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GROWTH CONDITIONS AND UNIQUENESS OF THE CAUCHY PROBLEM FOR THE EVOLUTIONARY INFINITY LAPLACIAN

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ABSTRACT: We study the Cauchy problem for the parabolic infinity Laplace equation. We prove a new comparison principle and obtain uniqueness of viscosity solutions in the class of functions with a polinomial growth at infinity, improving previous results obtained assuming a linear growth.

KEYWORDS: Infinity Laplacian, Cauchy problem, uniqueness, growth at infinity.
AMS SUBJECT CLASSIFICATION (2000): 35B05; 35K15; 35K55; 35K65.

1. Introduction

Arguably, the infinity Laplace equation is nowadays one of the most trendy nonlinear partial differential equations. This is due to the beautiful mathematical theory that has been put forward to understand it, starting from the pioneering work of Aronsson in the 1960's, but also to the recent finding that it is related to important applications in game theory, image processing and mass transfer problems. The modern approach through viscosity solutions goes back to [5] and [10], and [2] is an excellent survey on the subject, with plenty of clarifying examples.

Its parabolic counterpart is much less popular but lately started to attract the attention it probably also deserves. We are talking of the strongly degenerate equation

$$u_t - \Delta_\infty u = 0 \tag{1}$$

where

$$\Delta_\infty u := \left(D^2 u \frac{Du}{|Du|} \right) \cdot \frac{Du}{|Du|} \tag{2}$$

is the 1−homogeneous infinity Laplacian. The seminal paper of Juutinen and Kawohl [11], where basic results on the existence, uniqueness and regularity of solutions are collected, is the first attempt to systematically study (1). One of the issues touched in that paper concerns the associated Cauchy problem,

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namely

$$\begin{cases} u_t - \Delta_\infty u = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{3}$$

where $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a given initial datum and $N \geq 2$. The authors state a comparison principle and obtain the uniqueness assuming a *linear* growth of the solution as $|x| \to \infty$. The aim of this paper is to revisit the subject and improve the growth condition that guarantees the uniqueness. We prove a new comparison principle and obtain an existence and uniqueness result in the class of solutions with *polinomial* growth at infinity. Our approach has been strongly influenced by the papers [9] and [4], and the techniques we employ are a combination of those used there.

Already in the context of the heat equation, the relation between growth at infinity and the uniqueness for the Cauchy problem is pertinent, as shown by the celebrated one-dimensional counter-example of Tychonov (cf. [8, Ch. V, §5.1]). The optimal growth that guarantees the uniqueness is

$$u(x,t) \leq C e^{\alpha |x| h(|x|)} \qquad \text{as } |x| \to \infty,$$

where h is a positive and nondecreasing function such that

$$\int^\infty \frac{ds}{h(s)} = +\infty.$$

This growth is the best we can aim at here since the evolutionary infinity Laplace equation reduces to the heat equation in the case of only one space variable. We stress that we have no evidence supporting the optimality of polinomial growth.

Similar questions for problems involving equations that share some of the features of (1) have been studied in [1] and [4].

The appropriate notion of solution when dealing with (1) is that of viscosity solution. As there is more than one way of introducing the concept, we fix ideas in the next definition.

Definition 1.1. Let Q denote the strip $\mathbb{R}^N \times (0,T)$. An upper semicontinuous function $u(x,t)$ is a *viscosity subsolution* of (1) in Q if, whenever $(x_0,t_0) \in Q$ and $\varphi(x,t) \in C^2(Q)$ are such that

$$\varphi(x_0,t_0) = u(x_0,t_0)$$

and

$$\varphi(x,t) > u(x,t), \quad \forall\ (x,t) \in Q,\ (x,t) \neq (x_0,t_0),$$

then

$$\begin{cases} \varphi_t(x_0,t_0) \leq \Delta_\infty \varphi(x_0,t_0) & \text{if } D\varphi(x_0,t_0) \neq 0, \\ \varphi_t(x_0,t_0) \leq \Lambda\left(D^2\varphi(x_0,t_0)\right) & \text{if } D\varphi(x_0,t_0) = 0, \end{cases} \tag{4}$$

where $\Lambda\left(D^2\varphi(x_0,t_0)\right)$ denotes the largest eigenvalue of the Hessian matrix of φ at the point (x_0,t_0).

Analogously, a lower semicontinuous function $v(x,t)$ is a *viscosity supersolution* of (1) in Q if, whenever $(x_0,t_0) \in Q$ and $\varphi(x,t) \in C^2(Q)$ are such that

$$\varphi(x_0,t_0) = v(x_0,t_0)$$

and

$$\varphi(x,t) < v(x,t), \quad \forall\ (x,t) \in Q,\ (x,t) \neq (x_0,t_0),$$

then

$$\begin{cases} \varphi_t(x_0,t_0) \geq \Delta_\infty \varphi(x_0,t_0) & \text{if } D\varphi(x_0,t_0) \neq 0, \\ \varphi_t(x_0,t_0) \geq \lambda\left(D^2\varphi(x_0,t_0)\right) & \text{if } D\varphi(x_0,t_0) = 0, \end{cases} \tag{5}$$

where $\lambda\left(D^2\varphi(x_0,t_0)\right)$ denotes the smallest eigenvalue of the Hessian matrix of φ at the point (x_0,t_0).

We use the modifier *strict* when the inequalities in (4) and (5) are strict.

Finally, a continuous function $z(x,t)$ is a *viscosity solution* of (1) in Q if it is both a viscosity subsolution and a viscosity supersolution.

Throughout the paper, the following assumption on the initial datum will be in force:

$$u_0 \in C^0(\mathbb{R}^N) \quad \text{and} \quad \exists\, C_0 > 0,\ p > 1 :\ \frac{|u_0(x)|}{1+|x|^p} \leq C_0,\ \forall x \in \mathbb{R}^N. \tag{6}$$

We say that a function $v : \mathbb{R}^N \times [0,T) \to \mathbb{R}$ satisfies polinomial κ-growth at infinity if there exists $\kappa > 0$ such that

$$\lim_{|x|\to\infty} \frac{v(x,t)}{|x|^\kappa} = 0,$$

uniformly with respect to t.

The main result of the paper is the following theorem.

Theorem 1.2. *If the initial datum satisfies* (6) *then, in the class of functions with polynomial growth at infinity, there exists a unique viscosity solution* u *of* (1) *that satisfies the initial condition in the classical sense. Moreover, for some* $k>0$,

$$|u(x,t)|\leq k(1+|x|^p),\quad \forall (x,t)\in Q.$$

The paper is organized as follows. The next section is devoted to the proof of growth estimates that will be instrumental in the sequel and Section 3 collects a few technical lemmas. In Section 4 we obtain the comparison principle and the last section contains the proof of the main result.

2. Growth estimates

We start by showing that the growth imposed on the initial datum carries through to any solution of the problem. Throughout the paper, we denote by

$$z^*(x,t)=\limsup_{s\searrow 0}\,\{z(y,\tau):|x-y|\leq s,\ |t-\tau|\leq s\}$$

the upper envelope of a given function $z(x,t)$. The definition of the lower envelope $z_*(x,t)$ is analogous, with $\liminf$ replacing $\limsup$. Note that, in fact, $z^*(x,t)$ ($z_*(x,t)$, respectively) is nothing but the smallest (largest) upper (lower) semicontinuous function that lies above (below) $z(x,t)$.

Lemma 2.1. *Let* $u_0(x)$ *be given satisfying (6). If* $u(x,t)$ *and* $v(x,t)$ *are, respectively, a viscosity subsolution and a viscosity supersolution of* (1), *with polynomial growth, such that*

$$u^*(x,0)\leq u_0(x)\leq v_*(x,0),\quad \forall x\in\mathbb{R}^N, \tag{7}$$

then there exists $k>0$ *such that*

$$u(x,t)\leq k(1+|x|^p)\quad \textit{and}\quad v(x,t)\geq -k(1+|x|^p),\quad \forall (x,t)\in Q. \tag{8}$$

Proof: We only prove the first inequality in (8) since the other one is entirely similar. For a given arbitrary $\varepsilon>0$, define the function

$$\Upsilon(x,t)=\tilde{k}e^{\eta t}\left[(1+|x|^2)^{\frac{p}{2}}+\varepsilon(1+|x|^h)\right],$$

where $\tilde{k}$ and η are suitable positive constants to be fixed later, and h is chosen such that $h>\max\{2,p\}$.

We first show that Υ is a strict supersolution of (3) in the viscosity sense. Computing the derivatives, we find $\Upsilon_t(x,t) = \eta\Upsilon(x,t)$,

$$D\Upsilon(x,t) = \tilde{k}e^{\eta t}\left[p(1+|x|^2)^{\frac{p}{2}-1} + \varepsilon h|x|^{h-2}\right]x$$

and

$$\begin{aligned} D^2\Upsilon(x,t) &= \tilde{k}e^{\eta t}\left[p(p-2)(1+|x|^2)^{\frac{p}{2}-2} + \varepsilon h(h-2)|x|^{h-4}\right]x\otimes x \\ &\quad +\tilde{k}e^{\eta t}\left[p(1+|x|^2)^{\frac{p}{2}-1} + \varepsilon h|x|^{h-2}\right]\mathcal{I}. \end{aligned}$$

Noting that $D\Upsilon(x,t) = 0$ if, and only if, $x = 0$, fixing

$$\eta > \begin{cases} \max\left\{\dfrac{p(p-1)^2}{4(p-2)}, h(h-1)\right\} & \text{if } \ p>2 \\ \\ h(h-1) & \text{if } \ 1<p\le 2, \end{cases}$$

we deduce that $\Upsilon(x,t)$ is a strict supersolution of (1); in particular, if $D\Upsilon(x,t)\neq 0$,

$$\Upsilon_t(x,t) > \Delta_\infty\Upsilon(x,t). \tag{9}$$

Suppose now that $u(x,t)$ is a viscosity subsolution of (1), with polynomial κ-growth, and consider the difference

$$w(x,t) = u(x,t) - \Upsilon(x,t).$$

Setting $h>\kappa$, for any $\varepsilon>0$, there exists $R>0$ such that $w(x,t)<0$, for any (x,t) such that $|x|>R$. Our goal is to exclude that

$$\sup_{\mathbb{R}^N\times(0,T)} w(x,t) > 0; \tag{10}$$

indeed, otherwise, we have

$$u(x,t) \le \tilde{k}e^{\eta T}\left[(1+|x|^2)^{\frac{p}{2}} + \varepsilon(1+|x|^h)\right],$$

and letting $\varepsilon\to 0$ we deduce, setting $k=\tilde{k}e^{\eta T}2^{\frac{p}{2}-1}$, if $p\ge 2$ or $k=\tilde{k}e^{\eta T}$ if $1<p<2$,

$$u(x,t) \le k(1+|x|^p)$$

as desired.

Suppose the supremum in (10) is achieved at (x_0,t_0), with $t_0>0$. Then

$$u(x,t) - \Upsilon(x,t) < u(x_0,t_0) - \Upsilon(x_0,t_0)$$

and $u(x,t) < \Upsilon(x,t) + u(x_0,t_0) - \Upsilon(x_0,t_0)$, in a punctured neighborhood of (x_0,t_0). We can then test the equation with the function $\Upsilon(x,t)+u(x_0,t_0)-\Upsilon(x_0,t_0)$ and we get a contradiction to (9).

Finally, due to (6), we can fix $\tilde{k} \geq C_0$ such that

$$\begin{aligned} \Upsilon(x,0) &= \tilde{k}\left[(1+|x|^2)^{\frac{p}{2}} + \varepsilon(1+|x|^h)\right] \\ &\geq \tilde{k}\,(1+|x|^p) \\ &\geq u_0(x) \\ &\geq u^*(x,0) \end{aligned}$$

and a contradiction also follows for $t_0 = 0$.

■

Remark 2.2. A direct consequence of the lemma is that any solution of (3) with polynomial growth verifies, for some $k > 0$,

$$|u(x,t)| \leq k(1+|x|^p), \quad \forall (x,t) \in Q,$$

provided the initial datum satisfies the growth condition (6).

The following estimate will be instrumental in the sequel.

Lemma 2.3. *Let $u_0(x)$ be given satisfying (6). If $u(x,t)$ and $v(x,t)$ are, respectively, a viscosity subsolution and a viscosity supersolution of* (1)*, with polynomial growth and satisfying* (7)*, then there exists a constant $C > 0$ such that*

$$u(x,t) - v(y,t) \leq C\left(1+|x-y|^p\right), \quad \forall (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,T). \tag{11}$$

Proof: We define

$$\psi(x,y,t,s) = e^{\eta t}\left[k\left(1+|x-y|^2\right)^{\frac{p}{2}} + 2k + \rho_r\left(\sqrt{|x|^2+|y|^2}\right)\right],$$

where $\eta > 0$ and $(\rho_r)_{r>1}$ is a family of non-negative C^2−functions, nondecreasing in $\mathbb{R}^+$, such that

$$\rho_r(s) \equiv 0 \quad \text{if} \ \ 0 < s \leq r; \tag{12}$$

$$\lim_{s\to+\infty} \frac{\rho_r(s)}{s^p} = 2k; \tag{13}$$

and there exists $\sigma > 0$, independent of r, such that

$$\rho_r''(s) + 3\frac{\rho_r'(s)}{s} \leq \sigma(\rho_r(s)+1), \quad \forall s > 0. \tag{14}$$

An explicit possible choice of the family $(\rho_r)_{r>1}$ is exhibited in the remark after the proof of the lemma.

Next, we take
$$w(x,y,t,s) = u(x,t) - v(y,s)$$
and consider
$$\Phi(x,y,t,s) = w(x,y,t,s) - \psi(x,y,t,s). \tag{15}$$

We will prove (11), showing that
$$\sup_{(x,y,t)\in\mathbb{R}^N\times\mathbb{R}^N\times(0,T)} \Phi(x,y,t,t) \le 0.$$
Indeed, if this holds then, given $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$, we can choose $r>1$ sufficiently large such that
$$\rho_r\left(\sqrt{|x|^2+|y|^2}\right) = 0.$$
Thus,
$$\begin{aligned} u(x,t)-v(y,s) &\le e^{\eta t}\left[k\left(1+|x-y|^2\right)^{\frac{p}{2}} + 2k\right] \\ &\le e^{\eta T}\left[k\,2^{\frac{p-2}{2}}\left(1+|x-y|^p\right) + 2k\right] \\ &\le C\left(\eta,T,k,p\right)\left(1+|x-y|^p\right). \end{aligned}$$

Due to (6) and (7), we first note that $\Phi(x,y,0,0)\le 0$. We will reason by contradiction, assuming that
$$\sup_{(x,y,t,s)\in\mathbb{R}^N\times\mathbb{R}^N\times(0,T)\times(0,T)} \Phi(x,y,t,s) > 0.$$
We start with the remark that, due to (13), there exists $R>r$ such that
$$\frac{\rho_r(s)}{s^p} \ge k, \qquad \forall s \ge R.$$
From this and (8) in Lemma 2.1, it follows that
$$\begin{aligned} \Phi(x,y,t,s) &= u(x,t)-v(y,s)-\psi(x,y,t,s) \\ &< k(2+|x|^p+|y|^p) - 2k - k\left(|x|^2+|y|^2\right)^{\frac{p}{2}} \\ &\le k(2+|x|^p+|y|^p) - 2k - k\left(|x|^p+|y|^p\right) \\ &= 0, \qquad \forall (x,y,t,s)\ :\ \sqrt{|x|^2+|y|^2} \ge R. \end{aligned}$$

Since $w(x,y,t,s)$ is upper semicontinuous and $\psi(x,y,t,s)$ is smooth, the supremum of Φ is attained at a point $(\hat{x},\hat{y},\hat{t},\hat{t})$ in the interior of the *cylinder* of radius R, *i.e.*, such that $\sqrt{|\hat{x}|^2+|\hat{y}|^2}<R$.

Thus, for any $(x,\hat{y},t,\hat{s})$, with $(x,t)\neq(\hat{x},\hat{t})$, we have

$$u(x,t)-v(\hat{y},\hat{s})-\psi(x,\hat{y},t,\hat{s})<u(\hat{x},\hat{t})-v(\hat{y},\hat{s})-\psi(\hat{x},\hat{y},\hat{t},\hat{s})$$

or, putting $\varphi(x,t):=\psi(x,\hat{y},t,\hat{s})+u(\hat{x},\hat{t})-\psi(\hat{x},\hat{y},\hat{t},\hat{s})$,

$$u(x,t)<\varphi(x,t),\quad \forall\ (x,t)\in Q,\ (x,t)\neq(\hat{x},\hat{t}).$$

It is also obvious that $\varphi(\hat{x},\hat{t})=u(\hat{x},\hat{t})$.

Analogously, we obtain

$$v(y,s)>\vartheta(y,s),\quad \forall\ (y,s)\in Q,\ (y,s)\neq(\hat{y},\hat{s}),$$

for $\vartheta(y,s):=-\psi(\hat{x},y,\hat{t},s)+v(\hat{y},\hat{s})+\psi(\hat{x},\hat{y},\hat{t},\hat{s})$, and so we can use φ and ϑ in the definition, respectively, of viscosity subsolution and viscosity supersolution.

For this, we need to compute the derivatives of both φ and ϑ and we next present the relevant calculations (hereafter, we denote the distance of a point $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ to the origin by $d(x,y)=\sqrt{|x|^2+|y|^2}$).

$$D\varphi(x,t)=e^{\eta t}\left[kp\left(1+|x-\hat{y}|^2\right)^{\frac{p}{2}-1}(x-\hat{y})+\frac{\rho_r'(d)}{d}x\right];$$

$$D^2\varphi(x,t)=e^{\eta t}\left[kp\left(1+|x-\hat{y}|^2\right)^{\frac{p}{2}-1}\left(\mathcal{I}+\frac{(p-2)(x-\hat{y})\otimes(x-\hat{y})}{1+|x-\hat{y}|^2}\right)\right.$$
$$\left.+\rho_r''(d)\frac{x\otimes x}{d^2}+\frac{\rho_r'(d)}{d}\left(\mathcal{I}-\frac{x\otimes x}{d^2}\right)\right],$$

with $d=d(x,\hat{y})$;

$$D\vartheta(y,s)=-e^{\eta\hat{t}}\left[-kp\left(1+|\hat{x}-y|^2\right)^{\frac{p}{2}-1}(\hat{x}-y)+\frac{\rho_r'(\tilde{d})}{\tilde{d}}y\right];$$

$$D^2\vartheta(y,s)=-e^{\eta\hat{t}}\left[kp\left(1+|\hat{x}-y|^2\right)^{\frac{p}{2}-1}\left(\mathcal{I}+\frac{(p-2)(\hat{x}-y)\otimes(\hat{x}-y)}{1+|\hat{x}-y|^2}\right)\right.$$
$$\left.+\rho_r''(\tilde{d})\frac{y\otimes y}{\tilde{d}^2}+\frac{\rho_r'(\tilde{d})}{\tilde{d}}\left(\mathcal{I}-\frac{y\otimes y}{\tilde{d}^2}\right)\right],$$

with $\tilde{d} = d(\hat{x}, y)$.
We remark that

$$D\varphi(\hat{x}, t) = 0 \quad \Leftrightarrow \quad \begin{cases} \hat{x} = \hat{y} \\ \\ |\hat{x}|^2 + |\hat{y}|^2 \leq r^2 \end{cases} \quad \Leftrightarrow \quad D\vartheta(\hat{y}, s) = 0$$

and now split the proof into two cases.

Case 1. $\hat{x} = \hat{y}$ and $|\hat{x}|^2 + |\hat{y}|^2 \leq r^2$.

According to Definition 1.1, we have

$$\varphi_t(\hat{x}, \hat{t}) \leq \Lambda(D^2\varphi(\hat{x}, \hat{t}))$$

while

$$\vartheta_s(\hat{y}, \hat{s}) \geq \lambda(D^2\vartheta(\hat{y}, \hat{s})),$$

where $\Lambda(D^2\varphi(\hat{x}, \hat{t}))$ and $\lambda(D^2\vartheta(\hat{y}, \hat{s}))$ are the largest and the smaller eigenvalue of the matrices $D^2\varphi(\hat{x}, \hat{t})$ and $D^2\vartheta(\hat{y}, \hat{s})$, respectively. Since

$$D^2\varphi(\hat{x}, \hat{t}) = e^{\eta\hat{t}} kp\mathcal{I}$$

and

$$D^2\vartheta(\hat{y}, \hat{s}) = -e^{\eta\hat{t}} kp\mathcal{I},$$

we deduce that $\Lambda(e^{\eta\hat{t}} kp\mathcal{I}) = e^{\eta\hat{t}} kp$ and $\lambda(-e^{\eta\hat{t}} kp\mathcal{I}) = -e^{\eta\hat{t}} kp$. Hence, subtracting the previous inequalities, we deduce that

$$3\eta k e^{\eta\hat{t}} \leq 2kpe^{\eta\hat{t}}$$

and choosing $3\eta > 2p$ we get a contradiction.

Case 2. $\hat{x} \neq \hat{y}$ or $\hat{x} = \hat{y}$ and $|\hat{x}|^2 + |\hat{y}|^2 > r^2$.

We first observe that, by definition, the functions φ and ϑ satisfy, respectively, the inequalities

$$\varphi_t(\hat{x}, \hat{t}) \leq \left(D^2\varphi(\hat{x}, \hat{t}) \frac{D\varphi(\hat{x}, \hat{t})}{|D\varphi(\hat{x}, \hat{t})|} \right) \cdot \frac{D\varphi(\hat{x}, \hat{t})}{|D\varphi(\hat{x}, \hat{t})|}$$

and

$$\vartheta_s(\hat{y}, \hat{s}) \geq \left(D^2\vartheta(\hat{y}, \hat{s}) \frac{D\vartheta(\hat{y}, \hat{s})}{|D\vartheta(\hat{y}, \hat{s})|} \right) \cdot \frac{D\vartheta(\hat{y}, \hat{s})}{|D\vartheta(\hat{y}, \hat{s})|}.$$

In particular, since for any $g \in C^2$, $|\Delta_\infty g| \leq \|D^2 g\|_{L^\infty}$, we have that

$$\varphi_t(\hat{x}, \hat{t}) \leq |D^2\varphi(\hat{x}, \hat{t})| + |D^2\vartheta(\hat{y}, \hat{s})|,$$

observing that ϑ is a stationary supersolution. A simple computation yields

$$|D^2\varphi(\hat{x},\hat{t})| \leq e^{\eta\hat{t}}\Big[kp(p-1)(1+|\hat{x}-\hat{y}|^2)^{\frac{p}{2}-1} \\ +\rho_r''(\hat{d})\frac{|\hat{x}|^2}{\hat{d}^2}+\frac{\rho_r'(\hat{d})}{\hat{d}}\left(1+\frac{|\hat{x}|^2}{\hat{d}^2}\right)\Big]$$

and

$$|D^2\vartheta(\hat{y},\hat{s})| \leq e^{\eta\hat{t}}\Big[kp(p-1)(1+|\hat{x}-\hat{y}|^2)^{\frac{p}{2}-1} \\ +\rho_r''(\hat{d})\frac{|\hat{y}|^2}{\hat{d}^2}+\frac{\rho_r'(\hat{d})}{\hat{d}}\left(1+\frac{|\hat{y}|^2}{\hat{d}^2}\right)\Big],$$

with $\hat{d}=d(\hat{x},\hat{y})$ and recalling that the functions ρ_r are nondecreasing. Thus

$$|D^2\varphi(\hat{x},\hat{t})|+|D^2\vartheta(\hat{y},\hat{s})| \\ \leq e^{\eta\hat{t}}\left[2kp(p-1)(1+|\hat{x}-\hat{y}|^2)^{\frac{p}{2}-1}+\rho_r''(\hat{d})+3\frac{\rho_r'(\hat{d})}{\hat{d}}\right].$$

Applying (14), and since $(1+|\hat{x}-\hat{y}|^2)^{\frac{p}{2}-1}\leq(1+|\hat{x}-\hat{y}|^2)^{\frac{p}{2}}$, we deduce that

$$|D^2\varphi(\hat{x},\hat{t})|+|D^2\vartheta(\hat{y},\hat{s})| \\ \leq e^{\eta\hat{t}}\left[2kp(p-1)(1+|\hat{x}-\hat{y}|^2)^{\frac{p}{2}}+\sigma(\rho_r(\hat{d})+1)\right]\leq\alpha\psi(\hat{x},\hat{y},\hat{t},\hat{s}),$$

where $\alpha=\max\left\{\sigma,2p(p-1),\frac{\sigma}{2k}\right\}$. On the other hand, since

$$\varphi_t(\hat{x},\hat{t})=\psi_t(\hat{x},\hat{y},\hat{t},\hat{s})=\eta\psi(\hat{x},\hat{y},\hat{t},\hat{s}),$$

we get a contradiction choosing $\eta>\alpha$.

∎

Remark 2.4. We exhibit an explicit choice of the family $\rho_r(s)$, $r>1$, used in the proof above. For $p>2$, we can choose, for instance,

$$\rho_r(s)=2k(s-r)_+^p.$$

It is easy to see that assumptions (12) and (13) are satisfied; in order to prove that inequality (14) holds true, we note that, since $p>2$, $\rho_r\in C^2([0,+\infty))$, with

$$\rho_r'(s)=2kp(s-r)_+^{p-1} \qquad \text{and} \qquad \rho_r''(s)=2kp(p-1)(s-r)_+^{p-2}.$$

Thus, since $\rho_r(s) \equiv 0$ if $s \in (0, r]$, we have to prove (14) only for $s > r$. Denoting $\tau = s - r$, (14) is satisfied if and only if $\rho_r(\tau)$ verifies

$$\begin{cases} \rho_r''(\tau) + 3\dfrac{\rho_r'(\tau)}{\tau + r} \leq \sigma(\rho_r(\tau) + 1) \\ \rho_r(0) = \rho_r'(0) = \rho_r''(0) = 0. \end{cases}$$

Thus, we have to prove that, for any $\tau > 0$,

$$\sigma(\tau + r)(2k\tau^p + 1) - 2kp(p-1)\tau^{p-2}(\tau + r) - 6kp\tau^{p-1} \geq 0,$$

for a suitable choice of σ. This is equivalent to

$$\begin{aligned} & r\left[2\sigma k\tau^p + \sigma - 2kp(p-1)\tau^{p-2}\right] \\ & +\tau\left[2\sigma k\tau^p + \sigma - 2kp(p-1)\tau^{p-2} - 6kp\tau^{p-2}\right] \geq 0 \end{aligned}$$

and since

$$\frac{2kp(p-1)\tau^{p-2}}{2k\tau^p + 1} \leq (p-1)(4k)^{\frac{2}{p}}(p-2)^{\frac{p-2}{p}}$$

while

$$\frac{2kp(p+2)\tau^{p-2}}{2k\tau^p + 1} \leq (p+2)(4k)^{\frac{2}{p}}(p-2)^{\frac{p-2}{p}},$$

it is enough to choose

$$\sigma = (p+2)(4k)^{\frac{2}{p}}(p-2)^{\frac{p-2}{p}}.$$

On the other hand, if $1 < p \leq 2$, we can take $\rho_r(s) = \rho\left[(s-r)_+\right]$, with

$$\rho(\tau) = \begin{cases} 2k\mu(\tau) & \text{if } 0 \leq \tau \leq 1 \\ 2k\tau^p & \text{if } \tau \geq 1, \end{cases}$$

where $\mu(\tau) \in C^2([0,1])$ and satisfies

$$\mu(0) = 0\,, \quad \mu'(0) = 0\,, \quad \mu''(0) = 0\,,$$

$$\mu(1) = 1\,, \quad \mu'(1) = p\,, \quad \mu''(1) = p(p-1)\,.$$

We can choose, for example,

$$\mu(\tau) = \tau^3\left[\frac{(p-5)(p-4)}{2} - (p-5)(p-3)\tau + \frac{(p-3)(p-4)}{2}\tau^2\right].$$

Thus, since $\mu \geq 0$ and $\mu' \geq 0$ in $[0,1]$, if we denote by

$$\sigma_0 = \max_{\tau \in [0,1]} \{\mu''(\tau) + 3\mu'(\tau)\}$$

and we choose

$$\sigma > \max\left\{\sigma_0\,,\ \frac{2kp(p+2)}{1+2k}\right\},$$

it is easy to check that inequality (14) holds true.

3. Some auxiliary results

We now introduce a family of auxiliary functions. For $\varepsilon, \delta, \gamma > 0$ and $m > q > \max\{2, p\}$, define

$$\Psi(x,y,t) = \frac{|x-y|^m}{\varepsilon m} + \delta(|x|^q + |y|^q) + \frac{\gamma}{T-t}. \tag{16}$$

In the above expression, every term plays a different role in the construction of a suitable barrier for the function

$$w(x,y,t) = u(x,t) - v(y,t). \tag{17}$$

In fact, in order to prove that $w(x,y,t) - \Psi(x,y,t)$ can not achieve a positive maximum, the term $\delta(|x|^q + |y|^q)$ controls the behavior of w at infinity, using estimate (11) and the growth condition on the solution (see also Remark 2.2). On the other hand, $\frac{|x-y|^m}{\varepsilon m}$ acts as a penalization term if x is different from y, while the last term in (16) forces t to be smaller than T.

We now assume that

$$\limsup_{l\to 0}\ \{w(x,y,t)\ :\ |x-y| \leq l\} = a > 0. \tag{18}$$

Lemma 3.1. *Suppose that $u(x,t)$ and $v(y,t)$ are upper semicontinuous and lower semicontinuous, respectively, and satisfy estimates (7) and (11). Assume also that (18) is in force.*

Then, for each $\varepsilon > 0$, there exist $\gamma_0, \delta_0 > 0$ such that, for every $\delta < \delta_0$ and every $\gamma < \gamma_0$, the following assertions hold:

(i)
$$\sup_{\mathbb{R}^N\times\mathbb{R}^N\times(0,T)} [w(x,y,t) - \Psi(x,y,t)] > \frac{a}{2}; \tag{19}$$

(ii) *there exists a point $(\hat{x}, \hat{y}, \hat{t}) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,T)$ where the maximum of $w - \Psi$ is attained;*

(iii) *there exists a constant* $C_1 > 0$*, independent of* γ*,* δ *and* ε*, such that*

$$|\hat{x} - \hat{y}| \leq C_1. \tag{20}$$

Moreover,

(iv) $|\hat{x} - \hat{y}|^m = O(\varepsilon)$ *and* $(\delta|\hat{x}|)^q + (\delta|\hat{y}|)^q = O(\delta^{q-1})$*;*

(v) $0 < \hat{t} < T$*.*

Proof: (i) Fix $\varepsilon > 0$. Due to (18),there exists $(x_0, y_0, t_0) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, T)$ such that

$$w(x_0, y_0, t_0) > \frac{3a}{4} \qquad \text{and} \qquad \frac{|x_0 - y_0|^m}{\varepsilon m} < \frac{a}{8}.$$

Thus,

$$w(x_0, y_0, t_0) - \Psi(x_0, y_0, t_0) > \frac{3a}{4} - \frac{a}{8} - \delta(|x_0|^q + |y_0|^q) - \frac{\gamma}{T - t_0}.$$

Choosing $\delta_0 > 0$ and $\gamma_0 > 0$ such that

$$\delta_0(|x_0|^q + |y_0|^q) + \frac{\gamma_0}{T - t_0} \leq \frac{a}{8},$$

(19) holds.

(ii) Note that, by (8) and since $q > p$,

$$w(x, y, t) - \Psi(x, y, t) < 0,$$

for all (x, y) such that $|x|^2 + |y|^2 > R_1^2$, where R_1 is sufficiently large. Since $\Psi(x, y, t)$ is smooth and $w(x, y, t)$ is upper semicontinuous, we deduce that there exists $(\hat{x}, \hat{y}, \hat{t}) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, T)$ where

$$\max_{\mathbb{R}^N \times \mathbb{R}^N \times (0,T)} [w(x, y, t) - \Psi(x, y, t)]$$

is achieved.

(iii) Since $w(\hat{x}, \hat{y}, \hat{t}) - \Psi(\hat{x}, \hat{y}, \hat{t}) > 0$, we deduce, using (11), that

$$\frac{|\hat{x} - \hat{y}|^m}{\varepsilon m} + \delta(|\hat{x}|^q + |\hat{y}|^q) + \frac{\gamma}{T - \hat{t}} \leq C(1 + |\hat{x} - \hat{y}|^p). \tag{21}$$

Dropping positive terms and assuming $\varepsilon < 1$, we deduce

$$\frac{|\hat{x} - \hat{y}|^m}{m} \leq C(1 + |\hat{x} - \hat{y}|^p).$$

Since $m > p$ we conclude that

$$|\hat{x} - \hat{y}| \leq C_1,$$

for a constant C_1, clearly independent of δ, γ and ε.

(iv) Using (20) in (21), we deduce that

$$\frac{|\hat{x} - \hat{y}|^m}{\varepsilon m} + \delta(|\hat{x}|^q + |\hat{y}|^q) \leq C(1 + C_1^p) = \tilde{C}$$

and, consequently,

$$\frac{|\hat{x} - \hat{y}|^m}{m} \leq \tilde{C}\varepsilon \qquad \text{and} \qquad \delta(|\hat{x}|^q + |\hat{y}|^q) \leq \tilde{C}. \tag{22}$$

The conclusion follows.

(v) Suppose, *ad contrarium*, that $\hat{t} = 0$. Consider a vanishing sequence (ε_j) and the corresponding sequence of maximizers $(\hat{x}_j, \hat{y}_j, \hat{t}_j)$ for $w - \Psi$. Note that, by (19), and since $\Psi \geq 0$,

$$\frac{a}{2} \leq w(\hat{x}_j, \hat{y}_j, \hat{t}_j) - \Psi(\hat{x}_j, \hat{y}_j, \hat{t}_j) \leq u(\hat{x}_j, \hat{t}_j) - v(\hat{y}_j, \hat{t}_j).$$

By (22), we deduce that $|\hat{x}_j - \hat{y}_j| \to 0$ as $j \to \infty$ and, since also $\hat{t}_j \to 0$, passing to the limit, we obtain

$$\frac{a}{2} \leq \limsup_{j\to\infty} \left[u(\hat{x}_j, \hat{t}_j) - v(\hat{y}_j, \hat{t}_j)\right] \leq u^*(\hat{x}, 0) - v_*(\hat{x}, 0),$$

where $\hat{x} = \lim_{j\to\infty} \hat{x}_j$. The above inequality contradicts (7).

Finally, to exclude that $\hat{t} = T$, assume there exists a sequence $t_j \to T$. From (21), we deduce that $\frac{\gamma_j}{T - t_j}$ is uniformly bounded, which leads to a contradiction.

■

We finally introduce some further notation and state an important lemma in the context of viscosity solutions.

Definition 3.2. The parabolic super 2-jet of a continuous function z at a point $(w, r) \in \mathbb{R}^N \times (0, T)$, denoted by $\mathcal{P}^{2,+}(z(w, r))$, is the set of all $(\tau, q, Z) \in \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N$ such that

$$\begin{aligned} z(x,t) \ \leq \ & z(w,r) + \tau(t-r) \\ & + q \cdot (x-w) + \frac{1}{2} Z(x-w) \cdot (x-w) \\ & + o(|t-r| + |x-w|^2), \quad \forall\ (x,t) \in \mathbb{R}^N \times (0,T). \end{aligned}$$

Analogously,

$$(\hat{\tau}, \hat{q}, \hat{Z}) \in \mathcal{P}^{2,-}(z(w,r)) \quad \text{if} \quad (\hat{\tau}, \hat{q}, \hat{Z}) \in -\mathcal{P}^{2,+}(-z(w,r)).$$

The proof of the following lemma can be found in [6] (see also [7]).

Lemma 3.3. *Let $u(x,t)$ be upper semicontinuous and let $v(x,t)$ be lower semicontinuous. Let $\chi(x,y,t)$ be continuously differentiable in $t \in (0,T)$ and twice continuously differentiable in $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. Suppose that there exists $(\hat{x}, \hat{y}, \hat{t}) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,T)$ such that*

$$u(x,t) - v(y,t) - \chi(x,y,t) \leq u(\hat{x},\hat{t}) - v(\hat{y},\hat{t}) - \chi(\hat{x},\hat{y},\hat{t}),$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,T)$. Assume further that there exists $\omega > 0$ such that, for every $M > 0$, there is a $C_M > 0$ such that

$$\tau_1, \tau_2 \leq C_M$$

whenever

$$(\tau_1, q_1, X) \in \mathcal{P}^{2,+}(u(x,t)); \quad (\tau_2, q_2, Y) \in \mathcal{P}^{2,+}(-v(y,t));$$

$$|x - \hat{x}| + |y - \hat{y}| + |t - \hat{t}| \leq \omega;$$

and

$$|u(x,t)| + |q_1| + \|X\| \leq M; \qquad |v(y,t)| + |q_2| + \|Y\| \leq M.$$

Then, for each $\theta > 0$, there exist $X, Y \in \mathcal{S}^N$ such that

- $(\tau_1, D_x\chi(\hat{x},\hat{y},\hat{t}), X) \in \overline{\mathcal{P}}^{2,+}(u(\hat{x},\hat{t}))$;
- $(\tau_2, D_y\chi(\hat{x},\hat{y},\hat{t}), Y) \in \overline{\mathcal{P}}^{2,+}(-v(\hat{y},\hat{t}))$;
- $\tau_1 + \tau_2 = \chi_t(\hat{x},\hat{y},\hat{t})$;
- $-\left(\dfrac{1}{\theta} + \|A\|\right) \mathcal{I} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq A + \theta A^2$,

where $A = D^2_{(x,y)}\chi(\hat{x},\hat{y},\hat{t})$.

4. The comparison principle

The main tool to prove the uniqueness part of our main theorem is the following comparison principle that extends the result of [9] in the case of the infinity-Laplacian.

Theorem 4.1. *Let $u_0(x)$ be given satisfying (6). If $u(x,t)$ and $v(y,t)$ are, respectively, a viscosity subsolution and a viscosity supersolution of* (1)*, with polynomial growth and satisfying* (7)*, then there exists a modulus of continuity μ such that*

$$u(x,t) - v(y,t) \le \mu(|x-y|), \quad \forall (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,T). \tag{23}$$

Proof: Let us assume by contradiction that (23) is violated. Then (18) is in force and thus, since (11) holds due to Lemma 2.3, the conclusions of Lemma 3.1 are valid. Recalling that Ψ is defined by (16) and w by (17), we can then assume that there exists $(\hat{x}, \hat{y}, \hat{t})$ such that

$$\sup_{\mathbb{R}^N \times \mathbb{R}^N \times (0,T)} w(x,y,t) - \Psi(x,y,t) = w(\hat{x},\hat{y},\hat{t}) - \Psi(\hat{x},\hat{y},\hat{t}).$$

Hence, by Lemma 3.3, applied with $\chi(x,y,t) = \Psi(x,y,t)$, there exist $\tau_1, \tau_2 \in \mathbb{R}$ and $X, Y \in S^N$ such that

$$(\tau_1, D_x\Psi(\hat{x},\hat{y},\hat{t}), X) \in \overline{\mathcal{P}}^{2,+}(u(\hat{x},\hat{t})),$$

$$(\tau_2, -D_y\Psi(\hat{x},\hat{y},\hat{t}), Y) \in \overline{\mathcal{P}}^{2,-}(v(\hat{y},\hat{t})),$$

$$\tau_1 - \tau_2 = \Psi_t(\hat{x},\hat{y},\hat{t}),$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \theta A^2, \tag{24}$$

where $\theta > 0$ and

$$A = D^2_{(x,y)}\Psi(\hat{x},\hat{y},\hat{t}).$$

We now compute the derivatives, obtaining

$$D_x\Psi(x,y,t) = \frac{|x-y|^{m-2}}{\varepsilon}(x-y) + q\delta|x|^{q-2}x; \tag{25}$$

$$D_y\Psi(x,y,t) = -\frac{|x-y|^{m-2}}{\varepsilon}(x-y) + q\delta|y|^{q-2}y; \tag{26}$$

$$
\begin{aligned}
D^2_{xx}\Psi(x,y,t) &= \frac{|x-y|^{m-2}}{\varepsilon}\left[\mathcal{I}+(m-2)\frac{(x-y)}{|x-y|}\otimes\frac{(x-y)}{|x-y|}\right]\\
&\quad +q\delta|x|^{q-4}(\mathcal{I}|x|^2+(q-2)x\otimes x);
\end{aligned}
$$

$$
\begin{aligned}
D^2_{yy}\Psi(x,y,t) &= \frac{|x-y|^{m-2}}{\varepsilon}\left[\mathcal{I}+(m-2)\frac{(x-y)}{|x-y|}\otimes\frac{(x-y)}{|x-y|}\right]\\
&\quad +q\delta|y|^{q-4}(\mathcal{I}|y|^2+(q-2)y\otimes y);
\end{aligned}
$$

and

$$
D^2_{xy}\Psi(x,y,t) = -\frac{|x-y|^{m-2}}{\varepsilon}\left[\mathcal{I}+(m-2)\frac{(x-y)}{|x-y|}\otimes\frac{(x-y)}{|x-y|}\right].
$$

Thus,

$$
\begin{aligned}
D^2_{(x,y)}\Psi &= \frac{|x-y|^{m-2}}{\varepsilon}\begin{pmatrix}\mathcal{I} & -\mathcal{I}\\ -\mathcal{I} & \mathcal{I}\end{pmatrix}\\
&\quad +\frac{(m-2)|x-y|^{m-4}}{\varepsilon}(x-y)\otimes(x-y)\begin{pmatrix}\mathcal{I} & -\mathcal{I}\\ -\mathcal{I} & \mathcal{I}\end{pmatrix}\\
&\quad +q\delta\begin{pmatrix}|x|^{q-4}\left[\mathcal{I}|x|^2+(q-2)x\otimes x\right] & 0\\ 0 & |y|^{q-4}\left[\mathcal{I}|y|^2+(q-2)y\otimes y\right]\end{pmatrix}\\
&\leq (m-1)\frac{|x-y|^{m-2}}{\varepsilon}\begin{pmatrix}\mathcal{I} & -\mathcal{I}\\ -\mathcal{I} & \mathcal{I}\end{pmatrix}\\
&\quad +q(q-1)\delta\left[|x|^{q-2}+|y|^{q-2}\right]\begin{pmatrix}\mathcal{I} & 0\\ 0 & \mathcal{I}\end{pmatrix}.
\end{aligned}
$$

We next compute the above inequality at $(\hat{x},\hat{y},\hat{t})$, set

$$
\eta := (m-1)\frac{|\hat{x}-\hat{y}|^{m-2}}{\varepsilon}
$$

and

$$
\zeta := q(q-1)\delta\left[|\hat{x}|^{q-2}+|\hat{y}|^{q-2}\right],
$$

obtaining, from (24),

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \eta(2\theta\eta + 2\theta\zeta + 1) \begin{pmatrix} \mathcal{I} & -\mathcal{I} \\ -\mathcal{I} & \mathcal{I} \end{pmatrix} + \zeta(1+\theta\zeta) \begin{pmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I} \end{pmatrix}. \tag{27}$$

Our aim is to get a contradiction, taking the limit as δ goes to 0. We proceed splitting the analysis in two cases.

(i) Suppose that $\eta \to 0$ as $\delta \to 0$. Then, since u is a viscosity subsolution and v is a viscosity supersolution of (1),

$$\tau_1 \leq X \frac{D_x\hat{\Psi} D_x\hat{\Psi}}{|D_x\hat{\Psi}|^2} \qquad \text{and} \qquad \tau_2 \geq Y \frac{D_y\hat{\Psi} D_y\hat{\Psi}}{|D_y\hat{\Psi}|^2}$$

(where, to simplify, the notation $D_x\hat{\Psi}$ and $D_y\hat{\Psi}$ means the functions are evaluated at $(\hat{x}, \hat{y}, \hat{t})$). We get

$$0 < \frac{\gamma}{T^2} \leq \tau_1 - \tau_2 \leq \|X\| + \|Y\| \leq 2\eta(2\theta\eta + 2\theta\zeta + 1) + \zeta(1+\theta\zeta), \tag{28}$$

upon subtraction and recalling that

$$\tau_1 - \tau_2 = \Psi_t(\hat{x}, \hat{y}, \hat{t}) = \frac{\gamma}{(T-\hat{t})^2} \geq \frac{\gamma}{T^2}.$$

Now, we have

$$\begin{aligned} \lim_{\delta\to 0} \zeta &= q(q-1) \lim_{\delta\to 0} \delta \left[|\hat{x}|^{q-2} + |\hat{y}|^{q-2} \right] \\ &= q(q-1) \lim_{\delta\to 0} \delta^{3-q} \left[(\delta|\hat{x}|)^{q-2} + (\delta|\hat{y}|^{q-2}) \right] \end{aligned}$$

and we deduce, since $(\delta|\hat{x}|)^q + (\delta|\hat{y}|)^q = O(\delta^{q-1})$ by Lemma 3.1, that

$$\zeta \sim q(q-1)\delta^{3-q} O(\delta^{\frac{q-1}{q}(q-2)}) \longrightarrow 0 \quad \text{as } \delta \to 0 \tag{29}$$

since

$$3 - q + (1 - \frac{1}{q})(q-2) = \frac{2}{q} > 0.$$

Hence, the right hand side of (28) vanishes as $\delta \to 0$, while the left hand side is strictly positive, and we get a contradiction.

(ii) Alternatively, suppose that η does not vanish as $\delta \to 0$, *i.e.*, that, at least for a subsequence $\delta_n \to 0$,

$$(\hat{x} - \hat{y}) \longrightarrow b \neq 0.$$

We first note, using Lemma 3.1-(iv), that

$$\delta|\hat{x}|^{q-1} = (\delta|\hat{x}|)^{q-1}\delta^{2-q} = O(\delta^{\frac{q-1}{q}(q-1)})\delta^{2-q} = O(\delta^{\frac{1}{q}})$$

and, as a consequence, recalling (25) and (26),

$$D_x\hat{\Psi} \longrightarrow \frac{|b|^{m-2}b}{\varepsilon} \qquad \text{and} \qquad -D_y\hat{\Psi} \longrightarrow \frac{|b|^{m-2}b}{\varepsilon}.$$

Moreover, applying (27) to vectors $\xi \in \mathbb{R}^N \times \mathbb{R}^N$ such that $\xi = (\xi_1, \xi_1)$, with $0 \neq \xi_1 \in \mathbb{R}^N$, we obtain

$$(\xi_1, \xi_1) \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} \leq 2\zeta(1 + \theta\zeta)|\xi_1|^2,$$

since

$$(\xi_1, \xi_1) \begin{pmatrix} \mathcal{I} & -\mathcal{I} \\ -\mathcal{I} & \mathcal{I} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} = 0.$$

This, in particular, implies that, for all $\xi \in \mathbb{R}^N$ such that $|\xi| = 1$,

$$\xi^T X\xi - \xi^T Y\xi \leq 2\zeta(1 + \theta\zeta). \tag{30}$$

Due to the fact that

$$\tau_1 \leq X\frac{D_x\hat{\Psi}D_x\hat{\Psi}}{|D_x\hat{\Psi}|^2} \qquad \text{and} \qquad \tau_2 \geq Y\frac{D_y\hat{\Psi}D_y\hat{\Psi}}{|D_y\hat{\Psi}|^2},$$

subtracting, using (30) and passing to the limit with respect to δ, we reach the contradiction

$$\frac{\gamma}{T^2} \leq 2\lim_{\delta\to 0}\zeta(1 + \theta\zeta) = 0,$$

since, arguing as in (29), $\zeta = O(\delta^{\frac{2}{q}})$. ∎

5. Existence and uniqueness

In this final section, we prove the existence of a unique solution of the Cauchy problem (3), in the class of functions with polynomial growth (Theorem 1.2).

We say that $z(x,t)$ is a viscosity subsolution of the Cauchy problem (3) if it is a viscosity subsolution of (1) and satisfies the initial condition in the viscosity sense, *i.e.*,

$$\forall x \in \mathbb{R}^N, \quad \min\Big\{ z_t(x,0) - \Delta_\infty z(x,0),\, z^*(x,0) - u_0(x) \Big\} \leq 0. \tag{31}$$

For a supersolution, the definition is analogous with min replaced by max, z^* replaced by z_* and $\leq$ replaced by $\geq$ in (31).

Proof of Theorem 2.1: Our aim is to apply Perron's method to build a solution of the problem and we start with the construction of a suitable supersolution for (3). By the computations of the previous sections,

$$\overline{u}(x,t) = Ce^{\eta t}(1+|x|^2)^{\frac{p}{2}},$$

with $\eta > p(p-1)$, is a supersolution of (1), with polynomial $p-$growth at infinity. Moreover, up to choosing $C > C_0$ (introduced in (6)), we deduce that

$$u_0(x) \leq \overline{u}(x,0), \qquad \forall x \in \mathbb{R}^N.$$

On the other hand, $\underline{u}(x,t) = -\overline{u}(x,t)$ is a subsolution of the same type.

Let us consider the set

$$\mathcal{A} = \Big\{ v(x,t) \;:\; \underline{u}(x,t) \leq v(x,t) \leq \overline{u}(x,t) \\ \text{and } \; v(x,t) \text{ is a viscosity subsolution of (3)} \Big\},$$

and define

$$u(x,t) = \sup_{v \in A} v(x,t).$$

It is well known (*cf.* [3]) that $u(x,t)$ is a viscosity subsolution of (3) and, moreover, that $u^*(x,0) \leq u_0(x)$ in $\mathbb{R}^N$. On the other hand, Perron's method applies (see, for example, [7, Section 4]) and $u(x,t)$ is also a viscosity supersolution of (3). As above, it follows that $u_0(x) \leq u_*(x,0)$ in $\mathbb{R}^N$. Hence $u(x,t)$ is a viscosity solution of (1) such that

$$u^*(x,0) \leq u_0(x) \leq u_*(x,0), \quad \forall x \in \mathbb{R}^N$$

and the existence follows.

The uniqueness holds due to the comparison principle and the estimate on the growth at infinity of $u(x,t)$ is a consequence of Lemma 2.1 (as explained in Remark 2.2). ■

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