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A UNIVERSAL CONSTRUCTION IN GOURSAT CATEGORIES

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Resumo: We prove that the category of internal groupoids $\mathrm{Grd}(\mathcal{E})$ is a reflective subcategory of the category $\mathrm{Rg}(\mathcal{E})$ of internal reflexive graphs in a regular Goursat category $\mathcal{E}$ with coequalisers: this implies that the category $\mathrm{Grd}(\mathcal{E})$ is itself regular Goursat.

Introduction

The category $\mathrm{Grd}(\mathcal{E})$ of internal groupoids in a regular Mal'cev category $\mathcal{E}$ with coequalisers is known to be a reflective subcategory of the category of internal reflexive graphs [2]. This fact can also be seen as a consequence of the good properties of the commutator of congruences in a regular Mal'cev category: indeed, the reflection of an internal reflexive graph

$$\underline{X} = X_1 \underset{c}{\overset{d}{\underset{\longleftarrow e\,}{\rightrightarrows}}} X_0$$

is simply given by the quotient of X_1 by the commutator $[R[d], R[c]]$ of the kernel congruences $R[d]$ and $R[c]$ of the "domain" and the "codomain" arrows d and c.

In this note an explicit construction of the reflection is presented under the weaker assumption that $\mathcal{E}$ is a regular Goursat category with coequalisers. If regular Mal'cev categories are characterised by the 2-permutability of the composition of equivalence relations, so that $RS = SR$ for any equivalence relation R and S on an object A, regular Goursat categories satisfy the strictly weaker 3-permutability condition: $RSR = SRS$ [6]. When we started this work, we knew in advance that, in a regular Goursat category $\mathcal{E}$, the reflection could not be simply given by a regular quotient. Indeed, since the category $\mathrm{Rg}(\mathcal{E})$ of internal reflexive graphs is regular, this fact would have implied that any reflexive relation in $\mathcal{E}$ is an equivalence relation, which is one of the equivalent definitions of a Mal'cev category.

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This fact partially explains why the proof is more delicate at this level of generality: it is inspired by a construction due to Bourn [3], and uses the technique of "calculus of relations" developed in [7, 6].

In the first two sections we recall the basic definitions and properties of equivalence relations and their composition in regular Mal'cev and regular Goursat categories which will be needed in the subsequent sections. In Section 3 some important properties of internal categories and internal groupoids in a regular Goursat category are recalled. Section 4 is devoted to the main result, asserting that the category $\mathrm{Grd}(\mathcal{E})$ is reflective in $\mathrm{Rg}(\mathcal{E})$. In the last section this result is used to show that the category $\mathrm{Grd}(\mathcal{E})$ is regular Goursat whenever $\mathcal{E}$ is regular Goursat.

1. Meets and joins of equivalence relations

In this article $\mathcal{E}$ will always be a finitely complete *regular* category, that is a category endowed with a pullback-stable (regular epimorphism, monomorphism) factorization system. Regular categories provide a very natural context for working with relations since their composition exists and is associative.

A relation R from A to B is a subobject $(r_0, r_1) : R \rightarrowtail A \times B$. The opposite of R is the relation R° from B to A given by $(r_1, r_0) : R \rightarrowtail B \times A$. In particular, we can identify a morphism $f : A \to B$ with the relation $(1_A, f) : A \rightarrowtail A \times B$ and write f° for its opposite. So any relation R can be written $R = r_1 r_0^\circ$.

Remark 1.1. Consider the (regular epimorphism, monomorphism) factorization $f = i \cdot r$ of an arrow $f : A \to B$. Then:

1: $f^\circ f$ is the kernel pair of f, thus $1_A \leq f^\circ f$ and $1_A = f^\circ f$ if and only if f is a monomorphism.

2: $f f^\circ$ is (i, i), thus $f f^\circ \leq 1_A$ and $f f^\circ = 1_A$ if and only if f is a regular epimorphism.

3: $f f^\circ f = f$ and $f^\circ f f^\circ = f^\circ$.

Recall that an *equivalence* relation on an object A is a relation R from A to A that is *reflexive* ($1_A \leq R$), *symmetric* ($R^\circ \leq R$) and *transitive* ($RR \leq R$). An equivalence relation is called a *congruence* when it is the kernel pair of some morphism f, written $(f_0, f_1) : R[f] \rightarrowtail A \times A$. We can equivalently consider it the kernel pair of a regular epimorphism since $R[f] = R[r]$. We shall usually represent congruences as $R[r], R[s], \cdots$ where

$r, s, \cdots$ are regular epimorphisms. A diagram $R[f] \rightrightarrows A \twoheadrightarrow B$ is called an *exact fork* when (f_0, f_1) is the kernel pair of f, and f is the coequaliser of (f_0, f_1). A regular category is called *exact* when every equivalence relation is a congruence. We denote by $\mathrm{Equiv}(A)$ the poset of equivalence relations on A and by $\mathrm{Cong}(A)$ the poset of congruences.

Given relations R and S from A to B, the meet of R and S, $R \wedge S$, always exists and is given by the pullback of $R \to A \times B$ along $S \to A \times B$. For arbitrary morphisms $g : X \to A$ and $h : X \to B$ we have

$$(R \wedge S)g = Rg \wedge Sg \quad \text{and} \quad h^\circ(R \wedge S) = h^\circ R \wedge h^\circ S. \tag{1}$$

It is easy to see that $\mathrm{Equiv}(A)$ is closed for meets; meets also exist in $\mathrm{Cong}(A)$ since $R[r] \wedge R[s] = R[(r, s)]$. So, given a monomorphism $(m_0, m_1) : B \rightarrowtail M \times M$, we have

$$R[m_0] \wedge R[m_1] = 1_B \text{ and} \tag{2}$$

$$R[r] = R[m_0 \cdot r] \wedge R[m_1 \cdot r]. \tag{3}$$

While the meet of relations is quite easy to handle, the case for joins is less trivial since they do not exist in general. However, the join of congruences $R[r]$ and $R[s]$ exists in $\mathrm{Cong}(A)$ if and only if the pushout of (r, s)

$$\begin{tikzcd} A \arrow[r, two heads, "r"] \arrow[d, two heads, "s"'] & B \arrow[d, two heads, "u"] \\ C \arrow[r, two heads, "v"'] & D \end{tikzcd} \tag{4}$$

exists; when this is the case, one has $R[r] \vee R[s] = R[u \cdot r]$.

Regular categories with the additional property of having a commutative composition of equivalence relations (on any fixed object) are called *regular Mal'cev* categories [7]. Mal'cev categories have been intensively studied in the last few years (see [2], and the references therein), and many nice characterisations have been explored. We list some of them in:

Theorem 1.2. [7] *Let $\mathcal{E}$ be a regular category. The following statements are equivalent:*

- **i:** *$\mathcal{E}$ is a regular Mal'cev category.*
- **ii:** *$RS = SR$, for all $R, S \in \mathrm{Equiv}(A)$.*
- **iii:** *$RS = SR$, for all $R, S \in \mathrm{Cong}(A)$.*
- **iv:** *$RS \in \mathrm{Equiv}(A)$, thus $R \vee S = RS$, for all $R, S \in \mathrm{Equiv}(A)$.*
- **v:** *Every reflexive relation is an equivalence relation.*

In this work we will focus on the strictly weaker property of 3-permutability.

Definition 1.3. A regular category $\mathcal{E}$ is called a *regular Goursat* category when the equivalence relations in $\mathcal{E}$ are 3-permutable, i.e. $RSR = SRS$ for any pair of equivalence relations R and S on the same object.

Examples 1.4. A variety $\mathcal{V}$ of universal algebras is a Goursat category exactly when it is a 3-permutable variety: this property is known to be equivalent to a Mal'cev condition, namely the existence of two ternary terms $p(x, y, z)$ and $q(x, y, z)$ satisfying the identities $p(x, y, y) = x$, $q(x, x, y) = y$ and $p(x, x, y) = q(x, y, y)$. In particular the varieties of groups, rings, von Neumann regular rings, associative algebras, Heyting algebras and implication algebras are exact Goursat categories (see [10]).

So far, regular Goursat categories have not been investigated as well as regular Mal'cev categories. However, many interesting characterisations and properties were discovered in [7, 6, 11]: in particular, the regular Goursat version of Theorem 1.2 is:

Theorem 1.5. *Let $\mathcal{E}$ be a regular category. The following statements are equivalent:*

- **i:** *$\mathcal{E}$ is a regular Goursat category.*
- **ii:** *$RSR = SRS$, for all $R, S \in \mathrm{Equiv}(A)$.*
- **iii:** *$RSR = SRS$, for all $R, S \in \mathrm{Cong}(A)$.*
- **iv:** *$RSR \in \mathrm{Equiv}(A)$, thus $R \vee S = RSR$, for all $R, S \in \mathrm{Equiv}(A)$.*
- **v:** *For every reflexive relation E, EE° is an equivalence relation.*

In an exact Goursat category both joins in $\mathrm{Equiv}(A)$ and in $\mathrm{Cong}(A)$ coincide, i.e. $R[r]R[s]R[r] = R[u \cdot r]$ (see (4)). Therefore exact Goursat categories always admit pushouts of regular epimorphisms. For non-exact regular Goursat categories the above equality does not hold in general; not even when the join of $R[r]$ and $R[s]$ exists in $\mathrm{Cong}(A)$ (see Proposition 2.2).

2. Direct images

An alternative characterisation of regular Goursat categories is described by the preservation of equivalence relations through the image by a regular epimorphism.

Theorem 2.1. [6] *A regular category $\mathcal{E}$ is a regular Goursat category if and only if for any regular epimorphism $r : A \twoheadrightarrow B$ and $S \in \mathrm{Equiv}(A)$, the image $r(S) = rSr^\circ \in \mathrm{Equiv}(B)$ is also an equivalence relation.*

In a regular Goursat category $\mathcal{E}$, images do not necessarily preserve congruences. That is, given regular epimorphisms r and s, the image $r(R[s]) = T$ is an equivalence relation which is not necessarily a congruence. If there exists a commutative square $u \cdot r = v \cdot s$

$$\begin{tikzcd}
R[s] \arrow[r, two heads, "\overline{r}"] \arrow[rr, bend left, "R(r)"] \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] & T \arrow[r, dotted] \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] & R[u] \arrow[dl, shift right] \arrow[dl, shift left] \\
A \arrow[r, two heads, "r"] \arrow[d, "s"'] \arrow[dr, phantom, "A"] & B \arrow[d, "u"] & \\
C \arrow[r, two heads, "v"'] & D, &
\end{tikzcd} \tag{5}$$

then $T \leq R[u]$ and $R(r)$ is not a regular epimorphism in general. It is clear that in the exact Goursat context, we always have $T = R[u]$ for $u = \mathrm{coeq}(t_0, t_1)$ and $R(r)$ a regular epimorphism. In the non-exact regular Goursat case, we would like to know under which conditions $R(r)$ is a regular epimorphism.

Proposition 2.2. *Let $\mathcal{E}$ be a regular Goursat category and consider a commutative square of regular epimorphisms* (5)A*. The induced morphism $R(r) : R[s] \to R[u]$ is a regular epimorphism if and only if $R[r]R[s]R[r] = R[u \cdot r]$ (i.e. the join of $R[r]$ and $R[s]$ exists both in* $\mathrm{Equiv}(A)$ *and in* $\mathrm{Cong}(A)$ *and they coincide).*

Demonstração: We use the properties mentioned in Remark 1.1. If $R(r)$ is a regular epimorphism, then $r(R[s]) = R[u]$. So, $R[r]R[s]R[r] = r^{\circ}(rR[s]r^{\circ})r = r^{\circ}u^{\circ}ur = R[u \cdot r]$. Conversely, when $R[r]R[s]R[r] = R[u \cdot r]$, the image of $R[s]$ along r is:

$$r(R[s]) = rs^{\circ}sr^{\circ} = r(r^{\circ}rs^{\circ}sr^{\circ}r)r^{\circ} = rR[ur]r^{\circ} = rr^{\circ}u^{\circ}urr^{\circ} = R[u].$$

□

Corollary 2.3. *Consider the conditions of Proposition 2.2. Then $R(r)$ is a regular epimorphism if and only if $R(s)$ is a regular epimorphism.*

Demonstração: $R(s)$ is a regular epimorphism if and only if $R[v \cdot s] = R[s]R[r]R[s]$. But $v \cdot s = u \cdot r$ and the regular Goursat assumption gives

$$R[u \cdot r] = R[s]R[r]R[s] = R[r]R[s]R[r],$$

i.e. $R(r)$ is a regular epimorphism. □

This corollary can also be deduced from the
3-by-3 Lemma [11] Consider the following diagram in a regular Goursat category

$$\begin{tikzcd}
R[t] \arrow[r, shift left, "y_0"] \arrow[r, shift right, "y_1"'] \arrow[d, shift right, "t_0"'] \arrow[d, shift left, "t_1"] & R[s] \arrow[r, "R(r)"] \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "s_1"] & R[u] \arrow[d, shift right, "u_0"'] \arrow[d, shift left, "u_1"] \\
R[r] \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] \arrow[d, "t"'] & A \arrow[r, two heads, "r"] \arrow[d, "s"'] & B \arrow[d, "u"] \\
X \arrow[r, shift left, "x_0"] \arrow[r, shift right, "x_1"'] & C \arrow[r, "v"'] & D,
\end{tikzcd}$$

where all the corresponding squares "reasonably commute", so that in particular $r_i \cdot t_j = s_j \cdot y_i$. If the three columns and the middle row are exact, then the top row is exact if and only if the bottom row is exact.

Corollary 2.4. *Consider the conditions of Proposition 2.2. If the vertical morphisms are split epimorphisms and the diagram commutes with the splittings, then $R(r)$ is a regular epimorphism.*

Demonstração: If s and u are split epimorphisms, then $R(s)$ is also a split epimorphism. Thus, $R(r)$ is a regular epimorphism by Corollary 2.3. $\square$

3. Internal structures

In this section we shall often apply the Yoneda embedding to use elements in our arguments as if we were working in the category of sets.

One of the well known consequences of the 3-permutability is the modularity of the lattice of equivalence relations $\mathrm{Equiv}(A)$.

Proposition 3.1. [6] *Let $\mathcal{E}$ be a regular Goursat category. If $R, S, T \in \mathrm{Equiv}(A)$ are such that $R \leq T$, then $R \vee (S \wedge T) = (R \vee S) \wedge T$.*

For a variety of universal algebras, it was shown by Gumm [10] that congruence modularity is equivalent to satisfying the
Shifting Property: Let $R, S, T \in \mathrm{Equiv}(A)$ be such that $R \wedge S \leq T$. Then, from $(x, y) \in R$, $(t, z) \in R$, $(x, t) \in S$, $(y, z) \in S$ and $(x, y) \in T$

$$\begin{tikzcd}
x \arrow[r, dash, "S"] \arrow[d, dash, "R"'] \arrow[d, dash, bend right=60, "T"'] & t \arrow[d, dash, "R"] \\
y \arrow[r, dash, "S"'] & z
\end{tikzcd}$$

it follows that $(t, z) \in T$.

It is easy to see that any regular Goursat category satisfies the shifting property since $(t,z) \in S(R\wedge T)S = (R\wedge T)S(R\wedge T)$ implies the existence of elements a and b such that $(t,a)\in R\wedge T, (a,b)\in S$ and $(b,z)\in R\wedge T$. Then $(a,t),(t,z),(z,b)\in R$ implies that $(a,b)\in R$. So, $(a,b)\in R\wedge S\leq T$ and $(t,a),(a,b),(b,z)\in T$ gives $(t,z)\in T$.

Categories satisfying the shifting property were called *Gumm* categories in [5]. The internal structures in a Gumm category were studied in [4]: we recall some of their properties here below.

An (internal) *reflexive graph* is given by a diagram

$$\underline{X} = \begin{tikzcd} X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_0 \arrow[l, "e" description] \end{tikzcd}$$

such that $d\cdot e = c\cdot e = 1_{X_0}$. It is called a(n) *(internal) category* when there exists a multiplication morphism $m : X_1\times_{X_0} X_1 \to X_1$ such that

$$d(m(x,y)) = d(x) \text{ and } c(m(x,y)) = c(y), \tag{6}$$

$$m(e(d(x)),x) = x \text{ and } m(x,e(c(x))) = x, \tag{7}$$

$$m(x,m(y,z)) = m(m(x,y),z). \tag{8}$$

A category is called a(n) *(internal) groupoid* when there exists an inversion morphism $i : X_1 \to X_1$ such that $d(i(x)) = c(x), c(i(x)) = d(x), m(i(x),x) = e(c(x))$ and $m(x,i(x)) = e(d(x))$. Among categories, groupoids are characterised by the fact that $(\pi_0, m) = R[d]$ or, equivalently, that $(m,\pi_1) = R[c]$, where (π_0,π_1) denotes the two projections of the pullback of (d,c). We denote by $\mathrm{Rg}(\mathcal{E}), \mathrm{Cat}(\mathcal{E})$ and $\mathrm{Grd}(\mathcal{E})$ the categories of reflexive graphs, categories and groupoids in $\mathcal{E}$, respectively.

Proposition 3.2. [4] *Let $\mathcal{E}$ be a finitely complete category satisfying the shifting property and consider $\underline{X}\in\mathrm{Rg}(\mathcal{E})$. The following statements are equivalent:*

- **i:** *There exists a (necessarily unique) category structure on $\underline{X}$.*
- **ii:** *There exists a (necessarily unique) morphism $m : X_1\times_{X_0} X_1 \to X_1$ satisfying* (6) *and* (7).

Another helpful result, well known in the regular Mal'cev context, is:

Proposition 3.3. [4] *Let $\mathcal{E}$ be a finitely complete category satisfying the shifting property. Then $\mathrm{Grd}(\mathcal{E}) \hookrightarrow \mathrm{Rg}(\mathcal{E})$ is a full inclusion.*

4. The universal groupoid associated with a reflexive graph

In this section $\mathcal{E}$ represents a regular Goursat category with coequalisers. Given a reflexive graph $\underline{X} : X_1 \overset{\rightarrow}{\underset{\rightarrow}{\leftarrow}} X_0$ in $\mathcal{E}$, our aim is to construct the associated universal groupoid.

Let us begin with a slightly more general situation: a pair of regular epimorphisms $d : X_1 \twoheadrightarrow Y_0$ and $c : X_1 \twoheadrightarrow X_0$. We construct the following diagram

$$\begin{tikzcd}
 & R[d_0] \arrow[r, two heads, "\overline{\delta}"] \arrow[d, shift right=2, "p_0"'] \arrow[d, "\mu" description] \arrow[d, shift left=2, "p_1"] & R[q_0] \arrow[d, shift right, "\pi_0"'] \arrow[d, shift left, "m"] \\
 & R[d] \arrow[r, two heads, "\delta"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Q \arrow[d, shift right=2, "q_0"'] \arrow[d, shift left=2, "q_1"] \\
R[c] \arrow[r, shift left, "c_0"] \arrow[r, shift right, "c_1"'] & X_1 \arrow[r, two heads, "c"'] \arrow[u, "s_0" description] \arrow[d, "d"] & X_0, \arrow[u, "\varepsilon" description] \\
 & Y_0 &
\end{tikzcd} \tag{9}$$

where δ is the coequaliser of $(s_0 \cdot c_0, s_0 \cdot c_1)$, q_0, q_1 and ε are induced by the coequalisers δ and c and $\overline{\delta}$ is necessarily a regular epimorphism by Corollary 2.4. We obviously have a left vertical groupoid since it is the congruence $R[d]$. On the other hand, the lower right vertical diagram is a reflexive graph; we denote it by $\underline{D}$, as it is induced by $R[d]$. Note that there exists an inversion $i : R[d] \xrightarrow{\sim} R[d]$ from which we can deduce a morphism $j : Q \to Q$ on the right diagram such that $j \cdot j = 1_Q$, $j \cdot \delta = \delta \cdot i$, $q_0 \cdot j = q_1$ and $q_1 \cdot j = q_0$. Being already equipped with two morphisms π_0 and m, we would like to find a third morphism π_1 that will endow $\underline{D}$ with a category structure, which is consequently a groupoid structure with inversion j.

Lemma 4.1. *In diagram* (9) *we have* $R[p_0] \wedge R[\delta \cdot \mu] \leq R[\delta \cdot p_1]$.

Demonstração:

$$
\begin{array}{rcl}
R[p_0] \wedge R[\delta \cdot \mu] & \overset{(3)}{=} & R[d_0 \cdot p_0] \wedge R[d_1 \cdot p_0] \wedge R[\delta \cdot \mu] \\
& = & R[d_0 \cdot p_1] \wedge R[d_0 \cdot \mu] \wedge R[\delta \cdot \mu] \\
& \overset{(1)}{=} & R[d_0 \cdot p_1] \wedge \mu^\circ(\, R[d_0] \wedge R[\delta]\,)\mu \\
& \leq & R[d_0 \cdot p_1] \wedge \mu^\circ(\, R[d_1](R[d_0] \wedge R[\delta])R[d_1]\,)\mu \\
& \overset{d_1 \cdot \mu = d_1 \cdot p_1}{=} & R[d_0 \cdot p_1] \wedge p_1^\circ(\, R[d_1](R[d_0] \wedge R[\delta])R[d_1]\,)p_1 \\
& \overset{(1), RSR = R \vee S}{=} & p_1^\circ(\, R[d_0] \wedge (\, R[d_1] \vee (R[d_0] \wedge R[\delta])\,)\,)p_1 \\
& \overset{\text{Proposition 3.1}}{=} & p_1^\circ(\, (R[d_0] \wedge R[d_1]) \vee (R[d_0] \wedge R[\delta])\,)p_1 \\
& \overset{(2)}{=} & p_1^\circ(R[d_0] \wedge R[\delta])p_1 \\
& \leq & p_1^\circ R[\delta] p_1 \\
& = & R[\delta \cdot p_1]
\end{array}
$$

□

Proposition 4.2. *The right vertical diagram in* (9) *endows* $\underline{D}$ *with a groupoid structure.*

Demonstração: By the remarks above, the main difficulty in showing that $\underline{D}$ is a groupoid is to prove the existence of a morphism $\pi_1 : R[q_0] \to Q$ such that $\delta \cdot p_1 = \pi_1 \cdot \overline{\delta}$. Then, by using $j : Q \overset{\sim}{\to} Q$, we can prove that (π_0, π_1) is the pullback of (q_0, q_1). The proof of axioms (6) and (7) are quite straightforward. Then, $\underline{D}$ is a category by Proposition 3.2 with an inversion j, thus a groupoid.

We begin by considering $w = \mathrm{coeq}(\, (1, s_0 \cdot d_1) \cdot \delta_0\, ,\, (1, s_0 \cdot d_1) \cdot \delta_1\,)$ and the right vertical morphisms induced from the coequalisers w and δ in

$$
\begin{tikzcd}
R[\delta] \arrow[r, shift left, "(1, s_0 \cdot d_1) \cdot \delta_0"] \arrow[r, shift right, "(1, s_0 \cdot d_1) \cdot \delta_1"'] & R[d_0] \arrow[r, two heads, "w"] \arrow[d, shift right=3, "p_0"'] \arrow[d, shift left, "\mu"] \arrow[d, shift left=3, "p_1"] & W \arrow[d, shift right=3, "v_0"'] \arrow[d, shift left, "u"] \arrow[d, shift left=3, "v_1"] \\
& R[d] \arrow[u, shift right, "(1, s_0 \cdot d_1)"'] \arrow[r, two heads, "\delta"'] & Q. \arrow[u, shift right, "\sigma_0"']
\end{tikzcd}
$$

From the commutative downward squares we easily see that

$$R[\delta \cdot p_0] = R[w] \vee R[p_0] \;\; \text{by Corollary 2.4 and Proposition 2.2} \tag{10}$$

$$R[w] \leq R[w] \vee R[\mu] = R[\delta \cdot \mu] \;\; \text{by Corollary 2.4 and Proposition 2.2} \tag{11}$$

$$R[w] \leq R[v_1 \cdot w] = R[\delta \cdot p_1] \tag{12}$$

We have

$$\begin{array}{rcl} R[\overline{\delta}] & \overset{(3)}{=} & R[\pi_0 \cdot \overline{\delta}] \wedge R[m \cdot \overline{\delta}] \\ & = & R[\delta \cdot p_0] \wedge R[\delta \cdot \mu] \\ & \overset{(10)}{=} & (\, R[w] \vee R[p_0]\,) \wedge R[\delta \cdot \mu] \\ & \overset{(11),\text{Proposition } 3.1}{=} & R[w] \vee (\, R[p_0] \wedge R[\delta \cdot \mu]\,) \\ & \overset{\text{Lemma } 4.1}{\leq} & R[w] \vee R[\delta \cdot p_1] \\ & \overset{(12)}{=} & R[\delta \cdot p_1]. \end{array}$$

This implies that $\delta \cdot p_1 \leq \overline{\delta}$ as regular epimorphisms, i.e. there exists a morphism $\pi_1 : R[q_0] \to Q$ such that $\pi_1 \cdot \overline{\delta} = \delta \cdot p_1$ $\square$.

We now focus on the construction of the groupoid $\underline{D}$ when starting from an actual reflexive graph $\underline{X} : X_1 \overset{\rightarrow}{\underset{\rightarrow}{\leftarrow}} X_0$.

Proposition 4.3. *The inclusion* $U : \mathrm{Grd}(\mathcal{E}) \hookrightarrow \mathrm{Rg}(\mathcal{E})$ *admits a left adjoint functor* $F : \mathrm{Rg}(\mathcal{E}) \to \mathrm{Grd}(\mathcal{E})$ *defined by* $F(\underline{X}) = \underline{D}$ *as in diagram* (9).

Demonstração: By using the same notations as above, for any reflexive graph $\underline{X}$, the unit of the adjunction is given by $\eta_{\underline{X}} = (\delta \cdot (e \cdot d, 1), 1_{X_0})$

$$\begin{tikzcd} \underline{X} : \arrow[d, "\eta_{\underline{X}}"'] & X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "\delta\cdot(e\cdot d,1)"'] & X_0 \arrow[l, "e" description] \arrow[d, equal] \\ U(F(\underline{X})) : & Q \arrow[r, shift left=2, "q_0"] \arrow[r, shift right=2, "q_1"'] & X_0. \arrow[l, "\varepsilon" description] \end{tikzcd}$$

For the universal property, let us consider an arbitrary morphism of reflexive graphs $(\alpha_1, \alpha_0) : \underline{X} \to U(\underline{X'})$. We define a morphism $(\beta_1, \alpha_0) : \underline{D} \to \underline{X'}$ of reflexive graphs (which is necessarily a morphism between the groupoids by Proposition 3.3), where β_1 is the unique morphism with the property $\beta_1 \cdot \delta = \pi'_1 \cdot R(\alpha_1)$ given in

$$\begin{tikzcd} R[c] \arrow[r, shift left, "c_0"] \arrow[r, shift right, "c_1"'] & X_1 \arrow[r, "s_0"] \arrow[d, "\alpha_1"'] & R[d] \arrow[r, two heads, "\delta"] \arrow[d, "R(\alpha_1)"'] & Q \arrow[d, dotted, "\beta_1"] \\ & X'_1 \arrow[r, "(1,e'\cdot c')"'] & X'_1 \times_{X'_0} X'_1 \arrow[r, "\pi'_1"'] & X'_1, \end{tikzcd}$$

for $R(\alpha_1)$ the induced morphism from $R[d]$ to $R[d'] \cong X'_1 \times_{X'_0} X'_1$. Moreover, $\beta_1 \cdot \delta \cdot (e \cdot d, 1) = \pi'_1 \cdot R(\alpha_1) \cdot (e \cdot d, 1) = \pi'_1 \cdot (e' \cdot d', 1) \cdot \alpha_1 = \alpha_1$.

As for the uniqueness, suppose there exists another functor $(\varphi_1, \alpha_0) : \underline{D} \to \underline{X}'$ such that $\varphi_1 \cdot \delta \cdot (e \cdot d, 1) = \alpha_1$. The morphism $R(e \cdot d, 1) : R[d] \to R[d_0]$ is such that $p_1 \cdot R(e \cdot d, 1) = 1_{R[d]}$ and the equality $\beta_1 \cdot \delta \cdot (e \cdot d, 1) = \varphi_1 \cdot \delta \cdot (e \cdot d, 1)$ implies that $\beta_1 \times_{X_0'} \beta_1 \cdot \overline{\delta} \cdot R(e \cdot d, 1) = \varphi_1 \times_{X_0'} \varphi_1 \cdot \overline{\delta} \cdot R(e \cdot d, 1)$. Since

$$\begin{aligned}
\beta_1 \cdot \delta &= \beta_1 \cdot \delta \cdot p_1 \cdot R(e \cdot d, 1) \\
&= \beta_1 \cdot \pi_1 \cdot \overline{\delta} \cdot R(e \cdot d, 1) \\
&= \pi_1' \cdot \beta_1 \times_{X_0'} \beta_1 \cdot \overline{\delta} \cdot R(e \cdot d, 1) \\
&= \pi_1' \cdot \varphi_1 \times_{X_0'} \varphi_1 \cdot \overline{\delta} \cdot R(e \cdot d, 1) \\
&= \cdots \\
&= \varphi_1 \cdot \delta
\end{aligned}$$

and δ is an epimorphism, then we get $\beta_1 = \varphi_1$. $\square$

Remark 4.4. **1:** If $\underline{X} \in \mathrm{Grd}(\mathcal{E})$, then $F(\underline{X}) \cong \underline{X}$ by Proposition 4.3.

2: Given a reflexive graph $\underline{X} : X_1 \underset{c}{\overset{d}{\rightleftarrows}} X_0$, we have $\underline{D} \cong \underline{C}$, where $\underline{C}$ denotes the groupoid induced from the kernel pair $R[c]$ of the codomain c when taking the coequaliser $\gamma = \mathrm{coeq}(s_0 \cdot d_0, s_0 \cdot d_1)$. Using the universal property of $(\delta \cdot (e \cdot d, 1), 1_{X_0}) : \underline{X} \to U(F(\underline{X}))$ and the morphism $(\gamma \cdot (e \cdot c, 1), 1_{X_0}) : \underline{X} \to U(\underline{C})$, we get a unique functor $\underline{D} \to \underline{C}$. By exchanging the roles of d and c we obtain the inverse morphism $\underline{C} \to \underline{D}$.

3: It follows from the construction of the universal groupoid associated with a reflexive graph in a regular Goursat with coequalisers category that the congruence $R[\delta]$ on $R[d]$ also determines an equivalence relation $(R(d_0), R(d_1)) : R[\delta] \rightarrowtail R[c] \times R[c]$.

$$\begin{tikzcd}
R[\delta] \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] \arrow[d, shift right=2, "R(d_0)"'] \arrow[d, shift left=2, "R(d_1)"] & R[d] \arrow[r, two heads, "\delta"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Q \arrow[d, shift right=2, "q_0"'] \arrow[d, shift left=2, "q_1"] \\
R[c] \arrow[u, "R(s_0)"] \arrow[r, shift left, "c_0"] \arrow[r, shift right, "c_1"'] & X_1 \arrow[u, "s_0"] \arrow[r, two heads, "c"'] & X_0. \arrow[u, "\varepsilon"]
\end{tikzcd} \tag{13}$$

One then has a double equivalence relation $R[\delta]$ on $R[d]$ and $R[c]$: in universal algebra this is the well-known double congruence $\Delta_{R[d],R[c]}$ used to define the commutator $[R[d], R[c]]$ of $R[d]$ and $R[c]$ (see [10] and [12]).

5. $\mathrm{Grd}(\mathcal{E})$ is a regular Goursat category

In this section we prove that $\mathrm{Grd}(\mathcal{E})$ is a regular Goursat category for any regular Goursat category $\mathcal{E}$ with coequalisers. The crucial point is to show that $\mathrm{Grd}(\mathcal{E})$ has (regular epimorphism, monomorphism) factorizations, since $\mathrm{Grd}(\mathcal{E})$ is not closed under quotients in $\mathrm{Rg}(\mathcal{E})$ in general.

Given $(\psi_1, \psi_0) : \underline{X}' \to \underline{X} \in \mathrm{Rg}(\mathcal{E})$, for any regular category $\mathcal{E}$, it is clear that (ψ_1, ψ_0) is a regular epimorphism (a monomorphism) in $\mathrm{Rg}(\mathcal{E})$ if and only if ψ_1 and ψ_0 are regular epimorphisms (monomorphisms) in $\mathcal{E}$. If the functor $(\psi_1, \psi_0) : \underline{X}' \to \underline{X} \in \mathrm{Grd}(\mathcal{E})$ is a regular epimorphism in $\mathrm{Rg}(\mathcal{E})$, then $\psi_2 : X_1' \times_{X_0'} X_1' \to X_1 \times_{X_0} X_1$ is also a regular epimorphism in any regular Goursat category $\mathcal{E}$ by Corollary 2.4, so that (ψ_1, ψ_0) is a regular epimorphism in $\mathrm{Grd}(\mathcal{E})$. The converse statement holds in regular Mal'cev categories (Lemma 3.1 in [9]), and we shall now prove that it also holds in the regular Goursat context in Proposition 5.2 below.

Remark 5.1. For any regular Goursat category $\mathcal{E}$ with coequalisers, the functor $F : \mathrm{Rg}(\mathcal{E}) \to \mathrm{Grd}(\mathcal{E})$ preserves colimits since it is a left adjoint (Proposition 4.3). So, given a regular epimorphism $(\psi_1, \psi_0) : \underline{X}' \to \underline{X}$ in $\mathrm{Rg}(\mathcal{E})$, the image $F(\psi_1, \psi_0) = (F(\psi_1), \psi_0) : \underline{D}' \to \underline{D}$ is a regular epimorphism in $\mathrm{Grd}(\mathcal{E})$, where $F(\psi_1)$ is given by the unique morphism such that $F(\psi_1) \cdot \delta' = \delta \cdot R(\psi_1)$:

$$\begin{tikzcd}
R[c'] \arrow[r, shift left, "c_0'"] \arrow[r, shift right, "c_1'"'] \arrow[d, "R(\psi_1)"'] & X_1' \arrow[r, "s_0'"] \arrow[d, "\psi_1"] & R[d'] \arrow[r, two heads, "\delta'"] \arrow[d, "R(\psi_1)"] & Q' \arrow[d, dotted, "F(\psi_1)"] \\
R[c] \arrow[r, shift left, "c_0"] \arrow[r, shift right, "c_1"'] & X_1 \arrow[r, "s_0"'] & R[d] \arrow[r, two heads, "\delta"'] & Q.
\end{tikzcd}$$

By Corollary 2.4, $R(\psi_1) : R[d'] \twoheadrightarrow R[d]$ is a regular epimorphism and, consequently, $F(\psi_1)$ is a regular epimorphism in $\mathcal{E}$.

Proposition 5.2. *Let $\mathcal{E}$ be a regular Goursat category with coequalisers and consider a functor $(\varphi_1, \varphi_0) : \underline{X}' \to \underline{X}'' \in \mathrm{Grd}(\mathcal{E})$. Then (φ_1, φ_0) is a regular epimorphism in $\mathrm{Grd}(\mathcal{E})$ if and only if it is a regular epimorphism in $\mathrm{Rg}(\mathcal{E})$.*

Demonstração: If (φ_1, φ_0) is a regular epimorphism in $\mathrm{Rg}(\mathcal{E})$, then it is also a regular epimorphism in $\mathrm{Grd}(\mathcal{E})$ as already observed above.

For the converse, let (φ_1, φ_0) be a regular epimorphism in $\mathrm{Grd}(\mathcal{E})$, and we have to prove that φ_1 and φ_0 are regular epimorphisms in $\mathcal{E}$. We begin by considering the (regular epimorphism, monomorphism) factorizations $\varphi_1 =$

$\iota_1 \cdot \psi_1$ and $\varphi_0 = \iota_0 \cdot \psi_0$ and the induced reflexive graph $\underline{X} : X_1 \overset{\rightarrow}{\underset{\rightarrow}{\leftarrow}} X_0$ given in the following diagram

$$\begin{tikzcd}
R[\varphi_1] \arrow[r, shift left, two heads] \arrow[r, shift right, two heads] \arrow[dd, shift left=2] \arrow[dd, shift right=2] \arrow[dd, leftarrow] & X'_1 \arrow[rr, "\varphi_1"] \arrow[dr, two heads, "\psi_1"] \arrow[dd, shift right=2, "d'"'] \arrow[dd, leftarrow, "e'" description] \arrow[dd, shift left=2, "c'"] & & X''_1 \arrow[dd, shift right=2, "d''"'] \arrow[dd, leftarrow, "e''" description] \arrow[dd, shift left=2, "c''"] \\
& & X_1 \arrow[ur, tail, "\iota_1"'] \arrow[dd, shift right=2, "d"'] \arrow[dd, leftarrow, "e" description] \arrow[dd, shift left=2, "c"] & \\
R[\varphi_0] \arrow[r, shift left, two heads] \arrow[r, shift right, two heads] & X'_0 \arrow[rr, "\varphi_0"] \arrow[dr, two heads, "\psi_0"'] & & X''_0. \\
& & X_0 \arrow[ur, tail, "\iota_0"'] &
\end{tikzcd} \tag{14}$$

Since $\underline{X'}$ and $\underline{X''}$ are groupoids, then the images $F(\psi_1, \psi_0) : \underline{X'} \to \underline{D}$ and $F(\iota_1, \iota_0) : \underline{D} \to \underline{X''}$ (see Remark 4.4.1.) provide the commutative diagrams

$$\begin{tikzcd}
X'_1 \arrow[r, two heads, "\psi_1"] \arrow[rr, bend left, "F(\psi_1)"] \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] & X_1 \arrow[r, "\delta\cdot(e\cdot d,1)"] \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] & Q \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] \\
X'_0 \arrow[r, two heads, "\psi_0"'] & X_0 \arrow[r, equal] & X_0
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X_1 \arrow[r, two heads, "\delta\cdot(e\cdot d,1)"] \arrow[rr, bend left, "\iota_1"] \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] & Q \arrow[r, "F(\iota_1)"] \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] & X''_1 \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] \\
X_0 \arrow[r, equal] & X_0 \arrow[r, tail, "\iota_0"'] & X''_0.
\end{tikzcd}$$

With the first diagram, we see that $\delta \cdot (e \cdot d, 1)$ is a regular epimorphism since $F(\psi_1)$ is a regular epimorphism (Remark 5.1) and with the second diagram we see that it is also a monomorphism. Hence, $\underline{X} \cong \underline{D}$, i.e. $\underline{X}$ is actually a groupoid. To finish, we have both $(\varphi_1, \varphi_0) = (\iota_1, \iota_0) \cdot (\psi_1, \psi_0)$ and (ψ_1, ψ_0) regular epimorphisms in $\mathrm{Grd}(\mathcal{E})$, implying that $(\varphi_1, \varphi_0) \cong (\psi_1, \psi_0)$. $\square$

Proposition 5.3. *Let $\mathcal{E}$ be a regular Goursat category with coequalisers. Then* $\mathrm{Grd}(\mathcal{E})$ *is a regular Goursat category.*

Demonstração: The category $\mathrm{Grd}(\mathcal{E})$ is finitely complete because the limits are computed as in $\mathcal{E}$, and this latter category is finitely complete. Moreover, regular epimorphisms in $\mathrm{Grd}(\mathcal{E})$ are stable under pullbacks since they are composed by pairs of regular epimorphisms in $\mathcal{E}$ (Proposition 5.2) which are stable under pullbacks in $\mathcal{E}$. For any functor $(\varphi_1, \varphi_0) : \underline{X'} \to \underline{X''} \in \mathrm{Grd}(\mathcal{E})$, we can consider the diagram (14) and we see that (ψ_1, ψ_0) is a regular epimorphism in $\mathrm{Grd}(\mathcal{E})$. Hence, $\mathrm{Grd}(\mathcal{E})$ has (regular epimorphism, monomorphism) factorizations, and is a regular category.
To see that $\mathrm{Grd}(\mathcal{E})$ is a regular Goursat category it suffices to check that the image of an equivalence relation along a regular epimorphism is still an equivalence relation (see Proposition 2.1). This fact easily follows from the

description of regular epimorphisms, by using the fact that the regular image of an equivalence relation in $\mathcal{E}$ is an equivalence relation in $\mathcal{E}$. $\square$

Remark 5.4. This last result obviously implies that the category $\mathrm{Grd}^2(\mathcal{E})$ of double groupoids in a regular Goursat category $\mathcal{E}$ is again regular Goursat, since $\mathrm{Grd}^2(\mathcal{E})$ is nothing but the category of groupoids in the regular Goursat category $\mathrm{Grd}(\mathcal{E})$. By iterating this construction we conclude that the category $\mathrm{Grd}^n(\mathcal{E})$ of n-fold groupoids in $\mathcal{E}$ is regular Goursat.

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