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BOOLEAN FULL KRIPKE STRUCTURES ARE *ALG-UNIVERSAL*

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ABSTRACT: Every group is isomorphic to the automorphism group of a Kripke structure with Boolean part equal to a power set Boolean algebra. More generally, we prove that the category of Kripke structures with Boolean part equal to a power set Boolean algebra and morphisms with complete Boolean part is *alg-universal*, which means that it contains any category of universal algebras as a full subcategory.

KEYWORDS: Kripke structures, dynamic algebras, Boolean algebras with operators, automorphism groups, algebraic universality.
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1. Introduction

Every group is isomorphic to the automorphism group of a distributive lattice. This is a result of G. Birkhoff [1]. On the other hand, R. McKenzie and J. D. Monk stated that $\mathbb{Z}_3$, the cyclic group of order 3, is an automorphism group of no Boolean algebra [15]. Nevertheless, the author proved [16] that separable Kripke structures (which are heterogeneous algebras with a Boolean part, and special cases of dynamic algebras) are *alg-universal*. In particular, every group is the automorphism group of a separable Kripke structure.

Since every automorphism of Boolean algebras is a complete endomorphism, complete Boolean algebras (with complete homomorphisms) fail to represent all groups, more generally, they are not *alg-universal*. Nevertheless, here we are going to show that

Main Theorem. *The category BfKri (whose objects are the Kripke structures with Boolean part equal to a power set Boolean algebra and whose morphisms are the Kripke structure homomorphisms with complete Boolean part) is alg-universal, which means that it contains any category of universal algebras as a full subcategory.*

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Since Kripke structures are heterogeneous algebras with two underlying sets, their homomorphisms are pairs of mappings, being one of them a homomorphism of Boolean algebras. So, somehow the heterogeneous character of the Kripke structures (with Boolean part of the form $exp(Q)$) acting in the Boolean part of the automorphisms allows us to choose the right Boolean automorphisms in order to achieve the desired representations.

Some consequences of this theorem are mentioned in [16] (they are common to every *alg-universal* category).

The paper is organized as follows. In section 2 we recall some concepts related to dynamic algebras, including the relevant example of Kripke structures. In section 3 we recall some representation problems concerning universality. In section 4 we refer an almost full embedding from which the main result will be proved. In section 5 we recall the construction which produces the *alg-universality* of separable Kripke structures. In sections 6 and 7 we prove the Main Theorem.

2. On Kripke structures and dynamic algebras

Dynamic algebras were introduced by Kozen [10] and Pratt [17] to provide models of Propositional Dynamic Logic (PDL). For relation to Computer Science and examples see [17].

Following [17] a *dynamic algebra* is a two-sorted algebra $(\mathcal{B}, \mathcal{A}, \langle\rangle)$ where

$$\mathcal{B} = (B, \vee, \sim, 0) \ \text{ and } \ \mathcal{A} = (A, ;, \cup, *)$$

are one-sorted algebras with

$0:$ 0-ary operation (constant),

$\sim, *:$ 1-ary operations,

$\vee, ;, \cup:$ 2-ary operations

and $\langle\cdot,\cdot\rangle : A \times B \longrightarrow B$ mixed operation (named *diamond*)

satisfying the following conditions (where ";" is omitted, for brevity):

(i) $\mathcal{B}$ is a Boolean algebra;

(ii) $\langle a, 0\rangle = 0$ and $\langle a, p \vee q\rangle = \langle a, p\rangle \vee \langle a, q\rangle$;

(iii) $\langle a \cup b, p\rangle = \langle a, p\rangle \vee \langle b, p\rangle$;

(iv) $\langle ab, p\rangle = \langle a, \langle b, p\rangle\rangle$;

(v) $p \vee \langle aa^*, p\rangle \leq \langle a^*, p\rangle \leq p \vee \langle a^*, \sim p \wedge \langle a, p\rangle\rangle$

for $a, b \in A$ and $p, q \in B$.

Obs.: $(p \leq q)$ and $(p \wedge q)$ are abbreviations of $(p \wedge q = p)$ and $(\sim(\sim p \vee \sim q))$, *resp.*

Homomorphisms of dynamic algebras $h : (\mathcal{B}, \mathcal{A}, \langle\rangle) \longrightarrow (\mathcal{B}', \mathcal{A}', \langle\rangle)$ are defined in the usual way, as homomorphisms of two-sorted algebras, *i.e.*, $h = (h_1, h_2)$ with $h_1 : \mathcal{B} \longrightarrow \mathcal{B}'$, $h_2 : \mathcal{A} \longrightarrow \mathcal{A}'$ such that h_1 and h_2 preserve the mentioned one-sorted operations and, moreover, h preserves the mixed operation, that is, $h_1(\langle a, p\rangle) = \langle h_2(a), h_1(p)\rangle$ for every $a \in \mathcal{A}$ and every $p \in \mathcal{B}$.

Let DA denote the category of all dynamic algebras (whenever we consider a category mentioning only its objects, we assume that its morphisms are all the homomorphisms among those objects).

A dynamic algebra $(\mathcal{B}, \mathcal{A}, \langle\rangle)$ is said to be *separable* if for all $a, b \in \mathcal{A}$,

$$\langle a, p\rangle = \langle b, p\rangle \text{ for every } p \in \mathcal{B} \implies a = b.$$

In a dynamic algebra $(\mathcal{B}, \mathcal{A}, \langle\rangle)$ each action induces an operator

$$\langle a\rangle : \mathcal{B} \longrightarrow \mathcal{B}$$

by means of $\langle a\rangle(p) = \langle a, p\rangle$ $(p \in \mathcal{B})$. In separable dynamic algebras different actions induce different operators. So, a separable dynamic algebra can be seen as a Boolean algebra with operators [17].

Kripke structures, the traditional models of PDL, were presented in [17] as examples of dynamic algebras. They are defined as follows. The *full Kripke structure* on a given non-empty set S is the pair $(exp(S), exp(S\times S))$ where $exp(S)$ is the Boolean algebra of all subsets of S (with the usual set theoretical operations) and $exp(S\times S)$ is the set of all binary relations on S. The operations ;, $\cup$ and $*$ on $exp(S\times S)$ are the composition, the union and the reflective-transitive closure of binary relations, respectively. The reflective-transitive closure of $a \in exp(S\times S)$ is defined,as usually, by

$$a^* = \bigcup_{n=0}^{\infty} a^n,$$

where a^0 is the identity on S, and a^n $(n > 0)$ is the composition $a; a; \ldots; a$, n times. The diamond operation $\langle a, p\rangle$ (for $a \in exp(S\times S)$ and $p \in exp(S)$) is defined to be the pre-image of p under a,

$$\{s \in S : (s, s') \in a, \text{ for some } s' \in p\}.$$

Informally, the set S can be interpreted as the set of states of a computer, the subsets of S can be interpreted as propositions, the binary relations as computer programs. Then, "$s \in p$" can be interpreted as "*state* s *satisfies* *proposition p*", $(s, s') \in a$ as "*program a may run from initial state s to final state s'*", $a; b$ as "*execute program a, then program b*", $a \cup b$ as "*execute program a or program b non-deterministically*", a^* as "*execute program a zero or more times*", $\langle a, p \rangle$ as "*the proposition satisfied when a is executed and stops in a state satisfying p*".

Kripke structures are defined as the dynamic subalgebras of full Kripke structures. They intend to reflect the input-output behaviour of computer programs.

We are interested in a special case of Kripke structures. Those Kripke structures whose Boolean part is a power set Boolean algebra $exp(S)$. We will designate them by *Boolean full Kripke structures* (this class includes the full Kripke structures). Boolean full Kripke structures are separable dynamic algebras, though Kripke structures are not necessarily separable. Let us denote by *SKri* the full subcategory of *DA* whose objects are the separable Kripke structures and by *BfKri* the category whose objects are the Boolean full Kripke structures and whose morphisms are the morphisms of dynamic algebras with complete Boolean part.

The simplest examples of dynamic algebras are the so called Boolean-trivial dynamic algebras $(\mathcal{B}, \mathcal{A}, \langle\rangle)$ with $\mathcal{B} = \{0 = 1\}$, $\mathcal{A} = (A, ;, \cup, *)$ of the required type and $\langle a, 0 \rangle = 0$. They are, obviously, non-separable.

Denote by *FKri* the class of all finite full Kripke structures and by T the class of all Boolean-trivial dynamic algebras. Allow us to reuse the symbol *DA* to denote the variety of dynamic algebras.

The importance of the above two types of examples was shown by Pratt [17] in the establishment of the Theorem,

$$DA = \mathrm{HSP}(FKri \cup T),$$

where H, S and P stand for the closure under homomorphic images, subalgebras and products, respectively. Thus, *DA* is the smallest variety containing *FKri* and determined by a set of Boolean equations only (*i.e.*, equations on Boolean sort), namely, the set of Boolean equations satisfied in *FKri* .

More about dynamic algebras and Kripke structures can be found in [8], [9], [11], [12].

3. On algebraic universality

3.1. Representation problems. The representation problem for groups mentioned in the abstract may be put in terms of representation of categories. Given a group G define a one object category $\mathcal{C}$ whose set of endomorphisms is G, where the composition is defined according to the group operation (all the endomorphisms become automorphisms). Given a category $\mathcal{U}$, the question is, if there is a full embedding* of the category $\mathcal{C}$ into the category $\mathcal{U}$. If $\mathcal{U}$ is the category of Boolean algebras the answer is no. If it is the category of symmetric graphs, distributive lattices, topological spaces, separable Kripke structures, the answer is positive (*c.f.* [3],[1],[19],[4],[16]).

The similar question about the representation of monoids, "*Given a category* $\mathcal{U}$*, is every monoid* M *isomorphic to the monoid of all endomorphisms of some object of* $\mathcal{U}$?", is also a problem a representation of categories. We define the category $\mathcal{C}$ to be a one object category with endomorphism set equal to M and, like in the anterior case, composition defined according to the monoid operation. If we consider $\mathcal{U}$ to be the category of binary relations, the category of semigroups ot the category of topological spaces and open continuous mappings, the answer to the question if $\mathcal{C}$ is fully embbedable into $\mathcal{U}$ is positive.

This problem is generalized when we represent algebraic categories. A category is said to be *algebraic* if it is isomorphic to a category of some algebras of a given type and all their homomorphisms. A category is *algebraically universal* (briefly, *alg-universal*) if every algebraic category is fully embeddable into it. The category $Alg(\Delta)$ of all algebras of a given type Δ is alg-universal *iff* the sum of the arities of the operations of type Δ is not less than 2. The category *Graph* of all graphs is also alg-universal. Moreover, this category provides a criterion to decide about the alg-universality of any category, since

A category is alg-universal iff the category Graph fully embeds into it.

Under the set-theoretical assumption that there are not too many measurable cardinals, more precisely, that *"There exists a cardinal* α *such that every* α*-additive two-valued measure is trivial"* every concretizable category is algebraic (*cf.* [5], [13]). A category is *concretizable* if it admits a faithful functor

*We recall that a category $\mathcal{A}$ is fully embeddable into a category $\mathcal{B}$ if there is a full one-to-one functor $F : \mathcal{A} \to \mathcal{B}$, that is, the category $\mathcal{B}$ contains a full subcategory isomorphic to $\mathcal{A}$.

into the category *Set* (the category of all sets and all mappings). In [14] it was proved that concretizability coincides with algebraicity *iff* the last condition happens (*e.g.* the category of compact Hausdorff spaces and continuous mappings is non-algebraic under the negation of that condition). Therefore, under the above condition every alg-universal category also contains any concretizable category as a full subcategory.

The above results are contributions of Kučera, Hedrlín, Pultr and Trnková. A full account on alg-universality can be found in the Pultr and Trnková's monograph [18].

The category of dynamic algebras is, trivially, alg-universal since its full subcategory whose objects are the Boolean-trivial dynamic algebras is isomorphic to the category $Alg(2, 2, 1)$.

3.2. Remarks on the definition of algebraic category. The concept of algebraic category was introduced by Isbell in [7], where homogeneous algebras, that is, algebras with one underlying set only, were understood. Actually, the algebraicity doesn't need to be described by (full) operations, since each $Rel(\Delta)$, the category of all relational systems of type Δ and all their homomorphisms, is algebraic as shown by Hedrlín and Pultr in [6].

Here, we deal with dynamic algebras, which are heterogeneous algebras, that is, algebras with possibly many underlying sets (*c.f.* [21]). However, categories of heterogeneous algebras and all their homomorphisms are algebraic, too. It is enough to see each heterogeneous algebra $\mathcal{A}$ as relational system defined on the set-theoretical disjoint union of the underlying sets of $\mathcal{A}$ with a relation corresponding to each operation of A (the relation which is the operation seen as a relation) plus a unary relation per each, and equal to each one of the underlying sets of $\mathcal{A}$. The unary relations are added to choose the right relation homomorphisms.

4. Preliminary construction

Let $UndGraph_0$ denote the category of connected undirected graphs without loops with more than one vertex and all their homomorphisms. It is known that $UndGraph_0$ is alg-universal [18]. Therefore, to prove the alg-universality of some category $\mathcal{C}$, it is enough to full embed $UndGraph_0$ into $\mathcal{C}$. In our constructions, this full embedding will be a composition of two contravariant embeddings, say $\phi\circ\psi$ for $SKri$ and $\gamma\circ\psi$ for $BfKri$. The domain of the second embedding, ϕ (or γ), is restricted to the image $\psi(UndGraph_0^{op})$. So, we will recall the description of this category. This is the aim of this section (we follow the monograph [18]). The embedding ϕ is constructed in section 5 and the embedding γ is constructed in section 6. We preserve the notation used in this section in the forthcoming sections.

By $Comp$ we denote the category of compact Hausdorff spaces and continuous mappings. $Comp$ is not alg-universal, although its dual is almost alg-universal, that is, the required embedding is full, up to the constant morphisms, more exactly

Theorem 4.1 ([18]). *There exists an almost full embedding*

$$\psi : UndGraph_0^{op} \longrightarrow Comp.$$

Thus, by definition, we have

(i) ψ is one-to-one on objects and on morphisms;
(ii) For any morphism f of$UndGraph_0$, $\psi(f)$ is a non-constant continuous mapping;
(iii) For every pair of graphs (X, R), (X', R') belonging to $UndGraph_0$ and every non-constant morphism $g : \psi(X', R') \to \psi(X, R)$ in $Comp$ there exists a morphism $f : (X, R) \to (X', R')$ in $UndGraph_0$, such that $\psi(f) = g$.

In this case, $Comp$ is said to be dual to an almost alg-universal category. In order to describe the category $\psi(UndGraph^{op})$, we recall some definitions.

We define *continua* as connected compact Hausdorff spaces with more than one point. Such spaces are, consequently, infinite sets. A *Cook* continuum is a continuum D such that for any subcontinuum S of D, each continuous mapping $f : S \to D$ is either a constant mapping or the inclusion. Continua with this property were given the name Cook continuum, since it was H. Cook [2] who firstly constructed such a continuum. Each continuum has a countable pairwise disjoint system of its subcontinua. See the Appendix A of [18] and [20] for details.

We recall that a topological space D is said *rigid* if each continuous mapping $f : D \to D$ is either a constant mapping or the identity. The following property is satisfied:

(1)*Let X, X' be sets and D a rigid Hausdorff space. Denote by D^X the product space (i.e., with the topology of the pointwise convergence). Let a and b be distinct elements of D. Denote by c_a (resp. c'_a) the element of D^X (resp. $D^{X'}$) constantly equal to a. Consider, similarly, c_b and c'_b. Let $g : D^{X'} \to D^X$ be a continuous mapping such that $g(c'_a) = c_a$ and $g(c'_b) = g(c_b)$. Then, there exists a mapping $f : X \to X'$ such that $g(\alpha) = \alpha \circ f$ for all $\alpha \in D^{X'}$.*

Given a graph (X, R) we say that $Y \subseteq X$ is an *independent set* of (X, R) if no two vertices of Y are joined by an edge, *i.e.*, if $\{x, y\} \subseteq Y \Longrightarrow \{x, y\} \notin R$. A *characteristic* mapping $h_Y : X \longrightarrow \{0, 1\}$ of a set $Y \subseteq X$ is defined to be $h_Y(y) = 1$ for $y \in Y$ and $h_Y(y) = 0$ otherwise.

Now, we proceed with the construction of ψ, which is done after several steps, as follows:

- Choose A, B, C, H pairwise disjoint subcontinua of a *Cook continuum.* Thus, between two distinct continua of the system A, B, C, H no other continuous mappings are allowed than the constant ones. Moreover, each one of those spaces is rigid.
- Choose distinct elements $a_0, a_1 \in A$, $b_0, b_1 \in B$, $c_0, c_1 \in C$, $0, 1, a, b \in H$.
- Given a graph $(X, R) \in$ *UndGraph*$_0$ a compact Hausdorff space $\psi(X, R)$ is constructed considering:
  - the product space H^X;
  - c_a and c_b the elements of H^X constantly equal, respectively, to a and to b;
  - the subspace $\chi_R \subseteq \{0, 1\}^X \subseteq H^X$ consisting of the characteristic mappings of all independent sets of (X, R). The set χ_R is a closed subset of H^X ([18, VI.16.9]);
  - the following identifications in the topological sum

$$S = A \vee B \vee H^X \vee (\chi_R \times C)$$

$$\begin{array}{lll} a_0 & \sim & c_a, \\ b_0 & \sim & c_b, \\ h_Y & \sim & (h_Y, c_0) \text{ for all } h_Y \in \chi_R, \\ a_1 & \sim & b_1 \sim (h_Y, c_1) \text{ for all } h_Y \in \chi_R. \end{array}$$

- Define $\psi(X, R)$ as the quotient space $S/_\sim$. Since S is a finite sum of compact spaces, S is still compact. Then, $\psi(X, R)$ is compact and it is easily seen that it is Hausdorff, too.

To simplify the notation, allow us to suppose that A, B, H^X, $\chi_R \times C$ are subsets of $\psi(X,R)$ and $a_0 = c_a$, …, $a_1 = b_1 = (h_Y, c_1)$.

Let $f : (X,R) \longrightarrow (X', R')$ be a morphism in *UndGraph*$_0$,

$$\psi(f) = g : \psi(X', R') \longrightarrow \psi(X,R)$$

is defined as follows,

$$\begin{aligned} &g(z) = z \text{ for } z \in A \cup B, \\ &g(\alpha) = \alpha \circ f \text{ for } \alpha \in H^{X'}, \\ &g(h_Y, z) = (h_Y \circ f, z) \text{ for } (h_Y, z) \in \chi_{R'} \times C. \end{aligned}$$

The mapping g is correctly defined, since it preserves the identifications made in the definition of the space $\psi(X,R)$. It is straightforward that g is continuous. Clearly, ψ is a one-to-one functor. A detailed proof of the fullness can be found in [18, VI.16.8-VI.16.14]. For the sake of clarity, we present a sketch of that proof.

Let

$$g : \psi(X', R') \longrightarrow \psi(X,R)$$

be a non-constant continuous mapping. According to the following properties:

(2) *Let L be one of the spaces H, A, B, C. Let $d : L \longrightarrow \psi(X,R)$ be a non-constant continuous mapping. Then, either $L \in \{A, B\}$ and $d(z) = z$ for all $z \in L$, or $L = C$ and there exists precisely one independent set Y such that $d(z) = (h_Y, z)$ for all $z \in L$, or $L = H$ and $g(L) \subseteq H^X$;*

(3) *Let $d : H^{X'} \to \psi(X,R)$ be a non-constant continuous mapping. Then $d(H^{X'}) \subseteq H^X$;*

the mapping g cannot be constant neither on $A, B, H^{X'}$ nor $\{h_Y\} \times C$ for no $h_Y \in \chi_{R'}$. Thus, by (2) and (3) again $g(H^{X'}) \subseteq H^X$ and $g(z) = z$ for all $z \in A \cup B$. Therefore $g(c'_a) = c_a$ and $g(c'_b) = c_b$. By property (1) there exists a mapping $f : X \to X'$ such that $g(\alpha) = \alpha \circ f$ for all $\alpha \in H^{X'}$. Then, $g(h_Y, c_1) = (h_Y \circ f, c_1)$ for every $h_Y \in \chi_{R'}$, which means that f is a homomorphism of undirected graphs. Again, by property (2) we have $g(h_Y, z) = (h_Y \circ f, z)$ for all $h_Y \in \chi_{R'}$, $z \in C$. We conclude $\psi(f) = g$. □

Before proceeding we will fix some notation. By use of "*cl*" and "*b*" we denote, respectively, the closure and the boundary taken in the space which will index those keywords. Whenever the index space is Q, we won't use the index if there is no ambiguity.

5. Construction of the full embedding ϕ

$$\phi : \psi(\mathit{UndGraph}_{\mathbf{0}}^{op}) \longrightarrow \boldsymbol{SKri}^{op}$$

The construction of ϕ was presented in [16], where the proofs are made. It needs the previous consideration of some elements before the definition of the image of each object.

Let (X, R) be a graph of $\mathit{UndGraph}_0$. We denoted $Q = \psi(X, R)$ (we denoted also by Q the underlying set of $\psi(X, R)$). In order to define $\phi(Q)$ we were interested in a family of subsets of Q stable under pre-images under non-constant continuous mappings. For that purpose, we chose a particular family $\mathcal{T}$ of regularly open subsets of Q and proved that it is a basis of Q. We recall that an open subset is said regularly open (r.o. for brevity) if it is equal to the interior of its closure. The basis $\mathcal{T}$ determines the Boolean part and also the action part of $\phi(Q)$ and was refined in a way that, concerning the required fullness for ϕ, the Boolean part contributes to the definition of a convenient mapping and the action part forces the continuity of that mapping. We chose y_1, y_2 distinct points of $A\backslash\{a_0, a_1\}$ to be added to the structure of $\phi(Q)$ in a manner that the continuous mapping defined was not a constant one.

We divided the construction of $\mathcal{T}$ into 8 parts to describe the specific situation around the points $a_0, a_1, \ldots, h_Y$ (for $h_Y \in \chi_R$) in Q. That is, $\mathcal{T}$ was defined to be the union

$$\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_8,$$

where

$$\mathcal{T}_1 = \Big\{U_A \subseteq Q : U_A \text{ r.o. of } A,\ a_0, a_1 \notin cl_A(U_A),\ y_1, y_2 \notin b_A(U_A)\Big\},$$
$$\mathcal{T}_2 = \Big\{U_B \subseteq Q : U_B \text{ r.o. of } B,\ b_0, b_1 \notin cl_B(U_B)\Big\},$$
$$\mathcal{T}_3 = \Big\{U_{H^X} \subseteq Q : U_{H^X} \in \mathcal{T}_0,\ c_a, c_b \notin cl_{H^X}(U_{H^X}) \text{ and } cl_{H^X}(U_{H^X}) \bigcap \chi_R = \emptyset\Big\},$$

where $\mathcal{T}_0$ denotes the set

$$\Big\{\prod_{x\in X} U_x : U_x \text{ r.o. of } H, x \in X \text{ and only a finite number of } U_x\text{'s is different of } H\Big\},$$

$$\mathcal{T}_4 = \Big\{(U_{H^X} \bigcap \chi_R) \times U_C \subseteq Q : U_{H^X} \in \mathcal{T}_0, U_C \text{ is a r.o. of } C, c_0, c_1 \notin cl_C(U_C)\Big\},$$

$$\mathcal{T}_5 = \Big\{U_A \bigcup U_{H^X} \subseteq Q : U_A \text{ r.o. of } A,\ a_0 \in U_A, a_1 \notin cl_A(U_A), y_1, y_2 \notin b_A(U_A)$$
$$U_{H^X} \in \mathcal{T}_0,\ c_a \in U_{H^X},\ c_b \notin cl_{H^X}(U_{H^X}), cl_{H^X}(U_{H^X}) \bigcap \chi_R = \emptyset\Big\},$$

$$\mathcal{T}_6 = \Big\{U_B \bigcup U_{H^X} \subseteq Q : U_B \text{ r.o. of } B,\ b_0 \in U_B, b_1 \notin cl_B(U_B),$$
$$U_{H^X} \in \mathcal{T}_0,\ c_b \in U_{H^X}, c_a \notin cl_{H^X}(U_{H^X}), cl_{H^X}(U_{H^X}) \bigcap \chi_R = \emptyset\Big\},$$

$$\mathcal{T}_7 = \Big\{U_{H^X} \bigcup (U_{H^X} \bigcap \chi_R) \times U_C \subseteq Q : U_{H^X} \in \mathcal{T}_0,\ c_a, c_b \notin cl_{H^X}(U_{H^X}),$$
$$cl_{H^X}(U_{H^X}) \bigcap \chi_R = U_{H^X} \bigcap \chi_R \neq \emptyset, U_C \text{ r.o. of } C,\ c_0 \in U_C, c_1 \notin cl_C(U_C)\Big\},$$

$$\mathcal{T}_8 = \Big\{U_A \bigcup U_B \bigcup (\chi_R \times U_C) \subseteq Q : U_A \text{ r.o. of } A,\ a_1 \in U_A, a_0 \notin cl_A(U_A),$$
$$y_1, y_2 \notin b_A(U_A), U_B \text{ r.o. of } B,\ b_1 \in U_B, b_0 \notin cl_B(U_B), U_C \text{ r.o. of } C,\ c_1 \in U_C,$$
$$c_0 \notin cl_C(U_C)\Big\}.$$

Lemma 5.1 ([16])**.** *The pair* $(\mathcal{B}_Q, \mathcal{A}_Q)$*, where*

- $\mathcal{B}_Q$ *is the Boolean subalgebra of* $exp(Q)$ *generated by* $\mathcal{T} \bigcup \Big\{\{y_1\}, \{y_2\}\Big\}$*;*
- $\mathcal{A}_Q$ *is the subalgebra of* $exp(Q \times Q)$ *relative to the operations* $\cup, \circ$ *and* $*$ *generated by* $\Big\{\Delta_U : U \in \mathcal{T}\Big\} \bigcup \Big\{Q \times \{y_1\}, Q \times \{y_2\}\Big\}$

  for $\Delta_U = \Big\{(x, x) : x \in U\Big\}$

is a separable Kripke structure.

Lemma 5.2 ([16])**.** $\mathcal{A}_Q$ *is the set of binary relations on* Q *of the following type*

$$\Delta_G \cup G_1 \times \{y_1\} \cup G_2 \times \{y_2\}$$

for every G, G_1, G_2 *belonging to the subalgebra* $\mathcal{B}_{\mathcal{T}}$ *of* $(exp(Q), \cup, \cap)$ *generated by* $\mathcal{T}$*.*[†]

[†]Actually, it can be proved that the basis $\mathcal{T}$ is closed under finite intersections.

This is a consequence of the following facts: finite intersections of r.o. subsets are r.o. subsets and $b(G \cap G') \subseteq b(G) \cap b(G')$ for any subsets G, G' in a topological space. Actually, every subfamily $\mathcal{T}_i, i = 1, 2, 3, 4, 5, 6, 8$ is closed under finite intersections. Finite intersections of subsets of $\mathcal{T}_7$ belong to $\mathcal{T}_7$ if they contain elements of χ_R. Otherwise, they belong to $\mathcal{T}_3$. (Thanks to a remark of Prof. Júlia Vaz de Carvalho.)

The proof is an easy exercise in the Kripke structure operations. Notice, for future reference, that the elements of $\mathcal{B}_{\mathcal{T}}$ are open sets in Q. $\square$

Definition of $\boldsymbol{\phi}$ on objects: $\phi(Q) = (\mathcal{B}_Q, \mathcal{A}_Q)$.

Now, we define ϕ on morphisms. Let (X', R') be another graph in *UndGraph*$_0$. We denoted $Q' = \psi(X', R')$ and let $\mathcal{T}_0^{'}, \mathcal{T}_1^{'}, \ldots, \mathcal{T}_8^{'}, \mathcal{T}^{'}, c_a^{'}, c_b^{'}$. Let stand for (X', R') as $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_8, \mathcal{T}, c_a, c_b$ stand for (X, R).

Definition of $\boldsymbol{\phi}$ on morphisms: If $g : Q' \longrightarrow Q$ is a morphism of $\psi(\mathit{UndGraph}_0^{op})$, we defined

$$\phi(g) = (\phi_g^1, \phi_g^2) : (\mathcal{B}_Q, \mathcal{A}_Q) \longrightarrow (\mathcal{B}_{Q'}, \mathcal{A}_{Q'})$$

by

- $\phi_g^1(p) = g^{-1}(p)$ for $p \in \mathcal{B}_Q$;
- $\phi_g^2(\Delta_G) = \Delta_{g^{-1}(G)}$ for $\Delta_G \in \mathcal{A}_Q$;
- $\phi_g^2(G \times \{y\}) = g^{-1}(G) \times \{y\}$ for $G \times \{y\} \in \mathcal{A}_Q$;
- For the remaining elements of $\mathcal{A}_Q$ use (ii), (iii) and the fact that ϕ_g^2 preserves finite unions.

Lemma 5.3 ([16])**.** *$\phi(g) : \phi(Q) \longrightarrow \phi(Q')$ is a well defined homomorphism of dynamic algebras.*

Theorem 5.4 ([16])**.** *The functor $\phi : \psi(\mathit{UndGraph}_0^{op}) \longrightarrow \mathit{SKri}^{op}$ is a full embedding. As a consequence, the category of separable Kripke structures is alg-universal.*

Let us just recall how the fulness was produced. Given a homomorphism of dynamic algebras $(h_1, h_2) : (\mathcal{B}_Q, \mathcal{A}_Q) \longrightarrow (\mathcal{B}_{Q'}, \mathcal{A}_{Q'})$, the non-constant continuous mapping $g : Q' \longrightarrow Q$, such that $\phi(g) = (h_1, h_2)$ was defined by $g(x) = z$ $(x \in Q')$, where z was the unique element of the intersection

$$\bigcap_{F \in \mathcal{F}_x} F,$$

for $\mathcal{F}_x = \{F \in \mathcal{B}_Q : F \text{ is closed in } Q \text{ and } x \in h_1(F)\}$.

6. Construction of the embedding γ

$$\boldsymbol{\gamma : \psi(\mathit{UndGraph}_0^{op}) \longrightarrow \mathit{BfKri}^{op}}$$

This construction is made using many of the elements of the construction of ϕ. Therefore, we maintain, as far as possible, the same notation.

We begin to define γ on objects. Let (X, R) be a graph of *UndGraph*$_0$ and considerer the space $Q = \psi(X, R)$. Consider the basis $\mathcal{T}$ of the topological space Q and the two distinct elements $y_1, y_2 \in A\backslash\{a_0, a_1\}$ which were used for ϕ.

We recall, $\mathcal{B}_\mathcal{T}$ is the family of all the finite unions of elements of $\mathcal{T}$.

Lemma 6.1. *The pair* $\gamma(Q) = (exp(Q), \mathcal{A}_Q)$, *where*

$$\mathcal{A}_Q = \Big\{\Delta_G \cup G_1 \times \{y_1\} \cup G_2 \times \{y_2\} : G, G_1, G_2 \in \mathcal{B}_\mathcal{T}\Big\}$$

is a Boolean full Kripke structure on Q.

The proof that $\gamma(Q)$ is a Boolean full Kripke structure is reduced to the proof that $\mathcal{A}_Q$ is closed under the regular operations. But this is can be concluded by Lemma 5.2. □

Definition of γ on objects: $\gamma(Q) = (exp(Q), \mathcal{A}_Q)$.

Remark. We ask the reader to observe that the family $\mathcal{A}_Q$ is exactly the same that it was considered in the regular part of $\phi(Q)$ (*c.f.* section 5). For the Boolean part of $\gamma(Q)$, we choose $exp(Q)$, instead of $\mathcal{B}_Q$, the Boolean part of $\phi(Q)$.

Now, we define γ on morphisms. Let (X', R') be another graph in *UndGraph*$_0$. Let $Q', \mathcal{T}', c'_a, c'_b$ stand for (X', R') as $Q, \mathcal{T}, c_a, c_b$ stand for (X, R). Given a morphism $g : Q' \longrightarrow Q$ of $\psi(\mathit{UndGraph}_0^{op})$, we define

$$\gamma(g) = (\gamma_g^1, \gamma_g^2) : (exp(Q), \mathcal{A}_Q) \longrightarrow (exp(Q'), \mathcal{A}_{Q'})$$

as follows,

- $\gamma_g^1(p) = g^{-1}(p)$ for $p \in exp(Q)$;
- $\gamma_g^2\Big(\Delta_G \cup G_1 \times \{y_1\} \cup G_2 \times \{y_2\}\Big) = \Delta_{g^{-1}(G)} \cup g^{-1}(G_1) \times \{y_1\} \cup g^{-1}(G_2) \times \{y_2\}$ for $G, G_1, G_2 \in \mathcal{B}_\mathcal{T}$.

The definition of $\gamma(g)$ is similar to the definition of $\phi(g)$. The mapping $\gamma_g^1 : exp(Q) \rightarrow \exp(Q')$ is, obviously, a well defined complete homomorphism of Boolean algebras. Due to the fact that g is a special continuous mapping, it is of the form $g = \psi(f)$ for some (unique) morphism of graphs $f : (X, R) \longrightarrow (X', R')$, we can ensure that the pre-image under g of elements of the basis $\mathcal{T}$ is in $\mathcal{T}'$ [16, Lemma 5.5]. Then, exactly like for ϕ, $\gamma(g)$ is a well defined homomorphism of Kripke structures [16, Lemma 5.6].

Proposition 6.2. *The functor γ is an embedding.*

Proof. It is trivial to show that γ is a one-to-one functor on objects. To show that γ is also faithful we don't need to use the argument that Q is a Hausdorff space ([16, Proposition 5.7]). Consider

$$g, g' : Q' \longrightarrow Q,$$

non-constant continuous mappings such that $g \neq g'$. So, there exists $z \in Q'$ such that $g(z) \neq g'(z)$. Thus,

$$z \in \gamma_g^1(\{g(z)\}),\ z \notin \gamma_{g'}^1(\{g(z)\}).$$

Consequently, $\gamma(g) \neq \gamma(g')$. □

7. The embedding γ is full

Our aim is to prove that γ is also full. The proof, inspired in the proof of the fulness of the embedding ϕ, is simpler. We follow that proof step by step.

Let

$$(h_1, h_2) : (exp(Q), \mathcal{A}_Q) \longrightarrow (exp(Q'), \mathcal{A}_{Q'})$$

be a homomorphism of Kripke structures with h_1 complete. We are going to construct a mapping

$$g : Q' \longrightarrow Q$$

and show that g is a non-constant continuous mapping satisfying $\gamma(g) = (h_1, h_2)$.

Let $x \in Q'$. Consider

$$\mathcal{G}_x = \{G \subseteq Q : x \in h_1(G)\}.$$

We have,

Lemma 7.1. *The set* $I_x = \bigcap_{G \in \mathcal{G}_x} G$ *is a single set.*

Proof. I_x is a non-empty, otherwise we would have,

$$x \in \bigcap_{G \in \mathcal{G}_x} h_1(G) = h_1(\bigcap_{G \in \mathcal{G}_x} G) = \emptyset.$$

Now, let be $z \in I_x$. Then, if $x \notin h_1(\{z\})$, we have $x \in h_1(\{z\}^c)$, which means that $\{z\}^c \in \mathcal{G}_x$.[‡] Consequently, we get the contradiction $z \notin I_x$. Therefore, it happens $x \in h_1(\{z\})$. That is, $I_x = \{z\}$. □

Thus, we define $g(x)$ to be the unique element of the set I_x. Let us write

$$g(x) = \bigcap_{G \in \mathcal{G}_x} G.$$

Remark. We can deduce $g(x) = z$ *iff* $x \in h_1\{z\}$.

Proposition 7.2. *The mapping g is non-constant.*

Proof. We have,

$$\langle Q \times \{y_1\}, \{y_1\} \rangle = Q \text{ and } \langle Q \times \{y_2\}, \{y_2\} \rangle = Q.$$

Applying the homomorphism h to these equalities, necessarily we conclude

$$h_1(\{y_1\}) \neq \emptyset \text{ and } h_1(\{y_2\}) \neq \emptyset.$$

by the axioms of dynamic algebras. Taking one element x_1 in the first set and one element x_2 in the second set, we get $g(x_1) = y_1 \neq y_2 = g(x_2)$. □

Lemma 7.3. *For every $U \subseteq Q$ it happens,*

$$h_1(U) = g^{-1}(U).$$

[‡]Given a set S we denote by S^c the complement of S.

Proof. Let it be $U \subseteq Q$. Given $x \in h_1(U)$, since h_1 is complete, we have $x \in h_1(\{z\})$ for some $z \in U$. By last remark, this means, $g(x) = z$ for some $z \in U$. Thus, $h_1(U) \subseteq g^{-1}(U)$.

On the other hand, if $g(x) \in U$ then $g(x) = z$ for some $z \in U$. Therefore, $x \in h_1(\{z\}) \subseteq h_1(U)$. Thus, $g^{-1}(U) \subseteq h_1(U)$. □

Proposition 7.4. *The mapping g is continuous.*

Proof. It is enough to prove that pre-images of elements of $\mathcal{T}$ are open. Let $U \in \mathcal{T}$. By last Lemma we have $g^{-1}(U) = h_1(U)$. Therefore, it is enough to prove that $h_1(U)$ is open.

We have, $\langle \Delta_U, U^c \rangle = \emptyset$. Applying (h_1, h_2), we conclude $\langle h_2(\Delta_U), h_1(U^c) \rangle = = \emptyset$. Due to the form of the elements of $\mathcal{A}_{Q'}$ (*c.f.* Lemma 6.1), we have

$$\langle \Delta_G \cup G_1 \times \{y_1\} \cup G_2 \times \{y_2\}, (h_1(U))^c \rangle = \emptyset$$

for some open subsets G, G_1, G_2 in Q'. It then follows $G \cap (h_1(U))^c = \emptyset$, thus $G \subseteq h_1(U)$.

Besides, applying (h_1, h_2) to $\langle \Delta_U, U \rangle = U$, we have

$$\langle \Delta_G \cup G_1 \times \{y_1\} \cup G_2 \times \{y_2\}, h_1(U) \rangle = h_1(U).$$

Thus,

$$h_1(U) = \underbrace{\Big(G \cap h_1(U)\Big)}_{G} \cup \Big(G_1 \cap \delta_{y_1}^{h_1(U)}\Big) \cup \Big(G_2 \cap \delta_{y_2}^{h_1(U)}\Big).^{\S}$$

Consequently, $h_1(U)$ is open. □

Remark. Since $g : Q' \longrightarrow Q$ is a non-constant continuous mapping, we know the form of g. There is a homomorphism of graphs

$$f : (X, R) \longrightarrow (X', R')$$

such that $\psi(f) = g$ (*c.f.* section 4).

§We recall here the use of the Kronecker symbol used in [16]:

$$\delta_y^G = \begin{cases} Q' & \text{if } y \in G \\ \emptyset & \text{if } y \notin G \end{cases},$$

for $y \in Q'$ and $G \subseteq Q'$.

Now, we conclude the proof of the fulness of γ.

Proposition 7.5. $\gamma(g) = (h_1, h_2)$.

Proof. It remains to prove that $h_2 = \gamma_g^2$, more exactly, that $h_2(\Delta_G) = \Delta_{g^{-1}(G)}$ and $h_2(G \times \{y\}) = g^{-1}(G) \times \{y\}$. We decompose the proof in several steps. Some of the partial proofs are a repetition of the proofs made for ϕ. We repeat them here, for the sake of clarity.

1. $h_2(\Delta_Q) = \Delta_{Q'}$.

Let's suppose that $h_2(\Delta_Q) \not\subseteq \Delta_{Q'}$. Then, there exists $x, y \in Q'$, with $x \neq y$, such that $(x, y) \in h_2(\Delta_Q)$. Necessarily, $y \in \{y_1, y_2\}$.

Applying the homomorphism h to $\langle \Delta_Q, \{y\} \rangle = \{y\}$, we have

$$\langle h_2(\Delta_Q), h_1(\{y\}) \rangle = h_1(\{y\}).$$

Since $h_1(\{y\}) = g^{-1}(\{y\})$ and due to the form of g, $h_1(\{y\}) = \{y\}$ (recall that $y_1, y_2 \in A \backslash \{a_0, a_1\}$), it follows that

$$x \in \langle h_2(\Delta_Q), \{y\} \rangle = \{y\},$$

which is a contradiction. Consequently, $h_2(\Delta_Q) \subseteq \Delta_{Q'}$, *i.e.*,

$$h_2(\Delta_Q) = \Delta_G \text{ for some subset } G \subseteq Q'.$$

But, from $\langle \Delta_Q, Q \rangle = Q$, we have, successively,

$$\langle \underbrace{h_2(\Delta_Q)}_{\Delta_G}, \underbrace{h_1(Q)}_{Q'} \rangle = \underbrace{h_1(Q)}_{Q'} \Longrightarrow G \cap Q' = Q' \Longrightarrow G = Q'.$$

2. *Let* $\Delta_G \in \mathcal{A}_Q$. *Then* $h_2(\Delta_G) = \Delta_{g^{-1}(G)}$.

We have $\Delta_Q = \Delta_Q \cup \Delta_G$. Therefore, $h_2(\Delta_Q) = h_2(\Delta_Q) \cup h_2(\Delta_G)$, that is, $\Delta_{Q'} = \Delta_{Q'} \cup h_2(\Delta_G)$. Hence, $h_2(\Delta_G \subseteq \Delta_{Q'}$, that is,

$$h_2(\Delta_G) = \Delta_{G'}, \text{ for some } G' \subseteq Q'.$$

Applying h to $\langle \Delta_G, Q \rangle = G$, we get

$$\langle \Delta_{G'}, Q' \rangle = h_1(G).$$

Consequently, $G' = h_1(G) = g^{-1}(G)$.

3. For $G\times\{y\}\in\mathcal{A}_Q$, it holds $h_2(G\times\{y\})=g^{-1}(G)\times\{y\}$.

Applying (h_1,h_2) to $\langle G\times\{y\},\{y\}\rangle=G$, we get

$$\langle h_2(G\times\{y\}),\{y\}\rangle=g^{-1}(G),$$

due to Lemma 7.3.

Then $g^{-1}(G)\times\{y\}\subseteq h_2(G\times\{y\})$, that is $h_2(G\times\{y\})=g^{-1}(G)\times\{y\}\cup R$, for some subset $R\subseteq Q'\times Q'$ such that y is no second component of any of the binary pairs belonging to R.

Besides, applying (h_1,h_2) to $\langle G\times\{y\},\{y\}^c\rangle=\emptyset$, we conclude

$$\langle g^{-1}(G)\times\{y\}\cup R,\{y\}^c\rangle=\emptyset.$$

Consequently, $R=\emptyset$. $\square$

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