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LOCALIC REAL FUNCTIONS: A GENERAL SETTING

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ABSTRACT: A [semi-]continuous real function of a frame (locale) L has up to now been understood as a frame homomorphism from the frame $\mathfrak{L}(\mathbb{R})$ of reals into L [as a frame homomorphism (modulo some conditions) from certain subframes of $\mathfrak{L}(\mathbb{R})$ into L]. Thus, these continuities involve different domains. It would be desirable if all these continuities were to have $\mathfrak{L}(\mathbb{R})$ as a common domain. This paper demonstrates that this is possible if one replaces the codomain L by $\mathcal{S}(L)$ — the dual of the co-frame of all sublocales of L. This is a remarkable conception, for it eventually permits to have among other things the following: *lower semicontinuous + upper semicontinuous = continuous*. In this new environment we will have the same freedom in pointfree topology which so far was available only to the traditional topologists, for the lattice-ordered ring $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))$ may be viewed as the pointfree counterpart of the lattice-ordered ring $\mathbb{R}^X$ with X a topological space. Notably, we now have the pointfree version of the concept of an arbitrary *not necessarily continuous* function on a topological space. Extended real functions on frames are considered too.

KEYWORDS: Frame, locale, sublocale, frame of reals, frame continuous real function, lower semicontinuous, upper semicontinuous, lower regularization, upper regularization, insertion theorem, normal, monotonically normal, extremally disconnected, perfectly normal, countably paracompact.

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1. Introduction

A *continuous real function* on a frame L has up to now been understood as a frame homomorphism from the frame $\mathfrak{L}(\mathbb{R})$ of reals into L (definitions are given below). Similarly, *lower* and *upper semicontinuous real functions* on L have up to now been understood as frame homomorphisms (modulo some conditions) from certain subframes of $\mathfrak{L}(\mathbb{R})$ into L. The main disadvantage of

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these three types of continuities is that the involved functions have different domains. Our aim is to find a common framework for all those continuities. Even if – naively speaking – pointfree topology is supposed to be more general than the classical point-set topology, the parallel between functions and sets in point-set topology does not yet have a fine counterpart in pointfree topology. For instance, pointfree topology suffers of not having the concept of an *arbitrary not necessarily continuous* real function on L which would correspond to an *arbitrary* member of $\mathbb{R}^X$ with X a topological space. After this paper, the following quotation from [8, Chapter 1] will make sense in the pointfree setting:

> "The set $\mathrm{C}(X)$ of all continuous, real-valued functions on a topological space X will be provided with an algebraic structure and an order structure. Since their definitions do not involve continuity, we begin by imposing these structures on the collection $\mathbb{R}^X$ of all functions from X into the set $\mathbb{R}$ of real numbers. [...] In fact, it is clear that $\mathbb{R}^X$ is a commutative ring with unity element (provided that X is non empty). [...] The partial ordering on $\mathbb{R}^X$ is defined by: $f \geq g$ if and only if $f(x) \geq g(x)$ for all $x \in X$. [...] The set of all continuous functions from the topological space X into the space $\mathbb{R}$ is denoted $\mathrm{C}(X)$. [...] Therefore $\mathrm{C}(X)$ is a commutative ring, a subring of $\mathbb{R}^X$."

In actual fact, $\mathbb{R}^X$ has many other important substructures such as the lattices $\mathrm{LSC}(X)$ and $\mathrm{USC}(X)$ of all lower and upper semicontinuous real functions. Living all in $\mathbb{R}^X$, members of $\mathrm{LSC}(X) \cup \mathrm{USC}(X)$ are comparable and after making lattice or algebraic operations on them, formed in $\mathbb{R}^X$, they are still in $\mathbb{R}^X$ even though may travel far away from $\mathrm{LSC}(X) \cup \mathrm{USC}(X)$. We just want to be able to do the same in the pointfree topology. On the other hand, arbitrary members of $\mathbb{R}^X$ have their lower and upper semicontinuous regularizations (possibly in $\overline{\mathbb{R}}^X$, where $\overline{\mathbb{R}}$ stands for the extended reals). Even if all the individual concepts mentioned above do already exist in pointfree topology, one cannot manage with them so freely as described above due to the lack of a pointfree analogue of $\mathbb{R}^X$.

The aim of this paper is to remove that inconvenient situation and to exhibit an environment which would give pointfree topologists the same freedom which up to now was only available to the traditional topologists. Specifically, the pointfree counterpart of the lattice-ordered ring $\mathbb{R}^X$ will just be the lattice-ordered ring $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))$ where $\mathcal{S}(L)$ is the dual of the co-frame of all sublocales of L (recall that the set $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), M)$ of all frame homomorphisms from $\mathfrak{L}(\mathbb{R})$ into an arbitrary frame M is a lattice-ordered ring; see [2]). In order to motivate the idea of our approach, we first recall that the familiar (dual) adjunction

$$\mathsf{Top} \underset{\Sigma}{\overset{\mathcal{O}}{\rightleftarrows}} \mathsf{Frm}$$

between the categories of topological spaces and frames yields the bijection

$$\mathsf{Top}(X, \mathbb{R}) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)$$

where $\mathcal{O}X$ is the topology of the topological space X and $\mathbb{R}$ is endowed with its natural topology. The reason for that is that the spectrum $\Sigma\mathfrak{L}(\mathbb{R})$ is homeomorphic to the space $\mathbb{R}$ (cf. [15] and [2]). If we now observe that the set $\mathbb{R}^X$ is in an obvious bijection with $\mathsf{Top}((X, \mathcal{P}(X)), (\mathbb{R}, \tau))$ where τ is *any* topology on the set $\mathbb{R}$, we would, in particular, have a bijection

$$\mathbb{R}^X \simeq \mathsf{Top}((X, \mathcal{P}(X)), \mathbb{R})) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{P}(X)),$$

where $\mathbb{R}$ carries the natural topology.

Now, for a general frame L, the role of the lattice $\mathcal{P}(X)$ of *all* subspaces of X, is taken by the lattice $\mathcal{S}(L)$ of all sublocales of L, which justifies to think of the members of

$$\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))$$

as *arbitrary not necessarily continuous* real functions on the frame L.

We also notice that by the homeomorphism between $\Sigma\mathfrak{L}(\mathbb{R})$ and $\mathbb{R}$, real numbers are thought as frame homomorphisms from $\mathfrak{L}(\mathbb{R})$ to the two-point frame $\{0, 1\}$. In our new framework, *arbitrary* real function on L can be interpreted as $\mathcal{S}(L)$*-valued real numbers* (cf. [2, Remark 9]).

At this stage, the reader might have noticed what often happens in mathematics: one situation which superficially seems to generalize another one, may well be viewed as its particular case. Indeed, replacing L in $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$ by $\mathcal{S}(L)$ yields a larger class of morphisms due to the embedding $L \hookrightarrow \mathcal{S}(L)$ via $a \mapsto {\uparrow}a$. On the other hand, one may say that we deal with a particular case of $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), M)$ with $M = \mathcal{S}(L)$. Then we replace M by the larger frame $\mathcal{S}(M)$ and we are back to our framework. Instead of thinking as to whether replacing M by $\mathcal{S}(M)$ eventually stabilizes, we emphasize that the *point* of this paper is that in $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))$ we can do things that cannot be done within $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$, thereby having a lot of various possibilities which were not possible within $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$. (This is as simple as that: studying X and its powerset $\mathcal{P}(X)$ is not the same as studying an arbitrary set Y.) It is the aim of this paper to exhibit these possibilities. Of course, we shall utilize as far as possible results that hold for $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), M)$ with arbitrary M. For instance, we need not check that $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))$ is a lattice-ordered ring.

This paper is, in some sense, a continuation of our previous papers [9, 10, 11], opening new horizons for the research programme started with [18] and [14].

In Sections 2 and 3 we recall the needed background on the frame of sublocales and the frame of reals as well as on semicontinuous and continuous real functions on a frame. Section 4 provides details of how to generate continuous real functions on frames. Section 5 introduces the main idea of the paper: the ring of all real functions on a frame, while Section 6 presents the relations between the old notions and the newly established ones. Section 7 provides a general procedure of constructing the lower and upper regularizations of an arbitrary real function on a frame. We conclude, in Section 8, with a list of our insertion and extension theorems formulated in the new setting.

Our general references for frames and locales are [15] and [20].

Convention. If not otherwise stated, L stands for an arbitrary frame.

2. The frame of sublocales

We begin by briefly recording some familiar notions and standard results on sublocales that we shall need. We use the approach of [19] in terms of sublocale sets.

In pointfree topology the points of a space are regarded as secondary to its open sets. Accordingly, in pointfree topology spaces are represented by *generalized lattices of open sets*, called *frames*, abstractly defined as complete lattices L in which the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. In particular, a classical space X is represented by its lattice $\mathcal{O}(X)$ of open sets. Continuous maps are represented by *frame homomorphisms*, that is, those maps between frames that preserve arbitrary joins (hence 1, the top) and finite meets (hence 0, the bottom). Frm is then the corresponding category of *frames* and *frame homomorphisms*. The set of all morphisms from L into M is denoted by $\mathsf{Frm}(L, M)$.

The above representation is contravariant (continuous maps $f : X \to Y$ are represented by frame homomorphisms $h : \mathcal{O}(Y) \to \mathcal{O}(X)$). This is easily mended, in order to keep the geometric (topological) motivation, by considering, instead of Frm simply its opposite category. It is called the category of *locales* and *localic maps*, and we have "generalized continuous maps" $f : L \to M$ that are precisely frame homomorphisms $h : L \longleftarrow M$.

In the whole paper we keep the algebraic (frame) approach and reasoning. The reader should keep in mind that the geometric (localic) motivation reads backwards.

Being a Heyting algebra, each frame L has the implication $\to$ satisfying the standard equivalence $a \wedge b \leq c$ iff $a \leq b \to c$. The *pseudocomplement* of an $a \in L$ is the element

$$a^* = a \to 0 = \bigvee\{b \in L : a \wedge b = 0\}.$$

Then: $a \leq a^{**}$ and $(\bigvee A)^* = \bigwedge_{a \in A} a^*$ for all $A \subseteq L$. In particular, $(\cdot)^*$ is order-reversing.

A subset S of L is a *sublocale* of L if, whenever $A \subseteq L$, $a \in L$ and $b \in S$, then $\bigwedge S \in L$ and $a \to b \in S$. The intersection of sublocales is again a sublocale, so that the set of all sublocales is a complete lattice under inclusion. In fact, it is a co-frame, in which $\{1\}$ is the bottom and L is the top [19].

Convention. For notational reasons, we shall make the co-frame of all sublocales into a frame $\mathcal{S}(L)$ by considering the opposite ordering:

$$S_1 \leq S_2 \quad \Leftrightarrow \quad S_2 \subseteq S_1.$$

Thus, given $\{S_i \in \mathcal{S}(L) : i \in I\}$, we have

$$\bigvee_{i \in I} S_i = \bigcap_{i \in I} S_i \quad \text{and} \quad \bigwedge_{i \in I} S_i = \{\bigwedge A : A \subseteq \bigcup_{i \in I} S_i\}.$$

Then $\{1\}$ is the top element and L is the bottom element in $\mathcal{S}(L)$ that we just denote by 1 and 0, respectively. The pseudocomplement of S in $\mathcal{S}(L)$ will standardly be denoted by S^*.

Among the important examples of sublocales are the *closed sublocales*

$$\mathfrak{c}(a) = {\uparrow}a = \{b \in L : a \le b\}$$

and the *open sublocales*

$$\mathfrak{o}(a) = \{a \to b : b \in L\}$$

where $a \in L$. We shall freely use the following properties:

Properties 2.1. *For all $a, b \in L$ and $A \subseteq L$:*

(1) $\mathfrak{c}(a) \le \mathfrak{c}(b)$ *if and only if* $a \le b$,
(2) $\mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \wedge \mathfrak{c}(b)$,
(3) $\mathfrak{c}(\bigvee A) = \bigvee_{a \in A} \mathfrak{c}(a)$,
(4) $\mathfrak{c}(\bigwedge A) \le \bigwedge_{a \in A} \mathfrak{c}(a)$.

We note that the map $a \mapsto \mathfrak{c}(a)$ is a frame embedding $L \hookrightarrow \mathcal{S}(L)$. The subframe of $\mathcal{S}(L)$ consisting of all closed sublocales will be denoted by $\mathfrak{c}L$. Clearly, L and $\mathfrak{c}L$ are isomorphic. Denoting by $\mathfrak{o}L$ the subframe of $\mathcal{S}(L)$ generated by all $\mathfrak{o}(a)$, $a \in L$, the triple $(\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$ constitutes a biframe [3], the so called *sublocale biframe*.

Lemma 2.2. *Let $a, b \in L$. Then:*

(1) $\mathfrak{o}(a) \ge \mathfrak{c}(b)$ *if and only if* $a \wedge b = 0$,
(2) $\mathfrak{o}(a) \le \mathfrak{c}(b)$ *if and only if* $a \vee b = 1$,
(3) $\mathfrak{c}(a) = \mathfrak{o}(b)$ *if and only if a and b are complements of each other,*
(4) $\mathfrak{c}(a) \vee \mathfrak{o}(a) = 1$ *and* $\mathfrak{c}(a) \wedge \mathfrak{o}(a) = 0$.

Thus $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements to each other in $\mathcal{S}(L)$. More generally, we have:

Lemma 2.3. *Let $a \in L$ and $S \in \mathcal{S}(L)$. Then:*

(1) $S \wedge \mathfrak{c}(a) = 0$ *if and only if* $S \le \mathfrak{o}(a)$,
(2) $S \wedge \mathfrak{o}(a) = 0$ *if and only if* $S \le \mathfrak{c}(a)$,
(3) $S \vee \mathfrak{c}(a) = 1$ *if and only if* $\mathfrak{o}(a) \le S$,
(4) $S \vee \mathfrak{o}(a) = 1$ *if and only if* $\mathfrak{c}(a) \le S$.

Proof: To show (1), let $S \wedge \mathfrak{c}(a) = 0$. Then

$$\mathfrak{o}(a) = \mathfrak{o}(a) \vee (S \wedge \mathfrak{c}(a)) = \mathfrak{o}(a) \vee S,$$

hence $S \leq \mathfrak{o}(a)$. Conversely, if $S \leq \mathfrak{o}(a)$, then $0 \leq S \wedge \mathfrak{c}(a) \leq \mathfrak{o}(a) \wedge \mathfrak{c}(a) = 0$. The proof of (2) is similar to that of (1). To show (3), let $S \vee \mathfrak{c}(a) = 1$. Then

$$\mathfrak{o}(a) = \mathfrak{o}(a) \wedge (S \vee \mathfrak{c}(a)) = \mathfrak{o}(a) \wedge S,$$

hence $\mathfrak{o}(a) \leq S$. Conversely, if $\mathfrak{o}(a) \leq S$, then $1 = \mathfrak{o}(a) \vee \mathfrak{c}(a) \leq S \vee \mathfrak{c}(a) \leq 1$. Again, the proof of (4) is similar to that of (3). ■

Given a sublocale S of L, its *closure* and *interior* are defined, respectively, by

$$\overline{S} = \bigvee\{\mathfrak{c}(a) : \mathfrak{c}(a) \leq S\} = \mathfrak{c}(\bigwedge S)$$

and

$$S^\circ = \bigwedge\{\mathfrak{o}(a) : S \leq \mathfrak{o}(a)\}.$$

Proposition 2.4. *Let $S, T \in \mathcal{S}(L)$, $a \in L$ and $A \subseteq L$. Then:*

(1) $\overline{1} = 1$, $\overline{S} \leq S$, $\overline{\overline{S}} = \overline{S}$, *and* $\overline{S \wedge T} = \overline{S} \wedge \overline{T}$,
(2) $0^\circ = 0$, $S^\circ \geq S$, $S^{\circ\circ} = S^\circ$, *and* $(S \vee T)^\circ = S^\circ \vee T^\circ$,
(3) $S^\circ = \left(\overline{S^*}\right)^* = \mathfrak{o}(\bigwedge S^*)$,
(4) $\mathfrak{c}(a)^\circ = \mathfrak{o}(a^*)$,
(5) $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$.

Proof: (1) can be seen in [19], while (2) is dual to (1). For (3), we have

$$\begin{aligned} S^\circ &= \bigwedge\{\mathfrak{o}(a) : S \leq \mathfrak{o}(a)\} \\ &= \bigwedge\{\mathfrak{o}(a) : \mathfrak{c}(a) \wedge S = 0\} \\ &= \bigwedge\{\mathfrak{o}(a) : \mathfrak{c}(a) \leq S^*\} \\ &= (\bigvee\{\mathfrak{c}(a) : \mathfrak{c}(a) \leq S^*\})^* \\ &= \left(\overline{S^*}\right)^* \\ &= \mathfrak{o}(\bigwedge S^*). \end{aligned}$$

To show (4), we have $\mathfrak{c}(a)^\circ = \mathfrak{o}(\bigwedge \mathfrak{o}(a)) = \mathfrak{o}(a^*)$, since $a^* = a \to 0 \leq a \to b$ for every $b \in L$. To show (5), we have $\left(\overline{\mathfrak{o}(a)}\right)^* = \left(\overline{\mathfrak{c}(a)^*}\right)^* = \mathfrak{c}(a)^\circ = \mathfrak{o}(a^*)$ which yields $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$. ■

3. Frames of reals and their continuity notions

Notation. We write $L = \langle A \rangle$ if L is generated by $A \subseteq L$.

There are various equivalent definitions of the frame of reals (see e.g. [15] and [2, 4]). In [2, 4], the *frame* $\mathfrak{L}(\mathbb{R})$ *of reals* is the frame generated by all pairs $(p,q) \in \mathbb{Q} \times \mathbb{Q}$ satisfying the following relations:

(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,
(R2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \leq r < q \leq s$,
(R3) $(p,q) = \bigvee\{(r,s) : p < r < s < q\}$,
(R4) $\bigvee_{p,q \in \mathbb{Q}}(p,q) = 1$.

One writes: $(p,\text{—}) = \bigvee_{q \in \mathbb{Q}}(p,q)$ and $(\text{—},q) = \bigvee_{p \in \mathbb{Q}}(p,q)$.

As we shall deal with frames of lower and upper reals too, we take $(r,\text{—})$ and $(\text{—},r)$ as primitive notions. We thus adopt the equivalent description of $\mathfrak{L}(\mathbb{R})$ proposed in [17]. Specifically, the *frame of reals* $\mathfrak{L}(\mathbb{R})$ is the one having generators of the form $(r,\text{—})$ and $(\text{—},r)$ subject to the following relations:

(r1) $(r,\text{—}) \wedge (\text{—},s) = 0$ whenever $r \geq s$,
(r2) $(r,\text{—}) \vee (\text{—},s) = 1$ whenever $r < s$,
(r3) $(r,\text{—}) = \bigvee_{s>r}(s,\text{—})$,
(r4) $(\text{—},r) = \bigvee_{s<r}(\text{—},s)$,
(r5) $\bigvee_{r \in \mathbb{Q}}(r,\text{—}) = 1$,
(r6) $\bigvee_{r \in \mathbb{Q}}(\text{—},r) = 1$.

With $(p,q) = (p,\text{—}) \wedge (\text{—},q)$ one goes back to (R1)-(R4).

So, we have the frame of reals

$$\mathfrak{L}(\mathbb{R}) = \langle \{(r,\text{—}),(\text{—},r) : r \in \mathbb{Q},\ (r,\text{—}),(\text{—},r) \text{ satisfy (r1)—(r6) for all } r \in \mathbb{Q}\} \rangle\,,$$

and its two subframes:

$$\begin{aligned} \mathfrak{L}_u(\mathbb{R}) &= \langle \{(r,\text{—}) : r \in \mathbb{Q},\ (r,\text{—}) \text{ satisfy (r3) and (r5) for all } r \in \mathbb{Q}\} \rangle\,, \\ \mathfrak{L}_l(\mathbb{R}) &= \langle \{(\text{—},r) : r \in \mathbb{Q},\ (\text{—},r) \text{ satisfy (r4) and (r6) for all } r \in \mathbb{Q}\} \rangle\,. \end{aligned}$$

These subframes may be called, respectively, the *frame of upper reals* and the *frame of lower reals*. When dropping (r5) and (r6), we get the *extended* variants of frames just introduced, viz.:

$$\begin{aligned}
\mathfrak{L}(\overline{\mathbb{R}}) &= \langle\{(r,\text{—}),(\text{—},r) : r\in\mathbb{Q},\ (r,\text{—}),(\text{—},r) \text{ satisfy (r1)-(r4) for all } r\in\mathbb{Q}\}\rangle\,,\\
\mathfrak{L}_u(\overline{\mathbb{R}}) &= \langle\{(r,\text{—}) : r\in\mathbb{Q},\ (r,\text{—}) \text{ satisfy (r3) for all } r\in\mathbb{Q}\}\rangle\,,\\
\mathfrak{L}_l(\overline{\mathbb{R}}) &= \langle\{(\text{—},r) : r\in\mathbb{Q},\ (\text{—},r) \text{ satisfy (r4) for all } r\in\mathbb{Q}\}\rangle\,.
\end{aligned}$$

Members of

$$\begin{aligned}
\overline{\mathrm{lsc}}(L) &= \mathsf{Frm}(\mathfrak{L}_u(\overline{\mathbb{R}}),L),\\
\overline{\mathrm{usc}}(L) &= \mathsf{Frm}(\mathfrak{L}_l(\overline{\mathbb{R}}),L),\\
\overline{\mathrm{c}}(L) &= \mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}),L)
\end{aligned}$$

are called *extended*, respectively, *lower semicontinuous, upper semicontinuous*, and *continuous real functions* on L, while members of

$$\begin{aligned}
\mathrm{lsc}(L) &= \{f\in\mathsf{Frm}(\mathfrak{L}_u(\mathbb{R}),L) : \bigvee_{r\in\mathbb{Q}} \mathfrak{o}(f(r,\text{—}))=1\},\\
\mathrm{usc}(L) &= \{f\in\mathsf{Frm}(\mathfrak{L}_l(\mathbb{R}),L) : \bigvee_{r\in\mathbb{Q}} \mathfrak{o}(f(\text{—},r))=1\},\\
\mathrm{c}(L) &= \mathsf{Frm}(\mathfrak{L}(\mathbb{R}),L)
\end{aligned}$$

are called *lower semicontinuous, upper semicontinuous*, and *continuous real functions* on L [14].

Remark. The extra conditions in the definitions of $\mathrm{lsc}(L)$ and $\mathrm{usc}(L)$ come from [14], to which we refer for their role. The latter has been then exhibited in [9, 10, 11]. Their role will also be seen in this paper (cf. the proof of (4) of Proposition 6.1). Comparing with [14], we recall that, in this paper, we make the co-frame of sublocales into the frame $\mathcal{S}(L)$ by reversing the ordering.

Partial orderings. (1) The set $\overline{\mathrm{lsc}}(L)$ is partially ordered by

$$f_1\le f_2 \quad\Leftrightarrow\quad f_1(r,\text{—})\le f_2(r,\text{—}) \quad \text{for all } r\in\mathbb{Q}.$$

Under this ordering, $\overline{\mathrm{lsc}}(L)$ is closed under finite meets and arbitrary nonempty joins:

$$\begin{aligned}
(f_1\wedge f_2)(r,\text{—}) &= f_1(r,\text{—})\wedge f_2(r,\text{—}),\\
\left(\bigvee\mathcal{F}\right)(r,\text{—}) &= \bigvee_{f\in\mathcal{F}} f(r,\text{—}),
\end{aligned}$$

where $\varnothing\neq\mathcal{F}\subseteq\overline{\mathrm{lsc}}(L)$. The constant map with value 1 is the top, while there is no bottom in $\overline{\mathrm{lsc}}(L)$.

(2) The set $\overline{\mathrm{usc}}(L)$ is partially ordered by the reverse pointwise ordering:

$$f_1 \leq f_2 \quad \Leftrightarrow \quad f_2(\text{—}, r) \leq f_1(\text{—}, r) \quad \text{for all } r \in \mathbb{Q},$$

under which it is closed with respect to finite joins and arbitrary nonempty meets:

$$\begin{aligned} (f_1 \vee f_2)(\text{—}, q) &= f_1(\text{—}, q) \wedge f_2(\text{—}, q), \\ \left(\bigwedge \mathcal{F}\right)(\text{—}, r) &= \bigvee_{f \in \mathcal{F}} f(\text{—}, r), \end{aligned}$$

where $\varnothing \neq \mathcal{F} \subseteq \overline{\mathrm{usc}}(L)$. The constant map with value 1 is the bottom element, while there is no top element in $\overline{\mathrm{usc}}(L)$.

(3) The set $\overline{\mathrm{c}}(L)$ is partially ordered by

$$f_1 \leq f_2 \quad \Leftrightarrow \quad f_{1|\mathfrak{L}_u(\overline{\mathbb{R}})} \leq f_{2|\mathfrak{L}_u(\overline{\mathbb{R}})} \quad \Leftrightarrow \quad f_{2|\mathfrak{L}_l(\overline{\mathbb{R}})} \leq f_{1|\mathfrak{L}_l(\overline{\mathbb{R}})}.$$

Remark. There is an order-isomorphism $-(\cdot) : \overline{\mathrm{lsc}}(L) \to \overline{\mathrm{usc}}(L)$ defined by

$$(-f)(\text{—}, r) = f(-r, \text{—}) \quad \text{for all } r \in \mathbb{Q}.$$

When restricted to $\mathrm{lsc}(L)$ it becomes an isomorphism from $\mathrm{lsc}(L)$ onto $\mathrm{usc}(L)$. Its inverse, denoted by the same symbol, maps a $g \in \overline{\mathrm{usc}}(L)$ into $-g \in \overline{\mathrm{lsc}}(L)$ defined by $(-g)(r, \text{—}) = g(\text{—}, -r)$ for all $r \in \mathbb{Q}$, etc.

4. Generating frame homomorphisms by scales

A way of generating continuous real functions on frames by the so called scales has been described in detail in [2] with $\mathfrak{L}(\mathbb{R})$ being generated by pairs of rationals satisfying the relations (R1) – (R4) (cf. also [15, p. 127]). In what follows we decompose the investigations of [2] into two pieces so as to have ways of generating all the types of real functions on frames by means of scales. In what follows p, q, r, s stand for rationals.

Definition 4.1. A family $\mathcal{C} = \{c_r : r \in \mathbb{Q}\} \subseteq L$ is called an *extended scale* in L if

$$c_r \vee c_s^* = 1 \text{ whenever } r < s.$$

An extended scale is called a *scale* if

$$\bigvee_{r \in \mathbb{Q}} c_r = 1 = \bigvee_{r \in \mathbb{Q}} c_r^*.$$

Remark 4.2. An extended scale $\mathcal{C} = \{c_r : r \in \mathbb{Q}\}$ is necessarily an antitone family. Furthermore, if $\mathcal{C}$ consists of complemented elements, then $\mathcal{C}$ is an extended scale if and only if it is antitone. Indeed, if $\mathcal{C}$ is antitone and each c_r is complemented, then $c_r \vee c_s^* \geq c_r \vee c_r^* = 1$ whenever $r < s$.

Lemma 4.3. *Let $\mathcal{C} = \{c_r : r \in \mathbb{Q}\}$ be an extended scale in L and let*

$$f(r,\text{—}) = \bigvee_{s>r} c_s \quad \textit{and} \quad f(\text{—},r) = \bigvee_{s<r} c_s^*$$

for all $r \in \mathbb{Q}$. Then the following assertions hold:

(1) *The above two formulas determine an $f \in \overline{\mathrm{c}}(L)$.*
(2) *If $\mathcal{C}$ is a scale, then $f \in \mathrm{c}(L)$.*

Proof: (1) We must check that f turns conditions (r1)-(r4) into identities in L. To show (r1), let $q \leq p$. Then $f(p,\text{—}) \wedge f(\text{—},q) \leq c_p \wedge c_q^* \leq c_p \wedge c_p^* = 0$. As for (r2), if $p < r < s < q$, then $f(p,\text{—}) \vee f(\text{—},q) \geq c_r \vee c_s^* = 1$ by the definition of an extended scale. To show (r3), we have

$$\bigvee_{r>p} f(r,\text{—}) = \bigvee_{r>p} \bigvee_{s>r} c_s = \bigvee_{s>p} c_s = f(p,\text{—}).$$

Condition (r4) holds since

$$\bigvee_{s<q} f(\text{—},s) = \bigvee_{s<q} \bigvee_{r<s} c_r^* = \bigvee_{r<q} c_r^* = f(\text{—},q).$$

(2) Now we check (r5)-(r6) for $\mathcal{C}$ being a scale. As for (r5), we have

$$\bigvee_{r} f(r,\text{—}) = \bigvee_{r} \bigvee_{s>r} c_s = 1.$$

To have (r6), we observe that $\bigvee_r f(\text{—},r) = \bigvee_r \bigvee_{s<r} c_s^* = 1$. We have shown that $f \in \mathrm{c}(L)$. ■

Lemma 4.4. *Let $f, g \in \overline{\mathrm{c}}(L)$ be generated by the extended scales $\{c_r : r \in \mathbb{Q}\}$ and $\{d_r : r \in \mathbb{Q}\}$, respectively. Then:*

(1) *$f(r,\text{—}) \leq c_r \leq f(\text{—},r)^*$ for all $r \in \mathbb{Q}$,*
(2) *$f \leq g$ if and only if $c_r \leq d_s$ whenever $r > s$ in $\mathbb{Q}$.*

Proof: (1) We have

$$f(r,\text{—}) = \bigvee_{s>r} c_s \leq c_r \leq c_r^{**} \leq \bigwedge_{s<r} c_s^{**} = (\bigvee_{s<r} c_s^*)^* = f(\text{—},r)^*.$$

(2) First notice that if $r > s$, then

$$f(\text{—},r)^* = f(\text{—},r)^* \wedge (f(s,\text{—}) \vee f(\text{—},r)) = f(\text{—},r)^* \wedge f(s,\text{—}).$$

So, if $f \leq g$, then $c_r \leq f(\text{—}, r)^* \leq f(s, \text{—}) \leq g(s, \text{—}) \leq d_s$.

For the reverse implication, let $q > r > s$. Since $d_s^* \leq c_r^*$, we have

$$d_s^* \leq \bigvee_{r<q} c_r^* = f(\text{—}, q).$$

So, $g(\text{—}, q) = \bigvee_{s<q} d_r^* \leq f(\text{—}, q)$, i.e. $f \leq g$. ■

Even if we know that $\mathfrak{c}(a)^* = \mathfrak{o}(a)$, we keep at the notation in terms of $\mathfrak{c}$ in order to be in tune with Definition 4.1.

Lemma 4.5. *Let $\{d_r : r \in \mathbb{Q}\} \subseteq L$ be antitone. Then:*

(1) *$\{\mathfrak{c}(d_r) : r \in \mathbb{Q}\}$ is an extended scale in $\mathcal{S}(L)$,*

(2) *If $\bigvee_{r\in\mathbb{Q}} d_r = 1$ and $\bigvee_{r\in\mathbb{Q}} \mathfrak{c}(d_r)^* = 1$, then $\{\mathfrak{c}(d_r) : r \in \mathbb{Q}\}$ is a scale in $\mathcal{S}(L)$.*

Proof: (1) follows immediately by Remark 4.2. One detail for (2) is that $\bigvee_r \mathfrak{c}(d_r) = \mathfrak{c}(\bigvee_r d_r) = \mathfrak{c}(1) = 1$. ■

5. Localic real functions

In general topology one sometimes sees the phrase:

Let X be a topological space and let f be an arbitrary not necessarily continuous real-valued function on X.

In this section this will become possible in the pointfree setting.

Notation 5.1. *We let*

$$\begin{aligned} \mathrm{F}(L) &= \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L)) &= \mathrm{c}(\mathcal{S}(L)), \\ \overline{\mathrm{F}}(L) &= \mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathcal{S}(L)) &= \overline{\mathrm{c}}(\mathcal{S}(L)). \end{aligned}$$

Definition 5.2. An $F \in \mathrm{F}(L)$ will be called an *arbitrary real function* on L. We shall say that F is:

(1) *lower semicontinuous* if $F(r, \text{—})$ is a closed sublocale for all r, i.e. $F(\mathfrak{L}_u(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(2) *upper semicontinuous* if $F(\text{—}, r)$ is a closed sublocale for all r , i.e. $F(\mathfrak{L}_l(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

(3) *continuous* if $F(p, q)$ is a closed sublocale for all p, q, i.e. $F(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}(L)$.

Notation 5.3. We denote by

$$\mathrm{LSC}(L), \mathrm{USC}(L) \text{ and } \mathrm{C}(L)$$

the collections of all lower semicontinuous, upper semicontinuous, and continuous members of $F(L)$. If we replace $\mathfrak{L}_u(\mathbb{R})$, $\mathfrak{L}_l(\mathbb{R})$, and $\mathfrak{L}(\mathbb{R})$ by $\mathfrak{L}_u(\overline{\mathbb{R}})$, $\mathfrak{L}_l(\overline{\mathbb{R}})$, and $\mathfrak{L}(\overline{\mathbb{R}})$ in, respectively, (1), (2), and (3) above, we get the collections

$$\overline{\mathrm{LSC}}(L), \overline{\mathrm{USC}}(L), \text{ and } \overline{\mathrm{C}}(L)$$

of all *extended* lower semicontinuous, upper semicontinuous, and continuous members of $\overline{\mathrm{F}}(L)$. Of course, one has

$$\mathrm{C}(L) = \mathrm{LSC}(L) \cap \mathrm{USC}(L) \quad \text{and} \quad \overline{\mathrm{C}}(L) = \overline{\mathrm{LSC}}(L) \cap \overline{\mathrm{USC}}(L).$$

All the above collections of morphisms are partially ordered according to the definition of partial orderings in $\overline{\mathrm{lsc}}(L)$, $\overline{\mathrm{usc}}(L)$, and $\overline{\mathrm{c}}(L)$ where L is replaced by $\mathcal{S}(L)$. Thus, given $F, G \in \overline{\mathrm{F}}(L)$, one has

$$\begin{aligned} F \leq G \quad &\Leftrightarrow \quad F(r, \text{—}) \leq G(r, \text{—}) \quad \text{for all } r \in \mathbb{Q} \\ &\Leftrightarrow \quad G(\text{—}, r) \leq F(\text{—}, r) \quad \text{for all } r \in \mathbb{Q}. \end{aligned}$$

Remark 5.4. In addition to the discussion in the introductory section, we note that there also is a way of interpreting the above definitions from a bitopological point of view. Indeed, as explained in [14], $\mathrm{lsc}(L)$ corresponds bijectively to the biframe maps $(\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (\mathcal{S}(L), \mathfrak{o}L, \mathfrak{c}L)$ and, dually, for $\mathrm{usc}(L)$. Therefore

$$\begin{aligned} \mathrm{lsc}(L) &\simeq \mathbf{BiFrm}\,((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), (\mathcal{S}(L), \mathfrak{o}L, \mathfrak{c}L) \\ &\subseteq \mathbf{BiFrm}\,((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), (\mathcal{S}(L), \mathcal{S}(L), \mathcal{S}(L)) \\ &\simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L)) \;=\; \mathrm{F}(L) \end{aligned}$$

and, dually,

$$\begin{aligned} \mathrm{usc}(L) &\simeq \mathbf{BiFrm}\,((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), (\mathcal{S}(L), \mathfrak{c}(L), \mathfrak{o}L) \\ &\subseteq \mathbf{BiFrm}\,((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), (\mathcal{S}(L), \mathcal{S}(L), \mathcal{S}(L)) \\ &\simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L)) \;=\; \mathrm{F}(L). \end{aligned}$$

Given a complemented sublocale S of L, we define the *characteristic map* $\chi_S : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ by

$$\chi_S(r, \text{—}) = \begin{cases} 1 & \text{if } r < 0, \\ S^* & \text{if } 0 \leq r < 1, \\ 0 & \text{if } r \geq 1, \end{cases} \quad \text{and} \quad \chi_S(\text{—}, r) = \begin{cases} 0 & \text{if } r \leq 0, \\ S & \text{if } 0 < r \leq 1, \\ 1 & \text{if } r > 1, \end{cases}$$

for each $r \in \mathbb{Q}$. Then, as in the classical context, we have:

(1) $\chi_S \in \mathrm{LSC}(L)$ if and only if S is open,
(2) $\chi_S \in \mathrm{USC}(L)$ if and only if S is closed,
(3) $\chi_S \in \mathrm{C}(L)$ if and only if S is clopen.

6. Embedding of $\mathrm{lsc}(L)$ and $\mathrm{usc}(L)$ into $\mathrm{F}(L)$

In this section, we consider order-embeddings of $\overline{\mathrm{lsc}}(L)$, $\overline{\mathrm{usc}}(L)$, and $\overline{\mathrm{c}}(L)$ into $\overline{\mathrm{F}}(L)$. We begin with lower semicontinuity. Let $f \in \overline{\mathrm{lsc}}(L)$. Then $\{f(r,\text{—}) : r \in \mathbb{Q}\}$ is an antitone family and $\{\mathfrak{c}(f(r,\text{—})) : r \in \mathbb{Q}\}$ is then an extended scale in $\mathcal{S}(L)$ (cf. Remark 4.2). Thus, using Lemma 4.3, we can define

$$\Psi_l : \overline{\mathrm{lsc}}(L) \to \overline{\mathrm{LSC}}(L)$$

by the following two formulas:

$$\Psi_l(f)(r,\text{—}) = \bigvee_{s>r} \mathfrak{c}(f(s,\text{—})) \quad \text{and} \quad \Psi_l(f)(\text{—},r) = \bigvee_{s<r} \mathfrak{c}(f(s,\text{—}))^*.$$

We observe that $\Psi_l(f)(r,\text{—}) = \mathfrak{c}(f(r,\text{—})) \in \mathfrak{c}(L)$, so that $\Psi_l(f) \in \overline{\mathrm{LSC}}(L)$ indeed.

Dually, we define

$$\Psi_u : \overline{\mathrm{usc}}(L) \to \overline{\mathrm{USC}}(L)$$

by

$$\Psi_u(f) = -\Psi_l(-f).$$

An easy calculation shows that

$$\Psi_u(f)(r,\text{—}) = \bigvee_{s>r} \mathfrak{c}(f(\text{—},s))^* \quad \text{and} \quad \Psi_u(f)(\text{—},r) = \bigvee_{s<r} \mathfrak{c}(f(\text{—},s)).$$

Since $\Psi_u(f)(\text{—},r) = \mathfrak{c}(f(\text{—},r)) \in \mathfrak{c}(L)$, we indeed have $\Psi_u(f) \in \overline{\mathrm{USC}}(L)$.

Observe that $\{\mathfrak{c}(f(\text{—},r))^* : r \in \mathbb{Q}\}$ is an extended scale and that, thus, $\Psi_u(f)$ is being generated by it (cf. Lemma 4.3 and note that $\mathfrak{c}(f(-,r))^{**} = \mathfrak{c}(f(-,r)))$.

Finally, using Ψ_l and Ψ_u, we define

$$\Psi : \overline{\mathrm{c}}(L) \to \overline{\mathrm{C}}(L)$$

by

$$\Psi(f)(p,q) = \Psi_l(f)(p,\text{—}) \wedge \Psi_u(f)(\text{—},q).$$

Proposition 6.1. *The following assertions hold:*

1. $\Psi_l : \overline{\mathrm{lsc}}(L) \to \overline{\mathrm{LSC}}(L)$ *is a lattice isomorphism preserving arbitrary nonempty joins.*
2. $\Psi_u : \overline{\mathrm{usc}}(L) \to \overline{\mathrm{USC}}(L)$ *is a lattice isomorphism preserving arbitrary nonempty meets.*
3. $\Psi : \overline{\mathrm{c}}(L) \to \overline{\mathrm{C}}(L)$ *is a lattice isomorphism.*
4. *The restrictions* $\Psi_{l|\mathrm{lsc}(L)}, \Psi_{u|\mathrm{lsc}(L)}$, *and* $\Psi_{|\mathrm{c}(L)}$ *take values in* $\mathrm{LSC}(L)$, $\mathrm{USC}(L)$ *and* $\mathrm{C}(L)$, *respectively, and have the corresponding properties.*

Proof: To show (1), we first notice that Ψ_l is a bijection. Injectivity: if $\Psi_l(f) = \Psi_l(g)$, then $\mathfrak{c}(f(r,\text{—})) = \mathfrak{c}(g(r,\text{—}))$ for all $r \in \mathbb{Q}$, hence $f(r,\text{—}) = g(r,\text{—})$, so that $f = g$. Surjectivity: for any $F \in \overline{\mathrm{LSC}}(L)$, we have $\Psi_l(f) = F$ with $f(r,\text{—}) = \bigwedge F(r,\text{—})$ (i.e. $f = \mathfrak{c}^{-1} \circ F$ where $\mathfrak{c} : L \to \mathfrak{c}(L)$ is the frame isomorphism sending a to $\mathfrak{c}(a)$). Also, we have

$$\begin{aligned}\Psi_l(f \wedge g)(r,\text{—}) &= \mathfrak{c}((f \wedge g)(r,\text{—}))\\ &= \mathfrak{c}(f(r,\text{—}) \wedge g(r,\text{—}))\\ &= \mathfrak{c}(f(r,\text{—})) \wedge \mathfrak{c}(g(r,\text{—}))\\ &= \Psi_l(f)(r,\text{—}) \wedge \Psi_l(g)(r,\text{—})\end{aligned}$$

and

$$\Psi_l(\bigvee \mathcal{F})(r,\text{—}) = \mathfrak{c}((\bigvee \mathcal{F})(r,\text{—})) = \bigvee_{f\in\mathcal{F}} \mathfrak{c}(f(r,\text{—})) = \bigvee_{f\in\mathcal{F}} \Psi_l(f)(r,\text{—}).$$

Assertion (2) follows from (1), while combining (1) and (2) yields (3). Now, we move to the restriction $\Psi_{l|\mathrm{lsc}(L)}$. Assume $f \in \mathrm{lsc}(L)$. Then

$$\bigvee_{r\in\mathbb{Q}} \Psi_l(f)(r,\text{—}) = \bigvee_{r\in\mathbb{Q}} \mathfrak{c}(f(r,\text{—})) = \mathfrak{c}(1) = 1$$

and

$$\bigvee_{r\in\mathbb{Q}} \Psi_l(f)(\text{—},r) = \bigvee_{r\in\mathbb{Q}} \bigvee_{s<r} \mathfrak{c}(f(s,\text{—}))^* = \bigvee_{r\in\mathbb{Q}} \mathfrak{o}(f(r,\text{—})) = 1$$

(the latter equality is just the extra condition defining lower semicontinuity). Thus $\Psi_l(f) \in \mathrm{LSC}(L)$. The remaining cases follow from what has just been proved. ■

Due to the fact, that members of $\overline{\mathrm{lsc}}(L)$ and $\overline{\mathrm{usc}}(L)$ have different domains, they have so far been compared in terms of the relations of *minorization* and *majorization*. We shall now show that after embedding $\overline{\mathrm{lsc}}(L)$ and $\overline{\mathrm{usc}}(L)$ into

$\overline{\mathrm{F}}(L)$ those two relations become superfluous. We first recall that if $f \in \overline{\mathrm{lsc}}(L)$ and $g \in \overline{\mathrm{usc}}(L)$, then one says that f *minorizes* g (written: $f \lhd g$) iff

$$f(r,\text{—}) \wedge g(\text{—},s) = 0 \quad \text{for all } r > s \text{ in } \mathbb{Q}.$$

Clearly, $f \lhd g$ if and only if $f(r,\text{—}) \leq g(\text{—},r)^*$ for all $r \in \mathbb{Q}$. Further, one says that f *majorizes* g (written: $f \blacktriangleright g$) iff

$$f(r,\text{—}) \vee g(\text{—},s) = 1 \quad \text{for all } r < s \text{ in } \mathbb{Q}.$$

Proposition 6.2. *Let $f \in \overline{\mathrm{lsc}}(L)$ and $g \in \overline{\mathrm{usc}}(L)$. Then the following hold:*

(1) *$f \lhd g$ if and only if $\Psi_l(f) \leq \Psi_u(g)$.*
(2) *$f \blacktriangleright g$ if and only if $\Psi_l(f) \geq \Psi_u(g)$.*

Proof: (1) Recall that $\Psi_l(f)$ and $\Psi_u(g)$ are generated by the extended scales $\{\mathfrak{c}(f(r,\text{—}) : r \in \mathbb{Q}\}$ and $\{\mathfrak{c}(g(\text{—},r))^* : r \in \mathbb{Q}\}$. By the definition of $\lhd$ and Lemma 2.2(1) we have $f \lhd g$ if and only if $\mathfrak{c}(f(s,\text{—})) \leq \mathfrak{c}(g(\text{—},r))^*$ whenever $r > s$, which on account of Lemma 4.4(2) is equivalent to the statement that $\Psi_l(f) \leq \Psi_u(g)$.

(2) A similar argument applies except that the appeal to (1) of Lemma 2.2 is replaced by an application of Lemma 2.2(2). ■

We close this section by providing relations that hold between the characteristic functions $l_a \in \mathrm{lsc}(L)$ and $u_a \in \mathrm{usc}(L)$, $a \in L$, defined as follows:

$$l_a(r,\text{—}) = \begin{cases} 1 & \text{if } r < 0, \\ a & \text{if } 0 \leq r < 1, \\ 0 & \text{if } r \geq 1, \end{cases} \quad \text{and} \quad u_a(\text{—},r) = \begin{cases} 0 & \text{if } r \leq 0, \\ a & \text{if } 0 < r \leq 1, \\ 1 & \text{if } r > 1. \end{cases}$$

Properties 6.3. *For each $a \in L$ we have:*

$$\Psi_l(l_a) = \chi_{\mathfrak{o}(a)} \quad \textit{and} \quad \Psi_u(u_a) = \chi_{\mathfrak{c}(a)}.$$

7. Semicontinuous regularizations of localic real functions

We first recall from general topology that, given a topological space X and an *arbitrary not necessarily continuous function* $f : X \to \mathbb{R}$ one defines its lower and upper regularizations (also called lower and upper limit functions, respectively) as follows:

$$f_*(x) = \bigvee_{U \in \mathbb{U}_x} \bigwedge f(U) \quad \text{and} \quad f^*(x) = \bigwedge_{U \in \mathbb{U}_x} \bigvee f(U)$$

for all $x \in X$ where $\mathbb{U}_x$ is the system of all open neighbourhoods of x. Clearly, $f^* = -(-f)_*$ and both f_* and f^* may take values in $\overline{\mathbb{R}}$ (see [1, 5, 21], as well as [12] and [13] for the lattice-valued and the domain-valued cases, respectively).

In [10], the authors made some effort to define the corresponding concepts in the context of frame real functions but with serious limitations. Now, in our much wider framework we can overcome all those obstacles and have a nice theory being quite analogous to the classical one. We have choosen to use F° and F^- to denote the lower and upper regularizations of F rather than the standard notation F_* and F^*. This is to avoid confusion with the well established notation for morphisms in pointfree topology (cf. [15, p. 40]). As a matter of fact, our notation is even better than the standard one, for it emphasises the analogy between lower and upper regularizations and interior and closure operators (cf. Propositions 7.3 and 7.4 below as well as Properties 7.10).

We begin with the following which is actually a repetition of Lemma 4.5 in the context of antitone subfamilies of $\mathcal{S}(L)$.

Lemma 7.1. *Let $\{S_r : r \in \mathbb{Q}\} \subseteq \mathcal{S}(L)$ be an antitone family. Then the following assertions hold:*

(1) *$\{\overline{S_r} : r \in \mathbb{Q}\}$ is an extended scale in $\mathcal{S}(L)$.*
(2) *If $\bigvee_{r \in \mathbb{Q}} \overline{S_r} = 1 = \bigvee_{r \in \mathbb{Q}} \big(\overline{S_r}\big)^*$, then it is a scale in $\mathcal{S}(L)$.*

In particular, if $F \in \overline{\mathrm{F}}(L)$, then the assignment $r \longmapsto F(r,\text{—})$ is antitone and we, thus, have an extended scale $\{\overline{F(r,\text{—})} : r \in \mathbb{Q}\}$. Moreover, when $F \in \mathrm{F}(L)$, then

$$\bigvee_{r \in \mathbb{Q}} \big(\overline{F(r,\text{—})}\big)^* \geq \bigvee_{r \in \mathbb{Q}} F(r,\text{—})^* \geq \bigvee_{r \in \mathbb{Q}} F(\text{—},r) = 1.$$

To motivate our concepts of lower and upper regularizations of an arbitrary localic real function, we recall from [12, Proposition 5.3] (also see [13, Proposition 4.8]), that if $f : X \to \mathbb{R}$ is an arbitrary function (where X is a topological space) which is generated by an antitone family $\{F_r : r \in \mathbb{Q}\}$, that is: $f(x) = \sup\{r \in \mathbb{Q} : x \in F_r\}$, then the lower (resp., upper) regularization f_* (resp., f^*) of f is generated by the family $\{\overline{F_r} : r \in \mathbb{Q}\}$ (resp., $\{{F_r}^\circ : r \in \mathbb{Q}\}$). We now state the following:

Definition 7.2. The *lower regularization* F° of $F \in \overline{\mathrm{F}}(L)$ is defined by:

$$F^\circ(r,\text{—}) = \bigvee_{s>r} \overline{F(s,\text{—})} \quad \text{and} \quad F^\circ(\text{—},r) = \bigvee_{s<r} \left(\overline{F(s,\text{—})}\right)^*.$$

Dually, the *upper regularization* F^- of F is defined by:

$$F^- = \text{-}(\text{-}F)^\circ.$$

An easy calculation gives:

$$F^-(r,\text{—}) = \bigvee_{s>r} \left(\overline{F(\text{—},s)}\right)^* \quad \text{and} \quad F^-(\text{—},r) = \bigvee_{s<r} \overline{F(\text{—},s)}.$$

The following proposition shows that $(\cdot)^\circ : \overline{\mathrm{F}}(L) \to \overline{\mathrm{LSC}}(L)$ is actually an interior-like operator. In fact, the properties stated there resemble the properties of the classical interior operator.

Proposition 7.3. *The following hold for all* $F, G \in \overline{\mathrm{F}}(L)$*:*

(1) $\top^\circ = \top$*, where* $\top(\text{—},r) = 0$ *for all* $r \in \mathbb{Q}$*,*
(2) $F^\circ \le F$*,*
(3) $F^{\circ\circ} = F^\circ$*,*
(4) $(F \wedge G)^\circ = F^\circ \wedge G^\circ$.

Proof: (1) We first notice that if $s < r$, $\top(s,\text{—}) \vee \top(\text{—},r) = 1$, so that $\top(s,\text{—}) = 1$ for all s. Thus, $\top^\circ(\text{—},r) = \bigvee_{s<r}\left(\overline{\top(s,\text{—})}\right)^* = \overline{1}^* = 0 = \top(\text{—},r)$ for all r.

(2) We have $F^\circ(r,\text{—}) = \bigvee_{s>r} \overline{F(s,\text{—})} \le \bigvee_{s>r} F(s,\text{—}) = F(r,\text{—})$, hence $F^\circ \le F$.

(3) We only need to check that $F^\circ \le F^{\circ\circ}$. Given $r > s$ we have

$$\overline{F(r,\text{—})} \le \bigvee_{t>s} \overline{F(t,\text{—})} = F^\circ(s,\text{—}),$$

hence $\overline{F(r,—)} = \overline{\overline{F(r,—)}} \le \overline{F^\circ(s,—)}$. By recalling that $\{\overline{F(r,—)} : r \in \mathbb{Q}\}$ and $\{\overline{F^\circ(r,—)} : r \in \mathbb{Q}\}$ are scales that generate F° and $F^{\circ\circ}$, respectively, we get $F^\circ \le F^{\circ\circ}$ according to Lemma 4.4(2).

(4) Let us calculate

$$\begin{aligned}
(F^\circ \wedge G^\circ)(r,—) &= F^\circ(r,—) \wedge G^\circ(r,—) \\
&= \bigvee_{s>r} \overline{F(s,—)} \wedge \bigvee_{s>r} \overline{G(s,—)} \\
&= \bigvee_{s,t>r} \left(\overline{F(s,—)} \wedge \overline{G(t,—)}\right) \\
&= \bigvee_{s,t>r} \overline{F(s,—) \wedge G(t,—)} \\
&\le \bigvee_{s,t>r} \overline{F(s \wedge t,—) \wedge G(s \wedge t,—)} \\
&= \bigvee_{s>r} \overline{(F \wedge G)(s,—)} \\
&= (F \wedge G)^\circ(r,—),
\end{aligned}$$

while the reverse inequality is obvious. ■

As a corollary of Proposition 7.3 we have

$$\overline{\mathrm{LSC}}(L) = \{F \in \overline{\mathrm{F}}(L) : F = F^\circ\}$$

and

$$F^\circ = \bigvee \{G \in \overline{\mathrm{LSC}}(L) : G \le F\}.$$

For the sake of completeness we include the dual variant of Proposition 7.3 (showing that the operator $(\cdot)^- : \overline{\mathrm{F}}(L) \to \overline{\mathrm{USC}}(L)$ behaves like a closure operator).

Proposition 7.4. *The following hold for all $F, G \in \overline{\mathrm{F}}(L)$:*

(1) $\perp^- = \perp$, *where* $\perp(r,—) = 1$ *for all* $r \in \mathbb{Q}$,
(2) $F \le F^-$,
(3) $F^{--} = F^-$,
(4) $(F \vee G)^- = F^- \vee G^-$.

Also note that

$$\overline{\mathrm{USC}}(L) = \{F \in \overline{\mathrm{F}}(L) : F = F^-\}$$

and

$$F^- = \bigwedge\{G \in \overline{\mathrm{USC}}(L) : G \le F\}.$$

Both $(\cdot)^{\circ-}$ and $(\cdot)^{-\circ}$ are idempotent, i.e.

$$F^{\circ-\circ-} = F^{\circ-} \quad \text{and} \quad F^{-\circ-\circ} = F^{-\circ}.$$

Now we are going to discuss the connections between the lower and upper regularizations in the sense of [10] with those introduced above. Given $g \in \mathrm{usc}(L)$, we put $\downarrow_{\mathrm{LSC}}(g) = \{f \in \mathrm{lsc}(L) : f \lhd g\}$, and let

$$\begin{aligned} \mathrm{usc}^\circ(L) &= \{g \in \mathrm{usc}(L) : \downarrow_{\mathrm{LSC}}(g) \neq \varnothing\}, \\ \mathrm{lsc}^-(L) &= \{f \in \mathrm{lsc}(L) : \text{-}f \in \mathrm{usc}^\circ(L)\}. \end{aligned}$$

For each $g, \text{-}f \in \mathrm{usc}^-(L)$, we define

$$g^\circ = \bigvee(\downarrow_{\mathrm{LSC}}(g)) \quad \text{and} \quad f^- = \text{-}(\text{-}f)^\circ.$$

Proposition 7.5. *The following hold :*

(1) $\Psi_l(g^\circ) = \Psi_u(g)^\circ$ *for all* $g \in \mathrm{usc}^\circ(L)$,
(2) $\Psi_u(f^-) = \Psi_l(f)^-$ *for all* $f \in \mathrm{lsc}^-(L)$.

Proof: To show (1), let $g \in \mathrm{usc}^\circ(L)$. By Propositions 6.1(1) and 6.2:

$$\Psi_l(g^\circ) = \Psi_l(\bigvee_{\mathrm{lsc}(L)\ni h \lhd g} h) = \bigvee_{\Psi_l(h)\le \Psi_u(g)} \Psi_l(h) = \Psi_u(g)^\circ.$$

As always, (2) follows from (1) by duality. ■

Remark 7.6. In [10, Proposition 4.3] it is shown that, given $g, \text{-}f \in \mathrm{usc}^\circ(L)$, one has:

$$g^\circ(r, \text{—}) = \bigvee_{s>r} g(\text{—}, s)^* \quad \text{and} \quad f^-(\text{—}, r) = \bigvee_{s<r} f(s, \text{—})^*.$$

The above formulas make sense for arbitrary $g, \text{-}f \in \overline{\mathrm{usc}}(L)$ and Proposition 7.5 continues to hold in this more general setting.

Proposition 7.7. *The following hold:*

(1) $\Psi_l(g^\circ) = \Psi_u(g)^\circ$ *for all* $g \in \overline{\mathrm{usc}}(L)$,
(2) $\Psi_u(f^-) = \Psi_l(f)^-$ *for all* $f \in \overline{\mathrm{lsc}}(L)$.

Proof: To prove (1), we first observe that $g(—,s)^* \wedge g(—,s) = 0$, which yields $\mathfrak{c}(g(—,s)^*) \leq \mathfrak{c}(g(—,s))^*$. Thus,

$$\mathfrak{c}(g^\circ(r,—)) = \mathfrak{c}(\bigvee_{s>r} g(—,s)^*) = \bigvee_{s>r} \mathfrak{c}(g(—,s)^*) \leq \bigvee_{s>r} \mathfrak{c}(g(—,s))^*.$$

Since $\mathfrak{c}(g^\circ(r,—))$ is closed, we get

$$\mathfrak{c}(g^\circ(r,—)) \leq \overline{\bigvee_{s>r} \mathfrak{c}(g(—,s))^*} = \overline{\Psi_u(g)(r,—)}.$$

The above inequality for scales gives

$$\Psi_l(g^\circ) \leq \Psi_u(g)^\circ.$$

To get the reverse inequality we shall show that $\overline{\Psi_u(g)(r_1,—)} \leq \mathfrak{c}(g^\circ(r_2,—))$ whenever $r_1 > r_2$ (cf. Lemma 4.4). We have

$$\begin{aligned}
\overline{\Psi_u(g)(r_1,—)} &= \overline{\bigvee_{s>r_1} \mathfrak{c}(g(—,s))^*} \\
&\leq \overline{\mathfrak{c}(g(—,r_1))^*} \\
&= \mathfrak{c}(g(—,r_1)^*) \\
&\leq \bigvee_{s>r_2} \mathfrak{c}(g(—,s)^*) \\
&= \mathfrak{c}(\bigvee_{s>r_2} g(—,s)^*) \\
&= \mathfrak{c}(g^\circ(r_2,—)).
\end{aligned}$$

To have (2), given $f \in \overline{\mathrm{lsc}}(L)$, put $g = -f \in \overline{\mathrm{usc}}(L)$ into (1). ■

Proposition 7.8. *Let $F \in \mathrm{F}(L)$. The following hold:*

1. *If $\bigvee_{r\in\mathbb{Q}} \overline{F(r,—)} = 1$, then $F^\circ \in \mathrm{LSC}(L)$,*
2. *If $\bigvee_{r\in\mathbb{Q}} \overline{F(—,r)} = 1$, then $F^- \in \mathrm{USC}(L)$.*

Proof: We only prove (1), because (2) follows from (1) by the duality. So, we check that the conditions (r5) and (r6) hold for the extended scale $\{\overline{F(r,—)} : r \in \mathbb{Q}\}$. For (r5) we have

$$\bigvee_{r\in\mathbb{Q}} F^\circ(r,—) = \bigvee_{r\in\mathbb{Q}} \bigvee_{r>s} \overline{F(s,—)} = \bigvee_{r\in\mathbb{Q}} \overline{F(r,—)} = 1,$$

while for (r6):

$$\begin{aligned}
\bigvee_{r\in\mathbb{Q}} F^{\circ}(\text{—},r) &= \bigvee_{r\in\mathbb{Q}} \bigvee_{r<s} \left(\overline{F(s,\text{—})}\right)^{*} \\
&= \bigvee_{r\in\mathbb{Q}} \left(\overline{F(r,\text{—})}\right)^{*} \\
&\geq \bigvee_{r\in\mathbb{Q}} F(r,\text{—})^{*} \\
&\geq \bigvee_{r\in\mathbb{Q}} F(\text{—},r) = 1.
\end{aligned}$$

■

We also note the following:

Corollary 7.9. *Let $f, -g \in \mathrm{lsc}(L)$. Then:*

(1) $f^{-} \in \mathrm{usc}(L)$ *if and only if* $\Psi_l(f)^{-} \in \mathrm{USC}(L)$*,*
(2) $g^{\circ} \in \mathrm{lsc}(L)$ *if and only if* $\Psi_u(g)^{\circ} \in \mathrm{LSC}(L)$*.*

Proof: To show (1), let $f^{-} \in \mathrm{usc}(L)$. Then $\Psi_l(f)^{-} = \Psi_u(f^{-}) \in \mathrm{USC}(L)$ by Proposition 6.1(3). The reverse implication follows similarly, while (2) is a consequence of (1) when applied to $-g$. ■

Properties 7.10. For each complemented sublocale S of L the following hold:

$$(\chi_S)^{-} = \chi_{\overline{S}} \quad \text{and} \quad (\chi_S)^{\circ} = \chi_{S^{\circ}}.$$

In particular,

$$(\chi_{\mathfrak{c}(a)})^{-} = \chi_{\mathfrak{c}(a)}, \quad (\chi_{\mathfrak{o}(a)})^{-} = \chi_{\mathfrak{c}(a^{*})}, \quad (\chi_{\mathfrak{o}(a)})^{\circ} = \chi_{\mathfrak{o}(a)} \quad \text{and} \quad (\chi_{\mathfrak{c}(a)})^{\circ} = \chi_{\mathfrak{o}(a^{*})}$$

for every $a \in L$.

8. Appendix: some insertion and extension theorems revisited

We close with a brief illustration of how the framework introduced here provides nice formulations of the known important insertion and extension theorems on semicontinuous real functions [9, 10, 11]. Up to now (cf. Introduction), lower and upper semicontinuous real functions had different domains. With certain abuse of notation (related to the symbol $\leq$), we had, for instance, written $f \le h \le g$ to denote the situation in which $f \in \mathrm{lsc}(L)$, $g \in \mathrm{usc}(L)$, and $h \in \mathrm{c}(L)$ were such that $f \blacktriangleleft g$, $f \le h_{|\mathfrak{L}_l(\mathbb{R})}$, and $h_{|\mathfrak{L}_u(\mathbb{R})} \le g$ (cf. [11]). Now, with F, G, and H being the images of f, g, and h under the embeddings Ψ_l, Ψ_u, and Ψ, respectively, we just have $F \le H \le G$ where all the three morphisms act on the same domain $\mathfrak{L}(\mathbb{R})$ and take values in the frame $\mathcal{S}(L)$ and $\le$ denotes the partial order in $\mathrm{F}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))$. The proofs of all the theorems which follow remain the same.

We start with the pointfree version of the Katětov-Tong insertion theorem which (after [17]) was the initial motivation for our research programme started with [18] and [14]. We first need to recall some terminology.

Let

$$\mathcal{D}_L = \{(a, b) \in L \times L : a \vee b = 1\}.$$

Then L is called *normal* if there exists a function $\Delta : \mathcal{D}_L \to L$ such that $a \vee \Delta(a, b) = 1 = b \vee \Delta(a, b)^*$ for all $(a, b) \in \mathcal{D}_L$. The operator Δ is called a *normality operator*.

Theorem 8.1. *A frame L is normal if and only if, given an upper semicontinuous $G : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ and a lower semicontinuous $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ with $G \le F$, there exists a continuous $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $G \le H \le F$.*

Let $a, b \in L$ with $a \vee b = 1$. Then $\mathfrak{o}(b) \le \mathfrak{c}(a)$. Therefore, $\chi_{\mathfrak{c}(a)} \le \chi_{\mathfrak{o}(b)}$. Applying Theorem 8.1, there exists a continuous $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $\chi_{\mathfrak{c}(a)} \le H \le \chi_{\mathfrak{o}(b)}$. Hence (cf. [16]):

Corollary 8.2. *A frame L is normal if and only if for every $a, b \in L$ satisfying $a \vee b = 1$, there exists a continuous $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $\chi_{\mathfrak{c}(a)} \le H \le \chi_{\mathfrak{o}(b)}$.*

The existence of a continuous $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $\chi_{\mathfrak{c}(a)} \le H \le \chi_{\mathfrak{o}(b)}$ means that there exists $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $h((\text{–}, 0) \vee (1, \text{–})) = 0$, $h(0, \text{–}) \le a$ and $h(\text{–}, 1) \le b$. Thus, the corollary above is precisely Urysohn's Lemma for frames [6] (cf. [2, Prop. 5]).

Let S be a sublocale of L and let $c_S : L \to S$ with $c_S(x) = \bigwedge\{s \in S : x \le s\}$ be the corresponding frame quotient. We recall that an $f \in \mathrm{c}(S)$ has a *continuous extension* to L if there exists an $\tilde{f} \in \mathrm{c}(L)$ such that the following diagram commutes

$$\begin{tikzcd} & L \arrow[d, "c_S"] \\ \mathfrak{L}(\mathbb{R}) \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S \end{tikzcd}$$

i.e. $c_S \circ \tilde{f} = f$. Recall (see [19, Proposition 2.4]) that $\mathfrak{c}S \subseteq \mathfrak{c}L$. On the other hand, the frame quotient c_S induces another frame quotient $c_{\mathfrak{c}S} : \mathfrak{c}L \to \mathfrak{c}S$ making the following diagram commutative

$$\begin{tikzcd} L \arrow[r, "c_S"] \arrow[d, leftrightarrow, "\mathfrak{c}"] & S \arrow[d, leftrightarrow, "\mathfrak{c}"] \\ \mathfrak{c}L \arrow[r, "c_{\mathfrak{c}S}"] & \mathfrak{c}S \end{tikzcd}$$

i.e. $c_{\mathfrak{c}S}(\mathfrak{c}(a)) = \mathfrak{c}(c_S(a))$ for all $a \in L$.

We now say that $F \in \mathrm{C}(S)$ has a *continuous extension* to L if there exists an $\tilde{F} \in \mathrm{C}(L)$ such that the following diagram commutes

$$\begin{tikzcd} & \mathfrak{c}L \arrow[d, "c_{\mathfrak{c}S}"] \\ \mathfrak{L}(\mathbb{R}) \arrow[ur, "\tilde{F}"] \arrow[r, "F"] & \mathfrak{c}S \end{tikzcd}$$

i.e. $c_{\mathfrak{c}S} \circ \tilde{F} = F$. The next result provides the link between the old and the new approach to the extension problem.

Proposition 8.3. *Let S be a sublocale of L. Then $f \in \mathrm{c}(S)$ has a continuous extension to L if and only if $\mathfrak{c} \circ f$ has a continuous extension to L.*

Proof: Let $f \in \mathrm{c}(S)$ and $\overline{f} \in \mathrm{c}(L)$ be such that $c_S \circ \overline{f} = f$. Then $\mathfrak{c} \circ f \in \mathrm{C}(L)$ and

$$c_{\mathfrak{c}S} \circ \mathfrak{c} \circ \overline{f} = \mathfrak{c} \circ c_S \circ \overline{f} = \mathfrak{c} \circ f.$$

Conversely, let $f \in \mathrm{c}(S)$ and $F \in \mathrm{C}(L)$ be such that $c_{\mathfrak{c}S} \circ F = \mathfrak{c} \circ f$. Then $\mathfrak{c}^{-1} \circ F \in \mathrm{c}(L)$ and

$$c_S \circ \mathfrak{c}^{-1} \circ F = \mathfrak{c}^{-1} \circ c_{\mathfrak{c}S} \circ F = \mathfrak{c}^{-1} \circ \mathfrak{c} \circ f = f.$$

∎

The pointfree variant of Tietze extension theorem can now be stated as follows:

Theorem 8.4. *A frame L is normal if and only if for every closed sublocale S of L, each continuous $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(S)$ has a continuous extension $\tilde{F} : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $c_{\mathfrak{c}S} \circ \tilde{F} = F$.*

We now move to insertion and extension theorems for monotonically normal frames. Equip $\mathcal{D}_L = \{(a,b) \in L \times L : a \vee b = 1\}$ with the componentwise order inherited from $L^{op} \times L$. Then L is called *monotonically normal* if there exists a monotone normality operator. Further, the set

$$\mathrm{UL}(L) = \{(G,F) \in \mathrm{USC}(L) \times \mathrm{LSC}(L) : G \leq F\}$$

carries the componentwise order induced from $\mathrm{F}(L)^{op} \times \mathrm{F}(L)$. The following comes from [9, Theorem 5.4]. Even if the statements look similarly, this is a good place to repeat again how advantageous is the approach of considering $\mathcal{S}(L)$-valued morphisms. The reader should consult [9] to see how much effort is saved by moving from $\mathrm{usc}(L) \times \mathrm{lsc}(L)$ to $\mathrm{USC}(L) \times \mathrm{LSC}(L)$.

Theorem 8.5. *A frame L is monotonically normal if and only if there exists a monotone function $\Lambda : \mathrm{UL}(L) \to \mathrm{C}(L)$ such that $G \leq \Lambda(G,F) \leq F$ for all $(G,F) \in \mathrm{UL}(L)$.*

Given a sublocale S of L, a function $\Phi : \mathrm{C}(S) \to \mathrm{C}(L)$ is called an *extender* if $\Phi(F)$ extends F for all $F \in \mathrm{C}(S)$. Let S be a closed sublocale of L and $F \in \mathrm{C}(S)$. We define $F^l \in \mathrm{LSC}(L)$ and $F^u \in \mathrm{USC}(L)$ as follows:

$$F^l(r,\text{—}) = \begin{cases} 1 & \text{if } r < 0, \\ F(r,\text{—}) & \text{if } 0 \leq r < 1, \\ 0 & \text{if } r \geq 1, \end{cases}$$

$$F^l(\text{—},r) = \begin{cases} 0 & \text{if } r \leq 0, \\ \bigvee_{s<r} F(s,\text{—})^* & \text{if } 0 < r \leq 1, \\ 1 & \text{if } r > 1, \end{cases}$$

$$F^u(r,\text{—}) = \begin{cases} 1 & \text{if } r < 0, \\ \bigvee_{s>r} F(\text{—},s)^* & \text{if } 0 \leq r < 1, \\ 0 & \text{if } r \geq 1, \end{cases}$$

$$F^u(\text{—}, r) \quad = \quad \begin{cases} 0 & \text{if } r \leq 0, \\ F(\text{—}, r) & \text{if } 0 < r \leq 1, \\ 1 & \text{if } r > 1. \end{cases}$$

It is easy to check that $F^u \leq F^l$, i.e. $(F^u, F^l) \in \mathrm{UL}(L)$. The following is the reformulation of an extension theorem of [9, Theorem 6.4] in our new setting:

Theorem 8.6. *For L a frame the following are equivalent:*

(1) *L is monotonically normal.*
(2) *For each closed sublocale S of L there exists an extender $\Phi : \mathrm{C}(S) \to \mathrm{C}(L)$ such that for every closed sublocales S_1 and S_2 of L and $F_i \in \mathrm{C}(S_i)$ $(i = 1, 2)$ with $(F_1^u, F_1^l) \leq (F_2^u, F_2^l)$ one has $\Phi(F_1) \leq \Phi(F_2)$.*

Recall that a frame L is called *extremally disconnected* if $a^* \vee a^{**} = 1$ for all $a \in L$. We have the following (cf. [10]):

Theorem 8.7. *For L a frame the following are equivalent:*

(1) *L is extremally disconnected.*
(2) *Given lower semicontinuous $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ and upper semicontinuous $G : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ with $F \leq G$, there exists a continuous $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $F \leq H \leq G$.*
(3) *For every open sublocale S of L, each continuous $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(S)$ has a continuous extension $\tilde{F} : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $c_{\mathfrak{c}S} \circ \tilde{F} = F$.*

Let $F \in \mathrm{F}(L)$. We write $F \geq \mathbf{0}$ if $F(\text{—}, 0) = 0$. Similarly, $F \leq \mathbf{1}$ means that $F(1, \text{—}) = 0$. Also, we write $F > \mathbf{0}$ whenever $F(0, \text{—}) = 1$. The following two results come from [11]. We recall that a frame L is *perfectly normal* if for each $a \in L$ there is a countable subset $B \subseteq L$ such that $a = \bigvee B$ and $a \vee b^* = 1$ for all $b \in B$.

Theorem 8.8. *For L a frame the following are equivalent:*

(1) *L is perfectly normal.*
(2) *L is normal and for each lower semicontinuous $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ with $\mathbf{0} \leq F \leq \mathbf{1}$ there exists a continuous $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $\mathbf{0} < H \leq F$ and $H(0, \text{—}) = F(0, \text{—})$.*

(3) *For every closed sublocale* S *of* L, *each continuous* $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(S)$ *with* $\mathbf{0} \leq F \leq \mathbf{1}$ *has a continuous extension* $\tilde{F} \in \mathrm{C}(L)$ *such that* $\tilde{F}(0,1) \geq S$.

Finally, recall that a frame L is *countably paracompact* [7] if for every subset $\{a_n : n \in \mathbb{N}\} \subseteq L$ with $\bigvee_n a_n = 1$ there exists a subset $\{b_n : n \in \mathbb{N}\} \subseteq L$ such that $\bigvee_n b_n = 1$ and $a_n \vee b_n^* = 1$ for all n. As the last example we restate from [11] the following:

Theorem 8.9. *For a normal frame* L, *the following are equivalent:*

(1) L *is countably paracompact.*
(2) *For each lower semicontinuous* $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ *satisfying* $\mathbf{0} < F \leq \mathbf{1}$ *there exists a continuous* $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ *such that* $\mathbf{0} < H < F$.

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