DESCENT FOR COMPACT 0-DIMENSIONAL SPACES

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Dedicated to Walter Tholen on the occasion of his 60th birthday

Abstract: Using the reflection of the category \( C \) of compact 0-dimensional topological spaces into the category of Stone spaces we introduce a concept of a fibration in \( C \). We show that: (i) effective descent morphisms in \( C \) are the same as the surjective fibrations; (ii) effective descent morphisms in \( C \) with respect to the fibrations are all surjections.

Keywords: comma categories, effective descent, effective \( F \)-descent.


0. Introduction

Our original intention was to describe effective descent morphisms in the category \( C \) of compact 0-dimensional topological spaces by combining the following well-known facts:

- A compact 0-dimensional space is nothing but a set equipped with a surjection into a Stone space (see Theorem 2.1 for the precise formulation).
- The effective descent morphisms in the categories of sets and of Stone spaces are just surjections.

It is still the main purpose of the paper, although it turned out that:

- Not all pullbacks exist in \( C \). Therefore the definition of an effective descent morphism \( p \) in \( C \) should include the requirement: all pullbacks along \( p \) must exist (see Definition 3.2).
- When \( p \) is surjective, that requirement hold if and only if \( p \) is a fibration in a suitable sense (see Definition 2.2), which is very different from what is happening in the situations studied by H. Herrlich [1],

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and which makes the descent problem much easier. In a somewhat different situation, this is made clear in [3].

- The surjectivity requirement does not create any problem since it is independently forced by the reflection of isomorphisms by the pullback functor along an effective descent morphism.
- Therefore the problem of describing effective descent morphisms in $\mathcal{C}$ has an easy solution: Theorem $3.3$ says that they are the same as the surjective fibrations.
- However, this suggests a new question, namely, what are the effective descent morphisms with respect to fibrations? Fortunately there is a complete answer again: they are all surjections (Theorem $3.1$).
- In particular, even though the spaces we consider are not necessarily Hausdorff spaces, which prevents their convergence relations to be maps, our characterization of their effective descent morphisms avoids using the Reiterman-Tholen characterization of effective descent morphisms in the category of all topological spaces [4].

Accordingly, the paper is organized as follows:

Section 1 contains preliminary categorical observations with no topology involved. The ground category $\mathcal{C}$ there is constructed as a full subcategory in the comma category $(\mathcal{S} \downarrow \mathcal{U})$, where $\mathcal{U} : \mathcal{X} \to \mathcal{S}$ is a pullback preserving functor between categories with pullbacks, using also a distinguished class $\mathcal{E}$ of morphisms in $\mathcal{S}$. This class is also used to define what we call fibrations in $\mathcal{C}$. The sufficient conditions for a morphism to be an effective descent morphism (globally or with respect to the class of fibrations) given in Section 1 will become also necessary in the topological context of Sections 2 and 3.

Section 2 begins by recalling relevant topological concepts, presents the category of compact 0-dimensional spaces as a special case of $\mathcal{C}$ above, introduces fibrations of 0-dimendional spaces accordingly, and ends by proving that a surjective morphism in $\mathcal{C}$ admits all pullbacks along morphisms with the same codomain if and only if it is a fibration.

The purpose of Section 3 is to formulate and prove the two main results, namely the above mentioned Theorems $3.1$ and $3.3$.

1. Categorical framework

We fix the following data: categories $\mathcal{S}$ and $\mathcal{X}$ with pullbacks, a pullback preserving functor $\mathcal{U} : \mathcal{X} \to \mathcal{S}$ and a class $\mathcal{E}$ of morphisms in $\mathcal{S}$ that has the following properties:
• contains all isomorphisms;
• is pullback stable;
• is closed under composition;
• forms a stack (=coincides with its localization), which means that if

\[
\begin{array}{ccc}
  u & \rightarrow & v \\
  \downarrow & & \downarrow \\
  w & \rightarrow & w
\end{array}
\]

is a pullback diagram with \( w \) being an effective descent morphism, then \( u \in \mathbb{E} \Rightarrow v \in \mathbb{E} \).

Let \( C = C[\mathcal{X}, S, U, \mathbb{E}] \) be the full subcategory in the comma category \((S \downarrow U)\) with objects all triples \( A = (A_1, e_A, A_0) \), in which \( e_A : A_1 \rightarrow U(A_0) \) is in \( \mathbb{E} \); accordingly, a morphism \( A \rightarrow B \) in \( C \) is a pair \( f = (f_1, f_0) \), in which \( f_1 : A_1 \rightarrow B_1 \) and \( f_0 : A_0 \rightarrow B_0 \) are morphisms in \( S \) and \( \mathcal{X} \) respectively,

\[
\begin{array}{ccc}
  A_1 & \xrightarrow{e_A} & U(A_0) \\
  f_1 \downarrow & & \downarrow Uf_0 \\
  B_1 & \xrightarrow{e_B} & U(B_0)
\end{array}
\]

such that \( U(f_0)e_A = e_Bf_1 \).

**Definition 1.1.** A morphism \( f : A \rightarrow B \) in \((S \downarrow U)\) is said to be a fibration if the morphism

\[
< f_1, e_A > : A_1 \rightarrow B_1 \times_{U(B_0)} U(A_0)
\]

is in \( \mathbb{E} \).

**Observation 1.2.** If \( f : A \rightarrow B \) is a fibration, and \( B \) is in \( C \), then, since the class \( \mathbb{E} \) is pullback stable, \( A \) also is in \( C \).

**Proposition 1.3.** Let

\[
\begin{array}{ccc}
  D & \xrightarrow{q} & A \\
  g \downarrow & & \downarrow f \\
  E & \xrightarrow{p} & B
\end{array}
\]  \hspace{1cm} (1.1)

be a pullback diagram in \((S \downarrow U)\) with \( p : E \rightarrow B \) in \( C \). Then:

(a) If \( f \) is a fibration, then so is \( g \).

(b) If \( g \) is a fibration, and \( p_1 \) is an effective descent morphism, then \( f \) also is a fibration.
(c) If $p$ is a fibration and $A$ is in $C$, then $D$ is in $C$.
(d) If $E$ has the (weak left) cancellation property ($e', e \cdot e' \in E \Rightarrow e \in E$) and $p_1$ and $U(p_0)$ are in $E$ and $D$ is in $C$, then $A$ is in $C$.

Proof: Consider the diagram

\[
\begin{array}{ccc}
D_1 & \xrightarrow{q_1} & A_1 \\
\downarrow{d} & & \downarrow{h} \\
S & \xrightarrow{e_D} & T \\
\downarrow{s} & & \downarrow{t} \\
U(D_0) & \xrightarrow{U(g_0)} & U(A_0) \\
\downarrow{p_1} & & \downarrow{f_1} \\
E_1 & \xrightarrow{e_E} & B_1 \\
\downarrow{U(g_0)} & & \downarrow{U(f_0)} \\
U(E_0) & \xrightarrow{U(p_0)} & U(B_0) \\
\end{array}
\]

in which:
- the enveloping cube represents the diagram (1.1);
- $e_E s = U(g_0)s'$ and $e_B t = U(f_0)t'$ are pullbacks;
- $d =< g_1, e_D >$, $a =< f_1, e_A >$, and $h = p_1 \times U(q_0)$ are the suitable induced morphisms.

Since the front square $U(p_0)U(g_0) = U(f_0)U(g_0)$ and the quadrilaterals $e_E s = U(g_0)s'$ and $e_B t = U(f_0)t'$ are pullbacks, so is the quadrilateral $p_1 s = th$. Next, since $p_1 g_1 = f_1 q_1$ and $p_1 s = th$ are pullbacks, so is $hd = aq_1$. This proves (a).

(b): Since $p_1$ is an effective descent morphism and $p_1 s = th$ is a pullback, $h$ also is an effective descent morphism ([5]). Since $hd = aq_1$ is a pullback, this proves (b).
For (c) and (d), in order to use the same observations, let us “turn the diagram (1.1) around the diagonal connecting \( D \) and \( B \)”, i.e. let us reformulate (c) and (d) as follows:

\( (c') \) If \( f \) is a fibration and \( E \) is in \( \mathcal{C} \), then \( D \) is in \( \mathcal{C} \).

\( (d') \) If \( f_1 \) and \( U(f_0) \) are in \( \mathcal{E} \) and \( D \) is in \( \mathcal{C} \), then \( E \) is in \( \mathcal{C} \).

**Proof of (c’):**

- Since \( f \) is a fibration, \( a \) is in \( \mathcal{E} \).
- Since \( E \) is in \( \mathcal{C} \) and \( e_E s = U(g_0)s' \) is a pullback, \( s' \) is in \( \mathcal{E} \).
- Since \( a \) and \( s' \) are in \( \mathcal{E} \), so is \( e_D \), i.e. \( D \) is in \( \mathcal{C} \).

**Proof of (d’):**

- Since \( f_1 \) and \( U(f_0) \) are in \( \mathcal{E} \), so are \( g_1 \) and \( U(g_0) \).
- Since \( g_1, U(g_0) \) and \( e_D \) are in \( \mathcal{E} \), the cancellation property of (d’) implies that \( e_E \) is in \( \mathcal{E} \), as desired.

From Observation 1.2 and Proposition 1.3(a) we obtain:

**Corollary 1.4.** The category \( \mathcal{C} \) is closed in \( (\mathcal{S} \downarrow \mathcal{U}) \) under pullbacks along fibrations; that is, if (1.1) is a pullback diagram in \( (\mathcal{S} \downarrow \mathcal{U}) \) with \( f \) in \( \mathcal{C} \) and \( p \) being a fibration in \( \mathcal{C} \), then it is a pullback diagram in \( \mathcal{C} \).

When \( \mathcal{S} \) has coequalizers of equivalence relations, all effective descent morphisms in \( \mathcal{S} \) are regular epimorphisms. Using this fact it is easy to show that if \( p : E \rightarrow B \) is a morphism in \( (\mathcal{S} \downarrow \mathcal{U}) \), for which \( p_0 \) and \( p_1 \) are effective descent morphisms in \( \mathcal{X} \) and in \( \mathcal{S} \) respectively, then \( p \) itself is an effective descent morphism. After that, using Proposition 1.3 and Corollary 1.4 we obtain:

**Proposition 1.5.** If \( \mathcal{S} \) has coequalizers of equivalence relations and \( p : E \rightarrow B \) is a morphism in \( \mathcal{C} \), for which \( p_0 \) and \( p_1 \) are effective descent morphisms in \( \mathcal{X} \) and in \( \mathcal{S} \) respectively, then

(a) \( p \) is an effective \( F \)-descent morphism in \( \mathcal{C} \), where \( F \) is the class of all fibrations (in \( \mathcal{C} \)).

(b) if \( p \) is a fibration, then it is an effective descent morphism in \( \mathcal{C} \).
2. The category of compact 0-dimensional spaces

For a topological space $A$, we shall write $\text{Open}(A)$ for the set of open subsets in $A$ and $\text{Clopen}(A)$ for the set of those subsets in $A$ that are clopen, i.e. closed and open at the same time. Let us recall the definitions of the following full subcategories of the category $\mathcal{Top}$ of topological spaces:

- $\mathcal{Top}_0$, the category of $T_0$-spaces; a space $A$ is a $T_0$-space if, for every two distinct points $a$ and $a'$ in $A$, either there exists $U \in \text{Open}(A)$ with $a \in U$ and $a' \notin U$, or there exists $U \in \text{Open}(A)$ with $a' \in U$ and $a \notin U$. Note that $\mathcal{Top}_0$ is a reflective subcategory in $\mathcal{Top}$, with the reflection given by $A \mapsto A_0 = A/\sim$, where $a \sim a' \iff \forall U \in \text{Open}(A)(a \in U \iff a' \in U)$. (2.1)

- $0\text{-Dim}\mathcal{Top}$, the category of 0-dimensional spaces; a space is 0-dimensional, if it has a basis of clopen subsets, i.e. if every open subset in it can be presented as a union of clopen subsets.

- The category of compact 0-dimensional spaces, which is the category of interest in this paper, will be simply denoted by $\mathcal{C}$; hence

$$\mathcal{C} = \text{Comp}\mathcal{Top} \cap 0\text{-Dim}\mathcal{Top}$$

where $\text{Comp}\mathcal{Top}$ is the category of compact spaces.

- $\text{Stone}$, the category of Stone spaces = spaces that occur as Stone spaces of Boolean algebras = spaces that occur as limits of finite discrete spaces = compact Hausdorff 0-dimensional spaces = compact spaces $A$, such that for every two distinct points $a$ and $a'$ in $A$, there exists $U \in \text{Clopen}(A)$ with $a \in U$ and $a' \notin U$. The $T_0$-reflection (2.1) of course induces a reflection

$$\mathcal{C} \hookrightarrow \text{Stone}, A \mapsto A_0$$

The following theorem is a reformulation of well-known results (see also Example 3.3 in [2] for the same result for arbitrary topological spaces, which, together with other similar results was mentioned already in [1]):

**Theorem 2.1.** The category $\mathcal{C}$ of compact 0-dimensional spaces is equivalent to the category $\mathcal{C}[\mathcal{X}, \mathcal{S}, U, \mathcal{E}]$ (see Section 1), for $\mathcal{X} = \text{Stone}$, $\mathcal{S} = \text{Set}$, $U : \text{Stone} \to \text{Set}$ ($\text{Set}$ being the usual forgetful functor into the category of sets, and $\mathcal{E}$ being the class of all surjective maps. Under this equivalence a space $A$ corresponds to the triple $(A_1, e_A, A_0)$, in which $A_1$ is the underlying set of $A$, $A_0$ is the $T_0$-reflection of $A$, and $e_A : A_1 \to U(A_0)$ is the canonical map (and we write again $A = (A_1, e_A, A_0)$).

According to this theorem and Definition 1.1, we introduce:
Definition 2.2. A morphism $f : A \to B$ in $\mathcal{C}$ is said to be a fibration if so is the corresponding morphism in $\mathcal{C}[X, S, U, E]$ of Theorem 2.1, i.e. if for every $a$ in $A$ and $b$ in $B$ with $f(a) \sim b$ there exists $a'$ in $A$ with $a' \sim a$ and $f(a') = b$.

After that Proposition 1.3 helps to prove:

Theorem 2.3. Let $p : E \to B$ be a morphism in $\mathcal{C}$. If $p$ is surjective, then the following conditions are equivalent:

(a) every morphism $f : A \to B$ in $\mathcal{C}$ admits pullback along $p$;
(b) $p$ is a fibration.

Proof: (a)$\Rightarrow$(b): Suppose $p$ is not a fibration. This means that there are $e$ in $E$ and $b$ in $B$ with

$$p(e) \sim b \text{ and } (x \in p^{-1}(b) \Rightarrow \exists U_x \in \text{Clopen}(E)(x \in U_x \text{ and } e \notin U_x)).$$

We choose $U_x$ as in (2.3) for each $x$ in $p^{-1}(b)$, and consider two cases:

Case 1. There exists a finite subset $Y$ in $X$, for which

$$p^{-1}(b) \subseteq \bigcup_{x \in Y} U_x.$$

Case 2. There is no such $Y$.

In Case 1 we take

$$V = \bigcap_{x \in Y} (E \setminus U_x),$$

and observe that since $Y$ is finite, $V$ is clopen; and of course $V$ contains $e$ and has empty intersection with $p^{-1}(b)$. After that we take

$$A = \{n^{-1}|n = 1, 2, 3, \cdots\} \cup \{0\}$$

with the topology induced from the real line, and define $f : A \to B$ by $f(n^{-1}) = b$ and $f(0) = p(e)$. Suppose the pullback of $p$ and $f$ does exist, and let us write it as the diagram (1.1). Using the universal property of this pullback with respect to maps from a one-point space, we easily conclude that it is preserved by the forgetful functor into the category of sets. In particular, since $V$ contains $e$ and has empty intersection with $p^{-1}(b)$, we have

$$q(g^{-1}(E \setminus V)) = \{n^{-1}|n = 1, 2, 3, \cdots\}.$$

This is a contradiction since $g^{-1}(E \setminus V)$ being clopen in $D$ must be compact in it, while $\{n^{-1}|n = 1, 2, 3, \cdots\}$ is not compact in $A$. 
In Case 2 we take \( A = \{a\} \) to be a one-point space, and define \( f : A \to B \) by \( f(a) = b \). Then, using (1.1) as above, we observe that \( g(D) = p^{-1}(b) \) which is again a contradiction because now \( p^{-1}(b) \) is not compact.

That is, whenever \( p \) is not a fibration, there exists a morphism \( f : A \to B \) in \( C \) that has no pullback along \( p \).

\( (b) \Rightarrow (a) \) follows from Corollary 1.4 and Theorem 2.1.

3. \( \mathbb{F} \)-Descent and global descent

Let \( C \) be as in Section 2.

**Theorem 3.1.** The following conditions on a morphism \( p : E \to B \) in \( C \) are equivalent:

(a) \( p \) is an effective \( \mathbb{F} \)-descent morphism in \( C \);

(b) \( p \) is a surjective map.

**Proof:** (a) \( \Rightarrow \) (b): Suppose \( p \) is not surjective, and choose \( b \in B \setminus p(E) \). Let \( A \) be the equivalence class of \( b \) with respect to the equivalence relation \( \sim \) (see (2.1)). We take

\[ A' = (A \setminus \{b\}) \cup \{b\} \times \{1, 2\} \]

equipped with indiscrete topology, and define \( \alpha : A' \to A \) by \( \alpha(a) = a \), for \( a \in A \), and \( \alpha(b, 1) = b = \alpha(b, 2) \); then \( \alpha \) becomes a morphism \( (A', \alpha f) \to (A, f) \), where \( f : A \to B \) is the inclusion map, in the category \( \mathbb{F}(B) \) of fibrations over \( B \) (in \( C \)). Since the image of this morphism under the pullback functor \( p^* : \mathbb{F}(B) \to \mathbb{F}(E) \) is an isomorphism, \( p \) cannot be effective \( \mathbb{F} \)-descent morphism in \( C \).

(b) \( \Rightarrow \) (a): Let \((p_1, p_0) : (E_1, e_E, E_0) \to (B_1, e_B, B_0)\) be the morphism in \( C[\mathcal{X}, \mathcal{S}, U, \mathbb{E}] \) corresponding to \( p \) under the category equivalence of Theorem 2.1, where \( \mathcal{X} = \text{Stone}, \mathcal{S} = \text{Set}, U : \text{Stone} \to \text{Set} \) being the usual forgetful functor into the category of sets, and \( \mathbb{E} \) being the class of all surjective maps. Then \( p_1 \) is surjective and this makes \( p_0 \) surjective too. Since in both \( \text{Stone} \) and \( \text{Set} \) surjections are effective descent morphisms, this makes \( p \) an effective \( \mathbb{F} \)-descent morphism by Proposition 1.5(a).

Since \( C \) does not admit some pullbacks, we define effective (global-)descent morphisms in \( C \) as follows:

**Definition 3.2.** A morphism \( p : E \to B \) in \( C \) is said to be an effective descent morphism if every morphism \( f : A \to B \) in \( C \) admits pullback along
$p$, and the pullback functor

$$p^* : (C \downarrow B) \to (C \downarrow E)$$

is monadic.

**Theorem 3.3.** The following conditions on a morphism $p : E \to B$ in $\mathcal{C}$ are equivalent:

(a) $p$ is an effective descent morphism;
(b) $p$ is a surjective fibration.

**Proof:** (a)$\Rightarrow$(b): Surjectivity can be proved in the same way as in the proof of Theorem 3.1 (or even much simpler by considering the empty and one-point space instead of $A'$ and $A$ there). The fact that $p$ must be a fibration follows from the implication (a)$\Rightarrow$(b) of Theorem 2.3.

(b)$\Rightarrow$(a) can be deduced from Proposition 1.5(b) and Theorem 2.1 with the same arguments as in the proof of Theorem 3.1(b)$\Rightarrow$(a).

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**References**


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