Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 07–38

THE GEOMETRY OF A 3-QUASI-SASAKIAN MANIFOLD

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ABSTRACT: 3-quasi-Sasakian manifolds were studied systematically by the authors in a recent paper as a suitable setting unifying 3-Sasakian and 3-cosymplectic geometries. In this paper many geometric properties of this class of almost 3-contact metric manifolds are found. In particular, it is proved that the only 3-quasi-Sasakian manifolds of rank $4l+1$ are the 3-cosymplectic manifolds and any 3-quasi-Sasakian manifold of maximal rank is necessarily 3-α-Sasakian. Furthermore, the transverse geometry of a 3-quasi-Sasakian manifold is studied, proving that any 3-quasi-Sasakian manifold admits a canonical transversal, projectable quaternionic-Kähler structure and a canonical transversal, projectable 3-α-Sasakian structure.

KEYWORDS: 3-quasi-Sasakian structure, 3-cosymplectic manifold, 3-Sasakian manifold, foliation, quaternionic structure.
AMS SUBJECT CLASSIFICATION (2000): Primary 53C12, Secondary 53C25, 57R30.

1. Introduction

The well-known classes of 3-Sasakian and 3-cosymplectic manifolds belong to the wider family of almost 3-contact metric manifolds. Nevertheless, both classes sit also perfectly into the narrower class of 3-quasi-Sasakian manifolds which, as we will see, is a very natural framework for a unified study of the aforementioned geometries. A similar chain of inclusions takes place in the case of a single almost contact metric structure, whereas the class of quasi-Sasakian manifolds encloses both Sasakian and cosymplectic manifolds, but in the setting of 3-structures the interrelations between the triples of tensors produce key additional properties making the choice of the 3-quasi-Sasakian framework still more natural. 3-quasi-Sasakian manifolds were introduced long ago but their first systematic study was carried out by the authors in [5]. There, it was proven that in any 3-quasi-Sasakian manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ of dimension $4n+3$ the vertical distribution $\mathcal{V}$ generated by the three Reeb vector fields is completely integrable determining a canonical totally geodesic and Riemannian foliation. The characteristic vector fields obey the commutation relations $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ for any even permutation (α, β, γ) of $\{1, 2, 3\}$ and some $c \in \mathbb{R}$. Furthermore, it was shown that the ranks of the 1-forms

Received November 28, 2007.
The second author acknowledges financial support by CMUC.

η_1, η_2, η_3 coincide giving a single well-defined rank which falls into one of two possible families: $4l+3$ or $4l+1$ for some $0 \leq l \leq n$. A splitting theorem was proven for the manifolds in the first of the two families just for the case $c = 2$, under some additional hypotheses, while a sufficient condition for those of rank $4l + 1$ to be 3-cosymplectic was found. In this paper, beside obtaining many additional properties of 3-quasi-Sasakian manifolds, we strongly improve the splitting results previously found, and we succeed in studying the geometries of the spaces of leaves. In fact, it turned out that there are three distinct fundamental foliations for 3-quasi-Sasakian manifolds. The study of the transversal geometries with respect to those foliations allowed us to link 3-quasi-Sasakian manifolds to the more famous hyper-Kählerian and quaternionic-Kählerian sisters.

The article is organized as follows. In Section 2 we briefly recall the required preliminaries about almost contact metric geometry and 3-structures which are the two pillars supporting 3-quasi-Sasakian geometry. The most relevant results already known about 3-quasi-Sasakian manifolds are also summarized. In the third section we mainly prove that all 3-quasi-Sasakian manifolds of rank $4l+1$ are 3-cosymplectic. It follows that a 3-quasi-Sasakian manifold is Ricci-flat if and only if it is 3-cosymplectic. Such a corollary may be thought as an odd-dimensional analogue of the well-known fact that any quaternionic-Kähler manifold is Ricci-flat if and only if it is (locally) hyper-Kähler. Section 4 is devoted to the study of the complementary class: 3-quasi-Sasakian manifolds of rank $4l + 3$. We show that any 3-quasi-Sasakian manifold of maximal rank is 3-α-Sasakian (cf. [9]) and it is 3-Sasakian if and only if the constant c in the commutators $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ is equal to 2. Next, a new, much better splitting theorem is obtained (cf. [5], Theorem 5.6) proving that any 3-quasi-Sasakian manifold of rank $4l + 3$ with at least one of the almost product structures integrable is locally the product of a 3-α-Sasakian manifold and a hyper-Kählerian manifold. We do not make any assumption neither on the metric nor on the constant c. Finally, in Section 5 we study the geometries of the spaces of leaves corresponding to three fundamental foliations canonically associated to a 3-quasi-Sasakian manifold. We start by analyzing the vertical foliation $\mathcal{V}$. The use of an adapted connection (cf. [6]) derived from the Bott connection allows us to show that it exists a canonical transversal projectable almost quaternionic structure with respect to $\mathcal{V}$. The projectability of the structure tensors ϕ_α with respect to $\mathcal{V}$ characterizes the integrability of the horizontal distribution which turns out also to be a sufficient condition for the corresponding leaf space to be hyper-Kählerian and for the 3-quasi-Sasakian manifold to be η-Einstein. Next, we prove the integrability of a second fundamental distribution, denoted by $\mathcal{E}^{4m}$, in any 3-quasi-Sasakian manifold of rank $4l+3$. The leaves turn out to be hyper-Kählerian while the leaf space is 3-α-Sasakian. Finally, we prove that the integrability of the distribution $\mathcal{E}^{4m+3}=\mathcal{E}^{4m}\oplus\mathcal{V}$ gives rise to a foliation with 3-cosymplectic leaves whose leaf space has quaternionic-Kähler structure. In this way we generalize to the class of 3-quasi-Sasakian manifold a fundamental result proven by Ishiara [8] for 3-Sasakian manifolds with respect to the foliation $\mathcal{V}$ which revealed to be fundamental for the subsequent studies of Boyer, Galicki and many others giving to that class of manifolds their current relevance.

2. Preliminaries

An *almost contact manifold* is an odd-dimensional manifold M which carries a tensor field ϕ of type $(1,1)$, a vector field ξ, called *characteristic* or *Reeb vector field*, and a 1-form η satisfying $\phi^2=-I+\eta\otimes\xi$ and $\eta(\xi)=1$, where $I\colon TM\to TM$ is the identity mapping. From the definition it follows also that $\phi\xi=0$, $\eta\circ\phi=0$ and that the $(1,1)$-tensor field ϕ has constant rank $2n$ (cf. [3]). An almost contact manifold is said to be *normal* if the tensor field $N^{(1)}=[\phi,\phi]+2d\eta\otimes\xi$ vanishes identically. It is known that any almost contact manifold (M,ϕ,ξ,η) admits a Riemannian metric g such that $g(\phi\cdot,\phi\cdot)=g(\cdot,\cdot)-\eta\otimes\eta$ holds. This metric, in general not unique, is called a *compatible metric* and the manifold M together with the structure (ϕ,ξ,η,g) is called an *almost contact metric manifold*. As an immediate consequence one has $\eta=g(\xi,\cdot)$. The 2-form Φ on M defined by $\Phi(X,Y)=g(X,\phi Y)$ is called the *fundamental 2-form* of the almost contact metric manifold M. The following formula gives the expression of the covariant derivative of ϕ in terms of the remaining structure tensors in any almost contact metric manifold ([3]) and it will be useful in the sequel,

$$
\begin{aligned}
2g((\nabla_X\phi)Y,Z) &= 3d\Phi(X,\phi Y,\phi Z)-3d\Phi(X,Y,Z)+g(N^{(1)}(Y,Z),\phi X)\\
&\quad +N^{(2)}(Y,Z)\eta(X)+2d\eta(\phi Y,X)\eta(Z)-2d\eta(\phi Z,X)\eta(Y).
\end{aligned}
\tag{2.1}
$$

Almost contact metric manifolds such that both η and Φ are closed are called *almost cosymplectic manifolds* and almost contact metric manifolds such that $d\eta=\Phi$ are called *contact metric manifolds*. Finally, a normal almost cosymplectic manifold is called a *cosymplectic manifold* and a normal contact metric manifold is said to be a *Sasakian manifold*.

The notion of quasi-Sasakian structure, introduced by D. E. Blair in [2], unifies those of Sasakian and cosymplectic structures. A *quasi-Sasakian manifold* is a normal almost contact metric manifold such that $d\Phi = 0$. A quasi-Sasakian manifold M (or more generally an almost contact manifold) of dimension $2n+1$ is said to be of rank $2p$ (for some $p \leq n$) if $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$ on M, and to be of rank $2p+1$ if $\eta \wedge (d\eta)^p \neq 0$ and $(d\eta)^{p+1} = 0$ on M (cf. [2, 20]). It was proven in [2] that there are no quasi-Sasakian manifolds of even rank. Let the rank of M be $2p+1$. Then, the tangent bundle of M splits into two subbundles as follows: $TM = \mathcal{E}^{2p+1} \oplus \mathcal{E}^{2q}$, $p+q = n$, where

$$\mathcal{E}^{2q} = \{X \in TM \mid i_X\eta = 0 \text{ and } i_X d\eta = 0\}$$

and $\mathcal{E}^{2p+1} = \mathcal{E}^{2p} \oplus \langle\xi\rangle$, $\mathcal{E}^{2p}$ being the orthogonal complement of $\mathcal{E}^{2q} \oplus \langle\xi\rangle$ in TM. These distributions satisfy $\phi\mathcal{E}^{2p} = \mathcal{E}^{2p}$ and $\phi\mathcal{E}^{2q} = \mathcal{E}^{2q}$ (cf. [20]). Notice that the subspace ${\mathcal{E}_x}^{2q} = \{X \in T_xM \mid i_X\eta = 0 \text{ and } i_X d\eta = 0\}$ determined by $\mathcal{E}^{2q}$ in any point $x \in M$ coincides with the characteristic system defined by Cartan in [7] for an arbitrary differential form. The class of a differential form is one of the integral invariants defined by Cartan. The codimension $2p+1$ of ${\mathcal{E}_x}^{2q}$ is called by Cartan the *class* of η in x. It is easy to verify that when the class of η is constant the characteristic system has constant rank in any point and the determined distribution is integrable. This is the case in all important examples of quasi-Sasakian manifolds, such as Sasakian and cosymplectic manifolds. Thus, we will only consider, as Blair and Tanno implicitly did, quasi-Sasakian manifolds of constant class, i.e., of fixed (odd) rank. So, the rank of Blair and Tanno coincides with the class of Cartan.

Some useful properties of quasi-Sasakian manifolds will be now mentioned. For a quasi-Sasakian manifold we have the relation (cf. [17])

$$(\nabla_X\phi)\,Y = -g\,(\nabla_X\xi, \phi Y)\,\xi - \eta\,(Y)\,\phi\nabla_X\xi, \tag{2.2}$$

which generalizes the well-known conditions $\nabla\phi = 0$ and $(\nabla_X\phi)\,Y = g\,(X,Y)\,\xi - \eta\,(Y)\,X$ characterizing respectively cosymplectic and Sasakian manifolds. The quasi-Sasakian condition reflects also in some properties of curvature and of the Reeb vector field. In fact we have the following results.

Lemma 2.1 ([2],[17]). *Let (M,ϕ,ξ,η,g) be a quasi-Sasakian manifold. Then*

(i) *the Reeb vector field ξ is Killing and its integral curves are geodesics;*

(ii) *the Ricci curvature in the direction of* ξ *is given by*

$$\mathrm{Ric}\,(\xi) = \|\nabla\xi\|^2. \tag{2.3}$$

We now come to the main topic of our paper, i.e. 3-quasi-Sasakian geometry, which is framed into the more general setting of almost 3-contact geometry. An *almost* 3*-contact manifold* is a $(4n+3)$-dimensional smooth manifold M endowed with three almost contact structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3) satisfying the following relations, for any even permutation (α, β, γ) of $\{1, 2, 3\}$,

$$\begin{aligned} \phi_\gamma &= \phi_\alpha\phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta\phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\ \xi_\gamma &= \phi_\alpha\xi_\beta = -\phi_\beta\xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha. \end{aligned} \tag{2.4}$$

This notion was introduced by Y. Y. Kuo ([15]) and, independently, by C. Udriste ([22]). In [15] Kuo proved that given an almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$, there exists a Riemannian metric g compatible with each of them and hence we can speak of *almost contact metric* 3*-structures*. It is well known that in any almost 3-contact metric manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal with respect to the compatible metric g and that the structural group of the tangent bundle is reducible to $Sp\,(n) \times I_3$. Moreover, by putting $\mathcal{H} = \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ one obtains a $4n$-dimensional distribution on M and the tangent bundle splits as the orthogonal sum $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle$. We will call any vector belonging to the distribution $\mathcal{H}$ *horizontal* and any vector belonging to the distribution $\mathcal{V}$ *vertical*. An almost 3-contact manifold M is said to be *hyper-normal* if each almost contact structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ is normal.

A 3*-quasi-Sasakian manifold* is, by definition, an almost 3-contact metric manifold such that each structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is quasi-Sasakian. Important subclasses of the above defined class are the well-known 3-Sasakian and 3-cosymplectic manifolds. Many results about 3-quasi-Sasakian manifolds have been found in [5].

Theorem 2.2 ([5])**.** *Let* $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ *be a* 3*-quasi-Sasakian manifold. Then the distribution spanned by the Reeb vector fields* ξ_1*,* ξ_2*,* ξ_3 *is integrable and defines a totally geodesic and Riemannian foliation* $\mathcal{V}$ *of* M*. More in particular, we have, for an even permutation* (α, β, γ) *of* $\{1, 2, 3\}$*, that* $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ *for some* $c \in \mathbb{R}$*.*

According to Theorem 2.2, the geometry of 3-quasi-Sasakian manifolds with $c = 0$ and those with $c \neq 0$, is very different. This can be seen, for instance, in the notion of the "rank" of a 3-quasi-Sasakian manifold, which is well defined due to the following theorem.

Theorem 2.3 ([5]). *Let $(M,\phi_\alpha,\xi_\alpha,\eta_\alpha,g)$ be a 3-quasi-Sasakian manifold of dimension $4n+3$. Then the 1-forms η_1, η_2, η_3 have the same rank, which is called the rank of the 3-quasi-Sasakian manifold M. Furthermore, this rank is equal to $4l+1$ or $4l+3$, for some $l \leq n$, according to $c=0$ or $c\neq 0$ respectively.*

Now we collect some results on 3-quasi-Sasakian manifolds, which we will use in the sequel. As before, we refer the reader to [5] for the details.

Proposition 2.4. *In any 3-quasi-Sasakian manifold $(M,\phi_\alpha,\xi_\alpha,\eta_\alpha,g)$ we have:*

(i) $d\eta_\alpha(X,\xi_\beta)=0$ *for all* $X\in\Gamma(\mathcal{H})$ *and* $\alpha,\beta\in\{1,2,3\}$*;*
(ii) *every Reeb vector field* ξ_α *is an infinitesimal automorphism with respect to the distribution* $\mathcal{H}$*;*
(iii) $d\eta_\alpha=\frac{1}{2}\mathcal{L}_{\xi_\beta}\Phi_\gamma$, *for any even permutation* (α,β,γ) *of* $\{1,2,3\}$*;*
(iv) $d\eta_\alpha(X,\phi_\alpha Y)=d\eta_\beta(X,\phi_\beta Y)$ *for all* $X,Y\in\Gamma(\mathcal{H})$ *and* $\alpha,\beta\in\{1,2,3\}$*;*
(v) $d\eta_\alpha(\phi_\beta X,\phi_\beta Y)=-d\eta_\alpha(X,Y)$ *for all* $X,Y\in\Gamma(\mathcal{H})$ *and* $\alpha\neq\beta$*;*
(vi) $d\eta_\alpha(\phi_\beta X,Y)=d\eta_\gamma(X,Y)$ *for all* $X,Y\in\Gamma(\mathcal{H})$ *and for any even permutation* (α,β,γ) *of* $\{1,2,3\}$.

3. Further results on 3-quasi-Sasakian manifolds

In the following we will use the notation $\mathcal{E}^{4m}:=\{X\in\mathcal{H}\mid \text{ for some } \alpha\in\{1,2,3\}\ i_X d\eta_\alpha=0\}$, while $\mathcal{E}^{4l}$ will be the orthogonal complement of $\mathcal{E}^{4m}$ in $\mathcal{H}$, $\mathcal{E}^{4l+3}:=\mathcal{E}^{4l}\oplus\mathcal{V}$, and $\mathcal{E}^{4m+3}:=\mathcal{E}^{4m}\oplus\mathcal{V}$. It is easy to see that $\phi_\alpha(\mathcal{E}^{4m})=\mathcal{E}^{4m}$ and $\phi_\alpha(\mathcal{E}^{4l})=\mathcal{E}^{4l}$ for each $\alpha\in\{1,2,3\}$. Note also that, regarding to the definition of $\mathcal{E}^{4m}$, if $i_X d\eta_\alpha=0$ for *some* $\alpha\in\{1,2,3\}$ then by [5, Lemma 5.4] $i_X d\eta_\delta=0$ for *any* $\delta\in\{1,2,3\}$.

According to [5], we define for each $\alpha\in\{1,2,3\}$ two tensor fields of type $(1,1)$ ψ_α and θ_α on M. We put, for a 3-quasi-Sasakian manifold of rank $4l+3$,

$$\psi_\alpha X=\begin{cases}\phi_\alpha X, & \text{if } X\in\Gamma(\mathcal{E}^{4l+3});\\ 0, & \text{if } X\in\Gamma(\mathcal{E}^{4m});\end{cases}\qquad \theta_\alpha X=\begin{cases}0, & \text{if } X\in\Gamma(\mathcal{E}^{4l+3});\\ \phi_\alpha X, & \text{if } X\in\Gamma(\mathcal{E}^{4m}),\end{cases}$$

and for a 3-quasi-Sasakian manifold of rank $4l+1$,

$$\psi_\alpha X=\begin{cases}\phi_\alpha X, & \text{if } X\in\Gamma(\mathcal{E}^{4l});\\ 0, & \text{if } X\in\Gamma(\mathcal{E}^{4m+3});\end{cases}\qquad \theta_\alpha X=\begin{cases}0, & \text{if } X\in\Gamma(\mathcal{E}^{4l});\\ \phi_\alpha X, & \text{if } X\in\Gamma(\mathcal{E}^{4m+3}).\end{cases}$$

Note that, for each $\alpha \in \{1,2,3\}$ we have $\phi_\alpha = \psi_\alpha + \theta_\alpha$. We have given two different definitions of ψ_α and θ_α, depending on the two possible ranks (for each l) that correspond to the two types of 3-quasi-Sasakian manifolds. It should be noted, however, that in both cases ψ_α and θ_α coincide in the horizontal subbundle $\mathcal{H}$. Next, we define a new (pseudo-Riemannian, in general) metric $\bar{g}$ on M setting

$$\bar{g}\,(X,Y) = \begin{cases} -d\eta_\alpha\,(X,\phi_\alpha Y)\,, & \text{for } X,Y \in \Gamma(\mathcal{E}^{4l}); \\ g\,(X,Y)\,, & \text{elsewhere.} \end{cases}$$

Note that this definition is well-posed by virtue of Proposition 2.4. The metric $\bar{g}$ is in fact a compatible metric and $(\phi_\alpha, \xi_\alpha, \eta_\alpha, \bar{g})$ is a normal almost 3-contact metric structure, in general non-3-quasi-Sasakian (cf. [5]). Concerning the Levi Civita connection $\bar{\nabla}$ of the metric $\bar{g}$ we prove the following useful formula.

Proposition 3.1. *With the notation above, one has in a 3-quasi-Sasakian manifold*

$$\bar{\nabla}_X \xi_\alpha = -\psi_\alpha X \tag{3.1}$$

for any $X \in \Gamma\,(\mathcal{H})$ and $\alpha \in \{1,2,3\}$.

Proof: In the case $c = 0$ the result is an immediate consequence of [20, Lemma 2.3]. As for the case $c \neq 0$, using the same Lemma, we have

$$\bar{\nabla}'_\alpha \xi_\alpha = -\psi'_\alpha, \tag{3.2}$$

where

$$\psi'_\alpha X = \begin{cases} \phi_\alpha X, & \text{if } X \in \Gamma(\mathcal{E}^{4l} \oplus \langle \xi_\beta, \xi_\gamma \rangle); \\ 0, & \text{if } X \in \Gamma(\mathcal{E}^{4m} \oplus \langle \xi_\alpha \rangle) \end{cases}$$

and $\bar{\nabla}'_\alpha$ is the Levi-Civita connection associated to the compatible metric $\bar{g}'_\alpha$ defined by

$$\bar{g}'_\alpha\,(X,Y) = \begin{cases} -d\eta_\alpha\,(X,\phi_\alpha Y)\,, & \text{for } X,Y \in \Gamma(\mathcal{E}^{4l} \oplus \langle \xi_\beta, \xi_\gamma \rangle); \\ g\,(X,Y)\,, & \text{elsewhere.} \end{cases}$$

Note that $\psi_\alpha = \psi'_\alpha$ on $\Gamma(\mathcal{H})$. Now, considering $X \in \Gamma(\mathcal{H})$, we prove that

$$\bar{\nabla}'_{\alpha X} \xi_\alpha = \bar{\nabla}_X \xi_\alpha. \tag{3.3}$$

It should be noted that the metric $\bar{g}'_\alpha$, as well as $\bar{g}$, preserves the orthogonal decomposition $TM = \mathcal{H} \oplus \mathcal{V}$, whereas ξ_1, ξ_2, ξ_3 are only orthogonal and not orthonormal with respect to $\bar{g}'_\alpha$: indeed $\bar{g}'_\alpha(\xi_\beta, \xi_\beta) = \frac{c}{2}$. Then $\bar{g}|_{\mathcal{H}\times\mathcal{H}} = \bar{g}'_\alpha|_{\mathcal{H}\times\mathcal{H}}$

and $\bar{g}|_{\mathcal{H}\times\mathcal{V}} = \bar{g}'_\alpha|_{\mathcal{H}\times\mathcal{V}}$. Now, in order to prove (3.3), we show preliminarily that $\bar{\nabla}'_{\alpha X}\xi_\alpha, \bar{\nabla}_X\xi_\alpha \in \Gamma(\mathcal{H})$. Indeed,

$$\begin{aligned}2\bar{g}'_\alpha(\bar{\nabla}'_{\alpha X}\xi_\alpha, \xi_\delta) &= X(\bar{g}'_\alpha(\xi_\alpha, \xi_\delta)) + \xi_\alpha(\bar{g}'_\alpha(\xi_\delta, X)) - \xi_\delta(\bar{g}'_\alpha(X, \xi_\alpha)) \\ &\quad + \bar{g}'_\alpha([X, \xi_\alpha], \xi_\delta) + \bar{g}'_\alpha([\xi_\delta, X], \xi_\alpha) - \bar{g}'_\alpha([\xi_\alpha, \xi_\delta], X) = 0,\end{aligned}$$

since $\bar{g}'_\alpha(\xi_\alpha, \xi_\delta)$ is constant, $[\xi_\delta, \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H})$ for any δ, and $\mathcal{V}$ is integrable. Analogously, $\bar{g}_\alpha(\bar{\nabla}_X\xi_\alpha, \xi_\delta) = 0$. Then, using the definitions of $\bar{g}$ and $\bar{g}'_\alpha$, we have that for any $X, Y \in \Gamma(\mathcal{H})$,

$$\begin{aligned}2\bar{g}(\bar{\nabla}_X\xi_\alpha - \bar{\nabla}'_{\alpha X}\xi_\alpha, Y) &= X(\bar{g}(\xi_\alpha, Y)) + \xi_\alpha(\bar{g}(X, Y)) - Y(\bar{g}(\xi_\alpha, X)) \\ &\quad + \bar{g}([X, \xi_\alpha], Y) + \bar{g}([Y, X], \xi_\alpha) - \bar{g}([\xi_\alpha, Y], X) \\ &\quad - X(\bar{g}'_\alpha(\xi_\alpha, Y)) - \xi_\alpha(\bar{g}'_\alpha(X, Y)) + Y(\bar{g}'_\alpha(\xi_\alpha, X)) \\ &\quad - \bar{g}'_\alpha([X, \xi_\alpha], Y) - \bar{g}'_\alpha([Y, X], \xi_\alpha) + \bar{g}'_\alpha([\xi_\alpha, Y], X) \\ &= \xi_\alpha(\bar{g}(X, Y)) + \bar{g}([X, \xi_\alpha], Y) + g([Y, X], \xi_\alpha) \\ &\quad - \bar{g}([\xi_\alpha, Y], X) - \xi_\alpha(\bar{g}'_\alpha(X, Y)) - \bar{g}'_\alpha([X, \xi_\alpha], Y) \\ &\quad - g([Y, X], \xi_\alpha) + \bar{g}'_\alpha([\xi_\alpha, Y], X) = 0.\end{aligned}$$

Therefore we have that $\bar{\nabla}_X\xi_\alpha = \bar{\nabla}'_{\alpha X}\xi_\alpha = -\psi'_\alpha X = -\psi_\alpha X$ and (3.1) is proved. ■

Lemma 3.2. *In any 3-quasi-Sasakian manifold we have, for a cyclic permutation (α, β, γ) of $\{1, 2, 3\}$,*

$$\mathcal{L}_{\xi_\alpha} d\eta_\beta = c d\eta_\gamma.$$

Proof: From the Cartan formula for the Lie derivative it follows that $\mathcal{L}_{\xi_\alpha} d\eta_\beta = i_{\xi_\alpha} d^2\eta_\beta + d i_{\xi_\alpha} d\eta_\beta = d i_{\xi_\alpha} d\eta_\beta$, so that it is enough to find $i_{\xi_\alpha} d\eta_\beta$. By (i) of Proposition 2.4 we have, for any $X \in \Gamma(\mathcal{H})$,

$$(i_{\xi_\alpha} d\eta_\beta)(X) = 2d\eta_\beta(\xi_\alpha, X) = 0 = c\eta_\gamma(X).$$

Now, distinguishing the cases $c = 0$ and $c \neq 0$, one can verify that $i_{\xi_\alpha} d\eta_\beta = c\eta_\gamma$ also holds on $\Gamma(\mathcal{V})$, thus getting the result. ■

Lemma 3.3. *For any $X \in \Gamma(\mathcal{H})$ and $Y \in \Gamma(\mathcal{E}^{4m})$ we have $[X, Y] \in \Gamma(\mathcal{H})$.*

Proof: For any $\alpha \in \{1, 2, 3\}$ one has $\eta_\alpha([X, Y]) = -2d\eta_\alpha(X, Y) = (i_Y d\eta_\alpha)(X) = 0$, since $Y \in \Gamma(\mathcal{E}^{4m})$. Hence $[X, Y] \in \bigcap_{\alpha=1}^3 \ker(\eta_\alpha) = \mathcal{H}$. ■

Lemma 3.4. *Let $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold. Then the Reeb vector fields are infinitesimal automorphisms with respect to the distributions $\mathcal{E}^{4l}$ and $\mathcal{E}^{4m}$.*

Proof: Let us assume $c \neq 0$. Fixing an $\alpha \in \{1,2,3\}$, by [20, Lemma 2.2] we have that $[\xi_\alpha, \Gamma(\mathcal{E}^{4l})] \subset \Gamma(\mathcal{E}^{4l} \oplus \langle \xi_\beta, \xi_\gamma \rangle)$ and $[\xi_\alpha, \Gamma(\mathcal{E}^{4m})] \subset \Gamma(\mathcal{E}^{4m})$. Then the result follows from (ii) of Proposition 2.4. Analogously one obtains the claim for $c = 0$. ■

Proposition 3.5. *In any 3-quasi-Sasakian manifold we have*

$$\mathcal{L}_{\xi_\alpha} \phi_\beta = c\psi_\gamma, \tag{3.4}$$

for any cyclic permutation (α, β, γ) of $\{1,2,3\}$.

Proof: That (3.4) holds on $\mathcal{V}$ follows immediately from a direct computation and from the definitions of the tensors ψ_α. Next, for any $X \in \Gamma(\mathcal{H})$ we have

$$\begin{aligned}(\mathcal{L}_{\xi_\alpha}\phi_\beta)X &= [\xi_\alpha, \phi_\beta X] - \phi_\beta [\xi_\alpha, X] \\ &= \bar{\nabla}_{\xi_\alpha}\phi_\beta X - \bar{\nabla}_{\phi_\beta X}\xi_\alpha - \phi_\beta \bar{\nabla}_{\xi_\alpha} X + \phi_\beta \bar{\nabla}_X \xi_\alpha \\ &= (\bar{\nabla}_{\xi_\alpha}\phi_\beta)X - \bar{\nabla}_{\phi_\beta X}\xi_\alpha + \phi_\beta \bar{\nabla}_X \xi_\alpha.\end{aligned} \tag{3.5}$$

Now, using (2.1), we compute $\bar{\nabla}_{\xi_\alpha}\phi_\beta$. Taking into account the normality of $(\phi_\beta, \xi_\beta, \eta_\beta)$, (i) of Proposition 2.4 and the horizontality of X we have

$$2\bar{g}((\bar{\nabla}_{\xi_\alpha}\phi_\beta)X, Y) = 3d\bar{\Phi}_\beta(\xi_\alpha, \phi_\beta X, \phi_\beta Y) - 3d\bar{\Phi}_\beta(\xi_\alpha, X, Y). \tag{3.6}$$

If $Y = \xi_\delta$ for some $\delta \in \{1,2,3\}$, by (ii) of Proposition 2.4 and the integrability of the distribution spanned by ξ_1, ξ_2, ξ_3, we have

$$\begin{aligned}3d\bar{\Phi}_\beta(\xi_\alpha, X, \xi_\delta) &= \xi_\alpha(\bar{\Phi}_\beta(X, \xi_\delta)) + X(\bar{\Phi}_\beta(\xi_\delta, \xi_\alpha)) + \xi_\delta(\bar{\Phi}_\beta(\xi_\alpha, X)) \\ &\quad - \bar{\Phi}_\beta([\xi_\alpha, X], \xi_\delta) - \bar{\Phi}_\beta([X, \xi_\delta], \xi_\alpha) - \bar{\Phi}_\beta([\xi_\delta, \xi_\alpha], X) = 0,\end{aligned}$$

and, in the same way, we find $3d\bar{\Phi}_\beta(\xi_\alpha, \phi_\beta X, \phi_\beta \xi_\delta) = 0$, so that $(\bar{\nabla}_{\xi_\alpha}\phi_\beta)X \in \Gamma(\mathcal{H})$. Now we prove that

$$\bar{g}((\bar{\nabla}_{\xi_\alpha}\phi_\beta)X, Y) = (c-2)\bar{g}(\psi_\gamma X, Y) \tag{3.7}$$

for every $X, Y \in \Gamma(\mathcal{H})$. Indeed, by (3.6) we have

$$\begin{aligned} 2\bar{g}((\bar{\nabla}_{\xi_\alpha}\phi_\beta)X, Y) &= \xi_\alpha(\bar{\Phi}_\beta(\phi_\beta X, \phi_\beta Y)) + \phi_\beta X(\bar{\Phi}_\beta(\phi_\beta Y, \xi_\alpha)) \\ &\quad + \phi_\beta Y(\bar{\Phi}_\beta(\xi_\alpha, \phi_\beta X)) - \bar{\Phi}_\beta([\xi_\alpha, \phi_\beta X], \phi_\beta Y) \\ &\quad - \bar{\Phi}_\beta([\phi_\beta X, \phi_\beta Y], \xi_\alpha) - \bar{\Phi}_\beta([\phi_\beta Y, \xi_\alpha], \phi_\beta X) \\ &\quad - \xi_\alpha(\bar{\Phi}_\beta(X, Y)) - X(\bar{\Phi}_\beta(Y, \xi_\alpha)) - Y(\bar{\Phi}_\beta(\xi_\alpha, X)) \\ &\quad + \bar{\Phi}_\beta([\xi_\alpha, X], Y) + \bar{\Phi}_\beta([X, Y], \xi_\alpha) + \bar{\Phi}_\beta([Y, \xi_\alpha], X) \\ &= \xi_\alpha(\bar{\Phi}_\beta(\phi_\beta X, \phi_\beta Y)) - \bar{\Phi}_\beta([\xi_\alpha, \phi_\beta X], \phi_\beta Y) \\ &\quad - \bar{\Phi}_\beta([\phi_\beta X, \phi_\beta Y], \xi_\alpha) - \bar{\Phi}_\beta([\phi_\beta Y, \xi_\alpha], \phi_\beta X) \\ &\quad - \xi_\alpha(\bar{\Phi}_\beta(X, Y)) + \bar{\Phi}_\beta([\xi_\alpha, X], Y) \\ &\quad + \bar{\Phi}_\beta([X, Y], \xi_\alpha) + \bar{\Phi}_\beta([Y, \xi_\alpha], X). \end{aligned} \tag{3.8}$$

Because of the $\bar{g}$-orthogonal decomposition $\mathcal{H} = \mathcal{E}^{4l} \oplus \mathcal{E}^{4m}$ we can distinguish the cases (i) $X, Y \in \Gamma(\mathcal{E}^{4l})$, (ii) $X \in \Gamma(\mathcal{E}^{4l})$, $Y \in \Gamma(\mathcal{E}^{4m})$, (iii) $X \in \Gamma(\mathcal{E}^{4m})$, $Y \in \Gamma(\mathcal{E}^{4l})$, (iv) $X, Y \in \Gamma(\mathcal{E}^{4m})$. In the first case, taking into account that $\phi_\beta(\mathcal{E}^{4l}) = \mathcal{E}^{4l}$ and $[\xi_\alpha, \Gamma(\mathcal{E}^{4l})] \subset \Gamma(\mathcal{E}^{4l})$ (cf. Lemma 3.4) we get

$$\begin{aligned} 2\bar{g}((\bar{\nabla}_{\xi_\alpha}\phi_\beta)X, Y) &= (\mathcal{L}_{\xi_\alpha} d\eta_\beta)(\phi_\beta X, \phi_\beta Y) + \eta_\gamma([\phi_\beta X, \phi_\beta Y]) \\ &\quad - (\mathcal{L}_{\xi_\alpha} d\eta_\beta)(X, Y) - \eta_\gamma([X, Y]) \\ &= (\mathcal{L}_{\xi_\alpha} d\eta_\beta)(\phi_\beta X, \phi_\beta Y) - 2d\eta_\gamma(\phi_\beta X, \phi_\beta Y) \\ &\quad - (\mathcal{L}_{\xi_\alpha} d\eta_\beta)(X, Y) + 2d\eta_\gamma(X, Y). \end{aligned} \tag{3.9}$$

Continuing the computation and using Lemma 3.2 and (v) of Proposition 2.4, (3.9) becomes

$$\begin{aligned} 2\bar{g}((\bar{\nabla}_{\xi_\alpha}\phi_\beta)X, Y) &= c d\eta_\gamma(\phi_\beta X, \phi_\beta Y) - 2d\eta_\gamma(\phi_\beta X, \phi_\beta Y) - c d\eta_\gamma(X, Y) + 2d\eta_\gamma(X, Y) \\ &= -c d\eta_\gamma(X, Y) + 2d\eta_\gamma(X, Y) - c d\eta_\gamma(X, Y) + 2d\eta_\gamma(X, Y) \\ &= 2(2 - c) d\eta_\gamma(X, Y) \\ &= 2(c - 2)\bar{g}(\psi_\gamma X, Y). \end{aligned}$$

If we take $X \in \Gamma(\mathcal{E}^{4l})$ and $Y \in \Gamma(\mathcal{E}^{4m})$, then, due to the orthogonality between $\mathcal{E}^{4l}$ and $\mathcal{E}^{4m}$, (3.8) reduces to

$$\begin{aligned} 2\bar{g}((\bar{\nabla}_{\xi_\alpha}\phi_\beta)X, Y) &= -\bar{\Phi}_\beta([\phi_\beta X, \phi_\beta Y], \xi_\alpha) + \bar{\Phi}_\beta([X, Y], \xi_\alpha) \\ &= \eta_\gamma([\phi_\beta X, \phi_\beta Y]) - \eta_\gamma([X, Y]) = 0 \end{aligned}$$

by Lemma 3.3. Since $\bar{g}(\psi_\gamma X, Y) = 0$, we get (3.7). Next, arguing as above, one finds that (3.7) also holds for $X \in \Gamma(\mathcal{E}^{4m})$ and $Y \in \Gamma(\mathcal{E}^{4l})$. Finally, if $X, Y \in \Gamma(\mathcal{E}^{4m})$, by the definition of $\bar{g}$ and $d\Phi_\beta = 0$, one has

$$2\bar{g}((\bar{\nabla}_{\xi_\alpha}\phi_\beta)X, Y) = 3d\Phi_\beta(\xi_\alpha, \phi_\beta X, \phi_\beta Y) - 3d\Phi_\beta(\xi_\alpha, X, Y) = 0$$

which proves (3.7), since $\psi_\gamma X = 0$. Therefore we get that

$$(\bar{\nabla}_{\xi_\alpha}\phi_\beta)X = (c-2)\psi_\gamma X \tag{3.10}$$

for any $X \in \Gamma(\mathcal{H})$. Continuing the computation in (3.5), we obtain, by virtue of (3.10) and (3.1),

$$(\mathcal{L}_{\xi_\alpha}\phi_\beta)X = (c-2)\psi_\gamma X + \psi_\alpha\phi_\beta X - \phi_\beta\psi_\alpha X,$$

so that $(\mathcal{L}_{\xi_\alpha}\phi_\beta)X = c\phi_\gamma X$ if $X \in \Gamma(\mathcal{E}^{4l})$ and $(\mathcal{L}_{\xi_\alpha}\phi_\beta)X = 0$ if $X \in \Gamma(\mathcal{E}^{4m})$, from which the assertion follows. ■

Lemma 3.6. *In any* 3*-quasi-Sasakian manifold we have*

$$(\mathcal{L}_{\xi_\alpha}\Phi_\beta)(X, Y) = g(X, (\mathcal{L}_{\xi_\alpha}\phi_\beta)Y). \tag{3.11}$$

for all $X, Y \in \Gamma(TM)$ *and* $\alpha, \beta \in \{1, 2, 3\}$.

Proof: The assertion follows immediately from the fact that each ξ_α is Killing. ■

Theorem 3.7. *Every* 3*-quasi-Sasakian manifold of rank* $4l+1$ *is* 3*-cosymplectic.*

Proof: Using (iii) of Proposition 2.4 and (3.11), we have

$$2d\eta_\alpha(X, Y) = (\mathcal{L}_{\xi_\beta}\Phi_\gamma)(X, Y) = g(X, (\mathcal{L}_{\xi_\beta}\phi_\gamma)Y)$$

and the last term vanishes since, for $c = 0$, $\mathcal{L}_{\xi_\beta}\phi_\gamma = 0$ due to Proposition 3.5. ■

Corollary 3.8. *Any* 3*-quasi-Sasakian manifold is Ricci-flat if and only if it is* 3*-cosymplectic.*

Proof: That any 3-cosymplectic manifold is Ricci-flat has been proved in [6]. Conversely, if a 3-quasi-Sasakian manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is Ricci-flat, then by (2.3) we get $\nabla\xi_\alpha = 0$ for all $\alpha \in \{1, 2, 3\}$, hence $c = 0$. Thus applying Theorem 3.7 we get the result. ■

It should be remarked that Corollary 3.8 may be thought as an odd-dimensional analogue of the well-known fact that any quaternionic-Kähler manifold is Ricci-flat if and only if it is (locally) hyper-Kähler.

4. 3-quasi-Sasakian manifolds of rank $4l+3$

We recall that an *almost α-Sasakian manifold* ([11]) is an almost contact metric manifold satisfying $d\eta = \alpha\Phi$ for some $\alpha \in \mathbb{R}^*$. An almost α-Sasakian manifold which is also normal is called an *α-Sasakian manifold.* It is well-known that an almost contact metric manifold is α-Sasakian if and only if

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) \tag{4.1}$$

holds for all $X, Y \in \Gamma(TM)$, for some $\alpha \in \mathbb{R}^*$. From (4.1) it follows also that

$$\nabla_X \xi = -\alpha\phi X, \quad R_{XY}\xi = \alpha^2(\eta(Y)X - \eta(X)Y). \tag{4.2}$$

Since the fundamental 2-form of an α-Sasakian manifold is exact (in particular closed) then the manifold is quasi-Sasakian.

Now consider an almost 3-contact metric manifold $(M, \phi_\delta, \xi_\delta, \eta_\delta, g)$ of dimension $4n+3$, such that each structure is α-Sasakian, and suppose $d\eta_\delta = \alpha_\delta\Phi_\delta$ for any $\delta \in \{1,2,3\}$. Then, as it has been showed in [14], $\alpha_1 = \alpha_2 = \alpha_3 := a$, and we have

$$[\xi_\alpha, \xi_\beta] = \nabla_{\xi_\alpha}\xi_\beta - \nabla_{\xi_\beta}\xi_\alpha = -a\phi_\beta\xi_\alpha + a\phi_\alpha\xi_\beta = 2a\xi_\gamma. \tag{4.3}$$

Hence, M is 3-quasi-Sasakian with $c = 2a$ and maximal rank $4n+3$.

We will call an almost 3-contact metric manifold such that each structure is (almost) α-Sasakian simply by a *(almost) 3-α-Sasakian manifold.*

An example of these manifolds is given by the sphere $S^{4n+3}(r)$ of radius r, considered as a hypersurface in $\mathbb{H}^{n+1}$. Indeed, taking the quaternionic structure (J_1, J_2, J_3) on $\mathbb{H}^{n+1}$, one can define three vector fields on the sphere, $\xi_\alpha = -J_\alpha\nu$, ν being a unit normal of $S^{4n+3}(r)$. Next, one defines the tensor fields ϕ_α of type $(1,1)$ and the 1-forms η_α by requiring that, for any vector field X tangent to the sphere, $\phi_\alpha X$ and $\eta_\alpha(X)\nu$ are respectively the tangential and the normal component of $J_\alpha X$ to the sphere. Considering the induced Riemannian metric g, one obtains an almost 3-contact metric structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ which is 3-α-Sasakian, since it is hyper-normal and the fundamental 2-forms satisfy $d\eta_\alpha = \frac{1}{r}\Phi_\alpha$.

We prove that, in fact, strictly almost 3-α-Sasakian manifolds do not exist. This is a consequence of a generalization of the Hitchin Lemma, due to Kashiwada, which we now recall.

Lemma 4.1 ([13]). *Let (M^{4n}, J_α, G), $\alpha \in \{1,2,3\}$, be an almost hyper-Hermitian manifold such that each fundamental 2-form Ω_α satisfies $d\Omega_\alpha = 2\omega \wedge \Omega_\alpha$, for some 1-form ω. Then each J_α is integrable.*

Proposition 4.2. *Every almost* 3-α-*Sasakian manifold is necessarily* 3-α-*Sasakian.*

Proof: Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$, $\alpha \in \{1,2,3\}$, be an almost 3-α-Sasakian manifold and let us consider on the product manifold $M \times \mathbb{R}$ the almost Hermitian structures J_1, J_2, J_3 defined by

$$J_\alpha\left(X, f\frac{d}{dt}\right) = \left(\phi_\alpha X - f\xi_\alpha, \eta_\alpha(X)\frac{d}{dt}\right),$$

for any vector field X on M and any smooth function f on $M \times \mathbb{R}$, where we have denoted by t the global coordinate on $\mathbb{R}$. A straightforward computation shows that $J_\alpha J_\beta = -J_\beta J_\alpha = J_\gamma$ for an even permutation (α, β, γ) of $\{1,2,3\}$. Moreover, it is simple to check that the Riemannian metric $G = g + dt^2$ is compatible with respect to the hyper-complex structure (J_1, J_2, J_3). Computing the expressions of the fundamental 2-forms we find

$$\Omega_\alpha(X,Y) = \Phi_\alpha(X,Y), \quad \Omega_\alpha\left(X, \frac{d}{dt}\right) = -\eta_\alpha(X), \tag{4.4}$$

for all $X, Y \in \Gamma(TM)$ and $\alpha \in \{1,2,3\}$. From (4.4) it follows that

$$d\Omega_\alpha(X,Y,Z) = d\Phi_\alpha(X,Y,Z) = 0, \quad d\Omega_\alpha\left(X, Y, \frac{d}{dt}\right) = -\frac{2}{3}d\eta_\alpha(X,Y), \tag{4.5}$$

for every $X, Y, Z \in \Gamma(TM)$. In particular, we have that, for each $\delta \in \{1,2,3\}$, $d\Omega_\delta = 2\omega \wedge \Omega_\delta$, where $\omega = -\alpha dt$. By Lemma 4.1, this concludes the proof. ∎

We will prove that every 3-quasi-Sasakian manifold of maximal rank is necessarily 3-α-Sasakian. This will be an immediate consequence of the following result, which is an analogue of Theorem 3.7 for the class of 3-quasi-Sasakian manifolds which are not 3-cosymplectic.

Theorem 4.3. *Let* $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ *be a* 3-*quasi-Sasakian manifold such that* $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$, $c \neq 0$. *Let* $4l+3$ *be the rank of* M^{4n+3}. *Then, for each* $\alpha \in \{1,2,3\}$,

$$d\eta_\alpha(X,Y) = \frac{c}{2}g(X, \psi_\alpha Y) \tag{4.6}$$

for all $X, Y \in \Gamma(TM)$. *Consequently, on* $\mathcal{E}^{4l+3}$,

$$d\eta_\alpha = \frac{c}{2}\Phi_\alpha. \tag{4.7}$$

Proof: Using (3.11) and (3.4), for all $X, Y \in \Gamma(TM)$, we have

$$(\mathcal{L}_{\xi_\beta}\Phi_\gamma)(X,Y) = g(X, (\mathcal{L}_{\xi_\beta}\phi_\gamma)Y) = cg(X, \psi_\alpha Y).$$

On the other hand, by (iii) of Proposition 2.4, $\mathcal{L}_{\xi_\beta}\Phi_\gamma = 2d\eta_\alpha$ from which (4.6) follows. ■

Corollary 4.4. *Every 3-quasi-Sasakian manifold of maximal rank is necessarily 3-α-Sasakian.*

Remark 4.5. It should be emphasized that in general no analogue of Theorem 4.3, as well as of Corollary 4.4, holds for a quasi-Sasakian manifold. These properties are thus a special feature of 3-quasi-Sasakian manifolds.

Corollary 4.6. *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold. Then for each $\alpha \in \{1, 2, 3\}$*

$$\nabla\xi_\alpha = -\frac{c}{2}\psi_\alpha. \tag{4.8}$$

Proof: By Theorem 3.7 the assertion easily holds for $c = 0$, so that we can assume $c \neq 0$. Then by (2.1) we have $g((\nabla_X\phi_\alpha)\xi_\alpha, Z) = d\eta_\alpha(\phi_\alpha\xi_\alpha, X)\eta_\alpha(Z) - d\eta_\alpha(\phi_\alpha Z, X)\eta_\alpha(\xi_\alpha) = -d\eta_\alpha(\phi_\alpha Z, X)$, from which, applying (4.6), it follows that

$$g(\nabla_X\xi_\alpha, \phi_\alpha Z) = -d\eta_\alpha(\phi_\alpha Z, X) = -\frac{c}{2}g(\psi_\alpha X, \phi_\alpha Z).$$

Therefore, tacking into account the fact that $g(\nabla_X\xi_\alpha, \xi_\alpha) = 0 = -\frac{c}{2}g(\psi_\alpha X, \xi_\alpha)$, (4.8) is proved. ■

Corollary 4.7. *In any 3-quasi-Sasakian manifold one has, for each $\alpha \in \{1, 2, 3\}$,*

$$(\nabla_X\phi_\alpha)\,Y = \frac{c}{2}\left(\eta_\alpha(Y)\,\psi_\alpha^2 X - g\left(\psi_\alpha^2 X, Y\right)\xi_\alpha\right), \tag{4.9}$$

for any $X, Y \in \Gamma(TM)$.

Proof: It is a consequence of (2.2), (4.8) and the fact that $\phi_\alpha\psi_\alpha = \psi_\alpha^2$. ■

5. Transverse geometry of a 3-quasi-Sasakian manifold

In this section we study the leaf space of some foliations canonically associated to a 3-quasi-Sasakian manifold.

We start with the study of the 3-dimensional foliation $\mathcal{V}$ defined by the Reeb vector fields. Let ∇^B be the Bott connection associated to $\mathcal{V}$, that is the partial connection on the normal bundle $TM/\mathcal{V} \cong \mathcal{H}$ of $\mathcal{V}$ defined by

$$\nabla^B_V Z := [V, Z]_{\mathcal{H}}$$

for all $V \in \Gamma(\mathcal{V})$ and $Z \in \Gamma(\mathcal{H})$. Following [21] we may construct an adapted connection on $\mathcal{H}$ putting

$$\tilde{\nabla}_X Y := \begin{cases} \nabla^B_X Y, & \text{if } X \in \Gamma(\mathcal{V}); \\ (\nabla_X Y)_{\mathcal{H}}, & \text{if } X \in \Gamma(\mathcal{H}). \end{cases}$$

This connection can be also extended to a connection on all TM by requiring that $\tilde{\nabla}\xi_\alpha = 0$ for each $\alpha \in \{1,2,3\}$. Some properties of this global connection have been considered in [6] for any almost 3-contact metric manifold. Now combining Theorem 2.2 with [6, Theorem 3.6] we have:

Theorem 5.1. *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold. Then there exists a unique connection $\tilde{\nabla}$ on M satisfying the following properties:*

(i) $\tilde{\nabla}\eta_\alpha = 0$, $\tilde{\nabla}\xi_\alpha = 0$, *for each* $\alpha \in \{1,2,3\}$,
(ii) $\tilde{\nabla} g = 0$,
(iii) $\tilde{T}(X,Y) = 2\sum_{\alpha=1}^3 d\eta_\alpha(X,Y)\xi_\alpha$, *for all* $X, Y \in \Gamma(TM)$.

Furthermore, we have, for any cyclic permutation (α, β, γ),

$$(\tilde{\nabla}_X \phi_\alpha) Y = -c\left(\eta_\beta(X)\psi_\gamma(Y_{\mathcal{H}}) - \eta_\gamma(X)\psi_\beta(Y_{\mathcal{H}})\right).$$

Proof: Theorem 3.6 of [6] guarantees the existence and the uniqueness of a linear connection $\tilde{\nabla}$ on M such that $\tilde{\nabla}\xi_\alpha = 0$, $(\tilde{\nabla}_Z g)(X,Y) = 0$ for all $X, Y, Z \in \Gamma(\mathcal{H})$ and $\tilde{T}(X,Y) = 2\sum_{\alpha=1}^3 d\eta_\alpha(X,Y)\xi_\alpha$, $\tilde{T}(X,\xi_\alpha) = 0$ for all $X, Y \in \Gamma(\mathcal{H})$. This connection is explicitly defined as above. Since each ξ_α is Killing we have that $\tilde{\nabla}$ is metric ([6]). Moreover, (i) of Proposition 2.4 implies that each 1-form η_α is $\tilde{\nabla}$-parallel and, for the torsion tensor field, $\tilde{T}(X,\xi_\alpha) = 0 = \sum_{\delta=1}^3 2d\eta_\delta(X,\xi_\alpha)\xi_\delta$ for any $X \in \Gamma(\mathcal{H})$ (cf. [6]). Finally, from the integrability of $\mathcal{V}$ it follows also $\tilde{T}(\xi_\alpha,\xi_\beta) = [\xi_\beta,\xi_\alpha] = \sum_{\delta=1}^3 2d\eta_\delta(\xi_\alpha,\xi_\beta)\xi_\delta$. It remains to check the final part of the statement. We prove that

$$(\tilde{\nabla}_X \phi_\alpha) Y = \begin{cases} 0, & \text{for } X \in \Gamma(\mathcal{H}) \text{ or } X = \xi_\alpha \text{ or } Y \in \Gamma(\mathcal{V}); \\ -c\psi_\gamma Y, & \text{for } X = \xi_\beta,\ Y \in \Gamma(\mathcal{H}) \text{ and } (\alpha,\beta,\gamma) \text{ cyclic.} \end{cases} \tag{5.1}$$

Firstly, since the Reeb vector fields ξ_α are parallel with respect to $\tilde{\nabla}$, one has $(\tilde{\nabla}_X \phi_1)\xi_\alpha = 0$ for any $\alpha \in \{1,2,3\}$. Next, taking $X, Y \in \Gamma(\mathcal{H})$ we have, using (2.2),

$$\begin{aligned}
(\tilde{\nabla}_X\phi_1)Y &= (\nabla_X\phi_1 Y)_{\mathcal{H}} - \phi_1(\nabla_X Y)_{\mathcal{H}} \\
&= \nabla_X\phi_1 Y - \sum_{\alpha=1}^{3} g(\nabla_X\phi_1 Y, \xi_\alpha)\xi_\alpha - \phi_1\nabla_X Y + \sum_{\alpha=1}^{3} g(\nabla_X Y, \xi_\alpha)\phi_1\xi_\alpha \\
&= (\nabla_X\phi_1)Y + \sum_{\alpha=1}^{3} g(\phi_1 Y, \nabla_X\xi_\alpha)\xi_\alpha + g(\nabla_X Y, \xi_2)\xi_3 - g(\nabla_X Y, \xi_3)\xi_2 \\
&= -g(\nabla_X\xi_1, \phi_1 Y)\xi_1 - \eta_1(Y)\phi_1\nabla_X\xi_1 + g(\phi_1 Y, \nabla_X\xi_1)\xi_1 \\
&\quad + g(\phi_1 Y, \nabla_X\xi_2)\xi_2 + g(\phi_1 Y, \nabla_X\xi_3)\xi_3 - g(Y, \nabla_X\xi_2)\xi_3 + g(Y, \nabla_X\xi_3)\xi_2 \\
&= -g(\phi_1\nabla_X\xi_3 + \nabla_X\xi_2, Y)\xi_3 - g(\phi_1\nabla_X\xi_2 - \nabla_X\xi_3, Y)\xi_2 = 0.
\end{aligned}$$

Indeed, one has $\nabla_X\xi_2 = -\nabla_X\phi_1\xi_3 = -\phi_1\nabla_X\xi_3$, since (2.2) and the facts that ξ_1 is killing and $\mathcal{V}$ is totally geodesic imply

$$(\nabla_X\phi_1)\xi_3 = g(\nabla_X\xi_1, \xi_2)\xi_1 = -g(\nabla_{\xi_2}\xi_1, X)\xi_1 = 0.$$

Analogously, $\nabla_X\xi_3 = \phi_1\nabla_X\xi_2$. Finally, for any $Y \in \Gamma(\mathcal{H})$, by the definition of $\tilde{\nabla}$ and by (ii) of Proposition 2.4 one has

$$(\tilde{\nabla}_{\xi_1}\phi_1)Y = \nabla^B_{\xi_1}\phi_1 Y - \phi_1\nabla^B_{\xi_1}Y = [\xi_1, \phi_1 Y] - \phi_1[\xi_1, Y] = (\mathcal{L}_{\xi_1}\phi_1)Y = 0.$$

Similarly, using also Proposition 3.5 we have $(\tilde{\nabla}_{\xi_2}\phi_1)Y = (\mathcal{L}_{\xi_2}\phi_1)Y = -c\psi_3 Y$ and $(\tilde{\nabla}_{\xi_3}\phi_1)Y = (\mathcal{L}_{\xi_3}\phi_1)Y = c\psi_2 Y$. We have thus proved (5.1). Now, decomposing any vector fields X and Y in their horizontal and vertical components one easily gets the claimed formula for $\tilde{\nabla}\phi_\alpha$. ■

Using the constructions above, we prove the projectability of a 3-quasi-Sasakian structure. Indeed we know by Theorem 2.2 that a 3-quasi-Sasakian manifold M of dimension $4n+3$ is foliated by a 3-dimensional foliation $\mathcal{V}$ which, as we have seen, influences greatly the geometry of M. It can be very useful to know more about the space of leaves $M' = M/\mathcal{V}$ generated by this foliation, which is, under some assumptions of regularity, a $4n$-dimensional smooth manifold, more in general an orbifold. As $\mathcal{V}$ is a Riemannian foliation, the metric g projects along the leaves onto a Riemannian metric g' on M'. What we have to study is the (local) projectability of the tensor fields ϕ_α or, more in general, of the subbundle of $\mathrm{End}(TM)$ that they span. This question is solved by the following Theorem.

Theorem 5.2. *Every 3-quasi-Sasakian manifold admits a canonical projectable, transversal almost quaternionic-Hermitian structure.*

Proof: Let $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold of rank $4l+3$. We first notice that the distributions $\mathcal{E}^{4l}$ and $\mathcal{E}^{4m}$ are foliated objects, i.e. they locally project along the leaves of $\mathcal{V}$ onto two distributions on the space of leaves which we denote by $\mathcal{E}'^{4l}$ and $\mathcal{E}'^{4m}$, respectively. In order to prove this, let π be a local submersion defining the foliation $\mathcal{V}$. We note that, as $\mathcal{V}$ is a Riemannian foliation, π is in fact a Riemannian submersion. We have to prove that, given any two points x and y on the same fiber, one has

$$\pi_{*_x}(\mathcal{E}^{4l}_x) = \pi_{*_y}(\mathcal{E}^{4l}_y), \quad \pi_{*_x}(\mathcal{E}^{4m}_x) = \pi_{*_y}(\mathcal{E}^{4m}_y). \tag{5.2}$$

Firstly observe that from Lemma 3.4 it follows immediately that the Bott connection preserves the distributions $\mathcal{E}^{4l}$ and $\mathcal{E}^{4m}$. In particular these distributions are preserved by the parallel transport along vertical curves. Now, let $x, y \in M$ such that $\pi(x) = x' = \pi(y)$ and let γ be a leaf curve such that $\gamma(0) = x$ and $\gamma(1) = y$. Let τ denote the parallel transport with respect to the Bott connection along the curve γ. Then we preliminarily prove that the following diagram commutes on $\mathcal{E}^{4l}_x$,

$$\begin{tikzcd} \mathcal{H}_x \arrow[rr, "\tau"] \arrow[dr, "\pi_{*_x}"'] & & \mathcal{H}_y \arrow[dl, "\pi_{*_y}"] \\ & T_{x'}M' & \end{tikzcd}$$

Indeed, let $v \in \mathcal{E}^{4l}_x$ and $X : I \to \mathcal{H}$ be the unique vector field along γ such that $\nabla^B_{\gamma'} X \equiv 0$ and $X(0) = v$, so that $\tau(v) = X(1)$. Let Y' be any vector field on the base space and Y be the corresponding basic vector field on M. Then we have

$$\begin{aligned} \frac{d}{dt} g'(\pi_{*_{\gamma(t)}}(X(t)), Y'_{\pi(\gamma(t))}) &= \frac{d}{dt} g'(\pi_{*_{\gamma(t)}}(X(t)), \pi_{*_{\gamma(t)}}(Y_{\gamma(t)})) \\ &= \frac{d}{dt} g(X(t), Y_{\gamma(t)}) \\ &= g(\tilde{\nabla}_{\gamma'} X, Y)_{\gamma(t)} + g(X, \tilde{\nabla}_{\gamma'} Y)_{\gamma(t)} \\ &= g(\nabla^B_{\gamma'} X, Y)_{\gamma(t)} + g(X, \nabla^B_{\gamma'} Y)_{\gamma(t)} \\ &= g(X, \nabla^B_{\gamma'} Y)_{\gamma(t)}. \end{aligned}$$

Now, as for all $t \in I$ $\gamma'(t) \in \mathcal{V}_{\gamma(t)}$, $\gamma' = \sum_{\alpha=1}^{3} f_\alpha \xi_\alpha$ for some functions f_α, hence

$$\nabla^B_{\gamma'} Y = \sum_{\alpha=1}^{3} f_\alpha \nabla^B_{\xi_\alpha} Y = \sum_{\alpha=1}^{3} f_\alpha [\xi_\alpha, Y]_{\mathcal{H}} = 0$$

because Y is assumed to be basic. Therefore

$$g'_{\pi(\gamma(0))}(\pi_{*_{\gamma(0)}}(X(0)), Y'_{\pi(\gamma(0))}) = g'_{\pi(\gamma(1))}(\pi_{*_{\gamma(1)}}(X(1)), Y'_{\pi(\gamma(1))}),$$

that is

$$g'_{x'}(\pi_{*_x}(v), Y'_x) = g'_{x'}(\pi_{*_y}(\tau(v)), Y'_{x'}).$$

By the arbitrariness of Y' we conclude that $\pi_{*_x}(v) = (\pi_{*_y} \circ \tau)(v)$. Thus $\pi_{*_x}(\mathcal{E}^{4l}_x) = \pi_{*_y}(\tau(\mathcal{E}^{4l}_x)) = \pi_{*_y}(\mathcal{E}^{4l}_y)$ and arguing analogously for $\mathcal{E}^{4m}$ one has $\pi_{*_x}(\mathcal{E}^{4m}_x) = \pi_{*_y}(\mathcal{E}^{4m}_y)$. Hence (5.2) are proved and $\mathcal{E}^{4l}$, $\mathcal{E}^{4m}$ project to well-defined distributions $\mathcal{E}'^{4l}$, $\mathcal{E}'^{4m}$ which are also mutually orthogonal since the Riemannian metric g is bundle-like. We can now construct an almost quaternionic structure on the space of leaves M'. By an abuse of notation we will denote, for each $\alpha \in \{1, 2, 3\}$, by ψ_α and θ_α the restriction of ϕ_α to $\mathcal{E}^{4l}$ and $\mathcal{E}^{4m}$, respectively. Let $\bar{Q}$ be the subbundle of $\mathrm{End}(\mathcal{E}^{4l})$ spanned by ψ_1, ψ_2, ψ_3 and $\bar{\bar{Q}}$ be the subbundle of $\mathrm{End}(\mathcal{E}^{4m})$ spanned by $\theta_1, \theta_2, \theta_3$. For any $X \in \mathcal{E}^{4l}$ we have

$$(\nabla^B_{\xi_\alpha} \psi_\beta) X = [\xi_\alpha, \psi_\beta X]_{\mathcal{H}} - \psi_\beta [\xi_\alpha, X]_{\mathcal{H}} = (\mathcal{L}_{\xi_\alpha} \phi_\beta) X = c \psi_\gamma X,$$

by Lemma 3.4 and Proposition 3.5. Thus the Bott connection preserves the subbundle $\bar{Q}$ and this guarantees the projectability of $\bar{Q}$ onto an almost quaternionic structure $\bar{Q}' \subset \mathrm{End}(\mathcal{E}'^{4l})$ on the space of leaves of the foliation $\mathcal{V}$ (cf. [19]). For the subbundle $\bar{\bar{Q}}$ we can prove something more, namely that each θ_α is projectable. Indeed, for any $Y \in \mathcal{E}^{4m}$ we have

$$(\mathcal{L}_{\xi_\alpha} \theta_\beta) Y = [\xi_\alpha, \phi_\beta Y] - \phi_\beta [\xi_\alpha, Y] = (\mathcal{L}_{\xi_\alpha} \phi_\beta) Y = c \psi_\gamma Y = 0$$

again by Lemma 3.4 and Proposition 3.5. Thus each θ_α projects to a tensor field θ'_α defined on $\mathcal{E}'^{4m}$. Let us denote by $\bar{\bar{Q}}'$ the subbundle of $\mathrm{End}(\mathcal{E}'^{4m})$ that is spanned by $\theta'_1, \theta'_2, \theta'_3$. Since $TM' = \mathcal{E}'^{4l} \oplus \mathcal{E}'^{4m}$, from $\bar{Q}'$ and $\bar{\bar{Q}}'$ we can define an almost quaternionic structure on M' in the following way. Let $\psi'_1, \psi'_2, \psi'_3$ be a local basis for $\bar{Q}'$ defined on an open coordinate neighborhood U' . Then we define three tensor fields, defined on U', by

$$\phi'_\alpha := \begin{cases} \psi'_\alpha, & \text{on } \mathcal{E}'^{4l}, \\ \theta'_\alpha, & \text{on } \mathcal{E}'^{4m}, \end{cases}$$

for each $\alpha \in \{1,2,3\}$. Let Q' be the subbundle of $\mathrm{End}(TM')$ spanned by $\phi'_1, \phi'_2, \phi'_3$. Since in the overlapping of two coordinate neighborhoods U' and V' the matrix of the components of the $\phi'_\alpha|_{U'}$ with respect to the $\phi'_\alpha|_{V'}$ has the form

$$\begin{pmatrix} A & 0 \\ 0 & I_{4m} \end{pmatrix}$$

for some $A \in \mathrm{SO}(4l)$, we conclude that Q' defines an almost quaternionic-Hermitian structure on M'. ■

We now examine more explicitly the case when a 3-quasi-Sasakian manifold projects (locally) onto a hyper-Kählerian manifold.

Proposition 5.3. *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold. Then the following statements are equivalent:*

(i) *each structure tensor ϕ_α is projectable (locally) on the space of the leaves of $\mathcal{V}$;*
(ii) *for all $\alpha \in \{1,2,3\}$ $d\eta_\alpha = 0$ on $\mathcal{H}$;*
(iii) *the horizontal subbundle $\mathcal{H}$ is integrable;*
(iv) *M is locally a Riemannian product of a hyper-Kähler manifold and SO(3) (or SU(2)).*

Furthermore, if one of the above conditions holds, then the transverse manifold is hyper-Kählerian and the Ricci tensor of M is given by

$$\mathrm{Ric} = \frac{c^2}{2}\left(\eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3\right), \tag{5.3}$$

hence M is η-Einstein.

Proof: Each ϕ_α is projectable if and only if $(\mathcal{L}_{\xi_\beta}\phi_\alpha)X = 0$ for all $\beta \in \{1,2,3\}$ and $X \in \Gamma(\mathcal{H})$, which, by virtue of (3.11), is equivalent to the vanishing of $(\mathcal{L}_{\xi_\beta}\Phi_\alpha)(X,Y)$ for all $\beta \in \{1,2,3\}$ and $X, Y \in \Gamma(\mathcal{H})$. This proves the equivalence between (i) and (ii). Moreover, that (ii) is equivalent to (iii) is obvious. The equivalence of (iii) and (iv) is a consequence of the fact that $\mathcal{V}$ defines a Riemannian foliation with totally geodesic leaves. Now we know that the Riemannian metric g projects locally onto a Riemannian metric G on the space of leaves of $\mathcal{V}$ because each ξ_α is Killing. Moreover, by (i), the tensor fields ϕ_1, ϕ_2, ϕ_3 project to three tensor fields J_1, J_2, J_3 on M'^{4n} and it is easy to check that they satisfy the quaternionic relations. In fact (J_α, G) are Hermitian structures which are integrable because $N_\alpha = 0$. Finally, we compute the Ricci tensor of M. Using (5.5) and (4.8) one easily finds that

$\mathrm{Ric}(X,\xi_\alpha)=0$ for any $X\in\Gamma(\mathcal{H})$ and $\mathrm{Ric}(\xi_\alpha,\xi_\beta)=0$ for $\alpha\neq\beta$, whereas, by (2.3), $\mathrm{Ric}(\xi_\alpha,\xi_\alpha)=\frac{c^2}{2}$ for each $\alpha\in\{1,2,3\}$. Thus it remains to compute $\mathrm{Ric}(X,Y)$ for all $X,Y\in\Gamma(\mathcal{H})$. It is not restrictive to assume X and Y basic. Then by a well-known formula (cf. [1]) the Ricci tensor on the horizontal distribution $\mathcal{H}$ is related to the Ricci tensor on the transverse manifold by the following relation

$$\begin{aligned}\mathrm{Ric}(X,Y)=\mathrm{Ric}'(\pi_*X,\pi_*Y)+\frac{1}{2}\left(g\left(\nabla_X N,Y\right)+g\left(\nabla_Y N,X\right)\right)\\ -2\sum_{i=1}^{4n}g\left(A_X X_i,A_Y X_i\right)-\sum_{\alpha=1}^{3}g\left(T_{\xi_\alpha}X,T_{\xi_\alpha}Y\right),\end{aligned}$$

where, π is a (local) submersion defining the foliation $\mathcal{V}$, A and T are the O'Neill tensors, $\{X_i,\xi_\alpha\}$ is a local adapted orthonormal basis and $N=\sum_{\alpha=1}^{3}T_{\xi_\alpha}\xi_\alpha$. Now, the total geodesicity of $\mathcal{V}$ and the integrability of $\mathcal{H}$ yield, respectively, the vanishing of T and A. On the other hand $\mathrm{Ric}'=0$, since M' is hyper-Kählerian. Thus M is horizontally Ricci-flat and (5.3) is proven. ∎

Theorem 5.4. *Let $(M,\phi_\alpha,\xi_\alpha,\eta_\alpha,g)$ be a 3-quasi-Sasakian manifold of rank $4l+3$. Then the distribution $\mathcal{E}^{4m}$ is integrable and defines a foliation of dimension $4m$ whose leaves are hyper-Kählerian manifolds. Furthermore, the space of leaves of this foliation is 3-α-Sasakian.*

Proof: Let $X,Y\in\Gamma(\mathcal{E}^{4m})$. For each $\alpha\in\{1,2,3\}$ one has $\eta_\alpha([X,Y])=-2d\eta_\alpha(X,Y)=0$, hence $[X,Y]\in\Gamma(\mathcal{H})$. Moreover, for any $Z\in\Gamma(TM)$ and for any $\alpha\in\{1,2,3\}$,

$$\begin{aligned}d\eta_\alpha([X,Y],Z)=-3d^2\eta_\alpha(X,Y,Z)+X(d\eta_\alpha(Y,Z))+Y(d\eta_\alpha(Z,X))\\ +Z(d\eta_\alpha(X,Y))-d\eta_\alpha([Y,Z],X)-d\eta_\alpha([Z,X],Y)=0\end{aligned}$$

because $i_X d\eta_\alpha=i_Y d\eta_\alpha=0$. Thus $[X,Y]\in\Gamma(\mathcal{E}^{4m})$. In order to prove the second part of the statement, let N be a leaf of this foliation and let, for each $\alpha\in\{1,2,3\}$, J_α, Ω_α, G be the tensors on N obtained from ϕ_α, Φ_α, g by restriction. Then $(J_\alpha,\Omega_\alpha,G)$ defines an hyper-Hermitian structure on N which is integrable because its Nijenhuis tensor satisfies $[J_\alpha,J_\alpha]=[\phi_\alpha,\phi_\alpha]|_N=([\phi_\alpha,\phi_\alpha]+d\eta_\alpha\otimes\xi_\alpha)|_N=0$, since M is normal. We prove that the foliation $\mathcal{E}^{4m}$ is transversely 3-α-Sasakian. We begin observing that, for each $\alpha\in\{1,2,3\}$, the forms η_α and $d\eta_\alpha$ are projectable, since for all $V\in\Gamma(\mathcal{E}^{4m})$ we have $i_V\eta_\alpha=0$ and $i_V d\eta_\alpha=0$ by definition of $\mathcal{E}^{4m}$. Next, by Lemma 3.4, the Reeb vector fields ξ_1, ξ_2, ξ_3 are basic vector fields. More delicate is the projectability of the tensor fields ϕ_α. First note that as each 2-form $d\eta_\alpha$ is non-degenerate on $\mathcal{E}^{4l}$, it induces a musical isomorphism $(d\eta_\alpha)^\flat : X \mapsto d\eta_\alpha(X,\cdot)$ between $\mathcal{E}^{4l}$ and the 1-forms which vanish on $\mathcal{E}^{4m+3}$. We denote its inverse by $(d\eta_\alpha)^\sharp$. We prove that

$$\phi_\alpha X = -(d\eta_\beta)^\sharp (d\eta_\gamma)^\flat (X) \tag{5.4}$$

for all $X \in \Gamma(\mathcal{E}^{4l})$, where as usual (α,β,γ) is an even permutation of $\{1,2,3\}$. Indeed, let us consider $X, Y \in \Gamma(\mathcal{E}^{4l})$ such that $(d\eta_\gamma)^\flat(X) = (d\eta_\beta)^\flat(Y)$. The forms $d\eta_\gamma(X,\cdot)$ and $d\eta_\beta(-\phi_\alpha X,\cdot)$ vanish on $\Gamma(\mathcal{V})$ and coincide on $\Gamma(\mathcal{H})$ by virtue of (i) and (vi) in Proposition 2.4. It follows that $d\eta_\beta(Y,\cdot)$ coincides with $d\eta_\beta(-\phi_\alpha X,\cdot)$ and thus $Y + \phi_\alpha X \in \Gamma(\mathcal{E}^{4m})$, which implies that $Y = -\phi_\alpha X$. In order to prove that each ϕ_α is foliate with respect to the foliation $\mathcal{E}^{4m}$, it is sufficient to show that ϕ_α maps basic vector fields to basic vector fields. In view of (5.4) we prove in fact that for each $\delta \in \{1,2,3\}$ $(d\eta_\delta)^\flat$ (respectively, $(d\eta_\delta)^\sharp$) maps basic vector fields (respectively, basic 1-forms) to basic 1-forms (respectively, basic vector fields). Indeed let $X \in \Gamma(\mathcal{E}^{4l})$ be a basic vector field. Then we have immediately $i_V((d\eta_\delta)^\flat(X))) = d\eta_\delta(X,V) = 0$ for any $V \in \Gamma(\mathcal{E}^{4m})$. Next, we have to compute $i_V(d((d\eta_\delta)^\flat(X)))(Y)$ for all $Y \in \Gamma(\mathcal{E}^{4l})$. It is not restrictive to assume Y basic. Moreover for simplify the notation we put $\omega := (d\eta_\delta)^\flat(X)$. Then we have

$$\begin{aligned}
i_V(d((d\eta_\delta)^\flat(X)))(Y) &= 2d\omega(V,Y) = V(\omega(Y)) - Y(\omega(V)) - \omega([V,Y]) \\
&= V(d\eta_\delta(X,Y)) - Y(d\eta_\delta(X,V)) - d\eta_\delta(X,[V,Y]) \\
&= V(d\eta_\delta(X,Y)) - Y(d\eta_\delta(X,V)) - d\eta_\delta(X,[V,Y]) \\
&\quad - X(d\eta_\delta(V,Y)) + d\eta_\delta(Y,[V,X]) + d\eta_\delta(V,[X,Y]) \\
&= 3d^2\eta_\delta(X,Y,V) = 0,
\end{aligned}$$

for all $V \in \Gamma(\mathcal{E}^{4m})$, so that the 1-form $(d\eta_\delta)^\flat(X)$ is basic. Conversely, let ω be a basic 1-form which vanishes on $\mathcal{E}^{4m+3}$. Then we prove that, for each $\alpha \in \{1,2,3\}$, the vector field $X = (d\eta_\alpha)^\sharp(\omega)$ is basic, that is $[X,V] \in \Gamma(\mathcal{E}^{4m})$ for any $V \in \Gamma(\mathcal{E}^{4m})$. Since, by Lemma 3.3, $[X,V] \in \Gamma(\mathcal{H})$, the last condition is equivalent to require that $d\eta_\alpha([X,V],Y) = 0$ for any $Y \in \Gamma(\mathcal{E}^{4l})$. Without loss in generality we can assume Y to be a basic vector field. We have

$$\begin{aligned} d\eta_\alpha([X,V],Y) &= 3d^2\eta_\alpha(V,X,Y) - V(d\eta_\alpha(X,Y)) + d\eta_\alpha(X,[V,Y]) \\ &\quad - X(d\eta_\alpha(Y,V)) - Y(d\eta_\alpha(V,X)) + d\eta_\alpha([X,Y],V) \\ &= -V(d\eta_\alpha(X,Y)) = -V((d\eta_\alpha)^\flat(X)(Y)) = -V(\omega(Y)) \\ &= -V(\omega(Y)) + Y(\omega(V)) + \omega([V,Y]) = -2d\omega(V,Y) \\ &= -(i_V d\omega)(Y) = 0 \end{aligned}$$

since ω is basic. This proves that X is basic. Therefore by (5.4) we get the projectability of ϕ_α. Finally we show that $\mathcal{E}^{4m}$ is a Riemannian foliation, that is for any $V \in \Gamma(\mathcal{E}^{4m})$ $(\mathcal{L}_V g)|_{N(\mathcal{E}^{4m})} = 0$, where $N(\mathcal{E}^{4m}) = TM/\mathcal{E}^{4m}$ is the normal bundle of the foliation $\mathcal{E}^{4m}$ which is identified with $\mathcal{E}^{4l} \oplus \mathcal{V}$ via the Riemannian metric g. For any $V \in \Gamma(\mathcal{E}^{4m})$ and $Y, Y' \in \Gamma(\mathcal{E}^{4l})$

$$\begin{aligned} (\mathcal{L}_V g)(X,Y) &= V(g(X,Y)) - g([V,X]_{\mathcal{E}^{4l}},Y) - g(X,[V,Y]_{\mathcal{E}^{4l}}) \\ &= -V(d\eta_\alpha(X,\phi_\alpha Y)) + d\eta_\alpha([V,X],\phi_\alpha Y) + d\eta_\alpha(X,\phi_\alpha[V,Y]) \\ &= -V(d\eta_\alpha(X,\phi_\alpha Y)) + d\eta_\alpha([V,X],\phi_\alpha Y) + d\eta_\alpha(X,[V,\phi_\alpha Y]) \\ &= -(\mathcal{L}_V d\eta_\alpha)(X,\phi_\alpha Y) = 0, \end{aligned}$$

where we have used the projectability of $d\eta_\alpha$ and ϕ_α. Moreover, by Lemma 3.3 and Lemma 3.4 we get that $(\mathcal{L}_X g)(\xi_\delta, Y) = (\mathcal{L}_X g)(\xi_\delta, \xi_\rho) = 0$. Thus the situation is the following: for each $\alpha \in \{1,2,3\}$ η_α and $d\eta_\alpha$ project to a 1-form η'_α and a 2-form $\Phi'_\alpha = d\eta'_\alpha$; the vector field ξ_α projects to a vector field ξ'_α satisfying $\eta'_\alpha(\xi'_\alpha) = 1$ and $d\eta'_\alpha(\xi'_\alpha, \cdot) = 0$; the tensor field ϕ_α projects to a tensor field ϕ'_α such that $\phi'^2_\alpha = -I + \eta'_\alpha \otimes \xi'_\alpha$. Moreover the Riemannian metric g projects to a Riemannian metric g' compatible with each almost contact structure $(\phi'_\alpha, \xi'_\alpha, \eta'_\alpha)$. Then one easily checks that (2.4) hold. Finally, that this projected structure is in fact 3-α-Sasakian follows directly from Corollary 4.7. ■

Corollary 5.5. *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold. Then the curvature tensor field satisfies*

$$R_{XY}\xi_\alpha = 0, \quad R_{X\xi_\beta}\xi_\alpha = 0 \tag{5.5}$$

for any $X, Y \in \Gamma(\mathcal{E}^{4m})$ and any $\alpha, \beta \in \{1,2,3\}$.

Proof: The assertion follows by a straightforward computation using (4.8) and taking into account the integrability of the distribution $\mathcal{E}^{4m}$ and Lemma 3.4. ■

Theorem 5.6. *Every 3-quasi-Sasakian manifold of rank $4l+3$ admits a canonical transversal quaternionic-Kähler structure given by a foliation whose leaves are 3-cosymplectic manifolds.*

Proof: By the integrability of $\mathcal{E}^{4m}$ and (ii) of Proposition 2.4, it follows easily that the distribution $\mathcal{E}^{4m+3} = \mathcal{E}^{4m} \oplus \mathcal{V}$ is involutive, hence it defines a $(4m+3)$-dimensional foliation of M. Let N be a leaf of this foliation and $(\phi_\alpha^N, \xi_\alpha^N, \eta_\alpha^N, g^N)$ be the normal almost 3-contact metric structure on N obtained from M by restriction. Then since each 1-form η_α is closed on $\mathcal{E}^{4m+3}$ and $d\Phi_\alpha = 0$ we have that N is endowed with a canonical 3-cosymplectic structure. Next, that $\mathcal{E}^{4m+3}$ is a Riemannian foliation follows from the fact that each Reeb vector field is Killing and $\mathcal{L}_V g|_{\mathcal{E}^{4l}} = 0$, $V \in \Gamma(\mathcal{E}^{4m})$, (cf. Theorem 5.4). Finally, let Q be the subbundle of the endomorphism bundle $\mathrm{End}(\mathcal{E}^{4l})$ spanned by ϕ_1, ϕ_2, ϕ_3. Then, since each ϕ_α is foliate with respect to the foliation $\mathcal{E}^{4m}$ (cf. Theorem 5.4) and by (3.4), we have that the subbundle Q is projectable with respect to the foliation $\mathcal{E}^{4m+3}$. Arguing as in [8] one can prove that the space of leaves is in fact quaternionic-Kähler. ∎

Theorem 5.7. *Every 3-quasi-Sasakian manifold of rank $4l+3$ admits a canonical transversal hyper-Kähler structure given by a foliation whose leaves are 3-α-Sasakian manifolds.*

Proof: We first prove that the distribution $\mathcal{E}^{4l+3}$ is integrable and defines a Riemannian foliation of the 3-quasi-Sasakian manifold $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$. Let $Y, Y' \in \Gamma(\mathcal{E}^{4l})$. Then for any $X \in \Gamma(\mathcal{E}^{4m})$ we have

$$\begin{aligned}
0 &= 3d\Phi_\alpha(X, Y, Y') \\
&= X(\Phi_\alpha(Y, Y')) + Y(\Phi_\alpha(Y', X)) + Y'(\Phi_\alpha(X, Y)) - \Phi_\alpha([X, Y], Y') \\
&\quad - \Phi_\alpha([Y, Y'], X) - \Phi_\alpha([Y', X], Y) \\
&= (\mathcal{L}_X \Phi_\alpha)(Y, Y') - \Phi_\alpha([Y, Y'], X) \\
&= (\mathcal{L}_X g)(Y, \phi_\alpha Y') + g(Y, (\mathcal{L}_X \phi_\alpha)Y') - \Phi_\alpha([Y, Y'], X) \\
&= -g([Y, Y'], \phi_\alpha X),
\end{aligned}$$

where we have used the projectability of the metric g and of the tensor field ϕ_α with respect to the foliation $\mathcal{E}^{4m}$, proved in Theorem 5.4. It follows that $[Y, Y']$ is orthogonal to $\mathcal{E}^{4m}$ and hence belongs to $\mathcal{E}^{4l+3}$. Moreover, by Theorem 2.2 and (ii) of Proposition 2.4 we have also $[Y, \xi_\alpha] \in \Gamma(\mathcal{E}^{4l}) \subset \Gamma(\mathcal{E}^{4l+3})$ and $[\xi_\alpha, \xi_\beta] \in \Gamma(\mathcal{V}) \subset \Gamma(\mathcal{E}^{4l+3})$ for all $Y \in \Gamma(\mathcal{E}^{4l})$ and $\alpha, \beta \in \{1, 2, 3\}$.

Thus $\mathcal{E}^{4l+3}$ is integrable and it is easy to check, using (4.7), that the normal almost 3-contact metric structure induced from $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ on each leaf of $\mathcal{E}^{4l+3}$ is in fact 3-α-Sasakian. Now we pass to study the space of leaves of $\mathcal{E}^{4l+3}$. We prove that $\mathcal{E}^{4l+3}$ is a Riemannian foliation and that the tensor fields ϕ_1, ϕ_2, ϕ_3 locally project, together with g, to a hyper-Kähler structure on the space of leaves. For all $Y \in \Gamma(\mathcal{E}^{4l+3})$ and $X, X' \in \Gamma(\mathcal{E}^{4m})$ we have

$$\begin{aligned}(\mathcal{L}_Y\Phi_\alpha)(X, X') &= Y(\Phi_\alpha(X, X')) - \Phi_\alpha([Y, X], X') - \Phi_\alpha(X, [Y, X'])\\ &= X(\Phi_\alpha(X', Y)) + X'(\Phi_\alpha(Y, X)) + Y(\Phi_\alpha(X, X'))\\ &\quad - \Phi_\alpha([X, X'], Y) - \Phi_\alpha([X', Y], X) - \Phi_\alpha([Y, X], X')\\ &= 3d\Phi_\alpha(X, X', Y) = 0,\end{aligned}$$

thus each fundamental 2-form Φ_α projects to a 2-form Ω'_α which is closed since Φ_α is. Next we prove that also each tensor field ϕ_α is foliate, that is it maps basic vector fields to basic vector fields. By virtue of [6, Lemma 4.1] we have that, for an even permutation (α, β, γ) of $\{1, 2, 3\}$,

$$\phi_\alpha X = -(\Phi_\beta)^\sharp(\Phi_\gamma)^\flat X \tag{5.6}$$

for all $X \in \Gamma(\mathcal{E}^{4m})$, where, for each $\delta \in \{1, 2, 3\}$, $(\Phi_\delta)^\flat : X \mapsto \Phi_\delta(X, \cdot)$ is the musical isomorphism induced from Φ_δ between $\mathcal{E}^{4m}$ and the 1-forms which vanish on $\mathcal{E}^{4l+3}$, and $(\Phi_\delta)^\sharp$ denotes its inverse. Therefore, in order to prove that ϕ_α is foliate it is sufficient to check that, for each $\delta \in \{1, 2, 3\}$, $(\Phi_\delta)^\flat$ maps basic vector fields to basic 1-forms and, conversely, $(\Phi_\delta)^\sharp$ maps basic 1-forms to basic vector fields. Let $X \in \Gamma(\mathcal{E}^{4m})$ be a basic vector field. We have to show that the 1-form $\omega := (\Phi_\delta)^\flat X$ is basic, i.e. satisfies $i_Y\omega = i_Y d\omega = 0$ for all $Y \in \Gamma(\mathcal{E}^{4l+3})$. Indeed we have $i_Y\omega = \omega(Y) = \Phi_\delta(X, Y) = g(X, \phi_\delta Y) = 0$ since $\phi_\delta(\mathcal{E}^{4l+3}) \subset \mathcal{E}^{4l+3}$. Next, one has $i_Y d\omega(X') = 2d\omega(Y, X') = Y(\omega(X')) - X'(\omega(Y)) - \omega([Y, X']) = (\mathcal{L}_Y\Phi_\delta)(X, X') = 0$ for any $X' \in \Gamma(\mathcal{E}^{4m})$ (which is not restrictive to assume basic). Conversely, for any basic 1-form ω we have to show that the vector field $X := (\Phi_\delta)^\sharp(\omega)$ is basic, that is $[X, Y] \in \Gamma(\mathcal{E}^{4l+3})$ for any $Y \in \Gamma(\mathcal{E}^{4l+3})$. This last condition is equivalent to require that $\Phi_\delta([X, Y], X') = 0$ for any $X' \in \Gamma(\mathcal{E}^{4m})$. It is not restrictive to assume

X' basic. Then we have

$$\begin{aligned}\Phi_\delta([X,Y],X') &= -3d^2\Phi_\delta(X,Y,X') + X(\Phi_\delta(Y,X')) + Y(\Phi_\delta(X',X)) \\ &\quad + X'(\Phi_\delta(X,Y)) - \Phi_\delta([Y,X'],X) - \Phi_\delta([X',X],Y) \\ &= -V(\Phi_\delta(X,X')) \\ &= i_Y d\omega(X') = 0.\end{aligned}$$

It remains to prove that the Riemannian metric g is bundle-like. This follows easily from the projectability of Φ_α and ϕ_α. Indeed for any $Y \in \Gamma(\mathcal{E}^{4l})$ and $X, X' \in \Gamma(\mathcal{E}^{4m})$ we have

$$(\mathcal{L}_Y g)(X,X') = -(\mathcal{L}_Y \Phi_\alpha)(X, \phi_\alpha X') + g(X, (\mathcal{L}_Y \phi_\alpha)\phi_\alpha X') = 0,$$

whereas $(\mathcal{L}_{\xi_\alpha} g)(X,X') = 0$ since ξ_α is Killing. We denote by J'_α and g' the tensor field and the Riemannian metric induced on the space of leaves by each ϕ_α and by the Riemannian metric g. Then a straightforward computation yields that $(J'_\alpha, \Omega'_\alpha, g')$ is an almost hyper-Hermitian structure. Thus, the closedness of Ω'_1, Ω'_2, Ω'_3 imply, by the Hitchin Lemma ([10]), that $(J'_\alpha, \Omega'_\alpha, g')$ is in fact hyper-Kähler. ■

Corollary 5.8. *Let $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold of rank $4l+3$ with $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$, $c \neq 0$. Then M^{4n+3} is locally the Riemannian product of a 3-α-Sasakian manifold M^{4l+3}, where $\alpha = \frac{c}{2}$, and a hyper-Kählerian manifold M^{4m}, with $m = n - l$.*

Proof: The tangent bundle of M^{4n+3} splits up as the orthogonal sum of the Riemannian foliations $\mathcal{E}^{4l+3}$ and $\mathcal{E}^{4m}$. Because of the duality Riemannian-totally geodesic, $\mathcal{E}^{4l+3}$ and $\mathcal{E}^{4m}$ are also totally geodesic foliations. It follows that M^{4n+3} is the Riemannian product of a leaf M^{4l+3} of $\mathcal{E}^{4l+3}$ and a leaf M^{4m} of $\mathcal{E}^{4m}$. Tacking into account that ψ_α and ϕ_α agree on $\mathcal{E}^{4l+3}$ and applying (4.7), we have that $(\psi_\alpha, \xi_\alpha, \eta_\alpha, g)|_{\mathcal{E}^{4l+3}}$ is an almost 3-α-Sasakian structure over M^{4l+3}, where we have put $\alpha = \frac{c}{2}$. Hence, by Proposition 4.2, it is 3-α-Sasakian. Since θ_α agrees with ϕ_α on $\mathcal{E}^{4m}$, the maps $\theta_\alpha|_{\mathcal{E}^{4m}}$ define a quaternionic structure which is compatible with the metric $g|_{\mathcal{E}^{4m}}$. Finally, define the 2-forms Θ_α by $\Theta_\alpha(X,Y) = g(X, \theta_\alpha Y)$ for any $X, Y \in \Gamma(\mathcal{E}^{4m})$. We have $\Theta_\alpha = \Phi_\alpha|_{\mathcal{E}^{4m}}$ and hence $d\Theta_\alpha = 0$. By virtue of the mentioned Hitchin Lemma ([10]) the structure defined on M^{4m} turns out to be hyper-Kählerian. ■

Remark 5.9. Note that Corollary 5.8 strongly improves, both in the assumptions and in the results, the splitting theorem [5, Theorem 5.6]. It should be also emphasized that an analogous result does not hold for a quasi-Sasakian manifold.

A consequence of Corollary 5.8 is an improving of Theorem 5.2. Namely, under the assumption of regularity for the foliation $\mathcal{V}$, the space of leaves $M/\mathcal{V}$ is an almost quaternionic-Hermitian manifold which is the local Riemannian product of a quaternionic-Kähler manifold and a hyper-Kähler manifold.

Now using Corollary 5.8 we can compute the complete expression of the Ricci tensor in any 3-quasi-Sasakian manifold. Before, we prove the following preliminary result.

Proposition 5.10. *Every* 3-α-*Sasakian manifold of dimension* $4n+3$ *is an Einstein manifold with Einstein constant* $2\alpha^2(2n+1)$.

Proof: Let $(M,\phi_\delta,\xi_\delta,\eta_\delta,g)$, $\delta\in\{1,2,3\}$, be a 3-α-Sasakian manifold. Then by virtue of (4.2) and [17, Proposition 4.4] we have that $(\phi_\delta,\xi_\delta,\eta_\delta,g)$ can be obtained by a homothetic deformation of a 3-Sasakian structure $(\bar\phi_\delta,\bar\xi_\delta,\bar\eta_\delta,\bar g)$ given by

$$\bar\phi_\delta=\phi_\delta,\ \bar\xi_\delta=\frac{1}{\alpha}\xi_\delta,\ \bar\eta_\delta=\alpha\eta_\delta,\ \bar g=\alpha^2 g.$$

Then, since it is well known that any 3-Sasakian manifold is Einstein, we conclude that also the metric g is Einstein. For computing the Einstein constant λ we use (2.3) and (4.2) getting

$$\lambda=\lambda g(\xi_\delta,\xi_\delta)=\mathrm{Ric}(\xi_\delta,\xi_\delta)=\|\nabla\xi_\delta\|^2=2\alpha^2(2n+1).$$

■

Theorem 5.11. *In any* 3-*quasi-Sasakian manifold of dimension* $4n+3$ *the Ricci tensor is given by*

$$\mathrm{Ric}(X,Y)=\begin{cases}\frac{c^2}{2}(2n+1)g(X,Y), & \textit{if } X,Y\in\Gamma(\mathcal{E}^{4l+3});\\ 0, & \textit{elsewere.}\end{cases}\tag{5.7}$$

Proof: If $c=0$ then the manifold is 3-cosymplectic and hence Ricci-flat, so we can assume that $c\neq0$ and the 3-quasi-Sasakian manifold $(M,\phi_\alpha,\xi_\alpha,\eta_\alpha,g)$ in question has rank $4l+3$. Then, in view of Corollary 5.8, M is locally the Riemannian product of a $(4l+3)$-dimensional 3-α-Sasakian manifold M', with $\alpha=\frac{c}{2}$, and of a $4m$-dimensional hyper-Kähler manifold M''. Thus, the

Ricci tensor of M is nothing that the sum of the Ricci tensors of M' and M''. So, because of Proposition 5.10 and of the Ricci-flatness of M'', we get the assertion. ∎

Remark 5.12. For a 3-quasi-Sasakian manifold of rank $4l+3$ the expression of the Ricci tensor (5.7) can be written also in the more concise form

$$\mathrm{Ric}(X,Y) = \frac{c^2}{2}(2n+1)g(\pi X, Y), \tag{5.8}$$

where $\pi = -\psi_\alpha^2 + \eta_\alpha \otimes \xi_\alpha$ (independent of $\alpha \in \{1,2,3\}$) is the projection onto $\mathcal{E}^{4l+3}$.

Corollary 5.13. *Every* 3*-quasi-Sasakian manifold has non negative scalar curvature given by* $\frac{c^2}{2}(2n+1)(4l+3)$.

Corollary 5.14. *No* 3*-quasi-Sasakian manifold is* η*-Einstein unless the following cases:*

(i) 3-α*-Sasakian manifolds, which are Einstein with strictly positive scalar curvature;*
(ii) 3*-cosymplectic manifolds, which are Ricci-flat;*
(iii) 3*-quasi-Sasakian manifolds with integrable horizontal distribution (cf. Proposition 5.3), which are* η*-Einstein non-Einstein.*

Remark 5.15. Applying the Pasternack's refinement [18] of the classical Bott vanishing theorem to the Riemannian foliations $\mathcal{V}$, $\mathcal{E}^{4m}$, $\mathcal{E}^{4m+3}$ and $\mathcal{E}^{4l+3}$, considered in Theorem 5.2, Theorem 5.4, Theorem 5.6 and Theorem 5.7, respectively, we get the following topological obstructions to the existence of a 3-quasi-Sasakian structure of rank $4l+3$ on a manifold M of dimension $4n+3$: $\mathrm{Pont}^j(\mathcal{H}) = 0$ for all $j > 4n$, $\mathrm{Pont}^j(\mathcal{E}^{4l+3}) = 0$ for all $j > 4m$, $\mathrm{Pont}^j(\mathcal{E}^{4l}) = 0$ for all $j > 4m+3$ and $\mathrm{Pont}^j(\mathcal{E}^{4m}) = 0$ for all $j > 4l+3$, where $\mathrm{Pont}(\mathcal{H})$, $\mathrm{Pont}(\mathcal{E}^{4l+3}) = 0$, $\mathrm{Pont}(\mathcal{E}^{4l})$ and $\mathrm{Pont}(\mathcal{E}^{4m})$ denote, respectively, the Pontryagin algebras of the subbundles $\mathcal{H}$, $\mathcal{E}^{4l+3}$, $\mathcal{E}^{4l}$ and $\mathcal{E}^{4m}$ of the tangent bundle of M. Furthermore, the vanishing of these primary charactestic classes permits also the construction of some secondary characteristic classes as it is done in [16] and [19].

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