*and $\eta$ denotes a nonzero constant.*

**Proof:** Multiplying (11) by $u(t)$ with respect to the inner product $(.,.)$ and using integration by parts, we easily get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = -D_1\|\frac{\partial u}{\partial x}(t)\|^2 - \frac{D_2}{\beta}\Big(\int_0^t e^{-\frac{t-s}{\beta}}\frac{\partial u}{\partial x}(s)\,ds, \frac{\partial u}{\partial x}(t)\Big)\\ +(f(u(t),u(t-\tau)),u(t)).\end{aligned} \tag{19}$$

We study separately each term of the second member of the last equality. As $f(0,0)=0$,

$$f(u(t),u(t-\tau)) = \frac{\partial f}{\partial y}(0,\theta_1 u(t-\tau))u(t-\tau) + \frac{\partial f}{\partial x}(\theta_2 u(t),u(t-\tau))u(t),$$

with $\theta_1,\theta_2 \in [0,1]$, and

$$\Big(\frac{\partial f}{\partial y}(0,\theta_1 u(t-\tau))u(t-\tau),u(t)\Big) \leq \eta^2\|u(t-\tau)\|^2 + \frac{1}{4\eta^2}\Big(\frac{\partial f}{\partial y}\Big)^2_{max}\|u(t)\|^2,$$

for some nonzero constant $\eta$, we obtain for $(f(u(t),u(t-\tau)),u(t))$ the following estimate

$$(f(u(t),u(t-\tau)),u(t)) \leq \Big(\frac{1}{4\eta^2}\Big(\frac{\partial f}{\partial y}\Big)^2_{max} + \Big(\frac{\partial f}{\partial x}\Big)_{max}\Big)\|u(t)\|^2 + \eta^2\|u(t-\tau)\|^2. \tag{20}$$

For the second term of the second member of (19) holds the following representation

$$\Big(\int_0^t e^{-\frac{t-s}{\beta}}\frac{\partial u}{\partial x}(s)\,ds, \frac{\partial u}{\partial x}(t)\Big) = \frac{1}{2}\frac{d}{dt}\|\int_0^t e^{-\frac{t-s}{\beta}}\frac{\partial u}{\partial x}(s)\,ds\|^2 + \frac{1}{\beta}\|\int_0^t e^{-\frac{t-s}{\beta}}\frac{\partial u}{\partial x}(s)\,ds\|^2. \tag{21}$$

Considering (20) and (21) in (19), we get

$$\frac{d}{dt}\Big(\|u(t)\|^2 + 2D_1\int_0^t \|\frac{\partial u}{\partial x}(s)\|^2\,ds + \frac{D_2}{\beta}\|\int_0^t e^{-\frac{t-s}{\beta}}\frac{\partial u}{\partial x}(s)\,ds\|^2\Big)$$
$$\leq \max\{-\frac{2}{\beta}, 1, \frac{1}{2\eta^2}(\frac{\partial f}{\partial y})^2_{max} + 2(\frac{\partial f}{\partial x})_{max}\}\Big(\|u(t)\|^2 + 2D_1\int_0^t \|\frac{\partial u}{\partial x}(s)\|^2 ds$$
$$+\frac{D_2}{\beta}\|\int_0^t e^{-\frac{t-s}{\beta}}\frac{\partial u}{\partial x}(s)\,ds\|^2\Big) + 2\eta^2\|u(t-\tau)\|^2\,. \tag{22}$$

In order to get an estimate to $E_{g,u}(t)$, we point out that, from (22), we have

$$E'_{g,u}(t) \leq CE_{g,u}(t) + 2\eta^2\|u(t-\tau)\|^2, t > 0, \tag{23}$$

with $C$ given by (18).

Let us consider $t \in [0,\tau]$. From (23) we obtain

$$\frac{d}{dt}\Big(e^{-Ct}E_{g,u}(t) - 2\eta^2\int_0^t e^{-Cs}\|u(s-\tau)\|^2 ds\Big) \leq 0 \tag{24}$$

and then

$$E_{g,u}(t) \leq e^{Ct}\|u(0)\|^2 + 2\eta^2\int_0^t e^{C(t-s)}\|u(s-\tau)\|^2\,ds\,. \tag{25}$$

From inequality (25) we finally get for the energy $E_{g,u}(t)$, with $t \in [0,\tau]$,

$$E_{g,u}(t) \leq e^{C\tau}(1+2\eta^2\tau)\max_{s\in[-\tau,0]}\|u_0(s)\|^2\,. \tag{26}$$

Let us consider now $t \in [\tau, 2\tau]$. From (23) we obtain

$$\frac{d}{dt}\Big(e^{-Ct}E_{g,u}(t) - 2\eta^2\int_\tau^t e^{-Cs}\|u(s-\tau)\|^2 ds\Big) \leq 0 \tag{27}$$

and then

$$\begin{aligned} E_{g,u}(t) \leq\ & e^{C(t-\tau)}E_{g,u}(\tau) + 2\eta^2\int_\tau^t e^{C(t-s)}\|u(s-\tau)\|^2\,ds \\ & \leq e^{C\tau}E_{g,u}(\tau) + 2\eta^2\tau e^{C\tau}\max_{t\in[\tau,2\tau]}\|u(t-\tau)\|^2\,. \end{aligned} \tag{28}$$

Using (26) in (28) we deduce

$$E_{g,u}(t) \leq e^{2C\tau}(1+2\eta^2\tau)\max_{s\in[-\tau,0]}\|u_0(s)\|^2 + \tau 2\eta^2 e^{C\tau}\max_{s\in[0,\tau]}\|u(s)\|^2, \tag{29}$$

and again, using (26) in (29), we obtain

$$E_{g,u}(t) \leq e^{2C\tau}(1+2\eta^2\tau) \max_{s\in[-\tau,0]} \|u_0(s)\|^2 + \tau 2\eta^2 e^{2C\tau}(1+2\eta^2\tau) \max_{s\in[-\tau,0]} \|u_0(s)\|^2 . \tag{30}$$

Inequality (30) implies

$$E_{g,u}(t) \leq e^{2C\tau}(1+2\eta^2\tau)^2 \max_{s\in[-\tau,0]} \|u_0(s)\|^2 , t \in [\tau, 2\tau]. \tag{31}$$

Considering now that for $t \in [(m-2)\tau, (m-1)\tau]$ holds the inequality (17) with $m$ replaced by $m-1$, following the procedure described above, we easily get for $t \in [m\tau, (m+1)\tau]$, the inequality (17).

∎

The estimate (17) establishes an upper bound for

$$\|u(t)\|^2, \quad \int_0^t \|\frac{\partial u}{\partial x}(s)\|^2 \, ds, \ \| \int_0^t e^{-\frac{t-s}{\beta}} \frac{\partial u}{\partial x}(s) \, ds\|^2,$$

for each time $t \in [0, T]$, depending on $e^{Cm\tau}(1 + 2\eta^2\tau)^m$ when $t \in [(m-1)\tau, m\tau]$. The exponential term can be eliminated in some cases. Nevertheless, in those cases, we only get an upper bound for

$$\|u(t)\|^2, \quad \| \int_0^t e^{-\frac{t-s}{\beta}} \frac{\partial u}{\partial x}(s) \, ds\|^2.$$

In fact, if we use in the identity (19) the Friedrichs-Poincaré inequality we get

$$\begin{aligned} &\frac{d}{dt}\Big(\|u(t)\|^2 + \frac{D_2}{\beta}\| \int_0^t e^{-\frac{t-s}{\beta}} \frac{\partial u}{\partial x}(s) \, ds\|^2\Big) \\ &\leq \max\{-\frac{2}{\beta}, -\frac{2D_1}{(b-a)^2} + \frac{1}{2\eta^2}\big(\frac{\partial f}{\partial y}\big)^2_{max} + 2\big(\frac{\partial f}{\partial x}\big)_{max}\}\Big(\|u(t)\|^2 \\ &\quad + \frac{D_2}{\beta}\| \int_0^t e^{-\frac{t-s}{\beta}} \frac{\partial u}{\partial x}(s) \, ds\|^2\Big) + 2\eta^2\|u(t-\tau)\|^2, \end{aligned} \tag{32}$$

which replaces (22). If the behaviour of the reaction term depending on the solution at the present time $t$ is dominated by the diffusion, that is,

$$\frac{D_1}{(b-a)^2} - \big(\frac{\partial f}{\partial x}\big)_{max} > 0, \tag{33}$$

then we conclude

$$\frac{d}{dt}\Big(\|u(t)\|^2 + \frac{D_2}{\beta}\|\int_0^t e^{-\frac{t-s}{\beta}}\frac{\partial u}{\partial x}(s)\,ds\|^2\Big) \leq 2\eta^2\|u(t-\tau)\|^2 \tag{34}$$

with $\eta$ defined by

$$2\eta^2 = \frac{1}{2}\frac{\left(\frac{\partial f}{\partial y}\right)^2_{max}}{\frac{D_1}{(b-a)^2} - \left(\frac{\partial f}{\partial x}\right)_{max}}. \tag{35}$$

Attending that (34) holds, following the proof of Theorem 1, we can prove for

$$E_u(t) = \|u(t)\|^2 + \frac{D_2}{\beta}\|\int_0^t e^{-\frac{t-s}{\beta}}\frac{\partial u}{\partial x}(s)\,ds\|^2,$$

the next result:

**Theorem 2.** *Let $u$ be a solution of (11)- (13) with homogeneous boundary conditions. Let us suppose that*

$$u \in L^2(0,T,H_0^1(a,b)),\ \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2} \in L^2(0,T,L^2(a,b)),$$

*and (14)-(16) hold.*

*If (33), then, for $,m = 1,\ldots,M$, we have*

$$E_u(t) \leq (1+2\eta^2\tau)^m \max_{s\in[-\tau,0]} \|u_0(s)\|^2, t \in [(m-1)\tau, m\tau] \tag{36}$$

*where $2\eta^2$ is defined by (35), else, $m = 1,\ldots,M$,*

$$E_u(t) \leq e^{Cm\tau}(1+2\eta^2\tau)^m \max_{s\in[-\tau,0]} \|u_0(s)\|^2, t \in [(m-1)\tau, m\tau], \tag{37}$$

*holds, with*

$$C = \max\{-\frac{2}{\beta}, -\frac{2D_1}{(b-a)^2} + 2\big(\frac{\partial f}{\partial x}\big)_{max} + \frac{1}{2\eta^2}\big(\frac{\partial f}{\partial y}\big)^2_{max}\}, \tag{38}$$

*for every nonzero constant $\eta$.*

■

In what follows we establish the relation between the previously obtained estimates and the known estimates for the solution of the Volterra integro-differential equation (10), that is, the solution of (11) with the reaction term $f$ delay independent. In [2] the next result was proved for $E_u(t)$, but for $E_{g,u}(t)$ holds the following:

**Theorem 3.** *Let $u$ be a solution of (10)- (13) with homogeneous boundary conditions. Let us suppose that*

$$u \in L^2(0,T,H_0^1(a,b)),\ \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2} \in L^2(0,T,L^2(a,b)),$$

*the assumption (15) holds, and $f$ is continuously differentiable with $f(0)=0$.*
*If $f'_{max} = \max_{u\in[c,d]} f'(u) < 0$, then*

$$E_{g,u}(t) \leq \|u_0\|^2, t \in [0,T], \tag{39}$$

*else*

$$E_{g,u}(t) \leq e^{f'_{max}t}\|u_0\|^2, t \in [0,T]. \tag{40}$$

*For $E_u(t)$ holds*

$$E_u(t) \leq e^{2\max\{-\frac{1}{\beta}, f'_{max} - \frac{D_1}{(b-a)^2}\}t}\|u_0\|^2,\ t \in [0,T]. \tag{41}$$

■

From Theorem 3 we conclude that if $f' < 0$ then $E_{g,u}(t)$ is less or equal to $\|u_0\|^2$ and $E_u(t)$ goes to zero when $t \to +\infty$, else $E_{g,u}(t)$ remains bounded in bounded time intervals.

For the retarded Volterra integro-differential problems we are only able to conclude the following: $E_{g,u}(t)$ remains bounded in bounded time intervals with an exponential increasing factor being this factor eliminated when $E_u(t)$ is considered provided that (33) holds.

In what follows we study the stability of the solutions of the IBVP (11)-(13) when the initial condition (13) is perturbed. Let $u$ and P $\tilde{u}$ be solutions of the IBVP(11)- (13) with initial conditions $u_0$ and $\tilde{u}_0$ respectively. Let $w$ be defined by $w(x,t) = u(x,t) - \tilde{u}(x,t), x \in [a,b], t \in [0,T]$. Then $w$ is solution of the IBVP

$$\begin{aligned} \frac{\partial w}{\partial t}(x,t) &= D_1 \frac{\partial^2 w}{\partial x^2}(x,t) + \frac{D_2}{\beta}\int_0^t e^{-\frac{t-s}{\beta}} \frac{\partial^2 w}{\partial x^2}(x,s)\,ds \\ &+ f(u(x,t),u(x,t-\tau)) - f(\tilde{u}(x,t),\tilde{u}(x,t-\tau)),\ x \in (a,b), t > 0, \end{aligned} \tag{42}$$

$$w(a,t) = w(b,t) = 0,\ t \in (0,T], \tag{43}$$

$$w(x,t) = u_0(x,t) - \tilde{u}_0(x,t),\ x \in (a,b),\ t \in [-\tau,0]. \tag{44}$$

As for $f(u(x,t),u(x,t-\tau))-f(\tilde{u}(x,t),\tilde{u}(x,t-\tau))$ holds the representation

$$
\begin{aligned}
&f(u(x,t),u(x,t-\tau))-f(\tilde{u}(x,t),\tilde{u}(x,t-\tau))\\
&=\frac{\partial f}{\partial x}(u(x,t)+\theta_1 w(x,t),u(x,t-\tau))w(x,t)\\
&+\frac{\partial f}{\partial y}(\tilde{u}(x,t),\tilde{u}(x,t)+\theta_2 w(x,t-\tau))w(x,t-\tau)),\ \theta_1,\theta_2\in[0,1],
\end{aligned}
$$

following the proof of Theorem 1 it can be shown the next result.

**Theorem 4.** *Let $u$ and $\tilde{u}$ be solutions of (11)- (13) with initial conditions $u_0$ and $\tilde{u}_0$. Let us suppose that (14)-(16) hold (with $f(0,0)$ not necessarily equal to zero) and*

$$
u\,,\tilde{u}\in L^2(0,T,H^1(a,b)),\ \frac{\partial u}{\partial t},\frac{\partial \tilde{u}}{\partial t},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 \tilde{u}}{\partial x^2}\in L^2(0,T,L^2(a,b)).
$$

*Then*

$$
E_{g,w}(t)\le e^{mC\tau}(1+2\eta^2\tau)^m\max_{s\in[-\tau,0]}\|u_0(s)-\tilde{u}_0(s)\|^2\,,t\in[(m-1)\tau,m\tau],\quad (45)
$$

*for $m=1,\dots,M$, where $C$ is defined by (18) and $\eta$ denotes a nonzero constant.*

*If (33) then*

$$
E_w(t)\le(1+2\eta^2\tau)^m\max_{s\in[-\tau,0]}\|u_0(s)-\tilde{u}_0(s)\|^2\,,t\in[(m-1)\tau,m\tau],\qquad (46)
$$

*for $m=1,\dots,M$, where $2\eta^2$ is defined by (35), else*

$$
E_w(t)\le e^{Cm\tau}(1+2\eta^2\tau)^m\max_{s\in[-\tau,0]}\|u_0(s)-\tilde{u}_0(s)\|^2\,,t\in[(m-1)\tau,m\tau],\quad (47)
$$

*for $m=1,\dots,M$, with $C$ defined by (38) and $\eta$ represents an arbitrary nonzero constant.*

■

The stability of the IBVP (11)- (13) is consequence of Theorem 4. In fact, from the previous result, we conclude that if $\max_{s\in[-\tau,0]}\|u_0(s)-\tilde{u}_0(s)\|$ is small, then

$$
\|u(t)-\tilde{u}(t)\|^2,\ \int_0^t\|\frac{\partial u}{\partial x}(s)-\frac{\partial \tilde{u}}{\partial x}(s)\|^2\,ds,\ \|\int_0^t e^{-\frac{t-s}{\beta}}\Big(\frac{\partial u}{\partial x}(s)-\frac{\partial \tilde{u}}{\partial x}(s)\Big)\,ds\|^2
$$

remains small in bounded time intervals.

From Theorem 4 we also conclude that if the IBVP (11)- (13) has a solution $u$ then $u$ is unique. In fact, let $\tilde{u}$ be another solution. Then $\|u(t)-\tilde{u}(t)\|^2 \leq 0$ for $t \in [0,T]$, which means that $u(t) = \tilde{u}(t)$ in $L^2(a,b)$ for each time $t$, $t \in [0,T]$.

**2.2. Stationary states.** In this section our aim is to study the stability of the stationary states of (11) when $x \in \mathbb{R}$. In order to do that we consider the initial value problem (IVP)

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = D_1 \dfrac{\partial^2 u}{\partial x^2}(x,t) + \dfrac{D_2}{\beta} \displaystyle\int_0^t e^{-\frac{t-s}{\beta}} \dfrac{\partial^2 u}{\partial x^2}(x,s)\, ds \\ \qquad\quad + f(u(x,t), u(x,t-\tau)),\ x \in \mathbb{R}, t > 0, \\ u(x,t) = u_0(x,t), x \in \mathbb{R}, t \in [-\tau, 0]. \end{cases} \tag{48}$$

Let $u$ be a solution of (48) and let us suppose that $u$ and $u_0$ are smooth enough. Then $u$ satisfies the following equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) = \; & D_1 \frac{\partial^3 u}{\partial t \partial x^2}(x,t) + \frac{D_2}{\beta}\frac{\partial^2 u}{\partial x^2}(x,t) - \frac{D_2}{\beta^2}\int_0^t e^{-\frac{t-s}{\beta}} \frac{\partial^2 u}{\partial x^2}(x,s)\, ds \\ & + \frac{\partial}{\partial t} f(u(x,t), u(x,t-\tau)), \end{aligned} \tag{49}$$

and it is easy to show that $u$ is solution of the following IVP

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) & = D_1 \dfrac{\partial^3 u}{\partial t \partial x^2}(x,t) + \dfrac{D_2}{\beta}\dfrac{\partial^2 u}{\partial x^2}(x,t) + \dfrac{\partial}{\partial t} f(u(x,t), u(x,t-\tau)) \\ & \quad -\dfrac{1}{\beta}\Big(\dfrac{\partial u}{\partial t}(x,t) - D_1 \dfrac{\partial^2 u}{\partial x^2}(x,t) - f(u(x,t), u(x,t-\tau)),\Big) \\ & \qquad\qquad , x \in \mathbb{R}, t > 0, \\ \dfrac{\partial u}{\partial t}(x,t) & = \dfrac{\partial u_0}{\partial t}(x,t),\ x \in \mathbb{R}, t \in [-\tau, 0], \\ u(x,t) & = u_0(x,t), x \in \mathbb{R}, t \in [-\tau, 0]. \end{cases} \tag{50}$$

Let us suppose now that $u$ is solution of the IVP (50). Then, from the retarded Volterra integro-differential equation we get for $u$ the equivalent equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{1}{\beta}\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial t}\Big(D_1\frac{\partial^2 u}{\partial x^2}(x,t) + f(u(x,t),u(x,t-\tau))\Big)$$
$$+\frac{D_2}{\beta}\frac{\partial^2 u}{\partial x^2}(x,t) + \frac{1}{\beta}\Big(D_1\frac{\partial^2 u}{\partial x^2}(x,t) + f(u(x,t),u(x,t-\tau))\Big)$$

which allow us to conclude that $u$ satisfies

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= D_1\frac{\partial^2 u}{\partial x^2}(x,t) + \frac{D_2}{\beta}\int_0^t e^{-\frac{t-s}{\beta}}\frac{\partial^2 u}{\partial x^2}(x,s)\,ds + f(u(x,t),u(x,t-\tau)) \\ &\quad + e^{\frac{t}{\beta}}\Big(\frac{\partial u}{\partial t}(x,0) - \Big(D_1\frac{\partial^2 u}{\partial x^2}(x,0) + f(u(x,0),u(x,-\tau))\Big)\Big).\end{aligned} \tag{51}$$

From (51) we deduce that $u$ is solution of the IVP (48) provided $u_0$ satisfies the following equality

$$\frac{\partial u_0}{\partial t}(x,0) = D_1\frac{\partial^2 u_0}{\partial x^2}(x,0) + f(u_0(x,0),u_0(x,-\tau)),\ x\in\mathbb{R}. \tag{52}$$

In the next proposition we summarize the previous considerations:

**Proposition 1.** *If $u$ is solution of (48), $u$ and $u_0$ are smooth enough, then $u$ is solution of (50). Furthermore, if $u$ is solution of (50) then $u$ is solution of (48) provided $u_0$ satisfies (52).*

■

In what follows we prove the following proposition:

**Proposition 2.** *The IVP (48) with $f(u(x,t),u(x,t-\tau)) = Uu(x,t)(1-u(x,t-\tau))$ has the stationary states $u=0$ and $u=1$, respectively unstable and stable.*

**Proof:** Using Proposition 1 it is easy to show that $u$ is a stationary state of (48) if and only if $u$ satisfies the following

$$\frac{D_1+D_2}{\beta}u''(x) + \frac{1}{\beta}f(u(x),u(x)) = 0,\ x\in\mathbb{R},$$

or, equivalently

$$Z'(x) = F(Z(x)), x\in\mathbb{R}, \tag{53}$$

with $Z(x) = (z_1(x), z_2(x))$, $z_1(x) = u(x), z_2(x) = u'(x)$ and

$$F(Z(x)) = (z_2(x), -\frac{1}{D_1+D_2}f(z_1(x),z_2(x))), x\in\mathbb{R}.$$

The points $(0,0)$ and $(1,0)$ are equilibrium points of the phase portrait of (53) respectively unstable and stable. Then we conclude that $u=0$ and $u=1$ are stationary states of (48) for the logistic reaction term, respectively unstable and stable.

■

Proposition 2 allow us to conclude that the retarded Volterra integro-differential equation obtained combining a Fickian flux or a non Fickian flux or a flux with two contributions (Fick and non Fick), with a mass conservation law with a delay reaction term, has the same stationary states as the delay reaction-diffusion equation (obtained with the Fickian flux and a mass conservation law with the reaction term $f(u)$) and with the same qualitative behaviour.

## 3. A discrete retarded Volterra integro-differential model

The retarded Volterra integro-differential IBVP (11)-(13) are non linear and analytical expressions for their solutions are not known. Numerical methods are the only procedure to get, at least approximately, the solutions of such problems. In this section we propose a numerical method to compute the solution of (11)- (13). We study the stability of such method and discrete versions of the proved results for the continuous model are obtained.

In $[a,b]$ we introduce the grid $I_h=\{x_i, i=0,\dots,N\}$ with $x_0=a, x_N=b$ and $x_{i+1}=x_i+h, i=0,\dots,N-1$. Let $\Delta t$ be the temporal stepsize and $p\in\mathbb{N}$ such that $p=\dfrac{\tau}{\Delta t}$. In $[-\tau,T]$ we consider the grid $\{t_\ell, \ell=-p,\dots,\mathfrak{M}\}$ defined by

$$t_{-p}=-\tau, t_{\ell+1}=t_\ell+\Delta t,\ \ell=-j,\dots,\mathfrak{M}-1, t_{\mathfrak{M}}=T.$$

Let $u_h^{n+1}(x_i)$ be the fully discrete approximation to $u(x_i,t_{n+1})$ defined by

$$\begin{aligned} u_h^{n+1}(x_i) &= u_h^n(x_i)+\Delta t D_1 D_{2,x}u_h^{n+1}(x_i)+\Delta t^2\frac{D_2}{\beta}\sum_{\ell=1}^{n+1} e^{-\frac{t_{n+1}-t_\ell}{\beta}} D_{2,x}u_h^\ell(x_i) \\ &\quad +\Delta t f(u_h^{n+1}(x_i), u_h^{n+1-p}(x_i)), \end{aligned} \tag{54}$$

for $i=1,\dots,N-1, n=1,\dots,\mathfrak{M}-1$, and such that

$$u_h^n(x_0)=u_a(t_n),\ u_h^n(x_N)=u_b(t_n),\quad n=1,\dots,\mathfrak{M}, \tag{55}$$

$$u_h^n(x_i)=u_0(x_i,t_n),\quad i=0,\dots,N,\quad n=-p+1,\dots,0. \tag{56}$$

In (54) the difference operator $D_{2,x}$ is the usual second order centered finite difference operator

$$D_{2,x}v_h(x_i) = \frac{v_h(x_{i+1}) - 2v_h(x_i) + v_h(x_{i-1})}{h^2}1,\ i = 1, \ldots, N-1.$$

The stability analysis is established with respect to a $L^2$ discrete norm which is defined in what follows. By $L^2(I_h)$ we denote the space of grid functions $v_h$ such that $v_h(x_0) = v_h(x_N) = 0$. In $L^2(I_h)$ we introduce the inner product

$$(v_h, w_h)_h = h\sum_{i=1}^{N-1} v_h(x_i)w_h(x_i),\ \ v_h, w_h \in L^2(I_h). \tag{57}$$

By $\|.\|_{L^2(I_h)}$ we denote the norm induced by the inner product (57).

We introduce other notations:

$$(v_h, w_h)_{h,+} = h\sum_{i=1}^{N} v_h(x_i)w_h(x_i), \tag{58}$$

$$\|v_h\|_{L^2(I_{h,+})} = \left(h\sum_{i=1}^{N} v_h(x_i)^2\right)^{1/2}. \tag{59}$$

Let $D_{-x}$ be the usual backward finite difference operator. The following relations have a central role on the proof of the main result of this section - Theorem 5:

$$(D_{2,x}v_h, w_h)_h = -(D_{-x}v_h, D_{-x}w_h)_{h,+},\ \ v_h, w_h \in L^2(I_h), \tag{60}$$

$$\|v_h\|^2_{L^2(I_h)} \leq (b-a)^2\|D_{-x}v_h\|^2_{L^2(I_{h,+})},\ v_h \in L^2(I_h). \tag{61}$$

Identity (60) can be proved using summation by parts. The second relation is known as a discrete Poincaré-Friedrichs inequality.

The discrete version of the energy $E_u(t_n)$

$$E(u_h^{n+1}) = \|u_h^{n+1}\|^2_{L^2(I_h)} + \frac{D_2}{\beta}\|\Delta t^2\sum_{\ell=1}^{n+1} e^{-\frac{t_{n+1}-t_\ell}{\beta}} D_{-x}u_h^\ell\|^2_{L^2(I_{h,+})},$$

is studied in what follows. According to this remark, Theorem 5 can be seen as a discrete version of Theorem 2.

**Theorem 5.** *Let $u_h^{n+1}$ be defined by (54)-(56) with homogeneous boundary conditions and such that $u_h^\ell(x_i)\in[c,d], i=1,\dots,N-1, \ell=-p,\dots,\mathfrak{M}$. Let us suppose that the reaction term $f$ satisfies (16). If (33) then, for $2\eta^2$ defined by (35), $(m-1)p\le n\le mp$, without any restriction to $\Delta t$, we have*

$$E(u_h^n)\le (1+2\eta^2\tau)(1+2\eta^2\Delta t)\max_{\ell=-p,\dots,0}\|u_0(t_\ell)\|^2_{L^2(I_h)}\sum_{i=0}^{m-1}\big(2\eta^2\tau(m-i)\big)^i. \tag{62}$$

*Else, for $(m-1)p\le n\le mp$,*

$$E(u_h^n)\le (1+\tau)(1+\Delta t)\max_{\ell=-p,\dots,0}\|u_0(t_\ell)\|^2_{L^2(I_h)}\tilde{C}^{mp}\sum_{i=0}^{m-1}\big(\tau(m-i)\big)^i \tag{63}$$

*with*

$$\tilde{C}=\frac{1}{1-\Delta t\Big(2\Big(\big(\frac{\partial f}{\partial x}\big)_{max}-\frac{D_1}{(b-a)^2}\Big)+\big(\frac{\partial f}{\partial y}\big)^2_{max}\Big)}, \tag{64}$$

*provided that $\Delta t$ satisfies*

$$\Delta t<\frac{1}{2\Big(\big(\frac{\partial f}{\partial x}\big)_{max}-\frac{D_1}{(b-a)^2}\Big)+\big(\frac{\partial f}{\partial y}\big)^2_{max}}\ . \tag{65}$$

**Proof:** Multiplying (54) by $u_h^{n+1}$ with respect to the inner product $(.,.)_h$ and using summation by parts we get

$$\begin{aligned}\|u_h^{n+1}\|^2_{L^2(I_h)} &= (u_h^n,u_h^{n+1})_h-\Delta t D_1\|D_{-x}u_h^{n+1}\|^2_{L^2(I_{h,+})}\\ &\quad -\Delta t^2\frac{D_2}{\beta}\sum_{\ell=1}^{n+1}e^{-\frac{t_{n+1}-t_\ell}{\beta}}(D_{-x}u_h^\ell,D_{-x}u_h^{n+1})_{h,+}\\ &\quad +\Delta t(f(u_h^{n+1},u_h^{n+1-p}),u_h^{n+1})_h,\end{aligned} \tag{66}$$

where $f(u_h^{n+1},u_h^{n+1-p})(x_i)=f(u_h^{n+1}(x_i),u_h^{n+1-p}(x_i)), i=1,\dots,N-1.$

We compute now a new representation of the two last terms of the second member of (66). Analogously to the continuous case, for the last term we have

$$\begin{aligned}(f(u_h^{n+1},u_h^{n+1-p}),u_h^{n+1})_h &\le \Big(\big(\frac{\partial f}{\partial x}\big)_{max}+\frac{1}{4\eta^2}\big(\frac{\partial f}{\partial y}\big)^2_{max}\Big)\|u_h^{n+1}\|^2_{L^2(I_h)}\\ &\quad +\eta^2\|u_h^{n+1-p}\|^2_{L^2(I_h)},\end{aligned} \tag{67}$$

where $\eta \neq 0$ is an arbitrary constant. For the first mentioned term hods the following representation

$$\begin{aligned}&(\sum_{\ell=1}^{n+1} e^{-\frac{t_{n+1}-t_\ell}{\beta}} D_{-x}u_h^\ell, D_{-x}u_h^{n+1})_{h,+} = \frac{1}{2}\|\sum_{\ell=1}^{n+1} e^{-\frac{t_{n+1}-t_\ell}{\beta}} D_{-x}u_h^\ell\|^2_{L^2(I_{h,+})}\\&-\frac{e^{-2\frac{\Delta t}{\beta}}}{2}\|\sum_{\ell=1}^{n} e^{-\frac{t_n-t_\ell}{\beta}} D_{-x}u_h^\ell\|^2_{L^2(I_{h,+})} + \frac{1}{2}\|D_{-x}u_h^{n+1}\|^2_{L^2(I_{h,+})}.\end{aligned} \tag{68}$$

Considering in (66) the Poincaré-Friedrichs inequality (61), the Cauchy-Schwarz inequality, the upper bound (67) and the representation (68), we obtain

$$\begin{aligned}&\Big(1-\Delta t\Big(2\big(\frac{\partial f}{\partial x}\big)_{max} + \frac{1}{2\eta^2}\big(\frac{\partial f}{\partial y}\big)^2_{max} - \frac{2D_1}{(b-a)^2}\Big)\Big)\|u_h^{n+1}\|^2_{L^2(I_h)}\\&+\frac{D_2}{\beta}\|\Delta t\sum_{\ell=1}^{n+1} e^{-\frac{t_{n+1}-t_\ell}{\beta}} D_{-x}u_h^\ell\|^2_{L^2(I_{h,+})}\\&\leq \|u_h^n\|^2_{L^2(I_h)} + e^{-2\frac{\Delta t}{\beta}}\frac{D_2}{\beta}\|\Delta t\sum_{\ell=1}^{n} e^{-\frac{t_n-t_\ell}{\tau}} D_{-x}u_h^\ell\|^2_{L^2(I_{h,+})} + 2\eta^2\Delta t\|u_h^{n+1-p}\|^2_{L^2(I_h)}.\end{aligned} \tag{69}$$

If (33) then with $2\eta^2$ defined by (35) and without any restriction on the time step size $\Delta t$, we get

$$E(u_h^{n+1}) \leq E(u_h^n) + 2\eta^2\Delta t\|u_h^{n+1-p}\|^2_{L^2(I_h)}. \tag{70}$$

Else, for every nonzero constant $\eta$ we have, for $\Delta t$ satisfying (65),

$$E(u_h^{n+1}) \leq \bar{C}\Big(E(u_h^n) + 2\eta^2\Delta t\|u_h^{n+1-p}\|^2_{L^2(I_h)}\Big), \tag{71}$$

with

$$\bar{C} = \frac{1}{1-\Delta t C}$$

where $C$ is now defined by

$$C = \Big(2\big(\frac{\partial f}{\partial x}\big)_{max} + \frac{1}{2\eta^2}\big(\frac{\partial f}{\partial y}\big)^2_{max} - \frac{2D_1}{(b-a)^2}\Big).$$

Let us now consider the following inequality

$$E(u_h^{n+1}) \leq \tilde{C}\Big(E(u_h^n) + 2\eta^2\Delta t\|u_h^{n+1-p}\|^2_{L^2(I_h)}\Big), \tag{72}$$

which has as particular cases the two previous relations (70) and (71). Relation (72) can be considered only for $n = 1, \ldots, \mathfrak{M} - 1$.

In what follows we establish an estimate to $E(u_h^{n+1})$ depending on $\|u_0(t_\ell)\|^2_{L^2(I_h)}$ for $\ell = 1-p, 2-p, \ldots, 0$, and $E(u_h^1)$.

Let us consider $j = 2, \ldots, p$. From (72) we obtain

$$\begin{aligned} E(u_h^{j+1}) &\leq \tilde{C}^{j-1} E(u_h^1) + \tilde{C} 2\eta^2 \Delta t \sum_{i=2}^{j} \tilde{C}^{j-i} \|u_0(t_{i-p})\|^2_{L^2(I_h)} \\ &\leq \tilde{C}^{j-1} E(u_h^1) + 2\eta^2 \tilde{C}^{j-1} (j-1) \Delta t \max_{i=2,\ldots,j} \|u_0(t_{i-p})\|^2_{L^2(I_h)} \\ &\leq \tilde{C}^{j-1} (1 + 2\eta^2 \tau) \max\{E(u_h^1), \max_{i=2,\ldots,j} \|u_0(t_{i-p})\|^2_{L^2(I_h)}\}. \end{aligned} \tag{73}$$

As for $E(u_h^1)$ we have

$$\begin{aligned} &\Big(1 - \Delta t \Big( 2 \big(\frac{\partial f}{\partial x}\big)_{max} + \frac{1}{2\eta^2} \big(\frac{\partial f}{\partial y}\big)^2_{max} - \frac{2D_1}{(b-a)^2}\Big)\Big) \|u_h^1\|^2_{L^2(I_h)} \\ &\qquad + \frac{D_2}{\beta} \|\Delta t D_{-x} u_h^1\|^2_{L^2(I_{h,+})} \leq \|u_h^0\|^2_{L^2(I_h)} + 2\eta^2 \Delta t \|u_h^{1-p}\|^2_{L^2(I_h)}, \end{aligned} \tag{74}$$

that is

$$E(u_h^1) \leq \tilde{C}(1 + 2\eta^2 \Delta t) \max_{\ell = 1-p, 0} \|u_0(t_\ell)\|^2_{L^2(I_h)}, \tag{75}$$

with $\tilde{C}$ as before, from (73) we conclude, for $j = 1, \ldots, p$,

$$E(u_h^j) \leq \tilde{C}^j (1 + 2\eta^2 \tau)(1 + 2\eta^2 \Delta t) \max_{i=1,\ldots,j} \|u_0(t_{i-p})\|^2_{L^2(I_h)}. \tag{76}$$

Let us consider now $j = p+1, \ldots, 2p$. For $j = p+1$ we have

$$
\begin{aligned}
E(u_h^{p+1}) &\leq \tilde{C} E(u_h^p) + \tilde{C} 2\eta^2 \Delta t \|u_h^1\|^2_{L^2(I_h)} \\
&\leq \tilde{C} E(u_h^p) + \tilde{C} 2\eta^2 \Delta t E(u_h^1) \\
&\leq \tilde{C}^{p+1} (1 + 2\eta^2 \tau)(1 + 2\eta^2 \Delta t) \max_{\ell = 1-p, \ldots, 0} \|u_0(t_\ell)\|^2_{L^2(I_h)} \\
&+ \tilde{C}^2 2\eta^2 \Delta t (1 + 2\eta^2 \Delta t) \max_{\ell = 1-p, 0} \|u_0(t_\ell)\|^2_{L^2(I_h)} \\
&\leq \Big(\tilde{C}^{p+1} + \tilde{C}^2 2\eta^2 \Delta t\Big)(1 + 2\eta^2 \tau)(1 + 2\eta^2 \Delta t) \max_{\ell = 1-p, \ldots, 0} \|u_0(t_\ell)\|^2_{L^2(I_h)}.
\end{aligned} \tag{77}
$$

Analogously, it can be shown that for $j \in \{2, \ldots, p\}$ we have

$$
E(u_h^{p+j}) \leq \Big(\tilde{C}^{p+j} + j \tilde{C}^{j+1} 2\eta^2 \Delta t\Big)(1 + 2\eta^2 \tau)(1 + 2\eta^2 \Delta t) \max_{\ell = 1-p, \ldots, 0} \|u_0(t_\ell)\|^2_{L^2(I_h)}. \tag{78}
$$

Attending that for $j = p$ we have (78), we deduce from (72) the following estimate

$$
\begin{aligned}
E(u_h^{2p+1}) \leq \Big(&\tilde{C}^{2p+1} + \tilde{C}^{p+2} 2\eta^2 \Delta t \big(\sum_{m_1=1}^{p+1} 1\big) + \tilde{C}^3 \big(2\eta^2 \Delta t\big)^2\Big) \\
&(1 + 2\eta^2 \tau)(1 + 2\eta^2 \Delta t) \max_{\ell = 1-p, \ldots, 0} \|u_0(t_\ell)\|^2_{L^2(I_h)}.
\end{aligned} \tag{79}
$$

Similarly it can be shown for $j \in \{2, \ldots, p\}$

$$
\begin{aligned}
E(u_h^{2p+j}) \leq \Big(&\tilde{C}^{2p+j} + \tilde{C}^{p+j+1} 2\eta^2 \Delta t \sum_{m_1=1}^{p+j} 1 + \big(2\eta^2 \Delta t\big)^2 \tilde{C}^{j+2} \sum_{m_1=1}^{j} \sum_{m_2=1}^{m_1} 1\Big) \\
&(1 + 2\eta^2 \tau)(1 + 2\eta^2 \Delta t) \max_{\ell = 1-p, \ldots, 0} \|u_0(t_\ell)\|^2_{L^2(I_h)}.
\end{aligned} \tag{80}
$$

As (80) holds for $j=p$, we deduce from (72) the inequality

$$E(u_h^{3p+j}) \leq \Big(\tilde{C}^{3p+j} + \tilde{C}^{2p+j+1} 2\eta^2\Delta t \sum_{m_1=1}^{2p+j} 1 + \big(2\eta^2\Delta t\big)^2 \tilde{C}^{p+j+2} \sum_{m_1=1}^{p+j}\sum_{m_2=1}^{m_1} 1$$
$$+\big(2\eta^2\Delta t\big)^3 \tilde{C}^{j+3} \sum_{m_1=1}^{j}\sum_{m_2=1}^{m_1}\sum_{m_3=1}^{m_2} 1\Big)(1+2\eta^2\tau)(1+2\eta^2\Delta t) \max_{\ell=1-p,\dots,0} \|u_0(t_\ell)\|^2_{L^2(I_h)}.$$

It can be shown that

$$\begin{aligned}
E(u_h^{mp}) \leq \Big(&\tilde{C}^{mp} + \tilde{C}^{(m-1)p+1} 2\eta^2\Delta t \sum_{i=1}^{(m-1)p} 1 \\
&+\big(2\eta^2\Delta t\big)^2 \tilde{C}^{(m-2)p} \sum_{m_1=1}^{(m-2)p+2}\sum_{m_2=1}^{m_1} 1 \\
&+\big(2\eta^2\Delta t\big)^3 \tilde{C}^{(m-3)p+3} \sum_{m_1=1}^{(m-3)p}\sum_{m_2=1}^{m_1}\sum_{m_3=1}^{m_2} 1 \\
&+\big(2\eta^2\Delta t\big)^4 \tilde{C}^{(m-4)p+4} \sum_{m_1=1}^{(m-4)p}\sum_{m_2=1}^{m_1}\sum_{m_3=1}^{m_2}\sum_{m_4=1}^{m_3} 1 \\
&+\dots+\big(2\eta^2\Delta t\big)^{m-1} \tilde{C}^{p+m-1} \sum_{m_1=1}^{p}\sum_{m_2=1}^{m_1}\sum_{m_3=1}^{m_2}\sum_{m_4=1}^{m_3}\dots\sum_{m_{m-1}=1}^{m_{m-2}} 1\Big) \\
&(1+2\eta^2\tau)(1+2\eta^2\Delta t) \max_{\ell=1-p,\dots,0} \|u_0(t_\ell)\|^2_{L^2(I_h)}.
\end{aligned} \tag{81}$$

Similar relation holds for $E(u_h^{mp+j})$ with $j=1,\dots,p-1$.

We obtain now a practical estimate for $E(u_h^n)$ with $(m-1)p \leq mp$. From (81) we get

$$\begin{aligned}
E(u_h^n) &\leq (1+2\eta^2\tau)(1+2\eta^2\Delta t) \max_{\ell=1-p,\dots,0} \|u_0(t_\ell)\|^2_{L^2(I_h)} \tilde{C}^{mp} \\
&\qquad \sum_{i=0}^{m-1} \big(2\eta^2\Delta t p(m-i)\big)^i \\
&\leq (1+2\eta^2\tau)(1+2\eta^2\Delta t) \max_{\ell=1-p,\dots,0} \|u_0(t_\ell)\|^2_{L^2(I_h)} \tilde{C}^{mp} \\
&\qquad \sum_{i=0}^{m-1} \big(2\eta^2\tau(m-i)\big)^i,
\end{aligned}$$

which conclude the proof.

■

In the following result we establish the stability of (54)-(56) with respect to perturbations of the initial condition $u_0$.

**Theorem 6.** *Let $u_h^{n+1}, \tilde{u}_h^{n+1}$ be defined by (54)-(56) with initial condition $u_0$ and $\tilde{u}_0$ respectively. Let $w_h^n, n=-p,\ldots,\mathfrak{M}$ be defined by $w_h^n = u_h^n - \tilde{u}_h^n$, $n=-p,\ldots,\mathfrak{M}$. We assume that $u_h^{n+1}, \tilde{u}_h^{n+1}$ and the reaction term $f$ satisfies the assumptions of Theorem 5 being $f(0,0)$ not necessarily equal to zero. If (33) then, for $2\eta^2$ defined by (35), $(m-1)p \le n \le mp$, without any restriction to $\Delta t$, we have*

$$E(w_h^n) \le (1+2\eta^2\tau)(1+2\eta^2\Delta t) \max_{\ell=1-p,\ldots,0} \|u_0(t_\ell) - \tilde{u}_0(t_\ell)\|^2_{L^2(I_h)} \sum_{i=0}^{m-1} \big(2\eta^2\tau(m-i)\big)^i.$$

*Else, for $(m-1)p \le n \le mp$,*

$$E(w_h^n) \le (1+\tau)(1+\Delta t) \max_{\ell=1-p,\ldots,0} \|u_0(t_\ell) - \tilde{u}_0(t_\ell)\|^2_{L^2(I_h)} \tilde{C}^{mp} \sum_{i=0}^{m-1} \big(\tau(m-i)\big)^i,$$

*with $\tilde{C}$ defined by (64) and for $\Delta t$ satisfying (65).*

**Proof:** We start by remarking that $w_h^n$ satisfies

$$\begin{aligned} w_h^{n+1}(x_i) &= w_h^n(x_i) + \Delta t D_1 D_{2,x} w_h^{n+1} + \Delta t^2 \frac{D_2}{\beta} \sum_{\ell=1}^{n+1} e^{-\frac{t_{n+1}-t_\ell}{\beta}} D_{2,x} w_h^\ell(x_i) \\ &\quad + \Delta t f(u_h^{n+1}(x_i), u_h^{n+1-p}(x_i)) - f(\tilde{u}_h^{n+1}(x_i), \tilde{u}_h^{n+1-p}(x_i)), \end{aligned}$$

for $i=1,\ldots,N-1, n=1,\ldots,\mathfrak{M}-1$, with the boundary conditions

$$w_h^n(x_0) = w_h^n(x_N) = 0, \quad n=1,\ldots,\mathfrak{M},$$

and the initial conditions

$$u_h^n(x_i) = u_0(x_i,t_n), \quad i=0,\ldots,N, \quad n=-p+1,\ldots,0.$$

As

$$f(u_h^{n+1}(x_i), u_h^{n+1-p}(x_i)) - f(\tilde{u}_h^{n+1}(x_i), \tilde{u}_h^{n+1-p}(x_i))$$
$$= \frac{\partial f}{\partial x}(\theta_1 u_h^{n+1}(x_i) + (1-\theta_1)\tilde{u}_h^{n+1}(x_i), u_h^{n+1-p}(x_i))w_h^{n+1}(x_i)$$
$$+\frac{\partial f}{\partial y}(\tilde{u}_h^{n+1}(x_i), \theta_2\tilde{u}_h^{n+1-p}(x_i) + (1-\theta_2)u_h^{n+1-p}(x_i))w_h^{n+1-p}(x_i)),\ \theta_1, \theta_2 \in [0,1],$$

the proof of this result follows the proof of Theorem 5.

■

**Remark 1.** *In Theorems 5 and 6 in the definitions of $\tilde{C}$ and in the upper bound for $\Delta t$ arises the term $\big(\frac{\partial f}{\partial y}\big)^2_{max}$. An increasing in this term implies a decreasing of the upper bound to $\Delta t$. This behaviour is not natural attending that $\big(\frac{\partial f}{\partial y}\big)^2_{max}$ is associated with an explicit term of the discretization (54). Following the proof of these two results it can be shown that holds*

$$\Big(1 - \Delta t\Big(2\big(\frac{\partial f}{\partial x}\big)_{max} + 2\eta^2 - \frac{2D_1}{(b-a)^2}\Big)\Big)\|u_h^{n+1}\|^2_{L^2(I_h)}$$
$$+\frac{D_2}{\beta}\|\Delta t\sum_{\ell=1}^{n+1} e^{-\frac{t_{n+1}-t_\ell}{\beta}} D_{-x}u_h^\ell\|^2_{L^2(I_{h,+})}$$
$$\leq \|u_h^n\|^2_{L^2(I_h)} + e^{-2\frac{\Delta t}{\beta}}\frac{D_2}{\beta}\|\Delta t\sum_{\ell=1}^{n} e^{-\frac{t_n-t_\ell}{\tau}} D_{-x}u_h^\ell\|^2_{L^2(I_{h,+})}$$
$$+\frac{\Delta t}{2\eta^2}\big(\frac{\partial f}{\partial y}\big)^2_{max}\|u_h^{n+1-p}\|^2_{L^2(I_h)}.$$

*If (33) then holds*

$$\|u_h^{n+1}\|^2_{L^2(I_h)} + \frac{D_2}{\beta}\|\Delta t\sum_{\ell=1}^{n+1} e^{-\frac{t_{n+1}-t_\ell}{\beta}} D_{-x}u_h^\ell\|^2_{L^2(I_{h,+})}$$
$$\leq \|u_h^n\|^2_{L^2(I_h)} + e^{-2\frac{\Delta t}{\beta}}\frac{D_2}{\beta}\|\Delta t\sum_{\ell=1}^{n} e^{-\frac{t_n-t_\ell}{\tau}} D_{-x}u_h^\ell\|^2_{L^2(I_{h,+})}$$
$$+\Delta t\frac{\big(\frac{\partial f}{\partial y}\big)^2_{max}}{2\big(\frac{D_1}{(b-a)^2} - \big(\frac{\partial f}{\partial x}\big)_{max}\big)}\|u_h^{n+1-p}\|^2_{L^2(I_h)}.$$

*and then*

$$E(u_h^{n+1}) \leq E(u_h^n) + \Delta t \frac{\left(\frac{\partial f}{\partial y}\right)^2_{max}}{2\left(\frac{D_1}{(b-a)^2} - \left(\frac{\partial f}{\partial x}\right)\right)} \|u_h^{n+1-p}\|^2_{L^2(I_h)}.$$

*If (33) does not holds we conclude*

$$E(u_h^{n+1}) \leq \tilde{C}\Big(E_(u_h^n) + \Delta t \frac{\left(\frac{\partial f}{\partial y}\right)^2_{max}}{2\left(\frac{D_1}{(b-a)^2} - \left(\frac{\partial f}{\partial x}\right)_{max}\right)} \|u_h^{n+1-p}\|^2_{L^2(I_h)}\Big).$$

*with*

$$\tilde{C} = \frac{1}{1 - \Delta t\Big(2\big(\left(\frac{\partial f}{\partial x}\right)_{max} - \frac{2D_1}{(b-a)^2}\big) + 1\Big)}$$

*and provided that* $\Delta t$ *satisfies*

$$\Delta t < \frac{1}{2\big(\left(\frac{\partial f}{\partial x}\right)_{max} - \frac{2D_1}{(b-a)^2}\big) + 1}.$$

*Based on this considerations we conclude that Theorems 5 and 6 hold considering the convenient adaptations induced by the previous comments.*

## 4. Numerical results

In all numerical experiments that we present in this section we consider $f(u(x,t), u(x,t-\tau)) = Uu(x,t)(1-u(x,t-\tau))$. We start by illustrating the stability without any condition on the time stepsize when condition (33) holds. We consider $a = 0, b = 1$, $U = 0.05$ $D_1 = D_2 = \tau = \beta = 0.1$ and $h = 0.1$. In Figure 3 we plot the results obtained for

$$u_0(x) = \begin{cases} 1, \ x \leq 0.5 \\ 0, \ x > 0.5, \end{cases}$$

when $\Delta t$ increases. A stable behaviour was observed. As condition (33) holds this behaviour was expected.

We consider now $a = 0, b = 50$ and $u_0$ defined by (3). We took $D_1 = 0.1, D_2 = 0.3, \beta = 0.1$ and $\tau = 0.2$. In order to illustrate the stable behaviour of the method (54) when condition (33) does not holds, we consider in what follows $h = 0.1$ and $\Delta t = 0.05$. In this case the upper bound to $\Delta t$ is approximately $(2U + U^2)^{-1}$. Then we expect that the unstable behaviour arises at $U \simeq 3.59$. In Figure 4 we plot the numerical results that confirm our observation.

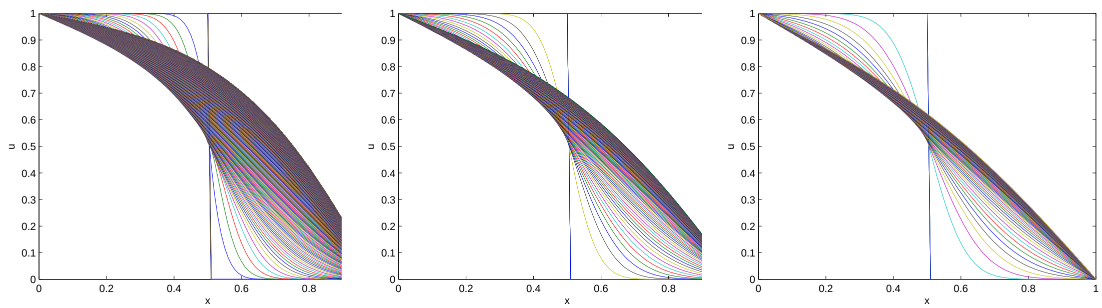

FIGURE 3. Numerical results obtained with method (54) for $\Delta t = 0.01$ (left), $\Delta t = 0.05$ (center) $\Delta t = 0.1$(right).

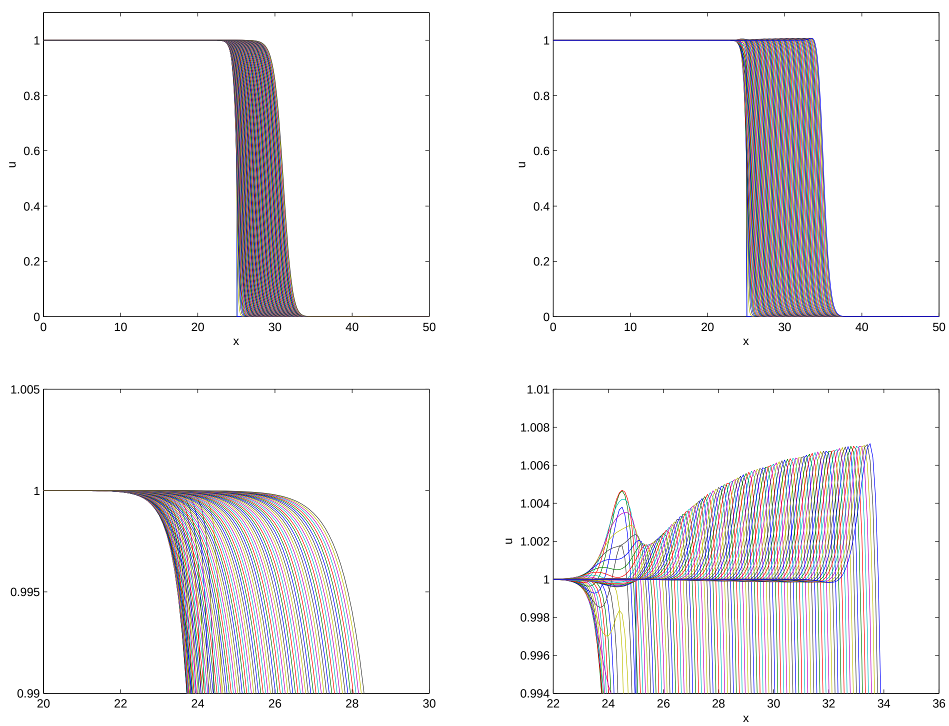

FIGURE 4. Numerical results obtained with method (54) for $\Delta t = 0.05$ and for different values of $U$: $U = 2$ (left), $U = 3.59$(right).In the second row we present a zoom of the pictures of the first row.

Finally we illustrate the behaviour of the proposed method when the condition (65) does not holds.In this case we can observe an unstable behaviour but a stable behaviour can be also observed. The obtained results are plotted in Figure 5.

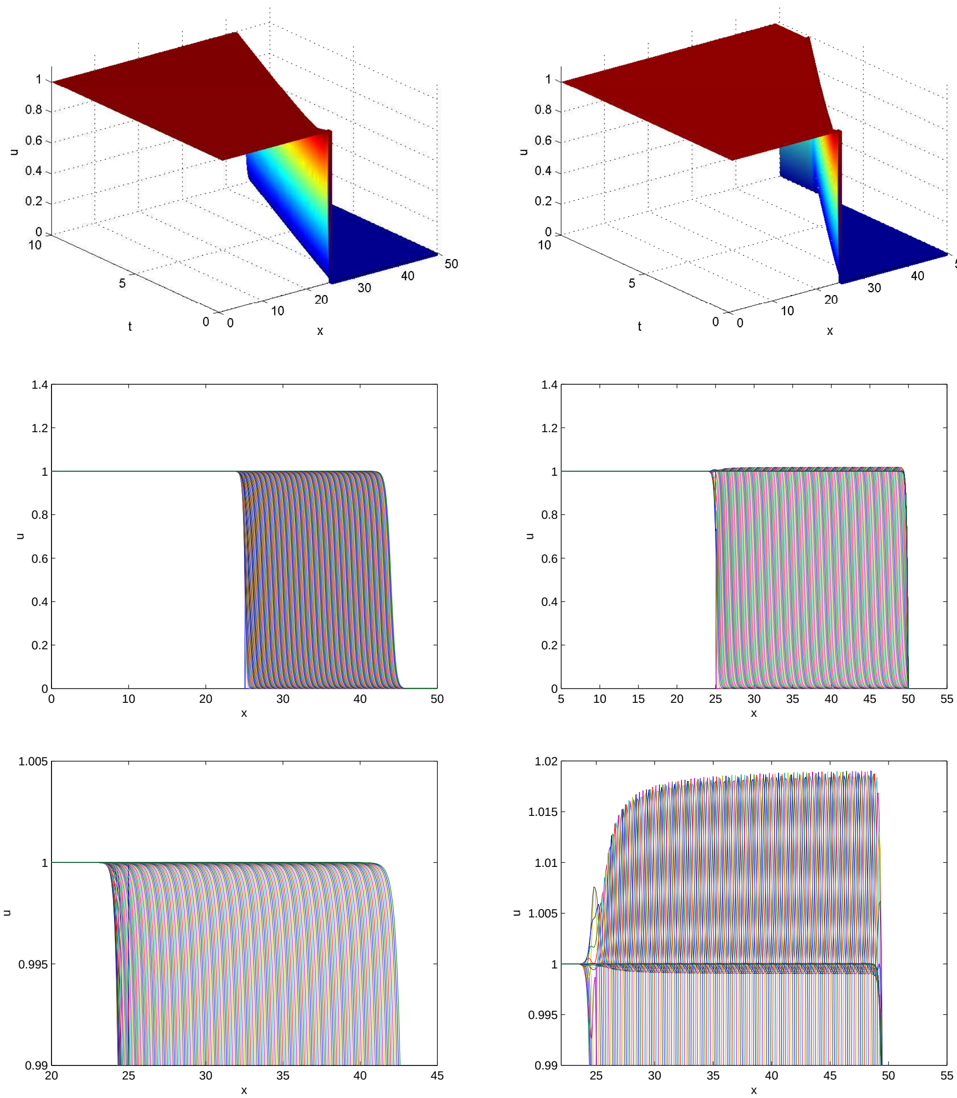


FIGURE 5. Numerical results obtained with method (54) for $\Delta t = 0.05$ and for different values of $U$: $U = 4.8, D_1 = D_2 = \tau = 0.1, \beta = 0.2$ (left), $U = 8, D_1 = \beta = 0.1, D_2 = 0.3, \tau = 0.2$(right). In the second row we present a zoom of the pictures of the first row and a $3D$ version of the pictures of the first row is in last rwo.

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