Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 08–54

DYNAMICS AND INTERPRETATION OF SOME INTEGRABLE SYSTEMS VIA MULTIPLE ORTHOGONAL POLYNOMIALS

D. BARRIOS ROLANÍA, A. BRANQUINHO AND A. FOULQUIÉ MORENO

Abstract: High-order non symmetric difference operators with complex coefficients are considered. The correspondence between dynamics of the coefficients of the operator defined by a Lax pair and its resolvent function is established. The method of investigation is based on the analysis of the moments for the operator. The solution of a discrete dynamical system is studied. We give explicit expressions for the resolvent function and, under some conditions, the representation of the vector of functionals, associated with the solution for the integrable systems.

Keywords: Orthogonal polynomials, differential equations, recurrence relations.
AMS Subject Classification (2000): Primary 37F05; Secondary 33C45.

1. Introduction and notation

1.1. Vector orthogonality. For a fixed $p \in \mathbb{Z}^+$, let us consider the sequence $\{P_n\}$ of polynomials given by the recurrence relation

$$\left.\begin{array}{l} xP_n(x) = P_{n+1}(x) + a_{n-p+1}P_{n-p}(x)\,, \quad n = p, p+1, \ldots \\ P_i(x) = x^i\,, \quad i = 0, 1, \ldots, p \end{array}\right\}, \qquad (1)$$

where we assume $a_j \neq 0$ for each $j \in \mathbb{N}$. For $m \in \mathbb{N}$ and $n = mp + i$, $i = 0, 1, \ldots, p-1$, we can write

$$\begin{cases} xP_{mp}(x) = P_{mp+1}(x) + a_{(m-1)p+1}P_{(m-1)p}(x) \\ \qquad\qquad \vdots \\ xP_{(m+1)p-1}(x) = P_{(m+1)p}(x) + a_{mp}P_{mp-1}(x)\,. \end{cases}$$

Received November 24, 2008.
The work of the first author was supported in part by Dirección General de Investigación, Ministerio de Educación y Ciencia, under grant MTM2006-13000-C03-02, and by Universidad Politécnica de Madrid and Comunidad Autónoma de Madrid, under Grant CCG07-UPM/000-1652. The work of the second author was supported by CMUC/FCT. The third author would like to thank UI Matemática e Aplicações from University of Aveiro.

This is, denoting $\mathcal{B}_m(x) = \big(P_{mp}(x), P_{mp+1}(x), \dots, P_{(m+1)p-1}(x)\big)^T$,

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

and $C_m = \operatorname{diag}\{a_{(m-1)p+1}, a_{(m-1)p+2}, \dots, a_{mp}\}$, we can rewrite (1) as

$$x\,\mathcal{B}_m(x) = A\,\mathcal{B}_{m+1}(x) + B\,\mathcal{B}_m(x) + C_m\,\mathcal{B}_{m-1}(x)\,, \quad m \in \mathbb{N}\,, \tag{2}$$

with initial conditions $\mathcal{B}_{-1} = (0, \dots, 0)^T$, $\mathcal{B}_0(x) = \big(1, x, \dots, x^{p-1}\big)^T$.

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients. It is well known that, given the recurrence relation (1), there exist p linear moment functionals $u^1, \dots, u^p$ from $\mathcal{P}$ to $\mathbb{C}$ such that for each $s \in \{0, 1, \dots, p-1\}$ the following orthogonality relations are satisfied,

$$u^i[x^j P_{mp+s}(x)] = 0 \ \text{ for } \ \begin{cases} j = 0, 1, \dots m\,, \ i = 1, \dots, s \\ j = 0, 1, \dots m-1\,, \ i = s+1, \dots, p \end{cases} \tag{3}$$

(see [4, Th. 3.2], see also [3]). In all the following, for each sequence $\{a_n\}$ in (1) we denote by $u^1, \dots, u^p$ a fixed set of moment functionals verifying (3).

We consider the space $\mathcal{P}^p = \{(q_1, \dots q_p)^T : q_i \text{ polynomial}, i = 1, \dots, p\}$ and the space $\mathcal{M}_{p\times p}$ of $(p\times p)$-matrices with complex entries. From the existence of the functionals $u^1, \dots, u^p$, associated with the recurrence relation (1), we can define the function $\mathcal{W} : \mathcal{P}^p \to \mathcal{M}_{p\times p}$ given by

$$\mathcal{W}\begin{pmatrix} q_1 \\ \vdots \\ q_p \end{pmatrix} = \begin{pmatrix} u^1[q_1] & \dots & u^p[q_1] \\ \vdots & \ddots & \vdots \\ u^1[q_p] & \dots & u^p[q_p] \end{pmatrix}. \tag{4}$$

In particular, for $m, j \in \{0, 1, \dots\}$ we have

$$\mathcal{W}\left(x^j \mathcal{B}_m\right) = \begin{pmatrix} u^1[x^j P_{mp}(x)] & \dots & u^p[x^j P_{mp}(x)] \\ \vdots & \ddots & \vdots \\ u^1[x^j P_{(m+1)p-1}(x)] & \dots & u^p[x^j P_{(m+1)p-1}(x)] \end{pmatrix}$$

and the orthogonality conditions (3) can be reinterpreted as

$$\mathcal{W}(x^j \mathcal{B}_m) = 0\,, \quad j = 0, 1, \dots, m-1\,. \tag{5}$$

For a fixed regular matrix, $M \in \mathcal{M}_{p\times p}$, we define the function

$$\mathcal{U} : \mathcal{P}^p \longrightarrow \mathcal{M}_{p\times p}$$

such that

$$\mathcal{U}\begin{pmatrix} q_1 \\ \vdots \\ q_p \end{pmatrix} = \mathcal{W}\begin{pmatrix} q_1 \\ \vdots \\ q_p \end{pmatrix} M\,, \quad \begin{pmatrix} q_1 \\ \vdots \\ q_p \end{pmatrix} \in \mathcal{P}^p\,, \tag{6}$$

being $\mathcal{W}$ given in (4). Briefly, we write

$$\mathcal{U} = \mathcal{W}M\,. \tag{7}$$

We say that $\mathcal{U}$, given by (6) and (7), is a *vector of functionals defined by the recurrence relation* (2). In this case, we say that $\{\mathcal{B}_m\}$ is the *sequence of vector polynomials orthogonal with respect to* $\mathcal{U}$.

More generally speaking, for any set $\{v^1, \ldots, v^p\}$ of linear functionals defined in the space $\mathcal{P}$ of polynomials, it is possible to define a function $\mathcal{U} : \mathcal{P}^p \longrightarrow \mathcal{M}_{p\times p}$ like (6). It is done in the following definition.

Definition 1. The function $\mathcal{U} : \mathcal{P}^p \longrightarrow \mathcal{M}_{p\times p}$ given by

$$\mathcal{U}\begin{pmatrix} q_1 \\ \vdots \\ q_p \end{pmatrix} = \begin{pmatrix} v^1[q_1] & \ldots & v^p[q_1] \\ \vdots & \ddots & \vdots \\ v^1[q_p] & \ldots & v^p[q_p] \end{pmatrix} M_{\mathcal{U}}$$

for each $(q_1, \ldots, q_p)^T \in \mathcal{P}^p$ is called *vector of functionals* associated with the linear functionals $v^1, \ldots, v^p$ and with the regular matrix $M_{\mathcal{U}} \in \mathcal{M}_{p\times p}$.

It is easy to see that, for any vector of functionals $\mathcal{U}$, the following properties are verified:

$$\mathcal{U}(\mathcal{Q}_1 + \mathcal{Q}_2) = \mathcal{U}(\mathcal{Q}_1) + \mathcal{U}(\mathcal{Q}_2)\,, \text{ for } \mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{P}^p, \tag{8}$$

$$\mathcal{U}(M\,\mathcal{Q}) = M\,\mathcal{U}(\mathcal{Q})\,, \text{ for } \mathcal{Q} \in \mathcal{P}^p \text{ and } M \in \mathcal{M}_{p\times p}\,. \tag{9}$$

In our case, if $\mathcal{U}$ is a vector of functionals defined by the recurrence relation (2), then $\mathcal{U}$ is associated with the moment functionals $u^1, \ldots, u^p$ defined by the recurrence relation (1). Therefore, the orthogonality conditions (5) are verified for $\mathcal{U}$. This is,

$$\mathcal{U}(x^j\mathcal{B}_m) = 0\,, \quad j = 0, 1, \ldots, m-1\,. \tag{10}$$

Using (8), (9) and the recurrence relation (2) we deduce

$$\begin{aligned}\mathcal{U}\left(x^m\mathcal{B}_m\right) &= \mathcal{U}\left(x^{m-1}A\mathcal{B}_{m+1} + x^{m-1}B\mathcal{B}_m + x^{m-1}C_m\mathcal{B}_{m-1}\right) \\ &= A\,\mathcal{U}\left(x^{m-1}\,\mathcal{B}_{m+1}\right) + B\,\mathcal{U}\left(x^{m-1}\,\mathcal{B}_m\right) + C_m\mathcal{U}\left(x^{m-1}\,\mathcal{B}_{m-1}\right)\,.\end{aligned}$$

Then, from (10), $\mathcal{U}\left(x^m \mathcal{B}_m\right) = C_m \mathcal{U}\left(x^{m-1} \mathcal{B}_{m-1}\right)$ and, iterating,

$$\mathcal{U}\left(x^m \mathcal{B}_m\right) = C_m C_{m-1} \cdots C_1 \mathcal{U}\left(\mathcal{B}_0\right) , \tag{11}$$

where $\mathcal{U}\left(\mathcal{B}_0\right) = \mathcal{W}\left(\mathcal{B}_0\right) M_{\mathcal{U}}$ and

$$\mathcal{W}\left(\mathcal{B}_0\right) = \begin{pmatrix} u^1[1] & \dots & u^p[1] \\ \vdots & \ddots & \vdots \\ u^1[x^{p-1}] & \dots & u^p[x^{p-1}] \end{pmatrix} \tag{12}$$

(see (4)). In the sequel we assume that $\mathcal{W}\left(\mathcal{B}_0\right)$ is a regular matrix. Then, the vector of functionals $\mathcal{U}$ associated with the linear functionals $u^1, \dots, u^p$ and with the matrix $\left(\mathcal{W}(\mathcal{B}_0)\right)^{-1}$ verifies

$$\left.\begin{array}{l} \mathcal{U}\left(x^j \mathcal{B}_m\right) = \Delta_m \delta_{mj} , \quad m = 1, 2, \dots, \quad j = 0, 1, \dots, m , \\ \Delta_m = C_m C_{m-1} \cdots C_1 , \ \ \mathcal{U}\left(\mathcal{B}_0\right) = I_p . \end{array}\right\} \tag{13}$$

At the same time, given $\left(q_1, \dots, q_p\right)^T \in \mathcal{P}^p$, for each $i = 1, \dots, p$ we can write

$$q_i(x) = \sum_{k=1}^{p} \alpha_{ik}^0 P_{k-1}(x) + \sum_{k=1}^{p} \alpha_{ik}^1 P_{p+k-1}(x) + \cdots + \sum_{k=1}^{p} \alpha_{ik}^m P_{mp+k-1}(x) , \ \alpha_{ik}^j \in \mathbb{C} ,$$

where $m = \max\{m_1, \dots, m_p\}$ and $\deg(q_i) \leq (m_i + 1)p - 1$ (we understand $\alpha_{ik}^j = 0$ when $j > m_i$). In other words,

$$\left(q_1, \dots, q_p\right)^T = \sum_{j=0}^{m} D_j \, \mathcal{B}_j ,$$

being $D_j = \left(\alpha_{ik}^j\right) \in \mathcal{M}_{p \times p}$, $j = 0, \dots, m$. Then, due to (8) and (9), we have that $\mathcal{U} : \mathcal{P}^p \longrightarrow \mathcal{M}_{p \times p}$ is determined by (13). In the following, for each sequence $\{a_n\}$ defining the sequence of polynomials $\{P_n\}$ in (1), we denote by $\mathcal{U}$ this fixed vector of functionals.

We will use the vectorial polynomials

$$\mathcal{P}_i = \mathcal{P}_i(x) = \left(x^{ip}, x^{ip+1}, \dots, x^{(i+1)p-1}\right)^T , \quad i = 0, 1, \dots .$$

(In particular, note that $\mathcal{P}_0 = \mathcal{B}_0$.) Also, for each vector of functionals $\mathcal{V}$ we will use the matrices $\mathcal{V}\left(\mathcal{P}_i\right)$. As in the scalar case (i.e. $p = 1$), we can define the moments associated with the vector of functionals $\mathcal{V}$.

Definition 2. For each $m = 0, 1, \dots$, the matrix $\mathcal{V}\left(x^m \mathcal{P}_0\right)$ is called moment of order m for the vector of functionals $\mathcal{V}$.

1.2. Connection with operator theory. If $\{a_n\}$ is a bounded sequence, then the infinite $(p+2)$-band matrix

$$J = \begin{pmatrix} 0 & 1 & & & & & & & \\ \vdots & \ddots & \ddots & & & & & & \\ 0 & \cdots & 0 & 1 & & & & & \\ a_1 & 0 & \cdots & 0 & 1 & & & & \\ & \ddots & \ddots & & \ddots & \ddots & & & \\ & & a_p & 0 & \cdots & 0 & 1 & & \\ & & & \ddots & \ddots & & \ddots & \ddots & \end{pmatrix} \tag{14}$$

induces a bounded operator in ℓ^2 with regard to the standard basis. In this case, we denote in the same way the operator and it matricial representation.

When J is a bounded operator, then $\{z : |z| > \|J\|\}$ is contained in the resolvent set. In this case we have

$$(zI - J)^{-1} = \sum_{n\geq 0} \frac{J^n}{z^{n+1}}, \quad |z| > \|J\| \tag{15}$$

(see [5, Th. 3, p. 211]). Being $e_0 = (I_p, 0_p, \ldots)^T$, we define the $(p\times p)$-matrix

$$\mathcal{R}_J(z) = \langle (zI - J)^{-1} e_0, e_0 \rangle = \sum_{n\geq 0} \frac{\langle J^n e_0, e_0\rangle}{z^{n+1}}, \quad |z| > \|J\|, \tag{16}$$

where we denote by $\langle M e_0, e_0 \rangle$, for an infinite matrix M, the finite matrix given by the formal product $e_0^T M e_0$, this is, the $(p \times p)$-matrix formed by the first p rows and the first p columns of M.

If we rewrite the matrix J given in (14) as a blocked matrix,

$$J = \begin{pmatrix} B & A & & & \\ C_1 & B & A & & \\ & C_2 & B & A & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

then another way to express (2) is

$$J \begin{pmatrix} \mathcal{B}_0 \\ \mathcal{B}_1 \\ \vdots \end{pmatrix} = x \begin{pmatrix} \mathcal{B}_0 \\ \mathcal{B}_1 \\ \vdots \end{pmatrix}.$$

Analogously, we have

$$J^n \begin{pmatrix} \mathcal{B}_0 \\ \mathcal{B}_1 \\ \vdots \end{pmatrix} = x^n \begin{pmatrix} \mathcal{B}_0 \\ \mathcal{B}_1 \\ \vdots \end{pmatrix}, \quad n \in \mathbb{N}, \tag{17}$$

being

$$J^n = \begin{pmatrix} J^n_{11} & J^n_{12} & \cdots \\ J^n_{21} & J^n_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

an infinite blocked matrix and J^n_{ij} the $(p \times p)$-block corresponding with the i row and the j column. In particular, the numerators on the right hand side of (16) are $J^n_{11} = e_0^T J^n e_0$.

Our main goal is to study the *discrete KdV equation*,

$$\dot{a}_n(t) = a_n(t) \left[\sum_{i=1}^{p} a_{n+i}(t) - \sum_{i=1}^{p} a_{n-i}(t) \right]. \tag{18}$$

Here and in the sequel we take $a_j = 0$ when $j \leq 0$. We know (see [2]) that (18) can be rewritten in Lax pair form,

$$\dot{J} = [J, M], \quad M = J_-^{p+1}, \tag{19}$$

where $J = J(t)$ is given by (14) for the sequence $\{a_n(t)\}$. Here, $[J, M] = JM - MJ$ is the *commutator of* J *and* M, and J_-^{p+1} is the infinite matrix $(\gamma_{ij})_{i,j}$ whose entry in the i-row and the j-column is $\gamma_{ij} = 0$, $i \leq j$ and $\gamma_{ij} = \beta_{ij}$, $i > j$, being $J^{p+1} = (\beta_{ij})_{i,j}$ the $(p+1)$-power of J.

1.3. The main results. For each $t \in \mathbb{R}$ we consider the vector of functionals $\mathcal{U} = \mathcal{U}_t$ defined by the recurrence relation (2) when the sequence $\{a_n(t)\}$ is used. This is,

$$\mathcal{U}_t = \mathcal{W}_t \left(\mathcal{W}_t(\mathcal{B}_0) \right)^{-1},$$

where $\mathcal{W}_t$ is given by (4) for the functionals $u_t^1, \ldots, u_t^p$. We are interested in to study the evolution of $\mathcal{R}_J(z)$ and the vector of functionals $\mathcal{U}_t$. Our main result is the following.

Theorem 1. *Assume that the sequence* $\{a_n(t)\}$, $n \in \mathbb{N}$, *is bounded, i.e. there exists* $M \in \mathbb{R}_+$ *such that* $|a_n(t)| \leq M$ *for all* $n \in \mathbb{N}$ *and* $t \in \mathbb{R}$, *and* $a_n(t) \neq 0$ *for all* $n \in \mathbb{N}$ *and* $t \in \mathbb{R}$. *Let* $\mathcal{U} = \mathcal{U}_t$ *be the vector of functionals* *associated with the recurrence relation (2). Then, the following conditions are equivalent:*

(a) $\{a_n(t)\}$ *is a solution of (18).*

(b) *For each* $m, k = 0, 1, \ldots,$ *we have*

$$\frac{d}{dt}\mathcal{U}\left(x^k\,\mathcal{P}_m\right) = \mathcal{U}\left(x^{k+1}\,\mathcal{P}_{m+1}\right) - \mathcal{U}\left(x^k\,\mathcal{P}_m\right)\mathcal{U}\left(x\,\mathcal{P}_1\right). \tag{20}$$

(c) *For each* $k = 0, 1, \ldots,$ *we have*

$$\frac{d}{dt}\mathcal{R}_J(z) = \mathcal{R}_J(z)\left[z^{p+1}I_p - \mathcal{U}(x\,\mathcal{P}_1)\right] - \sum_{k=0}^{p} z^{p-k}\,\mathcal{U}\left(x^k\,\mathcal{P}_0\right) \tag{21}$$

for all $z \in \mathbb{C}$ *such that* $|z| > \|J\|$.

When the conditions (a), (b), or (c) of Theorem 1 hold, then we can obtain explicitly the resolvent function in a neighborhood of $z = \infty$. We summarize this fact in the following result.

Theorem 2. *Under the conditions of Theorem 1, if* $\{a_n(t)\}$ *is a solution of (18) we have*

$$\mathcal{R}_J(z) = -e^{z^{p+1}t}S(z)e^{-\int J_{11}^{p+1}dt} \tag{22}$$

for each $z \in \mathbb{C}$ *such that* $|z| > \|J\|$, *where* $S(z) = (s_{ij}(z))$ *is the* $(p \times p)$*-matrix defined by*

$$s_{ij}(z) := \sum_{k=0}^{p} z^{p-k}\int\left(J_{11}^k\right)_{ij} e^{-z^{p+1}t}e^{\int\left(J_{11}^{p+1}\right)_{jj}dt}dt\,, \quad i, j = 1, \ldots, p\,, \tag{23}$$

and $(J_{11}^n)_{ij}$ *represents the entry corresponding to the row* i *and the column* j *in the* $(p \times p)$*-block* J_{11}^n.

Now we give a possible representation for the vector of functionals associated with the solution for the integrable systems (18). Using the above notation, we denote by $\mathcal{U}_0$ the vector of functionals defined by the recurrence relation (2) for the sequence $\{a_n(0)\}$. Then, $\mathcal{U}_0$ is associated with the moment functionals $u_0^1, \ldots, u_p^0$ defined by (1) and verifying (3) (for $t = 0$). We define the linear functionals $e^{x^{p+1}t}u_0^i$, $i = 1, \ldots, p$, as

$$\left(e^{x^{p+1}t}u_0^i\right)[x^j] = \sum_{k\geq 0}\frac{t^k}{k!}u_0^i[x^{(p+1)k+j}]\,, \quad j = 0, 1, \ldots. \tag{24}$$

In particular, f $J(0)$ is a bounded operator, then since [3, Th. 4, pag. 191] we know that

$$|u_0^i[x^{(p+1)k+j}]| \leq m_{ij}\|J(0)\|^{(p+1)k}$$

and the sum in the right-hand side of (24) is well defined. Then, in this case we have defined the vector of functionals, for all $(q_1, \ldots, q_p) \in \mathcal{P}^p$ by

$$\left(e^{x^{p+1}t}\mathcal{U}_0\right)\begin{pmatrix} q_1 \\ \vdots \\ q_p \end{pmatrix} = \begin{pmatrix} \left(e^{x^{p+1}t}u_0^1\right)[q_1] & \ldots & \left(e^{x^{p+1}t}u_0^p\right)[q_1] \\ \vdots & \ddots & \vdots \\ \left(e^{x^{p+1}t}u_0^1\right)[q_p] & \ldots & \left(e^{x^{p+1}t}u_0^p\right)[q_p] \end{pmatrix} \left(\mathcal{W}\left(\mathcal{B}_0\right)\right)^{-1},$$

where $\mathcal{W}(\mathcal{B}_0)$ is given in (12).

Theorem 3. *Let $\mathcal{U} = \mathcal{U}_t$ be the vector of functionals associated with the recurrence relation (2), and with the sequence of vector polynomials $\{\mathcal{B}_m\}$. If we have*

$$\mathcal{U}_t = \mathcal{W}_t\left(\mathcal{W}_t(\mathcal{B}_0)\right)^{-1}, \quad \textit{with} \quad \mathcal{W}_t = e^{x^{p+1}t}\mathcal{U}_0\,,$$

then the sequence $\{a_n(t)\}$, defined by (11), verify (18).

The rest of the paper is devoted to proving Theorems 1, 2, and 3. In section 2 we prove Theorem 1. $(a) \Leftrightarrow (b)$ is proved in subsection 2.1 and $(b) \Leftrightarrow (c)$ is proved in subsection 2.2. We dedicate section 3 to the proof of Theorem 2 and, finally, in section 4 we prove Theorem 3.

In the sequel we assume the conditions of theorems 1 and 2, i.e. in (14) we have a bounded matrix with entries $a_n(t)$, $n \in \mathbb{N}$, such that $a_n(t) \neq 0$, $n \in \mathbb{N}$, $t \in \mathbb{R}$.

2. Proof of Theorem 1

2.1. Evolution of the moments. In the following auxiliar result we determine the expression of the moment $\mathcal{U}(\mathcal{P}_n) = \mathcal{U}(x^n\,\mathcal{P}_0)$ in terms of the matrix J.

Lemma 1. *For each $n = 0, 1, \ldots$ we have $\mathcal{U}(x^n\,\mathcal{P}_0) = e_0^T J^n e_0$.*

Proof: We know that $\mathcal{U}(\mathcal{P}_0) = I_p$ (see (13)), then the moment of order 0 is $\mathcal{U}(\mathcal{P}_0) = e_0^T J^0 e_0$.

From (17), $\sum_{i\geq 1} J_{1i}^n\,\mathcal{B}_{i-1} = x^n\,\mathcal{B}_0$. Then, using (8), (9) and (13),

$$\mathcal{U}(x^n\,\mathcal{B}_0) = \sum_{i\geq 1} J_{1i}^n\,\mathcal{U}(\mathcal{B}_{i-1}) = J_{11}^n\,,$$

as we wanted to prove. ■

We need to analyze the matrix J^n and, in particular, the block J^n_{11}. We define

$$f_i := (0, \ldots, 0, \overset{(i)}{1}, 0, \ldots)\,, \quad i \in \mathbb{N}\,.$$

Then, for each $n \in \mathbb{N}$ the formal product $J^n f_i$ is the i-th column of matrix J^n. As in the case of a_j, we will assume $f_i = 0$ when $i \leq 0$.

Lemma 2. *With the above notation, for each $i \in \mathbb{N}$ and $m = 0, 1, \ldots$ we have*

$$J^m f_i = \sum_{k=0}^{m} A^{(m)}_{i,k} f_{i+k(p+1)-m}\,, \tag{25}$$

where

$$\left.\begin{aligned} A^{(m)}_{i,0} &= 1\,, \\ A^{(m)}_{i,k} &= \sum_{0 \leq j_0 \leq \cdots \leq j_{k-1} \leq m-k} \left(\prod_{r=0}^{k-1} a_{i+rp-j_r} \right), \quad k = 1, \ldots, m\,. \end{aligned}\right\} \tag{26}$$

Proof: We proceed by induction on m.

Firstly, for $m = 1$ we know

$$J f_i = f_{i-1} + a_i f_{i+p}\,, \quad i = 1, 2, \ldots\,. \tag{27}$$

(see (14)). Then, comparing (25) and (27) we deduce

$$A^{(1)}_{i,0} = 1 \ \text{ and } \ A^{(1)}_{i,1} = a_i\,, \ \text{ for } \ i \in \mathbb{N}$$

and, consequently, (25) holds.

Now, we assume that (25) is verified for a fixed $m \in \mathbb{N}$. Then, for $i \in \mathbb{N}$,

$$\begin{aligned} J^{m+1} f_i &= \sum_{k=0}^{m} A^{(m)}_{i,k} J f_{i+k(p+1)-m} \\ &= \sum_{k=0}^{m} A^{(m)}_{i,k} \left(f_{i+k(p+1)-m-1} + a_{i+k(p+1)-m} f_{i+k(p+1)-m+p} \right) \end{aligned}$$

$$= \sum_{k=0}^{m} A_{i,k}^{(m)} f_{i+k(p+1)-(m+1)} + \sum_{k=1}^{m+1} A_{i,k-1}^{(m)} a_{i+k(p+1)-p-(m+1)} f_{i+k(p+1)-(m+1)}$$
$$= \sum_{k=0}^{m+1} A_{i,k}^{(m+1)} f_{i+k(p+1)-(m+1)}\,, \quad i = 1, 2, \dots .$$

Comparing the coefficients of $f_{i+k(p+1)-(m+1)}$ in the above expression,

$$A_{i,k}^{(m+1)} = A_{i,k}^{(m)} + a_{i+k(p+1)-p-(m+1)} A_{i,k-1}^{(m)} = \sum_{0\le j_0\le\cdots\le j_{k-1}\le m-k} \left(\prod_{r=0}^{k-1} a_{i+rp-j_r} \right)$$
$$+ \sum_{0\le j_0\le\cdots\le j_{k-2}\le m-k+1} \left(\prod_{r=0}^{k-2} a_{i+rp-j_r} \right) a_{i+(k-1)(p+1)-m}\,, \tag{28}$$

where we are taking $A_{i,m+1}^{(m)} = A_{i,-1}^{(m)} = 0$. But

$$\sum_{0\le j_0\le\cdots\le j_{k-2}\le m-k+1} \left(\prod_{r=0}^{k-2} a_{i+rp-j_r} \right) a_{i+(k-1)(p+1)-m}$$
$$= \sum_{0\le j_0\le\cdots\le j_{k-1}\le m-k+1} \left(\prod_{r=0}^{k-1} a_{i+rp-j_r} \right) - \sum_{0\le j_0\le\cdots\le j_{k-1}\le m-k} \left(\prod_{r=0}^{k-1} a_{i+rp-j_r} \right) .$$

Then, substituting in (28) we arrive at (25) in $m+1$. ■

Remark . The coefficients $A_{i,k}^{(m)}$ have just been defined only for $m, i \in \mathbb{N}$ and $k = 0, 1, 2, \dots m$. In the sequel, we will take $A_{i,k}^{(m)} = 0$ for $k > m$, $k < 0$, or $i \le 0$.

Remark . In [2] the moments of the operator J were defined as

$$S_{kj} = \langle J^{(p+1)k+j-1} f_j, f_1 \rangle\,, \quad k \ge 0\,, \quad j = 1, 2, \dots, p\,.$$

Then, only the first row of $\mathcal{U}\left(x^{(p+1)k+j-1}\mathcal{P}_0\right)$ was used there. With our notation, these vectorial moments are

$$S_{kj} = A_{j,k}^{((p+1)k+j-1)}$$

(see [2, Th. 1, pag. 492]). As a consequence of Lemma 2, the so called *genetic sums associated to the sequence* $\{a_n\}$ can be expressed using (26).

From this we have

$$\sum_{i_1=1}^{j} a_{i_1} \sum_{i_2=1}^{i_1+p} a_{i_2} \cdots \sum_{i_k=1}^{i_{k-1}+p} a_{i_k} = \sum_{0 \le j_0 \le \cdots \le j_{k-1} \le kp+j-1} \left(\prod_{r=0}^{k-1} a_{j+rp-j_r} \right)$$

In other words, Lemma 2 extends the concepts of genetic sums and vectorial moments to the concept of matricial moments.

The next auxiliar result is used to prove the equivalence between (a) and (b) in Theorem 1. Moreover, this lemma has independent interest. In fact, the next result permits the inverse problem to be solved, restoring J from the resolvent operator (see (14) and (15)).

Lemma 3. *For $m, i, k \in \mathbb{N}$ we have*

$$A_{i,k}^{(m)} - A_{i-1,k}^{(m-1)} = a_i A_{i+p,k-1}^{(m-1)} . \tag{29}$$

Proof: Using (25), $A_{i,k}^{(m)} - A_{i-1,k}^{(m-1)}$

$$= \sum_{0 \le j_0 \le \cdots \le j_{k-1} \le m-k} \left(\prod_{r=0}^{k-1} a_{i+rp-j_r} \right) - \sum_{0 \le j_0 \le \cdots \le j_{k-1} \le m-k-1} \left(\prod_{r=0}^{k-1} a_{i-1+rp-j_r} \right)$$

$$= \sum_{0 \le j_0 \le \cdots \le j_{k-1} \le m-k} \left(\prod_{r=0}^{k-1} a_{i+rp-j_r} \right) - \sum_{1 \le j_0 \le \cdots \le j_{k-1} \le m-k} \left(\prod_{r=0}^{k-1} a_{i+rp-j_r} \right)$$

$$= \sum_{0 \le j_1 \le \cdots \le j_{k-1} \le m-k} a_i \left(\prod_{r=1}^{k-1} a_{i+rp-j_r} \right) ,$$

and so we get the desired result. ■

Now, we are ready to determine the main block of J^n.

Lemma 4.

$$J_{11}^{p+1} = \operatorname{diag}\left\{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_p\right\} . \tag{30}$$

Proof: It is sufficient to consider (25) for $m = p + 1$. In this case, the entries of column $J^{p+1} f_i$ corresponding to the first p rows are given by the coefficients $A_{i,k}^{(p+1)}$ when k is such that $i + k(p + 1) - (p + 1) \le p$, this is, $k = 1$. ■

As a consequence of the above expression, we have

$$e_0^T J_-^{p+1} e_0 = \mathcal{O}_p \,. \tag{31}$$

We are going to prove $(a) \Rightarrow (b)$ in Theorem 1. Assume that $\{a_n(t)\}$ is a solution of (18). Consequently, (19) is verifed. In the same way that in [1, p. 236], it is easy to see

$$\frac{d}{dt} J^n = J^n M - M J^n \,.$$

Then,

$$e_0^T \left(\frac{d}{dt} J^n \right) e_0 = e_0^T J^n M e_0 - e_0^T M J^n e_0 \,. \tag{32}$$

Let $m, k \in \{0, 1, \ldots\}$ be. For $n = mp + k$, from Lemma 1, and using the fact that $x^{mp} \mathcal{P}_0 = \mathcal{P}_m$, we can write

$$e_0^T \left(\frac{d}{dt} J^n \right) e_0 = \frac{d}{dt} \mathcal{U}\left(x^k \mathcal{P}_m \right) \,. \tag{33}$$

Moreover, using again Lemma 1, in the right-hand side of (20) we have

$$e_0^T J^{(m+1)p+k+1} e_0 - \left(e_0^T J^{mp+k} e_0 \right) \left(e_0^T J^{p+1} e_0 \right) \,. \tag{34}$$

In other words, because of (32)-(34) it is sufficient to prove

$$e_0^T J^n M e_0 - e_0^T M J^n e_0 = e_0^T J^{(m+1)p+k+1} e_0 - \left(e_0^T J^{mp+k} e_0 \right) \left(e_0^T J^{p+1} e_0 \right) \,. \tag{35}$$

Consider J^s as a blocked matrix. As was established in the proof of Lemma 1, we denote by J_{ij}^s the $(p \times p)$-block corresponding with the i row and the j column, and we use similar notation for M. Using (31), in the left-hand side of (35) we have:

i) $e_0^T M J^n e_0 = M_{11} J_{11}^n = \left(J_-^{p+1} \right)_{11} J_{11}^n = 0_p$, because M is a quasi-triangular matrix, this is, the blocks $M_{ij} = 0_p$ when $i \leq j$.

ii) $e_0^T J^n M e_0 = e_0^T J^n J_-^{p+1} e_0 = J_{11}^n \left(J_-^{p+1} \right)_{11} + \sum_{j \geq 2} J_{1j}^n \left(J_-^{p+1} \right)_{j1}$
$= \sum_{j \geq 1} J_{1j}^n \left(J_-^{p+1} \right)_{j1}$.

Then,

$$e_0^T J^n M e_0 - e_0^T M J^n e_0 = \sum_{j \geq 2} J_{1j}^n \left(J_-^{p+1} \right)_{j1} = \sum_{j \geq 2} J_{1j}^n J_{j1}^{p+1} \,. \tag{36}$$

In the right hand side of (35),

$$e_0^T J^{(m+1)p+k+1} e_0 - \left(e_0^T J^{mp+k} e_0\right)\left(e_0^T J^{p+1} e_0\right)$$
$$= -J_{11}^n J_{11}^{p+1} + J_{11}^{n+p+1} = -J_{11}^n J_{11}^{p+1} + \sum_{j\geq 1} J_{1j}^n J_{j1}^{p+1} = \sum_{j\geq 2} J_{1j}^n J_{j1}^{p+1}$$

and (20) is proved.

To show $(b) \Rightarrow (a)$ we need to know the derivatives of coefficients $A_{i,k}^{(m)}$. From the expression given in (25) for $J^m f_i$, the i-th column of J^m, we denote by $(J^m f_i)_p$ the vector in $\mathbb{C}^p$ given by the first p entries in this column, this is,

$$(J^m f_i)_p := \sum_{\substack{k=0 \\ 1\leq i+k(p+1)-m\leq p}}^{m} A_{i,k}^{(m)} f_{i+k(p+1)-m}\,. \tag{37}$$

Furthermore, as we saw in Lemma 1, another way to write (20) is

$$\dot{J}_{11}^m = J_{11}^{m+p+1} - J_{11}^m J_{11}^{p+1}\,. \tag{38}$$

The rest of the proof of $(b) \Rightarrow (a)$ is an immediate consequence of the following auxiliar result.

Lemma 5. *Assume that (20) holds. Then we have:*

- *For $i,k,m \in \mathbb{N}$ such that $1 \leq i+k(p+1)-m \leq p$,*

$$\dot{A}_{i,k}^{(m)} = -(a_{i-p+1} + \cdots + a_i)A_{i,k}^{(m)} - A_{i-p,k+1}^{(m+1)} + A_{i,k+1}^{(m+p+1)} \tag{39}$$

- *(18) holds for each $n \in \mathbb{N}$.*

Proof: We proceed by induction on i and n, proving (39) and (18) simultaneously.

1.- Firstly, we shall prove (39) for $i \in \{1,2,\ldots,p\}$. From (37), the derivative of the first p entries in $J^m f_i$ are given by

$$\sum_{\substack{k=0 \\ 1\leq i+k(p+1)-m\leq p}}^{m} \dot{A}_{i,k}^{(m)} f_{i+k(p+1)-m}\,.$$

Moreover, J_{11}^{p+1} is a diagonal block (see (30)). Then, the i-th column of $J_{11}^{m}J_{11}^{p+1}$ is $(a_1+\cdots+a_i)\left(J^m f_i\right)_p$. Since (38),

$$\begin{aligned}
\sum_{\substack{k=0\\ 1\leq i+k(p+1)-m\leq p}}^{m} \dot{A}_{i,k}^{(m)} f_{i+k(p+1)-m} &= \left(J^{m+p+1}f_i\right)_p-(a_1+\cdots+a_i)\left(J^m f_i\right)_p\\
&= \sum_{\substack{k=0\\ 1\leq i+k(p+1)-m-p-1\leq p}}^{m+p+1} A_{i,k}^{(m+p+1)} f_{i+k(p+1)-m-p-1}\\
&\quad -(a_1+\cdots+a_i)\sum_{\substack{k=0\\ 1\leq i+k(p+1)-m\leq p}}^{m} A_{i,k}^{(m)} f_{i+k(p+1)-m}\\
&= \sum_{\substack{k=0\\ 1\leq i+k(p+1)-m\leq p}}^{m+p} A_{i,k+1}^{(m+p+1)} f_{i+k(p+1)-m}\\
&\quad -(a_1+\cdots+a_i)\sum_{\substack{k=0\\ 1\leq i+k(p+1)-m\leq p}}^{m} A_{i,k}^{(m)} f_{i+k(p+1)-m}\,. \qquad (40)
\end{aligned}$$

In the right hand side of (40) there is no term corresponding to $k=m+1,\ldots,m+p$, because in these cases $i+k(p+1)-m>p$. Then,

$$\begin{aligned}
\sum_{\substack{k=0\\ 1\leq i+k(p+1)-m\leq p}}^{m} \dot{A}_{i,k}^{(m)} f_{i+k(p+1)-m}
&= \sum_{\substack{k=0\\ 1\leq i+k(p+1)-m\leq p}}^{m} \left[A_{i,k+1}^{(m+p+1)}-(a_1+\cdots+a_i)A_{i,k}^{(m)}\right] f_{i+k(p+1)-m}
\end{aligned}$$

and comparing the coefficients of $f_{i+k(p+1)-m}$ we deduce

$$\dot{A}_{i,k}^{(m)} = A_{i,k+1}^{(m+p+1)}-(a_1+\cdots+a_i)A_{i,k}^{(m)}\,,$$

which is (39).

2.- We assume that there exists $r \in \mathbb{N}$ such that (39) is verified for each $i = (r-1)p+1, \ldots rp$. We will show that, under this premise, (18) is verified, also, for each $n = i = (r-1)p+1, \ldots, rp$.

Take $k = 1$ and $m = rp+1$ in (39). Then, $1 \leq i + k(p+1) - m = i - (r-1)p \leq p$ and we have

$$\dot{A}_{i,1}^{(rp+1)} = -(a_{i-p+1} + \cdots + a_i)A_{i,1}^{(rp+1)} - A_{i-p,2}^{(rp+2)} + A_{i,2}^{((r+1)p+2)},$$

this is,

$$\sum_{j=0}^{rp} \dot{a}_{i-j} = -(a_{i-p+1} + \cdots + a_i)\sum_{j=0}^{rp} a_{i-j} - A_{i-p,2}^{(rp+2)} + A_{i,2}^{((r+1)p+2)}. \tag{41}$$

Moreover, taking $k = 1$ and $m = rp$ in (39) we have $1 \leq (i-1)+k(p+1)-m \leq p$ and, consequently,

$$\dot{A}_{i-1,1}^{(rp)} = -(a_{i-p} + \cdots + a_{i-1})A_{i-1,1}^{(rp)} - A_{i-p-1,2}^{(rp+1)} + A_{i-1,2}^{((r+1)p+1)},$$

this is,

$$\sum_{j=1}^{rp} \dot{a}_{i-j} = -(a_{i-p} + \cdots + a_{i-1})\sum_{j=1}^{rp} a_{i-j} - A_{i-p-1,2}^{(rp+1)} + A_{i-1,2}^{((r+1)p+1)}. \tag{42}$$

Subtracting (41) and (42), and taking into account (29), we arrive at

$$\dot{a}_i = a_i\left[(a_{i+1} + \cdots + a_{i+p}) - (a_{i-1} + \cdots + a_{i-p})\right],$$

which is (18) in $n = i$.

3.- Finally, we prove that if there exists $s \in \mathbb{N}$ such that (39) and (18) are verified for $n = i = 1, 2, \ldots, sp$, then (39) is verified also for $i + p$.

Take $i \in \mathbb{N}$ in the above conditions. Let $k, m \in \mathbb{N}$ be such that

$$1 \leq i + (k+1)(p+1) - (m+1) = (i-1) + (k+1)(p+1) - m \leq p. \tag{43}$$

Taking derivatives in (29),

$$\dot{a}_i A_{i+p,k}^{(m)} + a_i \dot{A}_{i+p,k}^{(m)} = \dot{A}_{i,k+1}^{(m+1)} - \dot{A}_{i-1,k+1}^{(m)}.$$

Therefore, from (39) and (18),

$$\begin{aligned} a_i \dot{A}_{i+p,k}^{(m)} &= -a_i\left[(a_{i+1} + \cdots + a_{i+p}) - (a_{i-1} + \cdots + a_{i-p})\right]A_{i+p,k}^{(m)} - a_{i-p}A_{i,k+1}^{(m+1)} \\ &\quad - (a_{i-p+1} + \cdots + a_i)A_{i,k+1}^{(m+1)} + (a_{i-p} + \cdots + a_{i-1})A_{i-1,k+1}^{(m)} + a_i A_{i+p,k+1}^{(m+p+1)} \\ &= -a_i(a_{i+1} + \cdots + a_{i+p})A_{i+p,k}^{(m)} - a_i A_{i,k+1}^{(m+1)} + a_i A_{i+p,k+1}^{(m+p+1)}. \end{aligned}$$

Then, we arrive at (39) for $i+p$.

We have verified (39) for $i+p$ when $i=1,\dots,sp$ is such that (43) holds. But (43) can be rewritten as

$$1 \le (i+p)+k(p+1)-m \le p\,. \qquad \blacksquare$$

2.2. Resolvent operator and moments. We are going to prove the equivalence between (b) and (c) in Theorem 1. Firstly, we assume that (20) is verified and we will show (21).

From Lemma 1 and (16),

$$\mathcal{R}_J(z) = \sum_{n\ge 0} \frac{e_0^T J^n e_0}{z^{n+1}} = \sum_{k=0}^{p-1}\left(\sum_{m\ge 0}\frac{\mathcal{U}\left(x^k\,\mathcal{P}_m\right)}{z^{mp+k+1}}\right), \quad |z|>\|J\|\,. \tag{44}$$

We define

$$\mathcal{R}_J^{(k)}(z) := \sum_{m\ge 0}\frac{\mathcal{U}(x^k\,\mathcal{P}_m)}{z^{mp+k+1}}\,, \quad k=0,1,\dots\,.$$

Then, from (44)

$$\mathcal{R}_J(z) = \sum_{k=0}^{p-1}\mathcal{R}_J^{(k)}(z)\,, \quad |z|>\|J\|\,,$$

and using (20)

$$\begin{aligned}\frac{d}{dt}\mathcal{R}_J(z) &= \sum_{k=0}^{p-1}\sum_{m\ge 0}\frac{\mathcal{U}\left(x^{k+1}\,\mathcal{P}_{m+1}\right)-\mathcal{U}\left(x^k\,\mathcal{P}_m\right)\mathcal{U}\left(x\,\mathcal{P}_1\right)}{z^{mp+k+1}}\\ &= \sum_{k=0}^{p-1}\sum_{m\ge 0}\frac{\mathcal{U}\left(x^{k+1}\,\mathcal{P}_{m+1}\right)}{z^{mp+k+1}} - \mathcal{R}_J(z)\mathcal{U}\left(x\,\mathcal{P}_1\right)\,,\end{aligned} \tag{45}$$

where

$$\begin{aligned}\sum_{m\ge 0}\frac{\mathcal{U}\left(x^{k+1}\,\mathcal{P}_{m+1}\right)}{z^{mp+k+1}} &= z^{p+1}\sum_{m\ge 0}\frac{\mathcal{U}\left(x^{k+1}\,\mathcal{P}_{m+1}\right)}{z^{(m+1)p+(k+1)+1}}\\ &= z^{p+1}\left[\mathcal{R}_J^{(k+1)}(z) - \frac{\mathcal{U}\left(x^{k+1}\,\mathcal{P}_0\right)}{z^{k+2}}\right]\\ &= z^{p+1}\,\mathcal{R}_J^{(k+1)}(z) - z^{p-k-1}\,\mathcal{U}\left(x^{k+1}\,\mathcal{P}_0\right)\,.\end{aligned}$$

Substituting in (45),

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{R}_J(z)\\
&\quad= -\mathcal{R}_J(z)\mathcal{U}\left(x\,\mathcal{P}_1\right) + z^{p+1}\sum_{k=0}^{p-1}\mathcal{R}_J^{(k+1)}(z) - \sum_{k=0}^{p-1} z^{p-k-1}\mathcal{U}\left(x^{k+1}\,\mathcal{P}_0\right)\\
&\quad= -\mathcal{R}_J(z)\mathcal{U}\left(x\,\mathcal{P}_1\right) + z^{p+1}\left[\mathcal{R}_J(z) + \mathcal{R}_J^{(p)}(z) - \mathcal{R}_J^{(0)}(z)\right]\\
&\qquad\qquad - \sum_{k=0}^{p-1} z^{p-k-1}\mathcal{U}\left(x^{k+1}\,\mathcal{P}_0\right)\,, \qquad (46)
\end{aligned}
$$

where it is easy to see that

$$
\mathcal{R}_J^{(p)}(z) = \mathcal{R}_J^{(0)}(z) - \frac{\mathcal{U}\left(\mathcal{P}_0\right)}{z}\,.
$$

From this we arrive at (21) .

In the second place, we prove $(c) \Rightarrow (b)$ in Theorem 1. For $z \in \mathbb{C}$ such that $|z| > \|J\|$ we have

$$
\begin{aligned}
z^{p+1}\mathcal{R}_J(z) &= \sum_{k=0}^{p-1}\sum_{m\geq 0}\frac{\mathcal{U}\left(x^k\,\mathcal{P}_m\right)}{z^{(m-1)p+k}}\\
&= \sum_{k=0}^{p-1}\frac{\mathcal{U}\left(x^k\,\mathcal{P}_0\right)}{z^{-p+k}} + \mathcal{U}\left(\mathcal{P}_1\right) + \sum_{m\geq 2}\frac{\mathcal{U}\left(\mathcal{P}_m\right)}{z^{(m-1)p}} + \sum_{k=1}^{p-1}\sum_{m\geq 1}\frac{\mathcal{U}\left(x^k\,\mathcal{P}_m\right)}{z^{(m-1)p+k}}\,.
\end{aligned}
$$

From this and the fact that

$$
\mathcal{U}\left(\mathcal{P}_{j+1}\right) = \mathcal{U}\left(x^p\,\mathcal{P}_j\right)\,, \quad j = 0, 1, \ldots\,,
$$

we obtain

$$
\begin{aligned}
z^{p+1}\,\mathcal{R}_J(z) &= \sum_{k=0}^{p}\frac{\mathcal{U}\left(x^k\,\mathcal{P}_0\right)}{z^{-p+k}} + \sum_{m\geq 0}\frac{\mathcal{U}\left(\mathcal{P}_{m+2}\right)}{z^{(m+1)p}} + \sum_{k=1}^{p-1}\sum_{m\geq 0}\frac{\mathcal{U}\left(x^k\,\mathcal{P}_{m+1}\right)}{z^{mp+k}}\\
&= \sum_{k=0}^{p}\frac{\mathcal{U}\left(x^k\,\mathcal{P}_0\right)}{z^{-p+k}} + \sum_{k=1}^{p}\sum_{m\geq 0}\frac{\mathcal{U}\left(x^k\,\mathcal{P}_{m+1}\right)}{z^{mp+k}}\\
&= \sum_{k=0}^{p}\frac{\mathcal{U}\left(x^k\,\mathcal{P}_0\right)}{z^{-p+k}} + \sum_{k=0}^{p-1}\sum_{m\geq 0}\frac{\mathcal{U}\left(x^{k+1}\,\mathcal{P}_{m+1}\right)}{z^{mp+k+1}}\,.
\end{aligned}
$$

Then, in the right-hand side of (21) we have

$$\mathcal{R}_J(z)\left[z^{p+1}I_p-\mathcal{U}(x\,\mathcal{P}_1)\right]-\sum_{k=0}^{p}z^{p-k}\mathcal{U}\left(x^k\,\mathcal{P}_0\right)$$
$$=-\mathcal{R}_J(z)\mathcal{U}(x\,\mathcal{P}_1)+\sum_{k=0}^{p-1}\sum_{m\geq 0}\frac{\mathcal{U}\left(x^{k+1}\,\mathcal{P}_{m+1}\right)}{z^{mp+k+1}}$$

and, consequently,

$$\frac{d}{dt}\mathcal{R}_J(z)=\sum_{k=0}^{p-1}\sum_{m\geq 0}\frac{\mathcal{U}\left(x^{k+1}\,\mathcal{P}_{m+1}\right)-\mathcal{U}\left(x^k\,\mathcal{P}_m\right)\mathcal{U}\left(x\,\mathcal{P}_1\right)}{z^{mp+k+1}} \tag{47}$$

(see (44)). Moreover, taking derivatives in (44) we have

$$\frac{d}{dt}\mathcal{R}_J(z)=\sum_{k=0}^{p-1}\sum_{m\geq 0}\frac{d}{dt}\frac{\mathcal{U}\left(x^k\,\mathcal{P}_m\right)}{z^{mp+k+1}}\,. \tag{48}$$

Comparing (47) and (48) in $|z|>\|J\|$ we arrive at (20), as we wanted to show.

3. Proof of Theorem 2

Let $\{a_n(t)\}$ be a solution of (18). In this section we assume that the conditions of Theorem 1 are verified. Therefore, (21) holds. Using this fact, we shall prove Theorem 2, obtaining a new expression for $\mathcal{R}_J(z)$. We remark that the right hand side of (22) is completely known, being the $(p\times p)$-blocks used in (23) given by Lemma 2. In particular, J_{11}^{p+1} is explicitly determined in Lemma 4.

Note that $\mathcal{U}\left(x\,\mathcal{P}_1\right)-z^{p+1}I_p=\mathcal{U}\left(x^{p+1}\,\mathcal{P}_0\right)-z^{p+1}I_p$ is a diagonal matrix (see Lemma 1 and (30)). Then, writing

$$\mathcal{R}_J(z)=\begin{pmatrix} r_{11}(z) & \cdots & r_{1p}(z)\\ \vdots & \ddots & \vdots\\ r_{n1}(z) & \cdots & r_{pp}(z)\end{pmatrix},$$

due to (21) we have for all $i,j=1,2,\ldots,p$,

$$\frac{d}{dt}r_{ij}(z)=\left[z^{p+1}-\left(J_{11}^{p+1}\right)_{jj}\right]r_{ij}(z)-\sum_{k=0}^{p}z^{p-k}\left(J_{11}^{k}\right)_{ij}, \tag{49}$$

where $(J_{11}^m)_{sj}$ denotes the entry corresponding to the row s and the column j of matrix J_{11}^m. It is well known that the solution of (49) is

$$r_{ij}(z) = -e^{z^{p+1}t} e^{-\int \left(J_{11}^{p+1}\right)_{jj} dt} \left[\sum_{k=0}^{p} z^{p-k} \int \left(J_{11}^{k}\right)_{ij} e^{-z^{p+1}t} e^{\int \left(J_{11}^{p+1}\right)_{jj} dt} dt \right] .$$

So, if we take

$$s_{ij} := \sum_{k=0}^{p} z^{p-k} \int \left(J_{11}^{k}\right)_{ij} e^{-z^{p+1}t} e^{\int \left(J_{11}^{p+1}\right)_{jj} dt} dt \,, \quad S := (s_{ij})_{i,j=1}^{p} \,,$$

we can express $\mathcal{R}_J(z)$ as the product of a diagonal matrix by the matrix S. This means we have proved (22).

4. Proof of the Theorem 3

Consider the matrix

$$\mathcal{S}_0 = \left(e^{x^{p+1}t} \, \mathcal{U}_0(\mathcal{B}_0) \right)^{-1} \tag{50}$$

and let $\mathcal{U}_t = \left(e^{x^{p+1}t} \, \mathcal{U}_0 \right) \mathcal{S}_0$ be the vector of functionals in the conditions of Theorem 3. We will prove that this vector of functionals verify (20). For $k, m = 0, 1, \ldots$, we know

$$\mathcal{U}_t \left(x^k \, \mathcal{P}_m \right) = \left(e^{x^{p+1}t} \, \mathcal{U}_0 \right) \left(x^k \, \mathcal{P}_m \right) \, \mathcal{S}_0 \,. \tag{51}$$

Because of $\mathcal{S}_0 \, \mathcal{S}_0^{-1} = I_p$ we have

$$\frac{d\mathcal{S}_0}{dt} = -\mathcal{S}_0 \frac{d\mathcal{S}_0^{-1}}{dt} \, \mathcal{S}_0 \,. \tag{52}$$

Moreover, from (24) and (50),

$$\frac{d\mathcal{S}_0^{-1}}{dt} = \begin{pmatrix} (e^{x^{p+1}t} u_0^1)[x^{p+1}] & \cdots & (e^{x^{p+1}t} u_0^p)[x^{p+1}] \\ \vdots & \ddots & \vdots \\ (e^{x^{p+1}t} u_0^1)[x^{2p}] & \cdots & (e^{x^{p+1}t} u_0^p)[x^{2p}] \end{pmatrix} = \left(e^{x^{p+1}t} \, \mathcal{U}_0 \right) \left(x \, \mathcal{P}_1 \right) \,. \tag{53}$$

Then, taking derivatives in (51), and taking into account (52) and (53),

$$\begin{aligned}
&\frac{d}{dt}\mathcal{U}_t\left(x^k\,\mathcal{P}_m\right)\\
&\quad=\left[\frac{d}{dt}\left(e^{x^{p+1}t}\,\mathcal{U}_0\right)\left(x^k\,\mathcal{P}_m\right)\right]\mathcal{S}_0-\left(e^{x^{p+1}t}\,\mathcal{U}_0\right)\left(x^k\,\mathcal{P}_m\right)\,\mathcal{S}_0\frac{d\mathcal{S}_0^{-1}}{dt}\,\mathcal{S}_0\\
&\quad=\left(e^{x^{p+1}t}\,\mathcal{U}_0\right)\left(x^{k+1}\,\mathcal{P}_{m+1}\right)\,\mathcal{S}_0-\mathcal{U}_0\left(x^k\,\mathcal{P}_m\right)\mathcal{S}_0\left(e^{x^{p+1}t}\,\mathcal{U}_0\right)\left(x\,\mathcal{P}_1\right)\,\mathcal{S}_0\\
&\quad=\mathcal{U}_t\left(x^{k+1}\,\mathcal{P}_{m+1}\right)-\mathcal{U}_t\left(x^k\,\mathcal{P}_m\right)\,\mathcal{U}_t\left(x\,\mathcal{P}_1\right)
\end{aligned}$$

and (20) holds.

Now, using the hypothesis we get from Theorem 1 that the sequence $\{a_n(t)\}$, defined by (11) with $\mathcal{U}_t(\mathcal{B}_0) = I_p$, verify (18).

References

[1] A.I. Aptekarev and A. Branquinho, *Padé approximants and complex high order Toda lattices*, J. Comput. Appl. Math. **155** (2003), 231-237.

[2] A. Aptekarev, V. Kaliaguine, and J. Van Isenghem, *The Genetic Sums' Representation for the Moments of a System of Stieltjes Functions and its Application*, Constr. Approx. **16** (2000), 487-524.

[3] V. Kaliaguine, *The operator moment problem, vector continued fractions and an explicit form of the Favard theorem for vector orthogonal polynomials*, J. Comput. Appl. Math. **65** (1995), 181-193.

[4] J. Van Iseghem, *Vector orthogonal relations. Vector QD-algorithm*, J. Comput. Appl. Math. **19** (1987), 141-150.

[5] K. Yosida, *Functional Analysis*, Springer-Verlag, 1995, Berlin.

D. Barrios Rolanía
Facultad de Informática, Universidad Politécnica de Madrid, 28660 Boadilla del Monte, Madrid, Spain.

E-mail address: dbarrios@fi.upm.es

A. Branquinho
CMUC, Department of Mathematics, University of Coimbra, Largo D. Dinis, 3001-454 Coimbra, Portugal.

E-mail address: ajplb@mat.uc.pt

A. Foulquié Moreno
Departamento de Matemática, Universidade de Aveiro, Campus de Santiago 3810, Aveiro, Portugal.

E-mail address: foulquie@ua.pt