THE THIRD COHOMOLOGY GROUP CLASSifies
DOUBLE CENTRAL EXTENSIONS

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Dedicated to Dominique Bourn on the occasion of his sixtieth birthday

Abstract: We characterise the double central extensions in a semi-abelian category in terms of commutator conditions. We prove that the third cohomology group $H^3(Z, A)$ of an object $Z$ with coefficients in an abelian object $A$ classifies the double central extensions of $Z$ by $A$.

Keywords: cohomology, categorical Galois theory, semi-abelian category, higher central extension, Baer sum.
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Introduction

The second cohomology group $H^2(Z, A)$ of a group $Z$ with coefficients in an abelian group $A$ is well-known to classify the central extensions of $Z$ by $A$ in the following manner. Any central extension $f$ of $Z$ by $A$ induces a short exact sequence

$$0 \longrightarrow A \xrightarrow{\ker f} X \xrightarrow{f} Z \longrightarrow 0$$

such that $axa^{-1}x^{-1} = 1$ for all $a \in A$ and $x \in X$. The elements of the group $H^2(Z, A)$ are equivalence classes of such central extensions; here two extensions $f: X \rightarrow Z$ and $f': X' \rightarrow Z$ are equivalent if and only if there exists a group (iso)morphism $x: X \rightarrow X'$ satisfying $f' \circ x = f$ and $x \circ \ker f = \ker f'$. The group structure on $H^2(Z, A)$ is given by the classical Baer sum—see for instance [21]. In [14], see also [5] and [8], this construction was extended categorically from the context of groups to semi-abelian categories [18]. Thus a similar interpretation of the second cohomology group also makes sense for, say, Lie algebras over a field, commutative algebras, non-unital rings, or (pre)crossed modules.

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The aim of the present work is to prove a two-dimensional version of this result, at once in a categorical context: we show that the third cohomology group $H^3(Z, A)$ of an object $Z$ with coefficients in an abelian object $A$ of a semi-abelian category $\mathcal{A}$ classifies the double central extensions in $\mathcal{A}$ of $Z$ by $A$. Thus the connections between two branches of non-abelian (co)homology are made explicit.

On one hand, there is the direction approach to cohomology established by Bourn and Rodelo [3, 4, 5, 9, 23]; here the cohomology groups $H^n\mathcal{A}$ of an internal abelian group $\mathcal{A}$ are described through direction functors, in such a way that any short exact sequence of internal abelian groups induces a long exact cohomology sequence. This concept of direction may be understood as follows. It is well-known that in a Barr exact context, $H^1\mathcal{A}$ can be interpreted in terms of $\mathcal{A}$-torsors. An $\mathcal{A}$-torsor is a generalised affine space over $\mathcal{A}$: a “group without unit” where any choice of a unit gives back $\mathcal{A}$—its direction. Further borrowing intuition from Affine Geometry, $H^1\mathcal{A}$ is described in terms of autonomous Mal’tsev operations with given direction $\mathcal{A}$. On level 2—the level which corresponds to the “third cohomology group” from the title—the direction functor theory is based on that of level 1: now $H^2\mathcal{A}$ is described in terms of internal groupoids with given direction $\mathcal{A}$. By means of higher order internal groupoids, the theory is inductively extended to higher levels $H^n\mathcal{A}$.

On the other hand, there is the approach to semi-abelian homology [1, 11] based on categorical Galois theory [2, 15] initiated by Janelidze [16, 17] and further worked out by Everaert, Gran and Van der Linden [10]. Here the basic situation is given by a semi-abelian category $\mathcal{A}$ and a Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$: the derived functors of the reflector $I: \mathcal{A} \to \mathcal{B}$ are computed in terms of higher Hopf formulae using the induced Galois structures of higher central extensions. In the specific case where $\mathcal{B}$ is the Birkhoff subcategory $\text{Ab}\mathcal{A}$ determined by the abelian objects in $\mathcal{A}$ and $I = \text{ab}$ is the abelianisation functor, we start from the Galois structure

$$\Gamma = (\mathcal{A} \xrightarrow{\text{ab}} \text{Ab}\mathcal{A}, |\text{Ext}\mathcal{A}|, |\text{ExtAb}\mathcal{A}|). \quad (\text{A})$$

The class of extensions $|\text{Ext}\mathcal{A}|$ (respectively $|\text{ExtAb}\mathcal{A}|$) consists of the regular epimorphisms in $\mathcal{A}$ (in $\text{Ab}\mathcal{A}$) and forms the class of objects of the category $\text{Ext}\mathcal{A}$ (or $\text{ExtAb}\mathcal{A}$) whose morphisms are commutative squares between extensions. The covers with respect to this Galois structure $\Gamma$ are exactly the central extensions in the sense of commutator theory: an extension $f: X \to Z$
is central if and only if $[R[f], \nabla_X] = \Delta_X$, i.e., the commutator of the kernel pair of $f$ with the largest relation $\nabla_X$ on $X$ is the smallest relation $\Delta_X$ on $X$. These central extensions, in turn, determine a reflective subcategory $\mathsf{CExt}_A$ of $\mathsf{Ext}_A$; the reflector $\text{centr}: \mathsf{Ext}_A \to \mathsf{CExt}_A$ which sends $f$ to the central extension $\text{centr} f: X/[R[f], \nabla_X] \to Z$ is the centralisation functor. Thus we obtain the Galois structure

$$\Gamma_1 = (\mathsf{Ext}_A \xrightarrow{\text{centr}} \mathsf{CExt}_A, |\mathsf{Ext}^2_A|, |\mathsf{Ext}^2_{\mathsf{Ab}}A|).$$

(B)

The classes $|\mathsf{Ext}^2_A|$ and $|\mathsf{Ext}^2_{\mathsf{Ab}}A|$ consist of double extensions in $A$ or in $\mathsf{Ab}.A$. A double extension is a commutative square

$$\begin{array}{ccc}
X & \xrightarrow{c} & C \\
\downarrow d & & \downarrow g \\
D & \xrightarrow{f} & Z
\end{array}$$

such that all its maps and the comparison map $(d, c): X \to D \times_Z C$ to the pullback of $f$ with $g$ are regular epimorphisms. The covers with respect to the Galois structure $\Gamma_1$ are used in the computation of the third homology functor $H_3(\cdot, \mathsf{ab}): A \to \mathsf{Ab}A$ (see [10]) and form the main subject of the present paper—they are the “double central extensions” from the title.

We start by recalling the main properties of the Galois structure $\Gamma_1$ in Section 1. In Section 2 we characterise the $\Gamma_1$-covers in terms of commutators (as Janelidze does in the category of groups [16] and Gran and Rossi do in the context of Mal’tsev varieties [13]) and in terms of internal pregroupoids [20]. Section 3 recalls Bourn and Rodelo’s definition of the third cohomology group in semi-abelian categories. We obtain a natural notion of direction for double extensions and show in Section 4 that the set $\mathsf{Centr}^2(Z, A)$ of equivalence classes of double central extensions of an object $Z$ by an abelian object $A$ carries a canonical abelian group structure. In Section 5 we conclude the paper with the isomorphism $H_3(Z, A) \cong \mathsf{Centr}^2(Z, A)$ between the third cohomology group of an object $Z$ with coefficients in an abelian object $A$ and the group $\mathsf{Centr}^2(Z, A)$.

We conjecture that this result may be generalised to higher degrees, so that also for $n > 2$ the $(n + 1)$-st cohomology group $H^{n+1}(Z, A)$ of $Z$ with coefficients in $A$ classifies the $n$-fold central extensions of $Z$ by $A$. This will be the subject of future work.
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1. Preliminaries

1.1. Semi-abelian categories. The basic context where we shall be working is that of semi-abelian categories [18]. Some examples are the categories $\text{Gp}$ of all groups, $\text{Rng}$ of non-unital rings, $\text{Lie}_K$ of Lie algebras over a field $K$, $\text{XMod}$ of crossed modules, and $\text{Loop}$ of loops. We briefly recall the main definitions.

A category is semi-abelian when it is pointed, Barr exact and Bourn protomodular and has binary coproducts. A category is pointed when it has a zero object $0$: a terminal object which is also initial. A Barr exact category is regular—finitely complete with pullback-stable regular epimorphisms and coequalisers of kernel pairs—and such that every internal equivalence relation is a kernel pair. A pointed and regular category is Bourn protomodular when the (regular) Short Five Lemma holds: given any commutative diagram of regular epimorphisms with their kernels

\[
\begin{array}{ccccccc}
K[f] & \xrightarrow{\ker f} & X & \xrightarrow{f} & Z \\
\downarrow{k} & & \downarrow{x} & & \downarrow{z} \\
K[f'] & \xrightarrow{\ker f'} & X' & \xrightarrow{f'} & Z'
\end{array}
\]

the morphisms $k$ and $z$ being isomorphisms implies that $x$ is an isomorphism.

Any semi-abelian category $\mathcal{A}$ is a Mal’tsev category: it is finitely complete and, in $\mathcal{A}$, every reflexive relation is an equivalence relation.

1.2. The commutator of equivalence relations. Let $R = (R, r_0, r_1)$ and $S = (S, s_0, s_1)$ be equivalence relations on an object $X$ of a Mal’tsev category $\mathcal{A}$. Let $R \times_X S$ denote the pullback of $r_1$ and $s_0$: 

\[
\begin{array}{ccccccc}
R \times_X S & \xrightarrow{p_S} & S \\
\downarrow{i_R} & & \downarrow{i_S} \\
R & \xleftarrow{r_1} & X \\
\end{array}
\]
The object \( R \times_X S \) “consists of” triples \((\alpha, \beta, \gamma)\) where \( \alpha R \beta \) and \( \beta S \gamma \). We say that \( R \) and \( S \) commute when there exists a connector between \( R \) and \( S \): a morphism \( p: R \times_X S \to X \) which satisfies \( p(\alpha, \alpha, \gamma) = \gamma \) and \( p(\alpha, \gamma, \gamma) = \alpha \) [6]; see also [1, Definition 2.6.1].

When \( \mathcal{A} \) is a semi-abelian category, the commutator of \( R \) and \( S \) [22], denoted by \([R, S]\), is the universal equivalence relation on \( X \) which, when divided out, makes them commute. More precisely, \([R, S]\) is the kernel pair \( R[\psi] \) of the map \( \psi \) in the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{r_0} & T \\
\downarrow{r} & & \downarrow{\psi} \\
R \times_X S & \xrightarrow{i_R} & X \\
\downarrow{i_S} & & \downarrow{s_1} \\
S & & \\
\end{array}
\]

where the dotted arrows denote the colimit of the outer square [1, Section 2.8]. The direct images \( \psi R \) and \( \psi S \) of \( R \) and \( S \) along the regular epimorphism \( \psi \) commute; hence \( R \) and \( S \) commute if and only if \([R, S] = \Delta_X \) [6, Proposition 4.2].

An equivalence relation \( R \) on an object \( X \) is central when it commutes with \( \nabla_X \)—when \([R, \nabla_X] = \Delta_X \).

**1.3. Central extensions.** Let \( \mathcal{A} \) be a semi-abelian category. For any object \( X \) of \( \mathcal{A} \) we may take the kernel of the \( X \)-component of the unit of the adjunction in \( \mathcal{A} \) to obtain a short exact sequence

\[
0 \longrightarrow [X] \xrightarrow{\mu_X} X \xrightarrow{\eta_X} abX \longrightarrow 0.
\]

Thus we acquire a functor \([-] : \mathcal{A} \to \mathcal{A} \) together with a natural transformation \( \mu : [-] \Rightarrow 1_{\mathcal{A}} \).

**Lemma 1.4.** The functors \( ab \) and \([-] \) preserve pullbacks of regular epimorphisms along split epimorphisms.

**Proof:** It is well-known that the functor \( ab \) has this property: see, for instance, [12]. Since kernels commute with pullbacks, it follows that the functor \([-] \) has the same property. \( \blacksquare \)
An extension $f : X \to Z$ is **central** (with respect to the Galois structure $\Gamma$ in diagram A) if and only if either one of the projections $p_0$ or $p_1$ of its kernel pair $(R[f], p_0, p_1)$ is a trivial extension, i.e., a pullback of an extension in $\text{AbA}$. It follows that $f$ is central if and only if the right hand side square in the diagram

\[
0 \longrightarrow [R[f]] \xrightarrow{\mu_{R[f]}} R[f] \xrightarrow{\eta_{R[f]}} \text{ab}R[f] \longrightarrow 0
\]

\[
0 \longrightarrow [X] \xrightarrow{\mu_X} X \xrightarrow{\eta_X} \text{ab}X \longrightarrow 0
\]

is a pullback or, equivalently, $[p_0]$ is an isomorphism. Hence the kernel of $[p_0]$, which is denoted by $[f]_1$, is zero if and only if $f$ is central. Considering $[f]_1$ as a normal subobject of $X$, the **centralisation functor** $\text{centr} : \text{ExtA} \to \text{CExtA}$, from the Galois structure $\Gamma_1$ in diagram B, takes the extension $f : X \to Z$ and maps it to the quotient $\text{centr}f : X/[f]_1 \to Z$ of $f : X \to Z$ by the extension $[f]_1 \to 0$.

This notion of centrality for extensions is compatible with the above-mentioned notion of centrality for equivalence relations. Indeed, an extension $f$ is central if and only if so is its kernel pair $R[f]$; see [12].

Given an object $Z$ and an abelian object $A$, a **central extension of $Z$ by $A$** is a central extension $f : X \to Z$ with kernel $K[f] = A$. The group of isomorphism classes of central extensions of $Z$ by $A$ is denoted $\text{Centr}^1(Z, A)$. Recall the following result from [14]:

**Proposition 1.5.** If $\mathcal{A}$ is a semi-abelian category and $Z$ is an object of $\mathcal{A}$ then the functor $\text{Centr}^1(Z, -) : \text{AbA} \to \text{Ab}$ preserves finite products.

**Proof:** We shall only repeat the main point of the construction behind [14, Proposition 6.1]. Let $a : A \to B$ be a morphism of abelian objects in $\mathcal{A}$ and $f : X \to Z$ a central extension of $Z$ by $A$. Let $A \oplus B$ denote the biproduct of $A$ with $B$ in $\text{AbA}$. The functor $\text{Centr}^1(Z, -)$ maps the equivalence class of $f$ to the equivalence class of the central extension $f'$ in the diagram with exact rows

\[
\begin{array}{ccc}
0 & \longrightarrow & A \oplus B \xrightarrow{\ker f \times 1_B} X \times B \xrightarrow{f_{\text{opr}} X} Z \longrightarrow 0 \\
\downarrow{[a, 1_B]} & & \downarrow{f'} \\
0 & \longrightarrow & B \xrightarrow{f'} X' \longrightarrow Z \longrightarrow 0.
\end{array}
\]
The extension \( f' \) is central as a quotient of the central extension \( f \circ \text{pr}_X \).

### 1.6. Double extensions as spans.

Recall that a **double extension (of an object \( Z \))** in a semi-abelian category is a commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{c} & C \\
\downarrow{d} & & \downarrow{g} \\
D & \xrightarrow{f} & Z \\
\end{array}
\]

such that all its maps and the comparison map \((d, c) : X \to D \times_Z C\) to the pullback of \( f \) with \( g \) are regular epimorphisms. Double extensions may be characterised in terms of spans in a slice category as follows.

**Definition 1.7.** A span \((X, d, c)\)

\[
\begin{array}{ccc}
X & c & \rightarrow & C \\
\downarrow{d} & & \downarrow{g} \\
D & \rightarrow & \rightarrow & Z \\
\end{array}
\]

in a regular category \( \mathcal{A} \)

- (1) **has global support** when \(!_D : D \to 1\) and \(!_C : C \to 1\) are regular epimorphisms;
- (2) is **aspherical** when also \((d, c) : X \to D \times C\) is a regular epimorphism.

**Proposition 1.8.** Let \( \mathcal{A} \) be a semi-abelian category. A commutative square \( D \) in \( \mathcal{A} \) is a double extension if and only if \((X, d, c)\) is an aspherical span in \( \mathcal{A} \downarrow Z \).

**Proof:** Since the terminal object of \( \mathcal{A} \downarrow Z \) is \( 1_Z : Z \to Z \), \((X, d, c)\) has global support whenever \( f \) and \( g \) are regular epimorphisms in \( \mathcal{A} \).

\[
\begin{array}{ccc}
X & \xrightarrow{c} & C \\
\downarrow{d} & & \downarrow{g} \\
D & \xrightarrow{f} & Z \\
\end{array}
\]

But the product of \( f \) and \( g \) in \( \mathcal{A} \downarrow Z \) is the map \( f \circ \text{pr}_D : D \times_Z C \to Z \) starting from the pullback of \( f \) and \( g \) in \( \mathcal{A} \), hence this span is aspherical if
and only if also the map \((d, c) : X \to D \times_{Z} C\) is a regular epimorphism in \(\mathcal{A}\)—which means that the square square \(D\) is a double extension.

1.9. Double central extensions. Let \(\mathcal{A}\) be a semi-abelian category. By definition, a double extension is central when it is a cover with respect to the Galois structure \(\Gamma_{1}\). Hence the double extension \(D\), considered as a map \((c, f) : d \to g\) in the category \(\text{Ext} \mathcal{A}\), is central if and only if the first projection

\[
\begin{array}{ccc}
R[c] & \xrightarrow{p_{0}} & X \\
\downarrow & & \downarrow \delta \\
R[(c,f)] & \xrightarrow{p_{0}} & D
\end{array}
\]

\[
\begin{array}{ccc}
R[c] & \xrightarrow{p_{0}} & X \\
\downarrow & & \downarrow \\
R[f] & \xrightarrow{p_{0}} & D
\end{array}
\]

of its kernel pair—the left hand side square—is a trivial extension with respect to \(\Gamma_{1}\). (Alternatively, one could use the square of second projections.) This means that the comparison map to its reflection into \(\text{CExt} \mathcal{A}\)—the right hand side square—is a pullback. For this to happen, the natural map \([R[(c, f)]_{1}] \to [d]_{1}\) must be an isomorphism. This, in turn, is equivalent to the square

\[
\begin{array}{ccc}
[R[d] \square R[c]] & \xrightarrow{[p_{0}]} & [R[d]] \\
\downarrow & & \downarrow [p_{0}] \\
[R[c]] & \xrightarrow{[p_{0}]} & [X]
\end{array}
\]

being a pullback, because \([R[(c, f)]_{1}]\) and \([d]_{1}\) are the kernels of the vertical maps above. Here \((R[d] \square R[c], p_{0}, p_{1})\) denotes the kernel pair of \(R[(c, f)]\); it consists of all quadruples \((\alpha, \beta, \gamma, \delta) \in X^{4}\) in the following configuration:

\[
\begin{bmatrix}
\alpha & c & \beta \\
\delta & d & \gamma
\end{bmatrix}
\]

\(d(\alpha) = d(\delta), c(\alpha) = c(\beta), c(\gamma) = c(\delta)\) and \(d(\gamma) = d(\beta)\).

2. Characterisation of double central extensions in terms of commutators

In this section we characterise the covers with respect to the Galois structure \(\Gamma_{1}\) in terms of internal pregroupoids in the sense of Kock [20]. This
characterisation turns out to be equivalent to the conditions given by Jane-
lidze in [16] and Gran and Rossi in [13]—and thus we prove a categorical
version of the next result.

Proposition 2.1. [13, 16] Let $A$ be a Mal’tsev variety. A double extension $D$
in $A$ is central if and only if $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$.

The concept of internal pregroupoid generalises internal groupoids in the
following manner: in a pregroupoid, the domain and codomain of a map may
live in different objects, and no identities need exist.

Definition 2.2. [19, 20] Let $A$ be a finitely complete category. A pre-
groupoid (also called a herdoid) $(X, d, c, p)$ in $A$ is a span $E$ with a partial
ternary operation $p$ on $X$ satisfying:

1. $p(\alpha, \beta, \gamma)$ is defined if and only if $c(\alpha) = c(\beta)$ and $d(\gamma) = d(\beta)$;
2. $dp(\alpha, \beta, \gamma) = d(\alpha)$ and $cp(\alpha, \beta, \gamma) = c(\gamma)$ if $p(\alpha, \beta, \gamma)$ is defined;
3. $p(\alpha, \alpha, \gamma) = \gamma$ if $p(\alpha, \alpha, \gamma)$ is defined, and $p(\alpha, \gamma, \gamma) = \alpha$ if $p(\alpha, \gamma, \gamma)$ is
defined;
4. $p(\alpha, \beta, p(\gamma, \delta, \epsilon)) = p(p(\alpha, \beta, \gamma), \delta, \epsilon)$ if either side is defined.

An “element” $\alpha$ of $X$ should be interpreted as a map $\alpha: d(\alpha) \to c(\alpha)$; its
domain $d(\alpha)$ is an element of $D$, while its codomain $c(\alpha)$ is an element of $C$.
The operation $p$ sends a composable triple

\[
d(\alpha) \xrightarrow{\alpha} c(\alpha) \\
\downarrow \delta \quad \beta \\
d(\gamma) \xrightarrow{\gamma} c(\gamma)
\]

to the dotted diagonal $\delta = p(\alpha, \beta, \gamma): d(\alpha) \to c(\gamma)$. In case the pregroupoid
is a groupoid (i.e., when the span is a reflexive graph so that $C = D$ and
identities exist), $p(\alpha, \beta, \gamma) = \gamma \circ \beta^{-1} \circ \alpha$.

We denote the category of (pre)groupoids in $A$ by $(\text{Pre})\text{Gd}A$.

Definition 2.3. Suppose that $A$ is regular. A pregroupoid $(X, d, c, p)$ has
global support or is aspherical whenever the span $(X, d, c)$ has global
support or is aspherical. This definition applies in the obvious way to internal
groupoids.
Suppose that $\mathcal{A}$ is a Mal’tsev category. As explained in the introduction of [6], an internal pregroupoid structure $p$ on a span $(X, d, c)$ is the same thing as a connector between the kernel pairs $R[c]$ and $R[d]$ of $c$ and $d$. Indeed, using that $\mathcal{A}$ is Mal’tsev, one shows that conditions (2) and (4) of Definition 2.2 are automatically satisfied: see Proposition 2.6.11 in [1] or Proposition 4.1 in [6]. Two equivalence relations admit at most one connector; hence, if it exists, a pregroupoid structure $p$ on a span $(X, d, c)$ is necessarily unique. In this case we shall say that the span $(X, d, c)$ is a pregroupoid and drop the structure $p$ from the notation.

Because of Proposition 1.8 which exhibits the close connection between double extensions in $\mathcal{A}$ and spans in a slice category $\mathcal{A} \downarrow Z$, we are also mostly interested in pregroupoids in slice categories. Asking that a span $(X, d, c)$ is a pregroupoid in $\mathcal{A} \downarrow Z$ amounts to asking that $(X, d, c)$ is a pregroupoid in $\mathcal{A}$: when $\mathcal{A}$ is semi-abelian, this happens precisely when the first equality $[R[d], R[c]] = \Delta_X$ of Proposition 2.1 holds.

**Definition 2.4.** Suppose that $\mathcal{A}$ is semi-abelian and let $Z$ be an object of $\mathcal{A}$. An aspherical (pre)groupoid $(X, d, c)$ in $\mathcal{A} \downarrow Z$ is central when $(d, c): X \to D \times_{Z} C$ is a central extension in $\mathcal{A}$.

Since $R[d] \cap R[c] = R[(d, c): X \to D \times_{Z} C]$, this makes the centrality of the aspherical pregroupoid $(X, d, c)$ equivalent to the second equality $[R[d] \cap R[c], \nabla_X] = \Delta_X$ of Proposition 2.1. And thus we proved:

**Proposition 2.5.** Let $\mathcal{A}$ be a semi-abelian category. A double extension $D$ in $\mathcal{A}$ satisfies $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$ if and only if the span $(X, d, c)$ is a central pregroupoid in the slice category $\mathcal{A} \downarrow Z$.

**Proposition 2.6.** In a semi-abelian category, condition $F$ is preserved and reflected by pullbacks of double extensions along double extensions.

**Proof:** The proof given in Section 4 of [13] in the context of Mal’tsev varieties is still valid in the present situation.

**Theorem 2.7.** Consider a double extension $D$ in a semi-abelian category $\mathcal{A}$. The following are equivalent:

1. $D$ is a double central extension;
2. $(X, d, c)$ is a central pregroupoid in $\mathcal{A} \downarrow Z$;
(3) \([R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]\).

**Proof:** By Proposition 2.5 we already know that (2) and (3) are equivalent. To see that (1) implies (3), suppose that \(D\) is a double central extension. Then either one of the projections of its kernel pair is trivial with respect to \(\Gamma_1\), meaning that it is a pullback of a double extension between central extensions (i.e., a morphism of the category \(\text{CExt} \mathcal{A}\)). This latter double extension satisfies the condition corresponding to \(F\); hence applying Proposition 2.6 twice shows that (3) holds.

Now we prove that (2) implies (1). The pregroupoid structure of \((X, d, c)\) is a connector \(p: R[c] \times_X R[d] \to X\). As explained in Subsection 1.9, we are to show that the outer square in the diagram

\[
\begin{array}{ccc}
[R[d] \square R[c]] & \xrightarrow{[p_0]} & [R[d]] \\
\downarrow & & \downarrow \\
[p_0] & & [p_0] \\
\downarrow & & \downarrow \\
[R[c] \times_X R[d]] & \xrightarrow{[p_0]} & [X] \\
\end{array}
\]

is a pullback. Here \(\pi: R[d] \square R[c] \to R[c] \times_X R[d]\) is defined by

\[
\begin{bmatrix}
\alpha & c & \beta \\
\delta & d & \gamma
\end{bmatrix} \mapsto (\alpha, \beta, \gamma).
\]

By Lemma 1.4 we know that the inner quadrangle is a pullback, hence it suffices that \([\pi]\) is an isomorphism. The left hand side square

\[
\begin{array}{ccc}
R[d] \square R[c] & \xrightarrow{\pi} & R[c] \times_X R[d] \\
\downarrow q & & \downarrow p \\
R[d] \cap R[c] & \xrightarrow{p_0} & X
\end{array}
\]

and

\[
\begin{array}{ccc}
[R[d] \square R[c]] & \xrightarrow{[p]} & [R[c] \times_X R[d]] \\
\downarrow [q] & & \downarrow [p] \\
[R[d] \cap R[c]] & \xrightarrow{[p_0]} & [X]
\end{array}
\]

are pullbacks.
where $q$ is defined by
\[
\begin{bmatrix}
\alpha & c & \beta \\
 d & d \\
\delta & c & \gamma
\end{bmatrix} \mapsto (p(\alpha, \beta, \gamma), \delta),
\]
is a pullback. Since $p_0$ is a split epimorphism we may again use Lemma 1.4 to show that also the right hand side square above is a pullback. It follows that $[\pi]$ is an isomorphism if and only if $[p_0]$ is an isomorphism, so that the internal pregroupoid $(X, d, c)$ is central if and only if $D$ is a double central extension.

3. The third cohomology group

In this section we translate the description of the second order direction functor and its associated cohomology groups, developed in [23] for Barr exact categories, to the context of semi-abelian categories. A similar translation was made in [23] for Moore categories (i.e., strongly protomodular semi-abelian categories) where the connection with $n$-fold crossed extensions is explored. Note that what we call the third cohomology group here is actually the second cohomology group in [23]; the dimension shift is there for historical reasons, in order to comply with the “non-abelian” numbering used in classical cohomology of groups. From now on, $A$ will denote a semi-abelian category and $Z$ a fixed object of $A$.

An aspherical (abelian) groupoid in $A \downarrow Z$ consists of a commutative diagram
\[
\begin{array}{c}
X \xleftarrow{d} \xrightarrow{i} Y \\
\downarrow{f \circ d = f \circ c} \\
\downarrow{f} \\
Z
\end{array}
\]
\[
\text{(G)}
\]
such that the top line is a groupoid in $A$, and both the morphisms $f$ and $(d, c): X \to R[f]$ are regular epimorphisms. Such an internal groupoid has an underlying double extension
\[
\begin{array}{c}
X \xrightarrow{c} Y \\
\downarrow{d} \\
Y \xrightarrow{f} Z
\end{array}
\]
\[
\text{(H)}
\]
We denote by $\text{Asph}(A \downarrow Z)$ the category of aspherical groupoids in $A \downarrow Z$. 

---

1. **Lemma 1.4**

2. **Moore categories**

3. **n-fold crossed extensions**
The category $\text{Mod}_{\mathbb{Z}}A$ of $\mathbb{Z}$-modules is the category $\text{Ab}(A \downarrow \mathbb{Z})$ of abelian groups in $A \downarrow \mathbb{Z}$. So, a $\mathbb{Z}$-module gives us a split exact sequence

$$0 \longrightarrow A \xrightarrow{\ker p} P \xrightarrow{p} \ker p \xrightarrow{s} Z \longrightarrow 0$$

where $A$ is an abelian object and $p$ is a split epimorphism (equipped with an additional structure making it an abelian group in $A \downarrow \mathbb{Z}$). Using the equivalence between split epimorphisms and internal actions [7], we can replace $P$ with a semi-direct product $Z \rtimes (A, \xi)$. For simplicity, we denote a $\mathbb{Z}$-module just by its induced $\mathbb{Z}$-algebra $(A, \xi)$.

In the context of semi-abelian categories, the direction functor from [23, Definition 3.7] determines a functor $d_Z : \text{Asph}(A \downarrow \mathbb{Z}) \to \text{Mod}_{\mathbb{Z}}A$ mapping an aspherical internal groupoid $G$ to the $\mathbb{Z}$-module $d_Z(G) = (A, \xi)$ defined by the downward pullback/upward pushout

$$R[(d, c)] \longrightarrow Z \times (A, \xi)$$

$$(1_X, 1_X) \longleftarrow \downarrow p \quad \downarrow s \quad \downarrow p$$

$X \xrightarrow{f \circ d} Z.$

More precisely, the pair $(p, s) : Z \times (A, \xi) \rightrightarrows Z$ arises as a pushout of $(1_X, 1_X)$ along $f \circ d$ but, using the properties of $G$, one may show that the square of downward arrows in $I$ is a pullback [4]. Thus we see that $A = K[p] = K[p_0] = K[(d, c)] = K[d] \cap K[c]$.

**Remark 3.1.** Suppose $(C, \otimes, E)$ is a symmetric monoidal category such that the following property holds:

$$\forall C \in C, \exists \overline{C} \in C : C \otimes \overline{C} \sim E, \quad (J)$$

where $\sim$ means “is connected to (by a zigzag)”. Then it is easy to check that the monoidal structure of $C$ induces an abelian group structure on the set $\pi_0C$ of its connected components (equivalence classes with respect to $\sim$). The addition is defined by $[C_1] + [C_2] = [C_1 \otimes C_2]$, the unit is $[E]$ and $-[C] = [\overline{C}]$.

It is shown in [4] that the fibres of $d_Z$ are symmetric monoidal categories with property $J$. The tensor product is called the Baer sum since it gives the Baer sum of (2-fold) extensions in the classical examples. So, for any $\mathbb{Z}$-module $(A, \xi)$, $\pi_0d_Z^{-1}(A, \xi)$ is an abelian group.

**Definition 3.2.** [23] Let $(A, \xi)$ be a $\mathbb{Z}$-module. The third cohomology group $H^3(\mathbb{Z}, (A, \xi))$ of $\mathbb{Z}$ with coefficients in $(A, \xi)$ is the abelian group
\[ \pi_0 d^{-1}_Z(A, \xi) \] of equivalence classes of aspherical internal groupoids in \( A \downarrow Z \) with direction \((A, \xi)\). This defines an additive functor

\[ H^3(Z, -): \text{Mod}_Z A \to \text{Ab}. \]

We are especially interested in the case of trivial \( Z \)-modules \((A, \tau)\), i.e., abelian objects \( A \) with the trivial \( Z \)-action \( \tau \). In this situation we write \( H^3(Z, A) \) for \( H^3(Z, (A, \tau)) \). The functor \( H^3(Z, -) \) restricts to an additive functor \( \text{Ab} A \to \text{Ab} \).

**Proposition 3.3.** The direction of an aspherical groupoid \( G \) in \( A \downarrow Z \) is a trivial \( Z \)-module \((A, \tau)\) in \( A \) if and only if \( G \) is a central groupoid.

**Proof:** Let us first suppose that \( d_Z(G) = (A, \tau) \). Then, \( d_Z(G) \), defined by \((p, s): Z \ltimes (A, \tau) \rightrightarrows Z \) in diagram \( I \), is the product projection with its canonical inclusion \((\text{pr}_Z, (1_Z, 0)): Z \times A \rightrightarrows Z \). It follows that the pullback \((p_0, (1_X, 1_X)): R[(d, c)] \rightrightarrows X \) is also a product projection with its canonical inclusion, namely \((\text{pr}_X, (1_X, 0)): X \times A \rightrightarrows X \). In particular, the splitting \((1_X, 1_X)\) is a normal monomorphism in \( A \), which by Theorem 5.2 in [6] (see also Corollary 6.1.8 in [1]) means that \( R[(d, c)] = R[d] \cap R[c] \) is central. Hence \([R[d] \cap R[c], \nabla_X] = \Delta_X \) and the groupoid is central.

Conversely, suppose that \( G \) is a central groupoid in \( A \downarrow Z \). By the same arguments as above we see that \((p_0, (1_X, 1_X))\) and hence \((p, s)\) are product projections with their canonical inclusions. It follows that \( A \) has a trivial \( Z \)-action \( \tau \).

**Corollary 3.4.** Let \( G \) be an aspherical groupoid in \( A \downarrow Z \) and let \( H \) be the corresponding double extension. Then \( H \) is a double central extension if and only if \( d_Z(G) \) is a trivial \( Z \)-module \((A, \tau)\) in \( A \).

Thus we see that the direction of a central internal groupoid \( G \) is just the intersection \( A = K[d] \cap K[c] \) of the kernels of \( d \) and \( c \); indeed, this object \( A \) is always abelian as the kernel of the central extension \((d, c)\). In view of this fact we can extend the concept of direction to double central extensions.

**4. The group of equivalence classes of double central extensions**

**Definition 4.1.** The direction of a double central extension \( D \) is the abelian object \( K[d] \cap K[c] \). This defines a functor

\[ D_Z: \text{CExt}_Z^2 A \to \text{Ab} A, \]
where $\text{CExt}^2_Z A$ denotes the category of double central extensions of the object $Z$ of $A$.

The fibre $D^{-1}A$ of this functor over an abelian object $A$ is the category of **double central extensions of $Z$ by $A$**. Two double central extensions of $Z$ by $A$ which are connected by a zigzag in $D^{-1}A$ are called equivalent. The equivalence classes form the set $\text{Centr}^2(Z, A) = \pi_0D^{-1}A$ of connected components of this category.

**Remark 4.2.** Depending on the context it might not be clear whether $\text{Centr}^2(Z, A)$ is indeed a set (rather than a proper class) but in any case Theorem 5.3 implies that $\text{Centr}^2(Z, A)$ is only as large as is $H^3(Z, A)$.

**Remark 4.3.** The double central extension $D$ induces a $3 \times 3$ diagram

$$
\begin{array}{ccc}
A & \rightarrow & K[d] \rightarrow K[g] \\
\downarrow & & \downarrow \\
K[c] & \rightarrow & X \rightarrow C \\
\downarrow & d & \downarrow g \\
K[f] & \rightarrow & D \rightarrow Z \\
\end{array}
$$

and the object $A$ in this diagram is the direction of $D$.

We now show that $\text{Centr}^2(Z, A)$ carries a canonical abelian group structure.

**Proposition 4.4.** Let $A$ be a semi-abelian category and let $Z$ be an object of $A$. Mapping an abelian object $A$ of $A$ to the set $\text{Centr}^2(Z, A)$ of equivalence classes of double central extensions of $Z$ by $A$ gives a finite product-preserving functor $\text{Centr}^2(Z, -) : \text{Ab}_A \rightarrow \text{Set}$.

**Proof:** Let $a : A \rightarrow B$ be a morphism of abelian objects in $A$ and $D$ a double central extension of $Z$ by $A$. Then $(d, c) : X \rightarrow D \times_Z C$ is a central extension of $D \times_Z C$ by $A$, and the construction of Proposition 1.5 yields a central extension $(d', c')$ of $D \times_Z C$ by $B$. The morphism $\text{Centr}^2(Z, a)$ now maps the equivalence class of $D$ to the class of the right hand side square below.
Indeed, since the left hand side square

\[
\begin{array}{cccccc}
X \times B & \xrightarrow{\text{copr}_X} & C \\
\downarrow_{d \circ \text{pr}_X} & & \downarrow_{g} \\
D \times_Z C & \xrightarrow{\text{pr}_D} & Z
\end{array}
\quad \begin{array}{cccccc}
B & \xrightarrow{} & 0 \\
\downarrow & & \downarrow \\
D & \xrightarrow{f} & Z
\end{array}
\quad \begin{array}{cccccc}
X' \xrightarrow{c'} & C \\
\downarrow_{(d',c')} & & \downarrow_{g} \\
D \times_Z C & \xrightarrow{\text{pr}_D} & Z
\end{array}
\]

—which arises from the regular epimorphism in the top sequence in \( \mathbf{C} \)—is a double central extension as the product of \( \mathbf{D} \) with the middle double central extension, so is its right hand side quotient. The functoriality of \( \text{Centr}^2(Z, -) \) now follows from the functoriality of \( \text{Centr}^1(Z, -) \).

It is clear that \( \text{Centr}^2(Z, -) \) preserves the terminal object: any double central extension with direction 0 is connected to

\[
\begin{array}{ccc}
Z & \xrightarrow{} & Z \\
\downarrow & & \downarrow \\
Z & \xrightarrow{} & Z.
\end{array}
\]

To show that \( \text{Centr}^2(Z, -) \) also preserves binary products, we must provide an inverse to the map

\[
(\text{Centr}^2(Z, \text{pr}_A), \text{Centr}^2(Z, \text{pr}_B)): \text{Centr}^2(Z, A \times B) \to \text{Centr}^2(Z, A) \times \text{Centr}^2(Z, B).
\]

This inverse is given by the product in the category \( \mathbf{CExt}^2_{Z, A} \) of double central extensions of \( Z \). Let indeed the two squares

\[
\begin{array}{cccc}
X & \xrightarrow{c} & C & \text{and} & X' & \xrightarrow{c'} & C' \\
\downarrow_{d} & & \downarrow_{g} & & \downarrow_{d'} & & \downarrow_{g'} \\
D & \xrightarrow{f} & Z & & D' & \xrightarrow{f'} & Z
\end{array}
\]


be double central extensions of $Z$ by $A$ and $B$, respectively. Then their product in $\text{CExt}^2_Z\mathcal{A}$ is the square

$$
\begin{array}{ccc}
X \times_Z X' & \xrightarrow{c \times Z c'} & C \times_Z C' \\
\downarrow{d \times_Z d'} & & \downarrow{g \circ \text{pr}_C} \\
D \times_Z D' & \xrightarrow{f \circ \text{pr}_D} & Z.
\end{array}
$$

In fact, this square represents a pregroupoid in $\mathcal{A} \downarrow Z$ as a product of two such pregroupoids, and the comparison map $(d \times_Z d', c \times_Z c')$ to the pullback is a central extension as a pullback of the central extension $(d, c) \times (d', c')$. Finally, the direction of this double central extension is the kernel of $(d \times_Z d', c \times_Z c')$, which is nothing but $A \times B$.

**Corollary 4.5.** The functor $\text{Centr}^2(Z, -)$ uniquely factors over the forgetful functor $\text{Ab} \rightarrow \text{Set}$ to yield a functor $\text{Centr}^2(Z, -): \text{Ab}\mathcal{A} \rightarrow \text{Ab}$.

**Proof:** Any abelian object of $\mathcal{A}$ carries a canonical internal abelian group structure; we just showed that the functor $\text{Centr}^2(Z, -)$ preserves such structures. See also Remark 5.5. □

5. $H^3(Z, A)$ and $\text{Centr}^2(Z, A)$ are isomorphic

**Proposition 5.1.** Let $\mathcal{A}$ be a finitely complete category. The forgetful embedding $\text{Gd}\mathcal{A} \rightarrow \text{PreGd}\mathcal{A}$ has a right adjoint $\text{gd}: \text{PreGd}\mathcal{A} \rightarrow \text{Gd}\mathcal{A}$. Moreover, when $\mathcal{A}$ is semi-abelian, $Z$ is an object of $\mathcal{A}$ and $A$ is an abelian object of $\mathcal{A}$, this adjunction restricts to the fibres of the direction functors $d_Z$ and $D_Z$

$$
d_Z^{-1}(A, \tau) \xrightarrow{\text{C}} \text{D}_Z^{-1}A. \quad (K)
$$

**Proof:** Given an internal pregroupoid $(X, d, c)$, the associated internal groupoid $\text{gd}(X, d, c)$ has as underlying reflexive graph

$$
R[c] \times_X R[d] \xrightarrow{\text{dom}} \xrightarrow{\text{id}} \xrightarrow{\text{cod}} X,
$$

where $\text{dom}$ and $\text{cod}$ are the first and third projections and $\text{id}$ is the diagonal. This reflexive graph is a groupoid: the composition maps a pair $(\alpha R[c] \beta R[d], \gamma R[c] \delta R[d])$ to the triple $(\alpha, p(\delta, \gamma, \beta), \epsilon)$, where $p$ is the pregroupoid structure of $(X, d, c)$. The $(X, d, c)$-component of the counit of the
adjunction is defined by the map

$$(p, d, c): (R[c] \times_X R[d], \text{dom}, \text{cod}) \to (X, d, c)$$

in $\text{PreGd}_A$; and given an internal groupoid

$$v \subseteq X \xleftarrow{\frac{d}{c}} Y$$

with inversion map $v$, the associated unit component is

$$X \xleftarrow{\frac{d}{c}} Y,$$

one easily checks that the triangular identities hold.

Corollary 3.4 implies that the embedding $\text{Gd}_A \hookrightarrow \text{PreGd}_A$ restricts to the fibres of $d_Z$ and $D_Z$. Now suppose that $D \in D_{Z}^{-1}A$; then $(X, d, c)$ is a central pregrouppid in $A \downarrow Z$ by Theorem 2.7, and $A = K[(d, c)]$. Using that the square

$$R[c] \times_X R[d] \xrightarrow{p} X$$

is a pullback, we see that $(\text{dom}, \text{cod})$ is a central extension and that $A = K[(\text{dom}, \text{cod})]$. Hence the groupoid $\text{gd}(X, d, c)$ in $A \downarrow Z$ is central, which by Proposition 3.3 means that it has direction $(A, \tau)$, that is, it is in the fibre $d_{Z}^{-1}(A, \tau)$—so the functor $\text{gd}$ also restricts to the fibres of the direction functors $d_Z$ and $D_Z$.

To see that these restrictions are still adjoint to each other, it suffices to prove that the components of the unit and the counit are in the fibre of $1_{(A, \tau)}$ (respectively $1_A$). This is the case, because both the square $\text{M}$ and the similar square corresponding to $\text{L}$ are pullbacks.
Remark 5.2. Consider an adjunction

\[ \begin{array}{c}
C & \xrightarrow{F} & D \\
\xleftarrow{G} & & \\
\end{array} \]

(1) The functors \( F \) and \( G \) induce functions \( \varphi: \pi_0 C \to \pi_0 D \), defined by \( \varphi[C] = [FC] \), and \( \gamma: \pi_0 D \to \pi_0 C \), defined by \( \gamma[D] = [GD] \), respectively.

(2) \( F \) being left adjoint to \( G \) implies that \( \varphi^{-1} = \gamma \), i.e., \( \pi_0 C \cong \pi_0 D \). In fact, \( (\varphi \circ \gamma)[D] = [FGD] = [D] \), for any object \( D \) of \( D \), since \( FGD \) is connected to \( D \) by the \( D \)-component of the counit of the adjunction; thus \( \varphi \circ \gamma = 1_{\pi_0 D} \). Similarly \( \gamma \circ \varphi = 1_{\pi_0 C} \), using the unit of the adjunction instead.

Now suppose that the category \( C \) carries a symmetric monoidal structure \( (\mathcal{C}, \otimes, E) \) as in Remark 3.1.

(3) \( \pi_0 C \) is an abelian group.

(4) \( \pi_0 D \) is an abelian group with addition given by

\[ [D_1] + [D_2] = [F(GD_1 \otimes GD_2)], \]

unit \( [FE] \) and \( -[D] = [F(\overline{GD})] \).

(5) The function \( \varphi \) is a group isomorphism with inverse \( \gamma \).

Theorem 5.3. In any semi-abelian category, the third cohomology group \( H^3(Z, A) \) of an object \( Z \) with coefficients in an abelian object \( A \) is isomorphic to the group \( \text{Centr}^2(Z, A) \) of equivalence classes of double central extensions of \( Z \) by \( A \).

Proof: By the unicity in Corollary 4.5, to show that the functors \( H^3(Z, -) \) and \( \text{Centr}^2(Z, -) \) are isomorphic as functors \( \text{Ab} A \to \text{Ab} \), it suffices to give a bijection between the underlying sets \( H^3(Z, A) \) and \( \text{Centr}^2(Z, A) \), natural in \( A \). Through Remark 5.2, the adjunction \( K \) from Proposition 5.1 induces the needed isomorphisms

\[ \varphi: H^3(Z, A) \to \text{Centr}^2(Z, A) \]

and \( \gamma: \text{Centr}^2(Z, A) \to H^3(Z, A) \).

Remark 5.4. We have \( \varphi: H^3(Z, A) \to \text{Centr}^2(Z, A): [G] \mapsto [H] \) and

\[ \gamma: \text{Centr}^2(Z, A) \to H^3(Z, A): [D] \mapsto [\text{gd}(D)], \]
where

\[ \text{gd}(D) = \begin{array}{c}
R[c] \times_X R[d] \\
\xrightarrow{\text{id}} \\
\xleftarrow{\text{dom}} \\
\xrightarrow{\text{cod}} \\
\xleftarrow{\text{id}} \\
\xrightarrow{\text{dom}} \\
\xleftarrow{\text{cod}} \\
\xrightarrow{\text{fod}=goc} \\
Z
\end{array} \]

such that \( \varphi \circ \gamma = 1_{\text{Centr}^2(Z,A)} \), because for any double central extension \( D \) of \( Z \) by \( A \), \( (\varphi \circ \gamma)[D] \) is equal to \([D]\) through \((p,d,c)\), the \( D \)-component of the counit of the adjunction \( K \)

\[ \begin{array}{c}
R[c] \times_X R[d] \xrightarrow{\text{cod}} X \\
d \xleftarrow{\text{dom}} \\
c \xrightarrow{\text{cod}} C \\
z \xrightarrow{\text{goc}} \\
X \xrightarrow{\text{fod}} Z \xrightarrow{\text{f}} \xrightarrow{\text{id}} \text{Z} \\
D \xrightarrow{\text{f}} \text{Z}
\end{array} \]

and \( \gamma \circ \varphi = 1_{H^3(Z,A)} \), since for any central internal groupoid \( G \), with inversion map \( v \) and direction \((A,\tau)\), \((\gamma \circ \varphi)[G]\) is equal to \([G]\) through \(((i \circ d,v,i \circ c),i)\), the \( G \)-component of the unit of the adjunction \( K \)

\[ \begin{array}{c}
X \xleftarrow{d} \xrightarrow{i} Y \\
\xleftarrow{c} \\
\xrightarrow{f} \xrightarrow{i} Z \\
\xleftarrow{f \circ d} \\
\xrightarrow{\text{dom}} \\
\xrightarrow{\text{cod}} \\
R[c] \times_X R[d] \xrightarrow{\text{id}} X \xrightarrow{\text{dom}} \text{X} \xrightarrow{\text{cod}} \text{X} \\
\xleftarrow{\text{cod}} \\
\xrightarrow{\text{dom}} \\
\xrightarrow{\text{id}} \\
\xleftarrow{\text{cod}} \\
\xrightarrow{\text{id}} \text{X}
\end{array} \]

**Remark 5.5.** We know that \( d^{-1}_Z(A,\tau) \) is a symmetric monoidal category with property \( J \) by Remark 3.1. The arguments in Remark 5.2 show how the addition on \( H^3(Z,A) \) is transported to an abelian group structure on
Centr^2(Z, A) as described in Remark 5.2, (4). This makes the connection between the canonical abelian group structure from Proposition 4.4 and Corollary 4.5 and the Baer sum on d_Z^1(A, τ) explicit.

References

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