Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 08–15

SESQUICATEGORY: A CATEGORY WITH A 2-CELL STRUCTURE

N. MARTINS-FERREIRA

ABSTRACT: For a given (fixed) category, we consider the category of all 2-cell structures (over it) and study some naturality properties. A category with a 2-cell structure is a sesquicategory; we use additive notation for the vertical composition of 2-cells; instead of a law for horizontal composition we consider a relation saying which pairs of 2-cells can be horizontally composed; for a 2-cell structure with every 2-cell invertible, we also consider a notion of commutator, measuring the obstruction for horizontal composition. We compare the concept of naturality in an abstract 2-cell structure with the example of internal natural transformations in a category of the form Cat(**B**), of internal categories in some category **B**, and show that they coincide. We provide a general construction of 2-cell structures over an arbitrary category, under some mild assumptions. In particular, the canonical 2-cell structures over groups and crossed-modules, respectively "conjugations" and "derivations", are instances of these general constructions. We define cartesian 2-cell structure and extend the notion of pseudocategory from the context of a 2-category (as in [6]) to the more general context of a sesquicategory. Some remarks on coherence are also given.

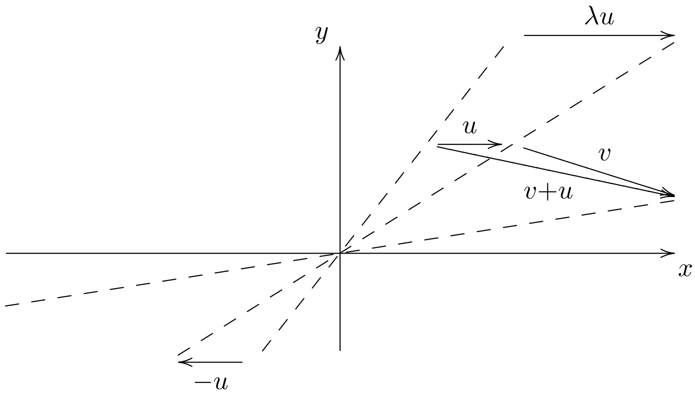
KEYWORDS: sesquicategory, 2-cell structure, cartesian 2-cell structure, natural 2-cell, pseudocategory.
AMS SUBJECT CLASSIFICATION (2000): Primary 18D05, 18D35; Secondary 18E05.

1. Introduction

In this article we use a different notation for the vertical composition of 2-cells: instead of the usual dot '$\cdot$' we use plus '$+$'. To support this we present the following analogy between geometrical vectors in the plane and 2-cells

Received March 20, 2008.
This is the result of several talks given by the author in Milan (Oct 2006), Coimbra (May 2007) and Haute-Bodeux (Jun 2007). The author thanks Professor G. Janelidze for much appreciated help of various kinds.

between morphisms in a category.



Two geometrical vectors in the plane can be added only if the end point of the second (u as in the picture above) is the starting point of the first one (v as in the picture) and in that case the resulting vector (the sum) goes from the starting point of the second to the end point of the first: exactly the same as with 2-cells

$$
\begin{tikzcd}[column sep=large]
\cdot \arrow[r, bend left=60, "\operatorname{dom} u"{name=U}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, "\operatorname{cod} v"'{name=D}] \arrow[Rightarrow, from=U, to=M, "u"] \arrow[Rightarrow, from=M, to=D, "v"] & \cdot
\end{tikzcd}
\longmapsto
\begin{tikzcd}[column sep=large]
\cdot \arrow[r, bend left=60, "\operatorname{dom} u"{name=U}] \arrow[r, bend right=60, "\operatorname{cod} v"'{name=D}] \arrow[Rightarrow, from=U, to=D, "v+u"] & \cdot
\end{tikzcd}\ ;
$$

In some sense the analogy still holds for scalar multiplication

$$
\begin{tikzcd}[column sep=large]
\cdot \arrow[r, "\lambda"] & \cdot \arrow[r, bend left=60, "\operatorname{dom} u"{name=U}] \arrow[r, bend right=60, "\operatorname{cod} u"'{name=D}] \arrow[Rightarrow, from=U, to=D, "u"] & \cdot \arrow[r, "\rho"] & \cdot
\end{tikzcd}
\longmapsto
\begin{tikzcd}[column sep=large]
\cdot \arrow[r, bend left=60, "\rho\operatorname{dom}(u)\lambda"{name=U}] \arrow[r, bend right=60, "\rho\operatorname{cod}(u)\lambda"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\rho u\lambda"] & \cdot
\end{tikzcd}\ ,
$$

and for inverses (in the case they exist)

$$
\begin{tikzcd}[column sep=large]
\cdot \arrow[r, bend left=60, "\operatorname{dom} u"{name=U}] \arrow[r, bend right=60, "\operatorname{cod} u"'{name=D}] \arrow[Rightarrow, from=U, to=D, "u"] & \cdot
\end{tikzcd}
\longmapsto
\begin{tikzcd}[column sep=large]
\cdot \arrow[r, bend left=60, "\operatorname{cod} u"{name=U}] \arrow[r, bend right=60, "\operatorname{dom} u"'{name=D}] \arrow[Rightarrow, from=U, to=D, "-u"] & \cdot
\end{tikzcd}\ .
$$

Concerning horizontal composition, there is still an analogy with some relevance: it is, in some sense, analogous to the cross product of vectors − in the sense that it raises in dimension (see the introduction of [1] and its references for further discussion on this). Given 2-cells, u and v

$$\begin{tikzcd} \cdot \arrow[r, bend left, "\mathrm{dom}\, u"] \arrow[r, bend right, "\mathrm{cod}\, u"'] \arrow[r, phantom, "\Downarrow u"] & \cdot \arrow[r, bend left, "\mathrm{dom}\, v"] \arrow[r, bend right, "\mathrm{cod}\, v"'] \arrow[r, phantom, "\Downarrow v"] & \cdot \end{tikzcd}$$

the horizontal composition $v \circ u$ should be a 3-cell, from the 2-cell

$$\mathrm{cod}\,(v)\, u + v\, \mathrm{dom}\,(u) \tag{1.1}$$

to the 2-cell

$$v\, \mathrm{cod}\,(u) + \mathrm{dom}\,(v)\, u. \tag{1.2}$$

In some cases (1.1) and (1.2) coincide (as it happens in a 2-category) and this is the reason why one may think of a horizontal composition, but it is an illusion; to overcome this we better consider a relation $v \circ u$ saying that the 2-cell v is natural with respect to u, defined as

$$v \circ u \Longleftrightarrow (1.1) = (1.2)\,,$$

in this sense, the horizontal composition is only defined for those pairs (v, u) that are in relation $v \circ u$, with the composite being then given by either (1.1) or (1.2) .

This is a geometrical intuition. An algebraic intuition is also provided in Proposition 1.

This article is organized as follows.

For a fixed category, $\mathbf{C}$, we define a 2-cell structure (over $\mathbf{C}$, as to make it a sesquicategory) and give a characterization of such a structure as a family of sets, together with maps and actions, satisfying some conditions. It generalizes the characterization of 2-Ab-categories as a family of abelian groups, together with group homomorphisms and laws of composition as given in [5] and [7] where the strong condition

$$D\,(x)\, y = xD\,(y)$$

is no longer required. A useful consequence is that the example of chain complexes, say of order 2, can be considered in this more general setting. Of course, this condition is equivalent to the naturality condition, and the results obtained in [5] and [7] heavily rest on this assumption, so one must be careful in removing it. For this we introduce and study the concept of a 2-cell being natural with respect to another 2-cell, and the concept of natural 2-cell, as one being natural with respect to all. Next we compare this notions when

$\mathbf{C}$ is a category of the form $\mathrm{Cat}(\mathbf{B})$, of internal categories in some category $\mathbf{B}$, and conclude that if the 2-cell structure is the canonical one (internal transformations, not necessarily natural) then a natural 2-cell corresponds to a natural transformation, and furthermore, it is sufficient to check if a given transformation is natural with respect to a particular 2-cell (from the "category of arrows"), to determine if it is natural.

We give a general process for constructing 2-cell structures in arbitrary categories, and for the purposes of latter discussions we will restrict our study to the 2-cell structures obtained this way. In order to argue that we are not restricting too much, we show that the canonical 2-cell structures over groups and crossed-modules, that are respectively "conjugations" and "derivations", are captured by this construction.

We introduce the notion of cartesian 2-cell structure, in order to consider 2-cells of the form $u \times_w v$ that are used in the coherence conditions involved in a pseudocategory.

At the end we extend the notion of pseudocategory from the context of a 2-category to the more general context of a category with a 2-cell structure (sesquicategory).

All the notions defined in [6]: pseudofunctor, natural and pseudo-natural transformation, modification, may also be extended in this way. However some careful is needed when dealing with coherence issues. For example MacLane's Coherence Theorem, saying that it suffices to consider the coherence for the *pentagon* and *middle triangle* is no longer true in general, since it uses the fact that α, λ, ρ are natural. One way to overcome this difficulty is to impose the naturality for α, λ, ρ in the definition, so that in [6] (introduction, definition of pseudocategory in a 2-category) instead of saying

"...α, λ, ρ are 2-cells (which are isomorphisms)..."

we have to say

"...α, λ, ρ are natural and invertible 2-cells ..."

We will not study deeply all the consequences of this. Instead we will restrict ourselves to the study of 2-cell structures such that all 2-cells are invertible (since the main examples are groups, abelian groups, 1-chain complexes and crossed modules) and hence the question of α, λ, ρ being invertible becomes intrinsic to the 2-cell structure. The issue of naturality is more delicate. To prove the results in [5], [7] and [9], we will only need λ and ρ to be natural with respect to each other, that is

$$\lambda \circ \lambda, \lambda \circ \rho, \rho \circ \lambda, \rho \circ \rho.$$

If interested in the Coherence Theorem, we can always use the reflexion

$$\text{2-cellstruct}(\mathbf{C}) \xrightarrow{I} \text{nat-2-cellstruct}(\mathbf{C})$$

of the category of 2-cell structures over $\mathbf{C}$ (sesquicategories "with base $\mathbf{C}$"), into the subcategory of natural 2-cell structures over $\mathbf{C}$ (2-categories "with base $\mathbf{C}$"), sending each 2-cell structure to its "naturalization"; which, if $\mathbf{C} = \mathbf{1}$, becomes the familiar reflexion of monoids into commutative monoids

$$\text{Mon} \xrightarrow{I} \text{CommMon}$$

and if restricting further to invertible 2-cells gives the reflection

$$\text{Grp} \xrightarrow{I} \text{Ab}$$

of groups into abelian groups.

All these considerations will appear in [9] when describing pseudocategories in weakly Mal'cev sesquicategories.

2. 2-cell structures and sesquicategories

Let $\mathbf{C}$ be a fixed category.

Definition 1 (2-cell structure)**.** *A 2-cell structure over* $\mathbf{C}$ *is a system*

$$\mathbf{H} = (H, \mathrm{dom}, \mathrm{cod}, 0, +)$$

where

$$H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Set$$

is a functor and

$$\begin{tikzcd} H \times_{\mathrm{hom}} H \arrow[r, "+"] & H \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathrm{hom}_{\mathbf{C}} \arrow[l, "0" description] \end{tikzcd}$$

are natural transformations, such that

$$(\mathrm{hom}_{\mathbf{C}}, H, \mathrm{dom}, \mathrm{cod}, 0, +)$$

is a category object in the functor category $Set^{C^{op}\times C}$ *or, in other words, an object in* $Cat\big(Set^{C^{op}\times C}\big)$.

Proposition 1. *Giving a 2-cell structure over a category* $\mathbf{C}$*, is to give, for every pair* (A,B) *of objects in* $\mathbf{C}$*, a set* $H(A,B)$*, together with maps*

$$H(A,B)\times_{\hom(A,b)} H \xrightarrow{+} H \begin{array}{c}\xrightarrow{\text{dom}}\\ \xleftarrow{0}\\ \xrightarrow[\text{cod}]{}\end{array} \hom(A,B),$$

and actions

$$\begin{array}{ccc} H(B,C)\times\hom(A,B) & \longrightarrow & H(A,C)\\ (x,f) & \longmapsto & xf\\ \hom(B,C)\times H(A,B) & \longrightarrow & H(A,C)\\ (g,y) & \longmapsto & gy \end{array}$$

satisfying the following conditions

$$\begin{gathered} \text{dom}(gy)=g\,\text{dom}(y)\ \ ,\ \ \text{dom}(xf)=\text{dom}(x)\,f \\ \text{cod}(gy)=g\,\text{cod}(y)\ \ ,\ \ \text{cod}(xf)=\text{cod}(x)\,f \\ g0_f=0_{gf}=0_gf \\ (x+x')\,f=xf+x'f\ ,\ g\,(y+y')=gy+gy' \end{gathered} \tag{2.1}$$

$$\begin{aligned} g'(gy) &= (g'g)\,y\ ,\ (xf)\,f'=x\,(ff') \\ g'(xf) &= (g'x)\,f \\ 1_Cx &= x=x1_B \end{aligned} \tag{2.2}$$

$$\begin{aligned} \text{dom}(0_f) &= f=\text{cod}(0_f) \\ \text{dom}(x+x') &= x'\ ,\ \text{cod}(x+x')=x \\ 0_{\text{cod}\,x}+x &= x=x+0_{\text{dom}\,x} \\ x+(x'+x'') &= (x+x')+x''. \end{aligned} \tag{2.3}$$

Proof: For every $f:A'\longrightarrow A$, $g:B\longrightarrow B'$ and $x\in H(A,B)$, write

$$H(f,g)(x)=gxf$$

and it is clear that the set of conditions (2.1) asserts the naturality of $\text{dom},\text{cod},0,+$; the set of conditions (2.2) asserts the functoriality of H and the set of conditions (2.3) asserts the axioms for a category. ■

Definition 2 (sesquicategory). *A* sesquicategory *is a pair* $(\mathbf{C},\mathbf{H})$ *where* $\mathbf{C}$ *is a category and* $\mathbf{H}$ *a 2-cell structure over it.*

Observation: A sesquicategory, as defined, is the same as a sesquicategory in the sense of Ross Street, that is, a category $\mathbf{C}$ together with a functor $\mathbf{H}$ into Cat, such that the restriction to Set gives $\hom_{\mathbf{C}}$, as displayed in the following picture

$$\begin{tikzcd} & Cat \arrow[d] \\ \mathbf{C}^{op} \times \mathbf{C} \arrow[ur, "\mathbf{H}"] \arrow[r, "\hom"'] & Set \end{tikzcd} .$$

Proposition 2. *A category* $\mathbf{C}$ *with a 2-cell structure*

$$\mathbf{H} = (H, \mathrm{dom}, \mathrm{cod}, 0, +),$$

is a 2-category if and only if the naturality condition

$$\mathrm{cod}\,(x)\,y + x\,\mathrm{dom}\,(y) = x\,\mathrm{cod}\,(y) + \mathrm{dom}\,(x)\,y \qquad \text{(naturality condition)}$$

holds for every $x \in H\,(B, C)$, $y \in H\,(A, B)$, *and every triple of objects* (A, B, C) *in* $\mathbf{C}$, *as displayed in the diagram below*

$$\begin{tikzcd} A \arrow[r, bend left, "\mathrm{dom}\,y"] \arrow[r, bend right, "\mathrm{cod}\,y"'] \arrow[r, phantom, "\Downarrow y"] & B \arrow[r, bend left, "\mathrm{dom}\,x"] \arrow[r, bend right, "\mathrm{cod}\,x"'] \arrow[r, phantom, "\Downarrow x"] & C \end{tikzcd} .$$

Proof: If $\mathbf{C}$ is a 2-category, the naturality condition follows from the horizontal composition of 2-cells and, conversely, given a 2-cell structure over $\mathbf{C}$, in order to make it a 2-category one has to define a horizontal composition and it is defined as

$$x \circ y = \mathrm{cod}\,(x)\,y + x\,\mathrm{dom}\,(y)$$

or

$$x \circ y = x\,\mathrm{cod}\,(y) + \mathrm{dom}\,(x)\,y$$

provided the naturality condition is satisfied for every appropriate x, y. The middle interchange law also follows from the naturality condition. ■

It may happen that the naturality condition does not hold for all possible x and y, but only for a few; thus the following definitions.

Let $\mathbf{C}$ be a category and $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$ a 2-cell structure over it.

Definition 3. *A 2-cell* $\delta \in H\,(A, B)$ *is* natural with respect to *a 2-cell* $z \in H\,(X, A)$, *when*

$$\mathrm{cod}\,(\delta)\,z + \delta\,\mathrm{dom}\,(z) = \delta\,\mathrm{cod}\,(z) + \mathrm{dom}\,(\delta)\,z;$$

in that case one writes $\delta \circ z$*.*

Definition 4. *A 2-cell* $\delta \in H(A,B)$ *is* natural *when it is natural with respect to all possible* $z \in H(X,A)$ *for all* $X \in \mathbf{C}$*, i.e.,* δ *is a natural 2-cell if and only if* $\delta \circ z$ *for all possible* z*.*

3. Examples

We shall now see how the above notions of naturality are related, in the case where $\mathbf{C} = Cat(\mathbf{B})$ for some category $\mathbf{B}$, with the 2-cell structure given by the internal (natural) transformations.

Example 1. *Consider* $\mathbf{C} = Cat(\mathbf{B})$ *the category of internal categories in some category* $\mathbf{B}$*. The objects are*

$$A = (A_0, A_1, d, c, e, m),\ B = (B_0, B_1, d, c, e, m),\ \ldots$$

and morphisms

$$f = (f_1, f_0) : A \longrightarrow B, \ldots$$

Consider the following 2-cell structure over $\mathbf{C}$*:*

$$\begin{aligned}
H(A,B) &= \{(k,t,h) \mid t : A_0 \to B_1; h,k \in \hom_{\mathbf{C}}(A,B); dt = h_0, ct = k_0\} \\
H(f,g)(k,t,h) &= (gkf, g_1 t f_0, ghf) \\
\operatorname{dom}(k,t,h) &= h \\
\operatorname{cod}(k,t,h) &= k \\
0_h &= (h, eh_0, h) \\
(k,t,h) + (h,s,l) &= (k, m\langle t,s\rangle, l)
\end{aligned}$$

where $f : A' \longrightarrow A, g : B \longrightarrow B', h,k,l : A \longrightarrow B$ *are morphisms in* $Cat(\mathbf{B})$ *and* $t,s : A_0 \longrightarrow B_1$ *are morphisms in* $\mathbf{B}$*.*
Observe that, in particular, for every $A = (A_0, A_1, d, c, e, m)$ *there is* $A^{\rightarrow} = (A_1, A_1, 1, 1, 1, 1)$ *and the two morphisms*

$$d^{\rightarrow} = (ed, d) : A^{\rightarrow} \longrightarrow A$$

and

$$c^{\rightarrow} = (ec, c) : A^{\rightarrow} \longrightarrow A.$$

Proposition 3. *In the context of the previous example, a 2-cell* $\mathbf{t} = (k, t, h) \in H(A, B)$ *is an internal natural transformation* $\mathbf{t} : h \longrightarrow k$ *if and only if it is natural with respect to the 2-cell*

$$\left(c^{\rightarrow}, 1_{A_1}, d^{\rightarrow}\right) \in H\left(A^{\rightarrow}, A\right).$$

Proof: Consider $\mathbf{t} = (k, t, h) \in H(A, B)$ and $\mathbf{z} = (g, z, f) \in H(X, A)$,

$$\begin{tikzcd}
\cdots & X_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] & X_0 \arrow[l] \arrow[dl, "z"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] \\
\cdots & A_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right, "h_1"'] \arrow[d, shift left, "k_1"] & A_0 \arrow[l] \arrow[dl, "t"'] \arrow[d, shift right, "h_0"'] \arrow[d, shift left, "k_0"] \\
\cdots & B_1 \arrow[r, shift left] \arrow[r, shift right] & B_0 \arrow[l]
\end{tikzcd}$$

by definition

$$\begin{aligned}
\mathbf{t} \circ \mathbf{z} \;&\Leftrightarrow\; (kg, k_1 z, kf) + (kf, tf_0, hf) = (kg, tg_0, hg) + (hg, h_1 z, hf) \\
&\Leftrightarrow\; (kg, m\langle k_1 z, tf_0\rangle, hf) = (kg, m\langle tg_0, h_1 z\rangle, hf) \\
&\Leftrightarrow\; m\langle k_1 z, tf_0\rangle = m\langle tg_0, h_1 z\rangle \qquad (3.1)
\end{aligned}$$

and also by definition t is an internal natural transformation when

$$m\langle k_1, td\rangle = m\langle tc, h_1\rangle \qquad (3.2)$$

which is equivalent to saying that (k, t, h) is natural relative to $\left(c^{\rightarrow}, 1_{A_1}, d^{\rightarrow}\right)$, as displayed below

$$\begin{tikzcd}
\cdots & A_1 \arrow[r, equal] \arrow[d, shift right, "ed"'] \arrow[d, shift left, "ec"] & A_1 \arrow[dl, "1"'] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\
\cdots & A_1 \arrow[r, shift left] \arrow[r, shift right] & A_0 \arrow[l]
\end{tikzcd} .$$

■

Corollary 1. *Every internal natural transformation is a natural 2-cell.*

Proof: Simply observe that

$$(3.2) \Longrightarrow (3.1)$$

since

$$\begin{aligned}
m\langle k_1, td\rangle z &= m\langle tc, h_1\rangle z \\
m\langle k_1 z, tdz\rangle &= m\langle tcz, h_1 z\rangle \\
m\langle k_1 z, tf_0\rangle &= m\langle tg_0, h_1 z\rangle .
\end{aligned}$$

■

The notion of a category with a 2-cell structure, besides giving a simple characterization of a 2-category as

$$\text{“2-category”} = \text{“sesquicategory”} + \text{“naturality condition”};$$

it also provides a powerful tool to construct examples in arbitrary situations.

Example 2. *Consider* $\boldsymbol{C}$ *a category and*

$$H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Mon$$

a functor into Mon, the category of monoids, together with a natural transformation

$$D : UH \times \hom_{\mathbf{C}} \longrightarrow \hom_{\mathbf{C}}$$

(where $U : Mon \longrightarrow Set$ *denotes the forgetful functor) satisfying*

$$\begin{aligned} D\,(0, f) &= f \\ D\,(x' + x, f) &= D\,(x', D\,(x, f)) \end{aligned}$$

for all $f : A \longrightarrow B$ *in* $\boldsymbol{C}$ *and* $x', x \in H\,(A, B)$*, with* 0 *the zero of the monoid* $H\,(A, B)$ *considered in additive notation.*
A 2-cell structure in $\boldsymbol{C}$ *is now given as*

$$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "D(x,f)"'{name=L}] & B \arrow[from=U, to=L, Rightarrow, "(x{,}f)"] \end{tikzcd}$$

with vertical composition

$$\begin{tikzcd} A \arrow[rr, bend left=60, "f"{name=U}] \arrow[rr, "D(x,f)"{name=M}] \arrow[rr, bend right=60, "D(x'{,}D(x{,}f))"'{name=L}] & & B \arrow[from=U, to=M, Rightarrow, "(x{,}f)"] \arrow[from=M, to=L, Rightarrow, "(x'{,}D(x{,}f))"] \end{tikzcd}$$

$$(x', D\,(x, f)) + (x, f) = (x' + x, f)$$

(well defined because $D(x'+x,f) = D(x', D(x,f))$*), with identity 2-cells*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow (0,f)"] & B \end{tikzcd}$$

well defined because $D(0,f) = f$*, and the left and right actions of morphisms in 2-cells,*

$$\begin{tikzcd} A' \arrow[r, "h"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "D(x,f)"'] \arrow[r, phantom, "\Downarrow (x,f)"] & B \arrow[r, "g"] & B' \end{tikzcd}$$

$$g(x,f)h = (gxf, gfh) = (H(h,g)(x), gfh).$$

If in addition,

$$D(y,g)x + yf = yD(x,f) + gx \tag{3.3}$$

for all x, y, f, g *pictured as*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "D(x,f)"'] \arrow[r, phantom, "\Downarrow (x,f)"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "D(y,g)"'] \arrow[r, phantom, "\Downarrow (y,g)"] & C \end{tikzcd},$$

then the result is a 2-category.

In some cases, the above example may even be pushed further.

Example 3. *Suppose the functor*

$$\hom_{\mathbf{C}} : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Set$$

may be extended to Mon, that is, there is a functor (denote it by map, and think of the underlying map of a homomorphism)

$$map : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Mon \xrightarrow{U} Set$$

with hom $\subseteq Umap$*, in the sense that* $\hom(A,B) \subseteq Umap(A,B)$ *naturally for every* $A, B \in \mathbf{C}$*;*
Now, given any functor

$$K : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Mon$$

and any natural transformation

$$D : K \longrightarrow map,$$

define

$$\begin{aligned} H(A,B) &= \{(x,f) \in UK(A,B) \times \hom(A,B) \mid D(x) + f \in \hom(A,B)\} \\ H(h,g)(x,f) &= (gxh, gfh) \end{aligned}$$

and obtain a functor $H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Set$. *With obvious* $\mathrm{dom}, \mathrm{cod}, 0, +,$ *a 2-cell structure in* $\mathbf{C}$ *is obtained as follows*

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "D(x)+f"'] \arrow[r, phantom, "\Downarrow (x,f)"] & B \end{tikzcd}$$

where $(x,f) \in H(A,B)$,
vertical composition: $(x', D(x) + f) + (x,f) = (x' + x, f)$
identity: $(0,f)$
left and right actions: $g(x,f)h = (gxh, gfh)$.
If in addition the property

$$D(y)x + gx + yf = yD(x) + yf + gx \tag{3.4}$$

is satisfied for all $(x,f) \in H_1(A,B)$ *and* $(y,g) \in H_1(A,C)$, *then the resulting structure is a 2-category.*

Remark 1. In particular, if $\mathbf{C}$ is an Ab-category, a 2-Ab-category as defined in [5] and [7] is obtained in this way; in that case the functor hom is in fact a functor

$$\hom : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Ab.$$

Giving a 2-cell structure is then to give a functor (usually required to be an Ab-functor) $H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Ab$, and a natural transformation $D : H \longrightarrow \hom$. This 2-cell structure makes $\mathbf{C}$ a 2-category (in fact a 2-Ab-category) if in addition the condition (3.4) is satisfied, which in the abelian context simplifies to $D(y)x = yD(x)$. Furthermore, as proved in [5], every 2-cell structure (if enriched in Ab) is obtained in this way.

3.0.1. *The example of Groups.* In the case of $\mathbf{C} = Grp$ the category of groups and group homomorphisms, the construction of Example 2 is so general that it includes the canonical 2-cells that are obtained if considering each group as a one object groupoid and each group homomorphism as a functor. In that case, as it is well known, a 2-cell

$$t : f \longrightarrow g$$

from the homomorphism f to the homomorphism g, both from the group A to the group B, is an element $t \in B$ such that

$$tf(x) = g(x)\, t \quad , \quad \text{for all } x \in A.$$

Now, given t and f, the homomorphism g is uniquely determined as

$$g(x) = tf(x)\, t^{-1} = {}^{t}f(x),$$

and hence, this particular 2-cell structure over Grp is an instance of Example 2 with Grp instead of Mon.

To see this just consider H the functor that projects the second argument

$$\begin{aligned} H : Grp^{op} \times Grp &\longrightarrow Grp \\ (A, B) &\longmapsto B \end{aligned}$$

and

$$\begin{aligned} D : B \times \hom(A, B) &\longrightarrow \hom(A, B) \\ (t, f) &\longmapsto {}^{t}f \end{aligned}$$

and it is a straightforward calculation to check that

$$\begin{aligned} D(0, f) &= f \\ D(t + t', f) &= D(t, D(t', f)) \end{aligned}$$

and also, since condition (3.3) is satisfied, the 2-cell structure is natural.

3.0.2. *The example of crossed modules.* In the case $\mathbf{C}$=$X - Mod$, the category of crossed modules, we have the canonical 2-cell structure given by derivations, and it is an instance of Example 3 with Grp instead of Mon:

The objects in X-Mod are of the form

$$A = \left(X \xrightarrow{d} B, \varphi : B \longrightarrow Aut(X) \right)$$

where $d : X \longrightarrow B$ is a group homomorphism, together with a group action of B in X denoted by $b \cdot x$ satisfying

$$\begin{aligned} d(b \cdot x) &= bd(x)b^{-1} \\ d(x) \cdot x' &= x + x' - x; \end{aligned}$$

a morphism $f : A \longrightarrow A'$ in X-Mod is of the form

$$f = (f_1, f_0)$$

where $f_1 : X \longrightarrow X'$ and $f_0 : B \longrightarrow B'$ are group homomorphisms such that

$$f_0 d = d' f_1$$

and

$$f_1(b \cdot x) = f_0(b) \cdot f_1(x).$$

Clearly there are functors

$$map : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Grp$$

sending (A, A') to the group of pairs (f_1, f_0) of maps (not necessarily homomorphisms) $f_1 : UX \longrightarrow UX'$ and $f_0 : UB \longrightarrow UB'$ such that

$$f_0 d = d' f_1,$$

with the group operation defined componentwise

$$(f_1, f_0) + (g_1, g_0) = (f_1 + g_1, f_0 + g_0).$$

Also there is a functor

$$M : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Grp$$

sending (A, A') to the group $M(A, A') = \{t \mid t : UB \longrightarrow UX' \text{ is a map}\}$, and a natural transformation

$$D : M \longrightarrow map$$

defined by

$$D(A, A')(t) = (td, dt).$$

Now, define $H(A, A')$ as

$$\{(t, f) \mid t \in M(A, A'), f = (f_1, f_0) : A \longrightarrow A', (td + f_1, dt + f_0) \in \hom(A, A')\}.$$

It is well known that the map $t : B \longrightarrow X'$ is such that

$$t(bb') = t(b) + f_0(b) \cdot t(b') \quad , \quad \text{for all } b, b' \in B,$$

while $(td + f_1, dt + f_0) \in \hom(A, A')$ asserts that the pair $(td + f_1, dt + f_0)$ is a morphism of crossed modules

$$\begin{tikzcd} X \arrow[r, "d"] \arrow[d, "td+f_1"'] & B \arrow[d, "dt+f_0"] \\ X' \arrow[r, "d"] & B' \end{tikzcd} \tag{3.5}$$

and it is equivalent to

- $dt + f_0$ is a homomorphism of groups
$$dt\,(bb') = d\,(t\,(b) + f_0\,(b) \cdot t\,(b'))$$
- $td + f_1$ is a homomorphism of groups
$$t\,(d\,(x)\,d\,(x')) = t\,(dx) + f_0 d\,(x) \cdot td\,(x')$$
- the square (3.5) commutes, which is trivial because $(f_1, f_0) \in \hom(A, A')$
- $(td + f_1)$ preserves the action of $(dt + f_0)$
$$t\left(bd\,(x)\,b^{-1}\right) = t\,(b) + f_0\,(b) \cdot t\,(d\,(x)) + f_0\left(bd\,(x)\,b^{-1}\right) \cdot (-t\,(b))\,.$$

3.0.3. *The commutator.* Previous examples apply to arbitrary (even large) categories, provided they admit the functors and the natural transformations as specified. Interesting examples also appear if one tries to particularize the category $\mathbf{C}$. For example if $\mathbf{C}$ has only one object, or if it is a preorder; the first case gives something that particularizes to a (strict) monoidal category (with fixed set of objects) in the presence of the naturality condition; while the second case gives something that particularizes to an enriched category over monoids.

The simplest case, when $\mathbf{C}$=$\mathbf{1}$, gives Monoids and Commutative Monoids under the naturality condition; so in particular, if considering only invertible 2-cell structures the result is Groups and Abelian Groups, respectively.

The well known reflection
$$Gr \xrightarrow{I} Ab,$$
accordingly to G. Janelidze, generalizes to a reflexion
$$\text{2-cellstruct}(\mathbf{C}) \xrightarrow{I} \text{nat-2-cellstruct}(\mathbf{C})$$
from the category of 2-cell structures over $\mathbf{C}$, into the subcategory of natural 2-cell structures over $\mathbf{C}$, sending each 2-cell structure to its “naturalization”;

and, under the assumption that all the 2-cells are invertible, one may consider for each

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow y"] & B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow x"] & C \end{tikzcd}$$

the commutator

$$\begin{aligned} [x,y] &= (c_1+d_2-d_1-c_2)(x,y) \\ &= c_1(x,y)+d_2(x,y)-d_1(x,y)-c_2(x,y) \end{aligned}$$

where

$$\begin{aligned} c_1(x,y) &= \operatorname{cod}(x)\,y\ ,\ c_2(x,y)=x\operatorname{cod}(y) \\ d_1(x,y) &= \operatorname{dom}(x)\,y\ ,\ d_2(x,y)=x\operatorname{dom}(y)\,, \end{aligned}$$

and the comparison with $0_{\operatorname{cod}(x)\operatorname{cod}(y)}$ tell us the obstruction that x and y offer to be composed horizontally.

We will not developed this concept further, at the moment we are only observing that in the case of **C** being an Ab-category (see [5],[7] and Remark 1) then the notion of commutator reduces to

$$[x,y]=D(x)\,y-xD(y)\,.$$

In fact the notion of 2-Ab-category (as introduced in [5]) may be pushed further in the direction of a sesquicategory enriched in any category **A** with a "forgetful" functor into Sets.

It is a simple generalization of Example 3 and it is as follows.

For a category **A** with a "forgetful" functor into Sets, $U:\mathbf{A}\longrightarrow$Sets, assume the existence of a functor

$$map:\mathbf{C}^{op}\times\mathbf{C}\longrightarrow\mathbf{A}$$

such that

$$\hom_C(A,B)\subseteq U map(A,B)$$

(as in Example 3).

If **A** were monoidal and **C** a category enriched in **A** then we would always be in the above conditions, simply by choosing $map=\hom$. It is then reasonably to say that, in this more general context, the category **C** is weakly enriched in **A** (for example, in this sense, Groups are weakly enriched in Groups, and every algebraic structure is weakly enriched in itself). In this conditions, we may be interested in considering only 2-cell structures over **C** that are "weakly enriched" in **A** in the same way as **C** is. This concept is obtained if considering only the 2-cell structures that are given by

$$H(A,B)=\{x\in UM(A,B)\mid U\operatorname{dom}x, U\operatorname{cod}x\in\hom(A,B)\}$$

for some $M, \mathrm{dom}, \mathrm{cod}$ being part of an internal category object in $\mathbf{A}^{\mathbf{C}^{op}\times\mathbf{C}}$, of the form

$$M\times_{map}M\xrightarrow{+}M\;\begin{array}{c}\xrightarrow{\mathrm{dom}}\\ \xleftarrow{0}\\ \xrightarrow[\mathrm{cod}]{}\end{array}\;map,$$

with the obvious restrictions after applying U.

It is interesting now to observe that in the case of $\mathbf{A}=Grp$ the result of this is precisely the construction of Example 3. If $\mathbf{A}=$Ab and also requiring M to be an Ab-functor, then the result is a 2-Ab-category if also adding the condition

$$D(x)\,y=xD(y)$$

for all appropriate x and y.

Next we formalize the category of 2-cell structures.

4. The category of 2-cell structures

For a fixed category **C**, there is the category 2-cellstruct(**C**) of all possible 2-cell structures over **C**, as well as the subcategory nat-2-cellstruct(**C**) of natural 2-cell structures over C and inv-2-cellstruct(C) of all the invertible 2-cell structures over C. The category 2-cellstruct(**C**) has a initial object (the discrete 2-cell structure) and a terminal object (the codiscrete 2-cell structure). If C is of the form Cat(B) for some category B, it also has the canonical 2-cell structure of internal transformations and the canonical natural 2-cell structure of internal natural transformations.

For the sake of a formal definition: the objects of 2-cellstruct(**C**) are of the form

$$\mathbf{H}=(H,\mathrm{dom},\mathrm{cod},0,+)$$

where

$$H:\mathbf{C}^{op}\times\mathbf{C}\longrightarrow Set$$

is a functor and

$$H \times_{\mathrm{hom}} H \xrightarrow{+} H \begin{array}{c} \xrightarrow{\mathrm{dom}} \\ \xleftarrow{0} \\ \xrightarrow[\mathrm{cod}]{} \end{array} \mathrm{hom}_{\mathbf{C}}$$

are natural transformations, such that

$$(\mathrm{hom}_{\mathbf{C}}, H, \mathrm{dom}, \mathrm{cod}, 0, +)$$

is a category object in the functor category $\mathrm{Set}^{C^{op}\times C}$, in other words is an object in $\mathrm{Cat}\big(Set^{C^{op}\times C}\big)$.
A morphism $\varphi : \mathbf{H} \longrightarrow \mathbf{H}'$ is a natural transformation

$$\varphi : H \longrightarrow H'$$

such that

$$\begin{aligned} \mathrm{dom}'\varphi &= \mathrm{dom} \\ \mathrm{cod}'\varphi &= \mathrm{cod} \\ \varphi 0 &= 0' \\ \varphi + &= +'(\varphi \times \varphi) . \end{aligned}$$

We will often write simply H to refer to a 2-cell structure, whenever confusion is unlikely to appear.

The purpose of describing 2-cellstruct($\mathbf{C}$), the category of all 2-cell structures over a given category $\mathbf{C}$, is the study of pseudocategories in $\mathbf{C}$. The notion of pseudocategory in a category $\mathbf{C}$ depends of the 2-cell structure considered over $\mathbf{C}$. For example, a pseudocategory in $\mathbf{C}$ with the codiscrete 2-cell structure is a precategory, while if considering the discrete structure it is a internal category. It seems to be interesting to study, for a given category $\mathbf{C}$, how the notion of pseudocategory changes from a precategory to a internal category by changing the 2-cell structure considered over $\mathbf{C}$. This topic is studied in [9] for the case of weakly Mal'cev sesquicategories.

Also, every morphism

$$\varphi : H \longrightarrow H' \tag{4.1}$$

in 2-cellstruct($\mathbf{C}$) induces a functor

$$PsCat\,(\mathbf{C}, H) \longrightarrow PsCat\,(\mathbf{C}, H') \tag{4.2}$$

from pseudocategories in $\mathbf{C}$ relative to the 2-cell structure H to pseudocategories in $\mathbf{C}$ relative to the 2-cell structure H'.

At this point it would be also interesting to study the notion of equivalent 2-cell structures, saying that (4.1) is an equivalence whenever (4.2) is. We choose to postpone it for a future work.

The notion of a pseudocategory ([6],[5]) rests in the construction of the induced 2-cells between pullback objects, thus the following definition.

5. Cartesian 2-cell structure

It will be useful to consider 2-cell structures such that the functor $H(D, \_) : \mathbf{C} \longrightarrow Set$ preserves pullbacks for every object D in $\mathbf{C}$, that is: the functor

$$H : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow Set,$$

giving a 2-cell structure to a category $\mathbf{C}$, has the following property

$$H\left(D, A \times_{\{f,g\}} B\right) \overset{\varphi}{\cong} \{(x,y) \in H(D,A) \times H(D,B) \mid fx = gy\}$$

for every object D in $\mathbf{C}$ and pullback diagram

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "f"] \\ A \arrow[r, "g"] & C \end{tikzcd},$$

where φ is required to be a natural isomorphism, that is, for every $h : D \longrightarrow D'$, the following square commutes

$$\begin{tikzcd} H(D, A \times_C B) \arrow[r, "\cong\varphi"] \arrow[d, "H(h,1)"'] & \{(x,y) \mid fx = gy\} \arrow[d] \\ H(D', A \times_C B) \arrow[r, "\cong\varphi"] & \{(x',y') \mid fx' = gy'\} \end{tikzcd}$$

or in other words, that

$$\langle x, y \rangle h = \langle xh, yh \rangle$$

as displayed in the diagram below

$$\begin{tikzcd} D' \arrow[r, "h"] & D \arrow[dr, dashed, "<x,y>"] \arrow[drr, Rightarrow, bend left, "y"] \arrow[ddr, Rightarrow, bend right, "x"'] & & \\ & & A \times_C B \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ & & A \arrow[r, "f"'] & C \end{tikzcd}.$$

In particular, for $D = A' \times_{C'} B'$, and appropriate x, y, z as in

$$\begin{tikzcd}
A' \arrow[d, Rightarrow, "x"'] & A' \times'_C B' \arrow[l, "\pi'_1"'] \arrow[r, "\pi'_2"] \arrow[d, Rightarrow, "x \times_z y"'] & B' \arrow[d, Rightarrow, "y"] & , & C' \arrow[d, Rightarrow, "z"] & , \\
A & A \times_C B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B & , & C &
\end{tikzcd}$$

it follows that $x \times_z y$ is the unique element (2-cell) in $H\left(A' \times_{C'} B', A \times_C B\right)$ satisfying

$$\begin{aligned}
\pi_2 \left(x \times_z y\right) &= y\pi'_2 \\
\pi_1 \left(x \times_z y\right) &= y\pi'_1 .
\end{aligned}$$

Let $\mathbf{C}$ be a category.

Definition 5 (cartesian 2-cell structure)**.** *A 2-cell structure* $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$ *over the category* $\mathbf{C}$ *is said to be* Cartesian *if the functor* $H(D, \_) : \mathbf{C} \longrightarrow Set$ *preserves pullbacks for every object* D *in* $\mathbf{C}$*.*

6. Pseudocategories

The notion of pseudocategory (as introduced in [6]) is only defined internally to a 2-category. Here we extend it to the more general context of a category with a 2-cell structure (or sesquicategory).

First consider three leading examples.

In any category $\mathbf{C}$, it is always possible to consider two different 2-cell structures, namely the discrete one, obtained when $H = \mathrm{hom}$ and $\mathrm{dom}, \mathrm{cod}, 0, +$ are all identities, and the codiscrete one, obtained when $H = \mathrm{hom} \times \mathrm{hom}$, dom is second projection, cod is first projection, 0 is diagonal and + is uniquely determined. A pseudocategory, in the first situation, becomes an internal category in $\mathbf{C}$, while in the second situation becomes a precategory in $\mathbf{C}$.

In the case of $\mathbf{C} = Cat$, and choosing the natural transformations to be the 2-cell structure, a pseudocategory becomes a pseudo-double-category (see [6]), which is at the same time a generalization of a double-category and a bicategory.

At this level of generality, it becomes clear that there is no particular reason why to prefer a specific 2-cell structure in a category instead of another.

For instance, in Top it is usually considered the 2-cell structure obtained from the homotopy classes of homotopies, but other may be consider as well.

Let $\mathbf{C}$ be a category with a cartesian 2-cell structure $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$.

Definition 6. *A pseudocategory in* $\mathbf{C}$*, with respect to the 2-cell structure* $(H, \mathrm{dom}, \mathrm{cod}, 0, +)$*, is a system*

$$(C_0, C_1, d, c, e, m, \alpha, \lambda, \rho)$$

where $(C_0, C_1, d, c, e, m, ...)$ *is a thin protocategory (see definition in* [8]*), and* α, λ, ρ *are natural and invertible 2-cells, in the sense that*

$$\alpha \in H(C_3, C_1) \;\; \text{and} \; \lambda, \rho \in H(C_1, C_1)$$

with

$$\begin{aligned} \mathrm{dom}(\alpha) &= mm_1 \;,\; \mathrm{cod}(\alpha) = mm_2 \\ \mathrm{dom}(\lambda) &= me_2 \;,\; \mathrm{dom}(\rho) = me_1 \;,\; \mathrm{cod}(\lambda) = 1_{C_1} = \mathrm{cod}(\rho) \end{aligned}$$

satisfying the following conditions

$$\begin{aligned} d\lambda &= 0_d = d\rho \\ c\lambda &= 0_c = c\rho \\ d\alpha &= 0_{d\pi_2 p_2} \;,\; c\alpha = 0_{c\pi_1 p_1} \end{aligned}$$

$$\lambda e = \rho e$$

$$m(\alpha \times 0_1) + \alpha(1 \times m \times 1) + m(0_1 \times \alpha) = \alpha(m \times 1 \times 1) + \alpha(1 \times 1 \times m) \tag{6.1}$$

$$m(\rho \times 0_1) + \alpha i_0 = m(0_1 \times \lambda). \tag{6.2}$$

Some remarks:

A 2-cell $x \in H(A, B)$ is invertible when there is a (necessarily unique) element

$$-x \in H(A, B)$$

such that $\mathrm{dom}(x) = \mathrm{cod}(-x)$, $\mathrm{cod}(x) = \mathrm{dom}(-x)$ and

$$x + (-x) = 0_{\mathrm{cod}(x)} \;,\; (-x) + x = 0_{\mathrm{dom}(x)};$$

A 2-cell $x \in H(A, B)$ is natural when

$$\mathrm{cod}(x)\, y + x\, \mathrm{dom}(y) = x\, \mathrm{cod}(y) + \mathrm{dom}(x)\, y$$

for every element $y \in H(X, A)$ for every object X in $\mathbf{C}$.

The 2-cells α, λ, ρ may also be presented as

$$\begin{tikzcd} C_3 \arrow[r, bend left, "mm_1"] \arrow[r, bend right, "mm_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & C_1 \end{tikzcd} \;,\quad \begin{tikzcd} C_1 \arrow[r, bend left, "me_2"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\lambda"] & C_1 \end{tikzcd} \;,\quad \begin{tikzcd} C_1 \arrow[r, bend left, "me_1"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\rho"] & C_1 \end{tikzcd}\;.$$

Equations (6.1) and (6.2) correspond to the internal versions of the famous MacLane's coherence pentagon and triangle, presented diagrammatically as follows

$$\begin{tikzcd} & \bullet \arrow[r, "m(0_{C_1}\times_{C_0}\alpha)"] \arrow[dl, "\alpha(1_{C_1}\times_{C_0}1_{C_1}\times_{C_0}m)"'] & \bullet \arrow[dr, "\alpha(1_{C_1}\times_{C_0}m\times_{C_0}1_{C_1})"] & \\ \bullet \arrow[drr, "\alpha(m\times_{C_0}1_{C_1}\times_{C_0}1_{C_1})"'] & & & \bullet \arrow[dl, "m(\alpha\times_{C_0}0_{C_1})"] \\ & & \bullet & \end{tikzcd} \tag{6.3}$$

$$\begin{tikzcd} \bullet \arrow[rr, "\alpha i_0"] \arrow[dr, "m(0_{C_1}\times_{C_0}\lambda)"'] & & \bullet \arrow[dl, "m(\rho\times_{C_0}0_{C_1})"] \\ & \bullet & \end{tikzcd} \tag{6.4}$$

and restated in terms of generalized elements as

$$\begin{tikzcd} & f(g(hk)) \arrow[r, "f\alpha_{g,h,k}"] \arrow[dl, "\alpha_{f,g,hk}"'] & f((gh)k) \arrow[dr, "\alpha_{f,gh,k}"] & \\ (fg)(hk) \arrow[dr, "\alpha_{fg,h,k}"'] & & & (f(gh))k \arrow[dll, "\alpha_{f,g,h}k"] \\ & ((fg)h)k & & \end{tikzcd} \tag{pentagon}$$

$$\begin{tikzcd} f(1g) \arrow[rr, "\alpha_{f,1,g}"] \arrow[dr, "f\lambda_g"'] & & (f1)g \arrow[dl, "\rho_f g"] \\ & fg & \end{tikzcd} \tag{midle triangle}$$

where $m\langle f, g\rangle = fg$.

As proved in [4] this two coherence conditions plus the naturality of α, λ, ρ are sufficient to show that every diagram involving instances of α, λ, ρ, possible nested with $m\,(- \times -)$, commutes; there are other such diagrams that still play an important role. They are the following

$$\begin{tikzcd} 1(fg) \arrow[rr, "\alpha_{1,f,g}"] \arrow[dr, "\lambda_{fg}"'] & & (1f)g \arrow[dl, "\lambda_f g"] \\ & fg & \end{tikzcd} \tag{6.5}$$

$$\begin{tikzcd} & 1(f1) \arrow[r, "\alpha_{1,f,1}"] \arrow[dl, "1\rho_f"'] & (1f)1 \arrow[dr, "\lambda_f 1"] & \\ 1f \arrow[drr, "\lambda"'] & & & f1 \arrow[dl, "\rho"] \\ & & f & \end{tikzcd} \tag{6.6}$$

$$\begin{tikzcd} f(g1) \arrow[rr, "\alpha_{f,g,1}"] \arrow[dr, "f\rho_g"'] & & (fg)1 \arrow[dl, "\rho_{fg}"] \\ & fg & \end{tikzcd} \tag{6.7}$$

and correspond, respectively, (when internalized) to the following equations

$$\begin{aligned} m\,(\lambda \times 0_{C_1}) + \alpha i_2 &= \lambda m, \\ \rho + m e_1 \lambda + \alpha\,(i_2 e_1 = i_1 e_2) &= \lambda + m e_2 \rho \\ \rho m + \alpha i_1 &= m\,(0_{C_1} \times \rho) \end{aligned}$$

and since the 2-cells are invertible, the above set of equations may be presented as

$$\begin{aligned} \alpha i_2 &= -m\,(\lambda \times 0_{C_1}) + \lambda m, \\ \alpha\,(i_2 e_1 = i_1 e_2) &= -m e_1 \lambda - \rho + \lambda + m e_2 \rho \\ \alpha i_1 &= -\rho m + m\,(0_{C_1} \times \rho)\,. \end{aligned}$$

Note that the definition of pseudocategory as introduced in [6] does not ask for naturality of α, λ, ρ. This is because in there we were assuming that the considered 2-cell structure was a 2-category, and hence every 2-cell was natural. It would be interesting to see what are the exact requirements about the naturality of α, λ, ρ in order to be able to prove MacLane's Coherence

Theorem, but we choose not to investigate it here and postpone it for a future work.

We observe that it is not necessary to ask for the naturality of α, λ, ρ in the sense defined above as to be natural with respect to all possible 2-cells. A quick look at the proof of MacLane's Coherence Theorem tells us that it is sufficient to consider naturality (of each one of α, λ, ρ) with respect to α, λ, ρ and instances of $m\,(u \times_{C_1} v)$ where u and v are α, λ, ρ or again of the form $m\,(\_ \times_{C_1} \_)$.

As mentioned in the introduction of this article, we will not concentrate on this problem since the main examples are 2-cell structures where every 2-cell is natural (as the examples of groups and crossed-modules above) and even if considering some 2-cell structure that it is not natural we may always use the "naturalization reflexion"

$$\text{2-cellstruct}(\mathbf{C}) \xrightarrow{I} \text{nat-2-cellstruct}(\mathbf{C}).$$

References

[1] Crans, S.E: A tensor product for Gray-categories, Theory and Applications of Categories, Vol.5, N^o.2, 1999, pp.12-69.

[2] Gray, J.W.: *Formal Category Teory: Adjointness for 2-categories*, Lecture Notes in Mathematics, Springer-Verlag, 1974

[3] Leinster, T.: *Higher Operads, Higher Categories*, London Mathematical Society Lecture Notes Series, Cambridge University Press, 2003 (electronic version).

[4] MacLane, S.: *Categories for the working Mathematician*, Springer-Verlag, 1998, 2^{nd} edition.

[5] Martins-Ferreira, N.: Internal Weak Categories in Additive 2-Categories with Kernels, Fields Institute Communications, Volume 43, p.387-410, 2004.

[6] Martins-Ferreira, N: Pseudo-categories, *JHRS*, Vol.1(1), 2006, pp.47-78.

[7] Martins-Ferreira, N.: The (tetra)category of pseudocategories in additive 2-categories with kernels, submited to Applied Categorical Structures, special volume CT2007.

[8] Martins-Ferreira, N.: Internal precategories in weakly Mal'cev categories, Ch. 7, PhD Thesis, 2008.

[9] Martins-Ferreira, N.: Pseudocategories in weakly Mal'cev sesquicategories, Ch. 9, PhD Thesis, 2008.

[10] Street, R.H.: Cosmoi of internal categories, Trans. Amer. Math. Soc. 258, 1980, 271-318

[11] Street, R.H.: Fibrations in Bicategories, Cahiers Topologie et Géométrie Diferéntielle Catégoriques, 21:111-120, 1980.

N. Martins-Ferreira
Polytechnic Institute of Leiria

E-mail address: `nelsonmf@estg.ipleiria.pt`
URL: `http://www.estg.ipleiria.pt/~nelsonmf`