Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 08–57

MATRIX INTERPRETATION OF MULTIPLE ORTHOGONALITY

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Abstract: In this work we give an interpretation of a $(s(d+1)+1)$-term recurrence relation in terms of type II multiple orthogonal polynomials. We rewrite this recurrence relation in matrix form and we obtain a three-term recurrence relation for vector polynomials with matrix coefficients. We present a matrix interpretation of the type II multi-ortogonality conditions. We state a Favard type theorem and the expression for the resolvent function associated to the vector of linear functionals. Finally a reinterpretation of the type II Hermite-Padé approximation in matrix form is given.

Keywords: Multiple-orthogonal polynomials, Hermite-Padé approximants, block tridiagonal operator, Favard type theorem.
AMS Subject Classification (2000): Primary 33C45; Secondary 39B42.

1. Introduction

Multiple orthogonal polynomials are a generalization of orthogonal polynomials in the sense that they satisfy orthogonality conditions with respect to a number of measures. Such polynomials arise, in a natural way, in the study of simultaneous rational approximation, and in particular for the study of Hermite-Padé approximation for a system of $d \in \mathbb{Z}^+$ Markov functions (see [12]). In this way, multiple orthogonal polynomials are intimately related to Hermite-Padé approximation. In the literature we can find a lot of examples of multiple orthogonal polynomials (see [1, 2, 3, 4, 8, 10, 14, 15]).

Let $\vec{n} = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$ which is called a *multi-index* with length $|\vec{n}| := n_1 + \cdots + n_d$ and let $\{u^1, \dots, u^d\}$ be a system of linear functionals $u^j : \mathbb{P} \to \mathbb{C}$ with $j = 1, 2, \dots, d$.

Definition 1. Let $\{P_{\vec{n}}\}$ be a sequence of polynomials where the degree of $P_{\vec{n}}$ is at most $|\vec{n}|$. We say that $\{P_{\vec{n}}\}$ is a *type II multiple orthogonal with respect*

Received December 2, 2008.
The work of the first author was supported by CMUC/FCT. The work of the second author was partially supported by "Acção do Prodep" reference 5.3/C/00187.010/03. The third author would like to thank UI Matemática e Aplicações from University of Aveiro.

to the system of linear functionals $\{u^1,\dots,u^d\}$ *and multi-index* $\vec{n}$, if

$$u^j\left(x^m P_{\vec{n}}\right)=0,\ \ m=0,1,\dots,n_j-1,\ \ j=1,\dots,d. \tag{1}$$

For the particular case in which the system of linear functionals is a system of positive Borel measures, μ_j, on $I_j\subset\mathbb{R}$, $j=1,\dots,d$, we have

$$u^j(x^k)=\int_{I_j} x^k d\mu_j,\ \ k\in\mathbb{N}\ ,\ \ j=1,\dots,d\,,$$

and the conditions of multi-orthogonality, (1), can be rewritten as

$$\int_{I_j} P_{\vec{n}}(x)x^k d\mu_j(x)=0,\ \ k=0,1,\dots,n_j-1,\ \ j=1,\dots,d\,.$$

Definition 2. A *multi-index* $\vec{n}=(n_1,\dots,n_d)\in\mathbb{Z}_+^d$ is said to be *normal* for the system of linear functionals $\{u^1,\dots,u^d\}$, if for any non trivial solution $P_{\vec{n}}$ of (1), the degree of $P_{\vec{n}}$ is equal to $|\vec{n}|$. When all the multi-indices of a given family are normal, we say that *the system of linear functionals* $\{u^1,\dots,u^d\}$ *is regular*.

In the works of K. Douak and P. Maroni [5], P. Maroni [11], V. Kaliaguine [9], J. Van Iseghem [16], and also in the work of V.N. Sorokin and J. Van Iseghem [13], we find that a sequence of type II multiple orthogonal polynomials with respect to the system of linear functionals $\{u^1,\dots,u^d\}$ and multi-index $\vec{n}=(n_1,\dots,n_d)\in\mathcal{I}$, where

$$\mathcal{I}=\{(0,0,\dots,0),(1,0,\dots,0),\dots,(1,1,\dots,1),\\ (2,1,\dots,1),\dots,(2,2,\dots,2),\dots\},$$

verify a $(d+2)$-term recurrence relation of type

$$xB_n=B_{n+1}+\sum_{k=0}^{d} a_{n-k}^n B_{n-k}\,.$$

They call such polynomials *d-orthogonal*, where d corresponds to the number of functionals.

In this work we consider sequences of type II multiple orthogonal polynomials for more general families of multi-indices, $\mathcal{J}$. We designate this multi-indices by quasi-diagonal. In section 2 we build the sets of quasi-diagonal multi-indices, $\mathcal{J}$. Next we give the type II multi-orthogonality conditions for a sequence of monic polynomials $\{B_n\}$ with respect to the system of linear functionals $\{u^1,\dots,u^d\}$ and a family of quasi-diagonal multi-indices, $\mathcal{J}$. We also prove that this sequence verifies a $(s(d+1)+1)$-term recurrence relation of type

$$x^s B_n = B_{n+s} + \sum_{k=0}^{s(d+1)-1} a_{n+s-1-k}^{n+s-1} B_{n+s-1-k} \, .$$

To finish this section, we rewrite the previous $(s(d+1)+1)$-term recurrence relation in matrix form and we obtain a three-term recurrence relation for vector polynomials with matrix coefficients. In section 3 we present an algebraic theory which enables us to operate with the new presented objects. Here, our main goal, is to present a matrix interpretation of the multi-ortogonality conditions presented in the section 2. Next we give a result of existence and uniqueness of a type II sequence of vector orthogonal polynomials with respect to a vector of linear functionals $\mathcal{U}$, and using a matrix three-term recurrence relations we establish a Favard type theorem. We remark that other characterization for sequences of orthogonal polynomials in terms of matrix three-term recurrence relations can be found in [6, 7]. In section 4 we express the resolvent function in terms of the matrix generating function associated to the vector of linear functionals. Finally, we give a reinterpretation of the type II multiple orthogonality, in terms of a Hermite-Padé approximation problem for the matrix generating function associated to the vector of linear functionals. We remark that Hermite-Padé approximation problems can be found for example in [12, 14].

2. Quasi-diagonal multi-indices

2.1. Definition and some examples. Now we construct the set of multi-indices, $\mathcal{J}$, that will be used in this work. We begin by considering blocks with sd elements of $\mathbb{Z}_+^d$ in the Table 1. The multi-indices $\left(k_i^1, \cdots, k_i^d\right)$ where

$n = \lvert\vec{n}\rvert$	$\vec{n} = (n_1, \ldots, n_d)$
0	$(0, \ldots, 0)$
1	$(1, 0, \ldots, 0)$
$\vdots$	$\vdots$
i	$(k_i^1, \ldots, k_i^d)$
$\vdots$	$\vdots$
$sd-1$	$(s, \ldots, s, s-1)$

TABLE 1. Pattern blocks

$i = 0, 1, \cdots, sd-1$ are defined by the following conditions:

- $k_{i+1}^j \geq k_i^j, \;\; i = 0, 1, \ldots, sd-2, \;\; j = 1, \ldots, d;$

- $k_i^{j+1} \leq k_i^j, \;\; i = 0, 1, \ldots, sd-1, \;\; j = 1, \ldots, d-1;$
- $\displaystyle\sum_{j=1}^{d} k_i^j = i, \;\; i = 0, 1, \ldots, sd-1, \;\; j = 1, \ldots, d;$
- $k_{sd-1}^j = \begin{cases} s, & j = 1, 2, \ldots, d-1 \\ s-1, & j = d. \end{cases}$

Now, we identify as the *pattern block*, $\mathcal{J}_0$, the set whose elements are the ones of any of the blocks presented in the Table 1, i.e,

$$\mathcal{J}_0 = \{(0, \ldots, 0), (1, 0, \ldots, 0), \ldots, (s, \ldots, s, s-1)\}.$$

From $\mathcal{J}_0$ we generate a sequence of sets which we denote by $\mathcal{J}_n$, $n \in \mathbb{N}$, according to the formula:

$$\mathcal{J}_n = \mathcal{J}_0 + n\{(s, \ldots, s)\}, \; n \in \mathbb{N}. \tag{2}$$

In this way we obtain a set of multi-indices, $\mathcal{J}$, given by

$$\mathcal{J} = \{\mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_n, \ldots\}.$$

Remark that for $s = 1$ we have that $\mathcal{J}_0$ is given by,

$$\mathcal{J}_0 = \{(0, \ldots, 0), (1, 0, \ldots, 0), (1, 1, \ldots, 0), \ldots, (1, \ldots, 1, 0)\},$$

whose *multi-indices* we designate by *diagonal*.

In each of the following examples, we build the possible pattern blocks, $\mathcal{J}_0$, and the sets of quasi-diagonal multi-indices obtained from each one.

Example 1. $s = 1$, $d = 2$. We identify as $\mathcal{J}_0$, i.e. the pattern block $\mathcal{J}_0 = \{(0, 0), (1, 0)\}$. Thus, by using the formula (2) the sequence of sets, $\mathcal{J}_n$, $n \in \mathbb{N}$, are given by:

$$\mathcal{J}_n \;=\; \mathcal{J}_0 + n\{(1, 1)\} = \{(n, n), (n+1, n)\}.$$

Example 2. $s = 3$, $d = 2$. Following the same idea, we identify as $\mathcal{J}_0$, i.e. the pattern block

$$\begin{aligned}
\mathcal{J}_0 &= \{(0, 0), (1, 0), (1, 1), (2, 1), (2, 2), (3, 2)\}, \\
\mathcal{J}_0 &= \{(0, 0), (1, 0), (2, 0), (2, 1), (3, 1), (3, 2)\}, \\
\mathcal{J}_0 &= \{(0, 0), (1, 0), (2, 0), (2, 1), (2, 2), (3, 2)\}, \\
\mathcal{J}_0 &= \{(0, 0), (1, 0), (1, 1), (2, 1), (3, 1), (3, 2)\}, \\
\mathcal{J}_0 &= \{(0, 0), (1, 0), (2, 0), (3, 0), (3, 1), (3, 2)\}.
\end{aligned}$$

Continuing in this manner, the sequence of sets, $\mathcal{J}_n$, $n \in \mathbb{N}$, obtained from the sets $\mathcal{J}_0$ provided above, are given using the formula $\mathcal{J}_n = \mathcal{J}_0 + 3n\{(1, 1)\}$, therefore, obtaining in each case:

$$\begin{aligned}
\mathcal{J}_n &= \{(3n,3n),(3n+1,3n),(3n+1,3n+1),\\
&\quad (3n+2,3n+1),(3n+2,3n+2),(3n+3,3n+2)\}\\
\mathcal{J}_n &= \{(3n,3n),(3n+1,3n),(3n+2,3n),\\
&\quad (3n+2,3n+1),(3n+3,3n+1),(3n+3,3n+2)\}\\
\mathcal{J}_n &= \{(3n,3n),(3n+1,3n),(3n+2,3n),\\
&\quad (3n+2,3n+1),(3n+2,3n+2),(3n+3,3n+2)\}\\
\mathcal{J}_n &= \{(3n,3n),(3n+1,3n),(3n+1,3n+1),\\
&\quad (3n+2,3n+1),(3n+3,3n+1),(3n+3,3n+2)\},\\
\mathcal{J}_n &= \{(3n,3n),(3n+1,3n),(3n+2,3n),\\
&\quad (3n+3,3n),(3n+3,3n+1),(3n+3,3n+2)\}.
\end{aligned}$$

2.2. Multi-orthogonality conditions of type II. We identify the vectors $\vec{n}=(n_1,\dots,n_d)\in\mathbb{Z}_+^d$ with $n\in\mathbb{Z}_0^+$, as in our sets of quasi-diagonal multi-indices, $\mathcal{J}$, there is an one-to-one correspondence, $\mathbf{i}$, between the sets $\mathbb{Z}_+^d$ and $\mathbb{Z}_0^+$ given by, $\mathbf{i}(\vec{n})=|\vec{n}|=n$.

Let us consider, $B_{\vec{n}}$, be a sequence of type II multiple orthogonal polynomial with respect to the system of linear functionals $\{u^1,\dots,u^d\}$ and multi-index $\vec{n}$. We identify $B_{\vec{n}}\equiv B_{|\vec{n}|}=B_n$.

Now we describe how to obtain the multi-orthogonality conditions of a sequence of monic type II multiple orthogonal polynomials, $\{B_n\}$, with respect to the system of linear functionals $\{u^1,u^2\}$ and quasi-diagonal multi-index $\mathcal{J}$, where $\mathcal{J}_0=\{(0,0),(1,0),(2,0),(2,1),(2,2),(3,2)\}$. By using the Definition 1, we have

$$\begin{aligned}
&u^1(B_1)=0,\\
&u^1(B_2)=0,u^1(xB_2)=0,\\
&u^1(B_3)=0,u^1(xB_3)=0,u^2(B_3)=0,\\
&u^1(B_4)=0,u^1(xB_4)=0,u^2(B_4)=0,u^2(xB_4)=0,\\
&u^1(B_5)=0,u^1(xB_5)=0,u^2(B_5)=0,u^2(xB_5)=0,u^1(x^2B_5)=0,\\
&u^1(B_6)=0,u^1(xB_6)=0,u^2(B_6)=0,u^2(xB_6)=0,u^1(x^2B_6)=0,\\
&\qquad\qquad u^2(x^2B_6)=0\,.
\end{aligned}$$

The monic polynomials $B_1,\dots,B_6$ are defined by the multi-orthogonality conditions in terms of $\{u^1,xu^1,x^2u^1,u^2,xu^2,x^2u^2\}$, this multi-orthogonality conditions appear with the order suggested by the pattern block, $\mathcal{J}_0$,

$$\{u^1,xu^1,u^2,xu^2,x^2u^1,x^2u^2\}\,.$$

Defining the linear functionals

$$v^1 := u^1,\ \ v^2 := xu^1,\ \ v^3 := u^2,\ \ v^4 := xu^2,\ \ v^5 := x^2u^1,\ \ v^6 := x^2u^2\,,$$

we have

$$\begin{aligned}
&v^1(B_1) = 0,\\
&v^1(B_2) = 0, v^2(B_2) = 0,\\
&v^1(B_3) = 0, v^2(B_3) = 0, v^3(B_3) = 0,\\
&v^1(B_4) = 0, v^2(B_4) = 0, v^3(B_4) = 0, v^4(B_4) = 0,\\
&v^1(B_5) = 0, v^2(B_5) = 0, v^3(B_5) = 0, v^4(B_5) = 0, v^5(B_5) = 0,\\
&v^1(B_6) = 0, v^2(B_6) = 0, v^3(B_6) = 0, v^4(B_6) = 0, v^5(B_6) = 0, v^6(B_6) = 0\,.
\end{aligned}$$

Similarly the monic polynomials $B_7, \ldots, B_{12}$ are defined by the multi-orthogonality conditions in terms of

$$\{u^1, xu^1, x^2u^1, u^2, xu^2, x^2u^2, x^3u^1, x^4u^1, x^5u^1, x^3u^2, x^4u^2, x^5u^2\}\,,$$

this multi-orthogonality conditions appear with the order suggested by the pattern block $\mathcal{J}_0$

$$\{u^1, xu^1, u^2, xu^2, x^2u^1, x^2u^2, x^3u^1, x^4u^1, x^3u^2, x^4u^2, x^5u^1, x^5u^2\}\,,$$

that can be written in terms of the linear functionals $v^1, \ldots, v^6$ as

$$\{v^1, v^2, v^3, v^4, v^5, v^6, x^3v^1, x^3v^2, x^3v^3, x^3v^4, x^3v^5, x^3v^6\}\,.$$

More precisely

$$\begin{aligned}
&v^1(B_{6\times1+1}) = 0, \ldots, v^6(B_{6\times1+1}) = 0,\ v^1(x^3B_{6\times1+1}) = 0,\\
&v^1(B_{6\times1+2}) = 0, \ldots, v^6(B_{6\times1+2}) = 0,\ v^\alpha(x^3B_{6\times1+2}) = 0,\ \alpha = 1, 2,\\
&v^1(B_{6\times1+3}) = 0, \ldots, v^6(B_{6\times1+3}) = 0,\ v^\alpha(x^3B_{6\times1+3}) = 0,\ \alpha = 1, 2, 3,\\
&v^1(B_{6\times1+4}) = 0, \ldots, v^6(B_{6\times1+4}) = 0,\ v^\alpha(x^3B_{6\times1+4}) = 0,\ \alpha = 1, 2, 3, 4,\\
&v^1(B_{6\times1+5}) = 0, \ldots, v^6(B_{6\times1+5}) = 0,\ v^\alpha(x^3B_{6\times1+5}) = 0,\ \alpha = 1, 2, 3, 4, 5,\\
&v^1((x^3)^iB_{6\times2+0}) = 0, \ldots, v^6((x^3)^iB_{6\times2+0}) = 0,\ i = 0, 1.
\end{aligned}$$

In general we can consider $n = 6r + k$ where $k = 0, 1, 2, 3, 4, 5$ and $r = 0, 1, \ldots,$ and we obtain the following type II multi-orthogonality conditions

$$\left\{\begin{array}{l} v^j((x^3)^iB_{6r+k}) = 0,\ \ i = 0, 1, \ldots, r-1,\ \ j = 1, 2, 3, 4, 5, 6\\ v^\alpha((x^3)^rB_{6r+k}) = 0,\ \ \alpha = 1, \ldots, k. \end{array}\right. \tag{3}$$

Let Γ be a linear functional acting on the the vector space of the polynomials $\mathbb{P}$ over $\mathbb{C}^6$, i.e., $\Gamma : \mathbb{P} \longrightarrow \mathbb{C}^6$, by

$$\Gamma(P(x)) := \left[v^1(P(x)), v^2(P(x)), v^3(P(x)), v^4(P(x)), v^5(P(x)), v^6(P(x))\right]^T.$$

The multi-orthogonality conditions (3), can be written in an equivalent way by

$$\begin{cases} \Gamma((x^3)^i B_{6r+k}) = 0_{6\times 1}, & i = 0, 1, \ldots, r-1 \\ v^\alpha((x^3)^r B_{6r+k}) = 0, & \alpha = 1, \ldots, k. \end{cases}$$

for any pattern block presented in Example 2, we can obtain a new set of linear functionals, $\{v^1, v^2, v^3, v^4, v^5, v^6\}$, of type $\{x^j u^k : j = 0, 1, 2,\ k = 1, 2\}$.

All of these new sets of linear functionals are respectively:

$$\{u^1, u^2, xu^1, xu^2, x^2u^1, x^2u^2\},\ \{u^1, xu^1, u^2, x^2u^1, xu^2, x^2u^2\},$$
$$\{u^1, u^2, xu^1, x^2u^1, xu^2, x^2u^2\},\ \{u^1, xu^1, x^2u^1, u^2, xu^2, x^2u^2\}.$$

Algorithm (Construction of linear functionals). *Let us consider the sequence of monic type II multiple orthogonal polynomials, $\{B_n\}$, with respect to the system of linear functionals $\{u^1, \ldots, u^d\}$ and family of quasi-diagonal multi-indices given in Table 1, $\mathcal{J} = \{\mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_n, \ldots\}$.*

Let $v^1 = u^1$, $v^i = x^{k^j_{i-1}} u^j$, $i = 2, \ldots, sd-1$ where j, for each i, is uniquely defined by the condition $k^j_i = k^j_{i-1} + 1$ and $v^{sd} = x^{s-1}u^d$. Hence, we have

$$v^i \in \{x^k u^j : k = 0, 1, \ldots, s-1,\ j = 1, 2, \ldots, d\},\ i = 1, 2, \ldots, sd.$$

Theorem 1. *The sequence of monic polynomials, $\{B_n\}$, where $n = sdr + k$, $k = 0, 1, \ldots, sd-1$ and $r = 0, 1, \ldots,$ is type II multiple orthogonal with respect to the regular system of linear functionals $\{u^1, \ldots, u^d\}$ and quasi-diagonal multi-index $\mathcal{J}$ if, and only if,*

$$\begin{cases} v^j((x^s)^m B_{sdr+i}) = 0,\ m = 0, 1, \ldots, r-1,\ j = 1, \ldots, sd \\ v^\alpha((x^s)^r B_{sdr+i}) = 0,\ \alpha = 1, \ldots, i \\ v^{i+1}((x^s)^r B_{sdr+i}) \neq 0, \end{cases} \tag{4}$$

where the linear functionals v^j, $j = 1, \ldots, sd$ are defined by the algorithm.

Proof: Let us consider the set of multi-indices

$$\mathcal{J}_0 = \{(0, \ldots, 0), (1, 0, \ldots, 0), \ldots, (k^1_i, \ldots, k^d_i), \ldots, (s, \ldots, s, s-1)\}.$$

The linear functionals $v^1, \ldots, v^{sd}$ are defined by the algorithm. We can verify that $v^1, \ldots, v^i \in \{x^k u^j,\ 0 \le k \le k^j_i - 1,\ j = 1, \ldots, d\}$, for $i = 1, \ldots, sd$. Using the multi-orthogonality conditions of the polynomial B_i and multi-index $(k^1_i, \ldots, k^d_i)$ we have that $v^j(B_i) = 0$, $j = 1, \ldots, i$, for $i = 1, \ldots, sd$.

We obtain the multi-orthogonality conditions for the polynomials B_{sd+i}, $i = 1, \ldots, sd$. Let us consider the multi-index $(k^1_i, \ldots, k^d_i) + s(1, \ldots, 1)$ and let $j \in \{1, \ldots, d\}$ be uniquely defined by the condition $k^j_i = k^j_{i-1} + 1$. We have

$$u^j(x^{k^j_{i-1}+s} B_{sd+i}) = 0 \Leftrightarrow x^{k^j_{i-1}} u^j(x^s B_{sd+i}) = 0 \Leftrightarrow v^i(x^s B_{sd+i}) = 0.$$

By the increasing structure of the multi-indices, B_{sd+i} complies with the multi-orthogonality conditions of $B_1, \dots, B_{sd+i-1}$, in other words, this is sufficient to identify that,

$$v^j(B_{sd+i}) = 0, \ \ j = 1, \dots, sd\,, \quad v^\alpha(x^s B_{sd+i}) = 0, \ \ \alpha = 1, \dots, i\,.$$

Following the same reasoning we have that B_{sdr+i} verify $v^i(x^{sr} B_{sdr+i}) = 0$, and so,

$$\begin{cases} v^j((x^s)^m B_{sdr+i}) = 0, \ \ m = 0, 1, \dots, r-1, \ \ j = 1, \dots, sd \\ v^\alpha((x^s)^r B_{sdr+i}) = 0, \ \ \alpha = 1, \dots, i. \end{cases}$$

Finally, we show that $v^{i+1}((x^s)^r B_{sdr+i}) \neq 0$. Let us suppose that,

$$\begin{cases} v^j((x^s)^m B_{sdr+i}) = 0, \ \ m = 0, 1, \dots, r-1, \ \ j = 1, \dots, sd \\ v^\alpha((x^s)^r B_{sdr+i}) = 0, \ \ \alpha = 1, \dots, i \\ v^{i+1}((x^s)^r B_{sdr+i}) = 0. \end{cases}$$

Then the polynomial B_{sdr+i} verify the multi-orthogonality conditions of the polynomial $B_{sdr+i+1}$ which contradicts the normality of the multi-indices. Hence, $v^{i+1}((x^s)^r B_{sdr+i}) \neq 0$.

Reciprocally, for $n = sdr + i$, $i = 1, \dots, sd$

$$\begin{cases} v^j((x^s)^m B_{sdr+i}) = 0, \ \ m = 0, 1, \dots, r-1, \ \ j = 1, \dots, sd \\ v^\alpha((x^s)^r B_{sdr+i}) = 0, \ \ \alpha = 1, \dots, i, \end{cases}$$

and considering that the degree of B_n is equal to n by the normality of each of the multi-indices which implies the uniqueness of the monic type II multiple orthogonal polynomial sequence, B_n, with respect to the system of linear functionals $\{u^1, \dots, u^d\}$ and quasi-diagonal multi-index n. ■

Let Γ be a linear functional acting on the the vector space of the polynomials $\mathbb{P}$ over $\mathbb{C}^{sd}$, i.e., $\Gamma : \mathbb{P} \longrightarrow \mathbb{C}^{sd}$, by

$$\Gamma(P(x)) := [\ v^1(P(x)) \ \ \dots \ \ v^{sd}(P(x))\ ]^T, \ \ n \in \mathbb{N}.$$

The multi-orthogonality conditions of type II (4), can be written in the equivalent way by

$$\begin{cases} \Gamma((x^s)^m B_{sdr+i}) = 0_{sd\times 1}, \ \ m = 0, 1, \dots, r-1 \\ v^\alpha((x^s)^r B_{sdr+i}) = 0, \ \ \alpha = 1, \dots, i \\ v^{i+1}((x^s)^r B_{sdr+i}) \neq 0. \end{cases} \tag{5}$$

2.3. The $(s(d+1)+1)$-term recurrence relation. Here we give the connection between a sequence of monic type II multiple orthogonal polynomials, $\{B_n\}$, with respect to the regular system of linear functionals $\{u^1, \dots, u^d\}$ and quasi-diagonal multi-index $\mathcal{J}$, and the $(s(d+1)+1)$-term recurrence relation.

Theorem 2. *Let* $\{B_n\}$ *be a monic type II multiple orthogonal polynomials sequence, with respect to a regular system of linear functionals* $\{u^1,\dots,u^d\}$ *and quasi-diagonal multi-index* $\mathcal{J}$*. Then, there are sequences* $(a_{n+s-1-k}^{n+s-1})\subset\mathbb{C}$, $k=0,1,\dots,s(d+1)-1$, *such that,*

$$x^s B_n(x)=B_{n+s}(x)+\sum_{k=0}^{s(d+1)-1} a_{n+s-1-k}^{n+s-1}B_{n+s-1-k}(x),\ \ n=sd,sd+1,\dots,$$

where $a_{n-sd}^{n+s-1}\neq 0$ *and* $B_0,B_1,\dots,B_{sd-1}$ *are given.*

Proof: As the sequence of monic polynomials $\{B_n\}$ is a basis of the vector space $\mathbb{P}$, for each $n\in\mathbb{N}$, there is an unique sequence $(a_j^{n+s-1})\subset\mathbb{C}$, such that:

$$x^sB_n=B_{n+s}+\sum_{j=0}^{n+s-1}a_j^{n+s-1}B_j\,.$$

Substituting n by $sdr+k$ where $k=0,1,\dots,sd-1$ and $r=0,1,\dots,$ in the above identity, we have

$$x^sB_{sdr+k}-B_{sdr+k+s}=\sum_{j=0}^{sdr+k+s-1}a_j^{sdr+k+s-1}B_j. \tag{6}$$

Let, $i=0,1,\dots$. Multiplying both members of the above identity by $(x^s)^i$ and applying the linear functional Γ, we have

$$\Gamma[(x^s)^{i+1}B_{sdr+k}]-\Gamma[(x^s)^iB_{sdr+k+s}]=\sum_{j=0}^{sdr+k+s-1}a_j^{sdr+k+s-1}\Gamma[(x^s)^iB_j]\,.$$

By the multi-orthogonality conditions (5), we have

$$0_{sd\times 1}=\sum_{j=0}^{sd(i+1)-1}a_j^{sdr+k+s-1}\Gamma[(x^s)^iB_j]\ \text{ for }\ i=0,\dots,r-2\,.$$

Let $i=0$, we have $0_{sd\times1}=\displaystyle\sum_{j=0}^{sd-1}a_j^{sdr+k+s-1}\Gamma(B_j)$, which leads us to the system of linear equations in matrix form:

$$[a_0^{sdr+k+s-1}\ \cdots\ a_{sd-1}^{sdr+k+s-1}]\begin{bmatrix} v^1(B_0) & \cdots & v^{sd}(B_0)\\ & \ddots & \vdots\\ & & v^{sd}(B_{sd-1})\end{bmatrix}=0_{sd\times1}.$$

Using, $v^1(B_0)\neq 0,\dots,v^{sd}(B_{sd-1})\neq 0$, we have $a_0^{sdr+k+s-1}=0,\dots,$ $a_{sd-1}^{sdr+k+s-1}=0$.

Let $i=1$, we have $0_{sd\times 1} = \displaystyle\sum_{j=sd}^{2sd-1} a_j^{sdr+k+s-1}\Gamma(x^sB_j)$, which leads us to the system of linear equations in matrix form:

$$\left[\begin{array}{ccc} a_{sd}^{sdr+k+s-1} & \cdots & a_{2sd-1}^{sdr+k+s-1}\end{array}\right]\left[\begin{array}{ccc} v^1(x^sB_{sd}) & \cdots & v^{sd}(x^sB_{sd}) \\ & \ddots & \vdots \\ & & v^{sd}(x^sB_{2sd-1})\end{array}\right] = 0_{sd\times 1.}$$

Using, $v^1(x^sB_{sd}) \neq 0, \ldots, v^{sd}(x^sB_{2sd-1}) \neq 0$, we have $a_{sd}^{sdr+k+s-1} = 0, \ldots, a_{2sd-1}^{sdr+k+s-1} = 0$.

Continuing in the same way, we obtain $a_{jsd}^{sdr+k+s-1} = 0, \ldots, a_{(j+1)sd-1}^{sdr+k+s-1} = 0,\ j=2,\ldots,r-2$.

Now, considering the multi-orthogonality conditions written in (5), given by

$$v^\alpha((x^s)^rB_{sdr+k}) = 0, \ \ \alpha = 1,\ldots,k\,,$$

and taking into account (6), we verify that

$$v^\alpha\left[(x^s)^{i+1}B_{sdr+k}\right] - v^\alpha\left[(x^s)^iB_{sdr+k+s}\right] = 0\,,$$

for $i=r-1$ and $\alpha = 1,\ldots,k$ which leads us to the system of linear equations in matrix form:

$$\left[a_{(r-1)sd}^{sdr+k+s-1} \ \ldots \ a_{(r-1)sd+k-1}^{sdr+k+s-1}\right] \times \left[\begin{array}{ccc} v^1((x^s)^{r-1}B_{(r-1)sd}) & \cdots & v^k((x^s)^{r-1}B_{(r-1)sd}) \\ & \ddots & \vdots \\ & & v^k((x^s)^{r-1}B_{(r-1)sd+k-1})\end{array}\right] = 0_{sd\times 1}\,.$$

Using, $v^1((x^s)^{r-1}B_{(r-1)sd}) \neq 0, \ldots, v^k((x^s)^{r-1}B_{(r-1)sd+k-1}) \neq 0$, we have $a_{(r-1)sd}^{sdr+k+s-1} = 0, \ldots, a_{(r-1)sd+k-1}^{sdr+k+s-1} = 0$. Hence, we have $a_0^{sdr+k+s-1} = \cdots = a_{(r-1)sd+k-1}^{sdr+k+s-1} = 0$. Then,

$$x^sB_{sdr+k} = B_{sdr+k+s} + \sum_{j=(r-1)sd+k}^{sdr+k+s-1} a_j^{sdr+k+s-1}B_j\,,$$

and the theorem is proved. ■

Definition 3. Let $\{B_n\}$ be a sequence of monic polynomials. The sequence $\{\mathcal{B}_n\}$ given by

$$\mathcal{B}_n = \left[\begin{array}{ccc} B_{nsd} & \cdots & B_{(n+1)sd-1}\end{array}\right]^T, \ \ n\in\mathbb{N}\,, \tag{7}$$

is said to be the *vector sequence of polynomials associated* to $\{B_n\}$.

Theorem 3. *Let $\{B_n\}$ be a monic sequence of polynomials. Then, the following conditions are equivalent:*

a) The sequence of polynomials $\{B_n\}$ verify the $(s(d+1)+1)$-term relation given by

$$x^s B_n(x) = B_{n+s}(x) + \sum_{k=0}^{s(d+1)-1} a_{n+s-1-k}^{n+s-1} B_{n+s-1-k}(x), \ \ n = sd, sd+1, \ldots,$$

where $a_{n-sd}^{n+s-1} \neq 0$ and $B_0, B_1, \ldots, B_{sd-1}$ are given.

b) The vector sequence of polynomials $\{\mathcal{B}_m\}$ associated to the sequence of polynomials $\{B_m\}$ verify a three-term recurrence relation with $sd \times sd$ matrix coefficients, $x^s \mathcal{B}_m(x) = \alpha_m^{s,d} \mathcal{B}_{m+1}(x) + \beta_m^{s,d} \mathcal{B}_m(x) + \gamma_m^{s,d} \mathcal{B}_{m-1}(x)$, $m = 0, 1, \ldots,$ with $\mathcal{B}_{-1} = 0_{sd\times 1}$ and $\mathcal{B}_0$ given, where the matrix coefficients $\alpha_m^{s,d}$, $\beta_m^{s,d}$ and $\gamma_m^{s,d}$ are respectively given by

$$\begin{bmatrix} 1 & & & \\ a_{(m+s)d}^{(m+s)d} & \ddots & & \\ \vdots & \ddots & 1 & \\ a_{(m+s)d}^{md+s(d+1)-2} & \cdots & a_{md+s(d+1)-2}^{md+s(d+1)-2} & 1 \end{bmatrix};$$

$$\begin{bmatrix} a_{md}^{md+s-1} & \cdots & a_{md+s-1}^{md+s-1} & 1 & & & \\ & & & \ddots & \ddots & & \\ \vdots & & \vdots & & & & \\ & & & & & a_{(m+s)d-2}^{(m+s)d-2} & 1 \\ a_{md}^{(m+s)d-1} & \cdots & a_{md+s-1}^{(m+s)d-1} & & \cdots & a_{(m+s)d-2}^{(m+s)d-1} & a_{(m+s)d-1}^{(m+s)d-1} \\ \vdots & & \vdots & & & \vdots & \vdots \\ a_{md}^{md+s(d+1)-2} & \cdots & a_{(m+s)d-1}^{md+s(d+1)-2} & & \cdots & a_{(m+s)d-2}^{md+s(d+1)-2} & a_{(m+s)d-1}^{md+s(d+1)-2} \end{bmatrix};$$

$$\begin{bmatrix} a_{(m-s)d}^{md+s-1} & \cdots & a_{md-1}^{md+s-1} \\ & \ddots & \vdots \\ & & a_{md-1}^{md+s(1+d)-2} \end{bmatrix}.$$

Proof: Taking into account the $(s(d+1)+1)$-term recurrence relation we obtain the matrix identity given by

$$x^s \begin{bmatrix} B_n \\ \vdots \\ B_{n+sd-1} \end{bmatrix} = \underline{\alpha}_n^{s,d} \begin{bmatrix} B_{n+sd} \\ \vdots \\ B_{n+2sd-1} \end{bmatrix} + \underline{\beta}_n^{s,d} \begin{bmatrix} B_n \\ \vdots \\ B_{n+sd-1} \end{bmatrix} + \underline{\gamma}_n^{s,d} \begin{bmatrix} B_{n-sd} \\ \vdots \\ B_{n-1} \end{bmatrix},$$

where the matrix coefficients $\underline{\alpha}_n^{s,d}$, $\underline{\beta}_n^{s,d}$ and $\underline{\gamma}_n^{s,d}$ are respectively given by:

$$\begin{bmatrix} 1 & & & \\ a_{n+sd}^{n+sd} & \ddots & & \\ \vdots & \ddots & 1 & \\ a_{n+sd}^{n+s(d+1)-2} & \cdots & a_{n+s(d+1)-2}^{n+s(d+1)-2} & 1 \end{bmatrix};$$

$$\begin{bmatrix} a_n^{n+s-1} & \cdots & a_{n+s-1}^{n+s-1} & 1 & & & \\ & & & \ddots & \ddots & & \\ \vdots & & \vdots & & & & \\ a_n^{n+sd-2} & \cdots & a_{n+s-1}^{n+sd-2} & & \cdots & a_{n+sd-2}^{n+sd-2} & 1 \\ a_n^{n+sd-1} & \cdots & a_{n+s-1}^{n+sd-1} & & \cdots & a_{n+sd-2}^{n+sd-1} & a_{n+sd-1}^{n+sd-1} \\ \vdots & & \vdots & & & \vdots & \vdots \\ a_n^{n+s(d+1)-2} & \cdots & a_{n+s-1}^{n+s(d+1)-2} & & \cdots & a_{n+sd-2}^{n+s(d+1)-2} & a_{n+sd-1}^{n+s(d+1)-2} \end{bmatrix};$$

$$\begin{bmatrix} a_{n-sd}^{n+s-1} & \cdots & a_{n-1}^{n+s-1} \\ & \ddots & \vdots \\ & & a_{n-1}^{n+s(1+d)-2} \end{bmatrix}.$$

Taking $n = md$ we obtain a three-term recurrence relation for vectors of polynomials $\{\mathcal{B}_m\}$ where $\mathcal{B}_m = [\ B_{msd} \ \cdots \ B_{(m+1)sd-1}\ ]^T,\ m \in \mathbb{N}$, given by

$$x^s \mathcal{B}_m = \alpha_m^{s,d} \mathcal{B}_{m+1} + \beta_m^{s,d} \mathcal{B}_m + \gamma_m^{s,d} \mathcal{B}_{m-1},\ m = 0, 1, \ldots$$

with initial conditions $\mathcal{B}_{-1} = 0_{sd\times 1}$ and $\mathcal{B}_0$, and matrix coefficients $\alpha_m^{s,d} = \underline{\alpha}_{md}^{s,d}$, $\beta_m^{s,d} = \underline{\beta}_{md}^{s,d}$ and $\gamma_m^{s,d} = \underline{\gamma}_n^{s,d}$. The converse is immediate. ■

3. Matrix interpretation of type II multi-orthogonality

In this section we present a matrix interpretation of the type II ortogonality conditions of a sequence of monic polynomials $\{B_n\}$, given in the Theorem 1, with respect to the regular system of linear functionals $\{u^1, \ldots, u^d\}$ and family of quasi-diagonal multi-indices, $\mathcal{J}$.

Let us consider the sequence of vectors of polynomials that we denote by

$$\mathbb{P}^{sd} = \{[\ P_1 \ \cdots \ P_{sd}\ ]^T : P_j \in \mathbb{P}\},$$

We denote by $\mathcal{M}_{sd\times sd}$ the set of $sd \times sd$ matrices with entries in $\mathbb{C}$.

Let $\{\mathcal{P}_j\}$ be a sequence of vectors of polynomials given by

$$\mathcal{P}_j = [\ x^{jsd} \ \cdots \ x^{(j+1)sd-1}\ ]^T,\ \ j \in \mathbb{N}. \tag{8}$$

Let $\{B_n\}$ be a sequence of polynomials, $\deg B_n = n$, $n \in \mathbb{N}$ and $\{\mathcal{B}_n\}$ where

$$\mathcal{B}_n = [\ B_{nsd} \ \cdots \ B_{(n+1)sd-1}\ ]^T,\ \ n \in \mathbb{N}.$$

It is easy to see that

$$\mathcal{B}_n = \sum_{j=0}^{n} B_j^n \, \mathcal{P}_j, \quad B_j^n \in \mathcal{M}_{sd\times sd}\,,$$

where the matrix coefficients B_j^n, $j = 0, 1, \ldots, n$ are uniquely determined.

Taking into account (8) we have that $\mathcal{P}_j = (x^{sd})^j \mathcal{P}_0$, $j \in \mathbb{N}$. Therefore, $\mathcal{B}_n = V_n(x^{sd})\mathcal{P}_0$, where V_n is a matrix polynomial of degree n and dimension sd, given by $V_n(x) = \sum_{j=0}^{n} B_j^n x^j, \;\; B_j^n \in \mathcal{M}_{sd\times sd}$.

Definition 4. Let $v^j : \mathbb{P} \to \mathbb{C}$ with $j = 1, \ldots, sd$ be linear functionals. We define the *vector of functionals* $\mathcal{U} = [\, v^1 \;\; \cdots \;\; v^{sd} \,]^T$ acting in $\mathbb{P}^{sd}$ over $\mathcal{M}_{sd\times sd}$, by

$$\mathcal{U}(\mathcal{P}) := (\mathcal{U}.\mathcal{P}^T)^T = \begin{bmatrix} v^1(P_1) & \cdots & v^{sd}(P_1) \\ \vdots & \ddots & \vdots \\ v^1(P_{sd}) & \cdots & v^{sd}(P_{sd}) \end{bmatrix},$$

where "." means the symbolic product of the vectors $\mathcal{U}$ and $\mathcal{P}^T$.

Now we define an operation called *left multiplication of a vector of functionals by a polynomial.*

Definition 5. Let $\widehat{A} = \sum_{k=0}^{l} A_k \, x^k$ be a matrix polynomial of degree l where $A_k \in \mathcal{M}_{sd\times sd}$ and $\mathcal{U}$ a vector of linear functionals. We define the vector of linear functionals, *left multiplication of* $\mathcal{U}$ *by a polynomial* $\widehat{A}$, and denote it by $\widehat{A}\mathcal{U}$, to the map of $\mathbb{P}^{sd}$ to $\mathcal{M}_{sd\times sd}$, defined by:

$$(\widehat{A}\mathcal{U})(\mathcal{P}) := (\widehat{A}\mathcal{U}.\mathcal{P}^T)^T = \sum_{k=0}^{l} (x^k\,\mathcal{U})(\mathcal{P})(A_k)^T\,.$$

Theorem 4. *A sequence of monic polynomials $\{B_m\}$, is type II multiple orthogonal with respect to the regular system of linear functionals $\{u^1, \ldots, u^d\}$ and family of quasi-diagonal multi-indices $\mathcal{J}$ if, and only if, the vector sequence of polynomials associated to $\{\mathcal{B}_m\}$ given by (7) verifies:*

$$\begin{array}{l} i)\;\; ((x^s)^k \mathcal{U})(\mathcal{B}_m) = 0_{sd\times sd}, \quad k = 0, 1, \ldots, m-1 \\ ii)\;\; ((x^s)^m \mathcal{U})(\mathcal{B}_m) = \Delta_m, \end{array} \tag{9}$$

where $\mathcal{U} = [\, v^1 \;\; \cdots \;\; v^{sd} \,]^T$, v^j, $j = 1, \ldots, sd$ are defined by the algorithm, and Δ_m is a regular upper triangular $sd \times sd$ matrix.

Proof: By Definition 4, we have

$$((x^s)^k\mathcal{U})(\mathcal{B}_m) = \begin{bmatrix} v^1((x^s)^k B_{msd}) & \cdots & v^{sd}((x^s)^k B_{msd}) \\ \vdots & \ddots & \vdots \\ v^1((x^s)^k B_{(m+1)sd-1}) & \cdots & v^{sd}((x^s)^k B_{(m+1)sd-1}) \end{bmatrix}.$$

Using the ortogonality conditions of type II in Theorem 1 we have the conditions (9), and reciprocally. ■

Definition 6. Let $\{\mathcal{B}_m\}$ be a vector sequence of polynomials where each $\mathcal{B}_m = [B_{m,1} \dots B_{m,sd}]^T$, $m \in \mathbb{N}$, such that $\mathcal{B}_m = \sum_{j=0}^{m} B_j^m \mathcal{P}_j$ where $B_j^m \in \mathcal{M}_{sd\times sd}$ and let $\mathcal{U} = [\ v^1 \ \cdots \ v^{sd}\ ]^T$ be the vector of linear functionals. We say that $\{\mathcal{B}_m\}$ *is type II multiple orthogonal* with respect to the vector of linear functionals $\mathcal{U}$ if

$$\begin{array}{l} i)\ \ ((x^s)^k\mathcal{U})(\mathcal{B}_m) = 0_{sd\times sd}, \ \ k = 0, 1, \dots, m-1 \\ ii)\ \ ((x^s)^m\mathcal{U})(\mathcal{B}_m) = \Delta_m, \end{array} \tag{10}$$

where Δ_m is a regular $sd \times sd$ matrix.

Lemma 1. *Let* $\{\mathcal{B}_m\}$ *be a vector sequence of polynomials where each* $\mathcal{B}_m = [\ B_{m,1} \ \cdots \ B_{m,sd}\ ]^T$, $m \in \mathbb{N}$, *such that* $\mathcal{B}_m = \sum_{j=0}^{m} B_j^m \mathcal{P}_j$ *where* $B_j^m \in \mathcal{M}_{sd\times sd}$. *If* B_m^m *is a regular matrix, for a* $m \in \mathbb{N}$, *then the set of polynomials* $\{B_{m,1}, \dots, B_{m,sd}\}$ *is linearly independent.*

Proof: Let $\alpha_i \in \mathbb{R}$, $i = 1, 2, \dots, sd$, such that

$$\alpha_1 B_{m,1} + \cdots + \alpha_{sd} B_{m,sd} = 0, \ \text{ i.e.,} \quad [\ \alpha_1 \ \cdots \ \alpha_{sd}\ ] \begin{bmatrix} B_{m,1} \\ \vdots \\ B_{m,sd} \end{bmatrix} = 0\,.$$

And so, $\alpha\mathcal{B}_m = 0$, with $\alpha = [\alpha_1 \dots \alpha_{sd}]$. Hence,

$$\alpha \sum_{j=0}^{m} B_j^m \mathcal{P}_j = 0, \ \text{ i.e.,} \quad \sum_{j=0}^{m} \alpha B_j^m \mathcal{P}_j = 0\,.$$

As $\{1, \dots, x^{(m+1)sd-1}\}$ is a linearly independent set of functions, we have

$$\alpha B_j^m = 0, \ \ j = 0, 1, \dots, m\,.$$

If B_m^m is a regular matrix then $\alpha = 0_{1\times sd}$, as was our purpose to show. ■

Lemma 2. *Let $\{\mathcal{B}_m\}$ be a vector sequence of polynomials where each* $\mathcal{B}_m = \left[\, B_{m,1} \;\; \cdots \;\; B_{m,sd} \,\right]^T$, $m \in \mathbb{N}$, *such that* $\mathcal{B}_m = \sum_{j=0}^{m} B_j^m \mathcal{P}_j$ *where* $B_j^m \in \mathcal{M}_{sd\times sd}$. *If B_m^m is a regular matrix, for all $m \in \mathbb{N}$, then the set of polynomials* $\{B_{m,j}, j = 1, \ldots, sd, m \in \mathbb{N}\}$ *, is linearly independent.*

Proof: It is sufficient to prove for each $m \in \mathbb{N}$ that the set of polynomials $\{B_{k,j}, j = 1, \ldots, sd, k = 0, 1, \ldots, m\}$ is linearly independent. Let

$$\begin{aligned}
\alpha &= \left[\, \alpha_1 \;\; \cdots \;\; \alpha_{sd} \,\right], \;\; \alpha_i \in \mathbb{R} \\
&\;\vdots \\
\beta &= \left[\, \beta_1 \;\; \cdots \;\; \beta_{sd} \,\right], \;\; \beta_i \in \mathbb{R} \\
\gamma &= \left[\, \gamma_1 \;\; \cdots \;\; \gamma_{sd} \,\right], \;\; \gamma_i \in \mathbb{R}\,.
\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{i=1}^{sd} \alpha_i B_{0,i} + \cdots + \sum_{i=1}^{sd} \beta_i B_{m-1,i} + \sum_{i=1}^{sd} \gamma_i B_{m,i} = 0\,, \\
&\alpha \mathcal{B}_0 + \cdots + \beta \mathcal{B}_{m-1} + \gamma \mathcal{B}_m = 0\,, \\
&\alpha(B_0^0 \mathcal{P}_0) + \cdots + \beta(B_0^{m-1}\mathcal{P}_0 + \cdots + B_{m-1}^{m-1}\mathcal{P}_{m-1}) \\
&\qquad\qquad + \gamma(B_0^m \mathcal{P}_0 + \cdots + B_m^m \mathcal{P}_m) = 0\,, \\
&(\alpha B_0^0 + \cdots + \beta B_0^{m-1} + \gamma B_0^m)\mathcal{P}_0 + \cdots \\
&\qquad\qquad + (\beta B_{m-1}^{m-1} + \gamma B_{m-1}^m)\mathcal{P}_{m-1} + \gamma B_m^m \mathcal{P}_m = 0\,.
\end{aligned}$$

As $\{1, x, \ldots, x^{(m+1)sd-1}\}$ is a linearly independent set of functions, we have

$$\left\{\begin{array}{l}
\alpha B_0^0 + \cdots + \beta B_0^{m-1} + \gamma B_0^m = 0 \\
\quad\vdots \\
\beta B_{m-1}^{m-1} + \gamma B_{m-1}^m = 0 \\
\gamma B_m^m = 0.
\end{array}\right. .$$

Using the regularity of the matrices $B_0^0, \ldots, B_m^m$ we obtain that $\gamma = 0_{1\times sd}$, $\beta = 0_{1\times sd}, \ldots, \alpha = 0_{1\times sd}$ and so the set of polynomials $\{B_{k,j}, j = 1, \ldots, sd, k = 0, 1, \ldots, m\}$ is linearly independent. ■

Definition 7. Let $\{\mathcal{B}_m\}$ be a vector sequence of polynomials where $\mathcal{B}_m = [B_{m,1} \; \cdots \; B_{m,sd}]^T$, $m \in \mathbb{N}$, such that $\mathcal{B}_m = \sum_{j=0}^{m} B_j^m \mathcal{P}_j$ where $B_j^m \in \mathcal{M}_{sd\times sd}$. We say that $\{\mathcal{B}_m\}$ *is a free vector sequence* if B_m^m is a regular matrix for $m \in \mathbb{N}$.

Lemma 3. *Let $\{\mathcal{B}_m\}$ be a vector type II multiple orthogonal polynomials sequence, with respect to the vector of linear functionals $\mathcal{U}$. Let us consider*

$\mathcal{Q}_m = \mathcal{C}_m \mathcal{B}_m$, $m \in \mathbb{N}$ where $\mathcal{C}_m$ are $sd \times sd$ regular matrices. Then $\{\mathcal{Q}_m\}$ is also type II multiple orthogonal polynomial sequence, with respect to the vector of linear functionals $\mathcal{U}$.

Proof: Let $\{\mathcal{B}_m\}$ be a vector type II multiple orthogonal polynomials sequence, with respect to the vector of linear functionals $\mathcal{U}$, i.e.,
$$((x^s)^k \mathcal{U})(\mathcal{B}_m) = \Delta_m \delta_{k,m}, \;\; k = 0, 1, \ldots, m, \;\; m \in \mathbb{N} \,,$$
where Δ_m is a regular $sd \times sd$ matrix. From
$$((x^s)^k \mathcal{U})(\mathcal{B}_m) = ((x^s)^k \mathcal{U})((\mathcal{C}_m)^{-1} \mathcal{C}_m \mathcal{B}_m) = (\mathcal{C}_m)^{-1}((x^s)^k \mathcal{U})(\mathcal{Q}_m) \,,$$
we have
$$(\mathcal{C}_m)^{-1}((x^s)^k \mathcal{U})(\mathcal{Q}_m) = \Delta_m \delta_{k,m}, \;\; k = 0, 1, \ldots, m, \;\; m \in \mathbb{N} \,,$$
hence
$$((x^s)^k \mathcal{U})(\mathcal{Q}_m) = \mathcal{C}_m \Delta_m \delta_{k,m}, \;\; k = 0, 1, \ldots, m, \;\; m \in \mathbb{N} \,,$$
where $\mathcal{C}_m \Delta_m$ is a regular $sd \times sd$ matrix. Hence, the vector sequence of polynomials, $\{\mathcal{Q}_m\}$, is type II multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$. ∎

Example 3. Let $\{\mathcal{B}_m\}$ be a vector type II multiple orthogonal polynomials sequence, with respect to the vector of linear functionals $\mathcal{U}$ and $\{\widehat{\mathcal{B}}_m\}$ a vector sequence of polynomials with $\widehat{\mathcal{B}}_m = (B_0^0)^{-1} \mathcal{B}_m$, $m \in \mathbb{N}$, where the matrix B_0^0 is such that $\mathcal{B}_0 = B_0^0 \, \mathcal{P}_0$. The vector sequence of polynomials $\{\widehat{\mathcal{B}}_m\}$ is also type II multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$. In fact, being $\{\mathcal{B}_m\}$ a vector sequence type II multiple orthogonal polynomials, with respect to the vector of linear functionals $\mathcal{U}$, we have
$$((x^s)^k \mathcal{U})(\mathcal{B}_m) = \Delta_m \delta_{k,m}, \;\; k = 0, 1, \ldots, m, \;\; m \in \mathbb{N} \,,$$
where Δ_m is a regular $sd \times sd$ matrix , i.e.,
$$((x^s)^k \mathcal{U})(\widehat{\mathcal{B}}_m) = (B_0^0)^{-1} \Delta_m \delta_{k,m}, \;\; k = 0, 1, \ldots, m, \;\; m \in \mathbb{N} \,,$$
where $(B_0^0)^{-1} \Delta_m$ is a regular $sd \times sd$ matrix. Hence, the vector sequence of polynomials $\{\widehat{\mathcal{B}}_m\}$ is type II multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$.

Example 4. Let $\{\mathcal{B}_m\}$ be a vector sequence type II multiple orthogonal polynomials, with respect to the vector of linear functionals $\mathcal{U}$ and $\{\breve{\mathcal{B}}_m\}$ a vector sequence of polynomials with $\breve{\mathcal{B}}_m = \Delta_m^{-1} \mathcal{B}_m$, $m \in \mathbb{N}$. The vector sequence of polynomials $\{\breve{\mathcal{B}}_m\}$ is also type II multiple orthogonal, with respect to the vector of linear functionals $\mathcal{U}$. In fact, being $\{\mathcal{B}_m\}$ a vector sequence type II multiple orthogonal polynomials, with respect to the vector of linear functionals $\mathcal{U}$, we have

$$((x^s)^k\mathcal{U})(\mathcal{B}_m) = \Delta_m \delta_{k,m}, \ \ k = 0, 1, \ldots, m, \ \ m \in \mathbb{N}\,,$$

where Δ_m is a regular $sd \times sd$ matrix, i.e.,

$$((x^s)^k\mathcal{U})(\breve{\mathcal{B}}_m) = I_{sd\times sd}\, \delta_{k,m}, \ \ k = 0, 1, \ldots, m, \ \ m \in \mathbb{N}\,,$$

and so the vector sequence of polynomials, $\{\breve{\mathcal{B}}_m\}$, is type II multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$.

Now we introduce the notions of *moments* and *Hankel matrices* by blocks associated to the vector of linear functionals $\mathcal{U}$.

Definition 8. We define the *the moments of order* $j \in \mathbb{N}$ associated to the vector of linear functionals $(x^s)^k\mathcal{U}$, by

$$\mathcal{U}_j^k := ((x^s)^k\mathcal{U})(\mathcal{P}_j) = \begin{bmatrix} v^1(x^{jsd+ks}) & \cdots & v^{sd}(x^{jsd+ks}) \\ \vdots & \ddots & \vdots \\ v^1(x^{(j+1)sd+ks-1}) & \cdots & v^{sd}(x^{(j+1)sd+ks-1}) \end{bmatrix}. \tag{11}$$

Definition 9. We define *Hankel matrices* by

$$\mathcal{H}_m = \begin{bmatrix} \mathcal{U}_0^0 & \cdots & \mathcal{U}_0^m \\ \vdots & \ddots & \vdots \\ \mathcal{U}_m^0 & \cdots & \mathcal{U}_m^m \end{bmatrix}, \ \ m \in \mathbb{N}, \tag{12}$$

where $\mathcal{U}_j^k$ are the moments of order j associated to the vector of linear functionals $(x^s)^k\mathcal{U}$ given by (11).

Definition 10. The vector of linear functionals $\mathcal{U}$ is said to be *regular* if $\det \mathcal{H}_m \neq 0$, $m \in \mathbb{N}$, where $\mathcal{H}_m$ is given by (12).

Theorem 5. *Let $\mathcal{U}$ be a vector of linear functionals. Then $\mathcal{U}$ is regular if, and only if, given a sequence of regular $sd \times sd$ matrices, (Δ_m), there is a unique free vector sequence $\{\mathcal{B}_m\}$ where $\mathcal{B}_m = [\ B_{m,1} \ \cdots \ B_{m,sd}\ ]^T$, $m \in \mathbb{N}$, such that*

i) $((x^s)^k\mathcal{U})(\mathcal{B}_m) = 0_{sd\times sd}, \ \ k = 0, 1, \ldots, m-1$
ii) $((x^s)^m\mathcal{U})(\mathcal{B}_m) = \Delta_m,$

i.e, $\{\mathcal{B}_m\}$ is type II multiple orthogonal polynomial sequence, with respect to the vector of linear functionals $\mathcal{U}$.

Proof: Let $\{\mathcal{B}_m\}$, $\mathcal{B}_m = [\ B_{m,1} \ \cdots \ B_{m,sd}\ ]^T$, $m \in \mathbb{N}$, be a vector sequence of polynomials, such that $\mathcal{B}_m = \sum_{j=0}^{m} B_j^m \mathcal{P}_j$ where $B_j^m \in \mathcal{M}_{sd\times sd}$. By the multi-orthogonality conditions (10) the vector sequence of polynomials $\{\mathcal{B}_m\}$

is type II multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$ if for $k = 0, \ldots, m-1$

$$((x^s)^k \mathcal{U})(\mathcal{B}_m) = ((x^s)^k \mathcal{U})(\sum_{j=0}^{m} B_j^m \mathcal{P}_j) = \sum_{j=0}^{m} B_j^m ((x^s)^k \mathcal{U})(\mathcal{P}_j) = 0_{sd\times sd}\,,$$

and for all $m \in \mathbb{N}$,

$$((x^s)^m \mathcal{U})(\mathcal{B}_m) = ((x^s)^m \mathcal{U})(\sum_{j=0}^{m} B_j^m \mathcal{P}_j) = \sum_{j=0}^{m} B_j^m ((x^s)^m \mathcal{U})(\mathcal{P}_j) = \Delta_m. \quad (13)$$

In matrix form we have,

$$\begin{bmatrix} B_0^m & \cdots & B_m^m \end{bmatrix} \begin{bmatrix} \mathcal{U}_0^0 & \cdots & \mathcal{U}_0^m \\ \vdots & \ddots & \vdots \\ \mathcal{U}_m^0 & \cdots & \mathcal{U}_m^m \end{bmatrix} = \begin{bmatrix} 0_{sd\times sd} & \cdots & 0_{sd\times sd} & \Delta_m \end{bmatrix}.$$

Supposing the regularity of the vector of linear functionals $\mathcal{U}$, we have

$$\begin{bmatrix} B_0^m & \cdots & B_m^m \end{bmatrix} = \begin{bmatrix} 0_{sd\times sd} & \cdots & 0_{sd\times sd} & \Delta_m \end{bmatrix} \begin{bmatrix} \mathcal{U}_0^0 & \cdots & \mathcal{U}_0^m \\ \vdots & \ddots & \vdots \\ \mathcal{U}_m^0 & \cdots & \mathcal{U}_m^m \end{bmatrix}^{-1}.$$

Therefore,

$$\mathcal{B}_m = \begin{bmatrix} 0_{sd\times sd} & \cdots & 0_{sd\times sd} & \Delta_m \end{bmatrix} \begin{bmatrix} \mathcal{U}_0^0 & \cdots & \mathcal{U}_0^m \\ \vdots & \ddots & \vdots \\ \mathcal{U}_m^0 & \cdots & \mathcal{U}_m^m \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{P}_0 \\ \vdots \\ \mathcal{P}_m \end{bmatrix}.$$

Taking $m = 0$ in (13), we have $B_0^0 \mathcal{U}_0^0 = \Delta_0$.
Using the regularity of the matrices $\mathcal{U}_0^0$ and Δ_0 we have that B_0^0 is a regular matrix. Similarly, taking $m = 1$ in (13), we have

$$\begin{cases} B_0^1 \mathcal{U}_0^0 + B_1^1 \mathcal{U}_1^0 = 0_{sd\times sd} \\ B_0^1 \mathcal{U}_0^1 + B_1^1 \mathcal{U}_1^1 = \Delta_1, \end{cases}, \text{ i.e., } B_1^1(\mathcal{U}_1^1 - \mathcal{U}_1^0 (\mathcal{U}_0^0)^{-1} \mathcal{U}_0^1) = \Delta_1\,.$$

Using the regularity of the $\mathcal{U}$ and by the triangular structure by blocks, we have $\det(\mathcal{U}_1^1 - \mathcal{U}_1^0(\mathcal{U}_0^0)^{-1}\mathcal{U}_0^1) \neq 0$, and so B_1^1 is a regular matrix.

Using the same argument we can conclude that B_m^m is a regular matrix and so $\{\mathcal{B}_m\}$ is a free vector sequence.

Reciprocally and in a similar way if B_m^m, $m \in \mathbb{N}$, is regular we obtain a regularity of the $\mathcal{U}$. ∎

In section 2 we have proved that a sequence of monic type II multiple orthogonal polynomials, $\{B_n\}$, with respect to the regular system of linear functionals $\{u^1, \ldots, u^d\}$ and quasi-diagonal multi-index $\mathcal{J}$ verify a $(s(d+1)+1)$-term recurrence relation and we rewrote this recurrence relation in matrix form, obtaining a three-term recurrence relation for vector polynomials with matrix coefficients. Now we prove the converse of this result which is called the *Favard type theorem.*

Theorem 6. *Let $\{B_n\}$ be a sequence of monic type II multiple orthogonal polynomials, with respect to a regular system of linear functionals $\{u^1,\dots,u^d\}$ and quasi-diagonal multi-index $\mathcal{J}$ and let $\mathcal{U}=[v^1\,\dots\,v^{sd}]^T$ be the vector of linear functionals where v^j, $j=1,\dots,sd$ are defined by the algorithm. Then, the following conditions are equivalent:*
a) The vector sequence of polynomials $\{\mathcal{B}_m\}$ is type II multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$, i.e.,

$$((x^s)^k\mathcal{U})(\mathcal{B}_m)=\Delta_m\,\delta_{k,m},\ \ k=0,1,\dots,m,\ \ m\in\mathbb{N}, \tag{14}$$

where Δ_m is a regular upper triangular $sd\times sd$ matrix given by

$$\Delta_m=\gamma_m^{s,d}\cdots\gamma_1^{s,d}\Delta_0,\ \ m=1,2,\dots,$$

and Δ_0 is an upper triangular $sd\times sd$ matrix.
b) There exist sequences of $sd\times sd$ matrices $(\alpha_m^{s,d})$, $(\beta_m^{s,d})$ and $(\gamma_m^{s,d})$, $m\in\mathbb{N}$, with $\gamma_m^{s,d}$ regular upper triangular matrix such that $\mathcal{B}_m$ is defined by the three-term recurrence relation with $sd\times sd$ matrix coefficients given by

$$x^s\mathcal{B}_m(x)=\alpha_m^{s,d}\,\mathcal{B}_{m+1}(x)+\beta_m^{s,d}\,\mathcal{B}_m(x)+\gamma_m^{s,d}\,\mathcal{B}_{m-1}(x),\ \ m=0,1,\dots \tag{15}$$

with $\mathcal{B}_{-1}=0_{d\times 1}$ and $\mathcal{B}_0$ given.

Proof: $a)\Rightarrow b)$. It proven in the Theorem 3.
$b)\Rightarrow a)$. We build a vector of linear functionals $\mathcal{U}$ that verifies (14) defined uniquely taking into account its moments $\mathcal{U}_m^k$ from the conditions:

$$\mathcal{U}(\mathcal{B}_0)=\Delta_0\,,\ \ \mathcal{U}(\mathcal{B}_j)=0_{sd\times sd},\ \ j=1,2,\dots. \tag{16}$$

As $\{\mathcal{P}_m\}$ is a basis of $\mathbb{P}^{sd}$, for each $m\in\mathbb{N}$, there is an unique sequence $(B_j^m)\subset\mathcal{M}_{sd\times sd}$, such that, $\mathcal{B}_m=\displaystyle\sum_{j=0}^{m}B_j^m\mathcal{P}_j$.

- Let $k=0$. We have
  $\mathcal{U}(\mathcal{B}_0)=B_0^0\mathcal{U}(\mathcal{P}_0)$ and so $\mathcal{U}_0^0=(B_0^0)^{-1}\mathcal{U}(\mathcal{B}_0)\,,$
  $\mathcal{U}(\mathcal{B}_m)=\sum_{j=0}^m B_j^m\mathcal{U}(\mathcal{P}_j)$, i.e., $\mathcal{U}_m^0=-\sum_{j=0}^{m-1}(B_m^m)^{-1}B_j^m\mathcal{U}_j^0\,,\ m=1,2,\dots.$
- Let $k=1,2,\dots$. Using (15) we have

$$(x^s)^k\mathcal{B}_m=\alpha_m^{s,d}x^{s(k-1)}\mathcal{B}_{m+1}+\beta_m^{s,d}x^{s(k-1)}\mathcal{B}_m+\gamma_m^{s,d}x^{s(k-1)}\mathcal{B}_{m-1}\,.$$

For $m=0$ we have

$$\mathcal{U}((x^s)^k\mathcal{B}_0)=\alpha_0^{s,d}\mathcal{U}(x^{s(k-1)}\mathcal{B}_1)+\beta_0^{s,d}\mathcal{U}(x^{s(k-1)}\mathcal{B}_0),$$

i.e.,

$$\mathcal{U}_0^k = (B_0^0)^{-1} \times \left[\alpha_0^{s,d} B_1^1 \mathcal{U}_1^{s(k-1)} + (\alpha_0^{s,d} B_0^1 + \beta_0^{s,d} B_0^0)\right] \mathcal{U}_0^{s(k-1)} .$$

For $m = 1$ we have

$$\mathcal{U}((x^s)^k \mathcal{B}_1) = \alpha_1^{s,d} \mathcal{U}(x^{s(k-1)} \mathcal{B}_2) + \beta_1^{s,d} \mathcal{U}(x^{s(k-1)} \mathcal{B}_1) + \gamma_1^{s,d} \mathcal{U}(x^{s(k-1)} \mathcal{B}_0) ,$$

i.e.,

$$\mathcal{U}_1^k = (B_1^1)^{-1} \left[\alpha_1^{s,d} B_2^2 \mathcal{U}_2^{s(k-1)} + (\alpha_1^{s,d} B_1^2 + \beta_1^{s,d} B_1^1) \mathcal{U}_1^{s(k-1)}\right] + (B_1^1)^{-1} \left[(\alpha_1^{s,d} B_0^2 + \beta_1^{s,d} B_0^1 + \gamma_1^{s,d} B_0^0) \mathcal{U}_0^{s(k-1)} - B_0^1 \mathcal{U}_0^k\right] .$$

For $m \leq k$, we have

$$\mathcal{U}((x^s)^k \mathcal{B}_m) = \alpha_m^{s,d} \mathcal{U}(x^{s(k-1)} \mathcal{B}_{m+1}) + \beta_m^{s,d} \mathcal{U}(x^{s(k-1)} \mathcal{B}_m) + \gamma_m^{s,d} \mathcal{U}(x^{s(k-1)} \mathcal{B}_{m-1}),$$

$$\mathcal{U}((x^s)^k \mathcal{B}_m) = \alpha_m^{s,d} \sum_{j=0}^{m+1} B_j^{m+1} \mathcal{U}_j^{k-1} + \beta_m^{s,d} \sum_{j=0}^{m} B_j^m \mathcal{U}_j^{k-1} + \gamma_m^{s,d} \sum_{j=0}^{m-1} B_j^{m-1} \mathcal{U}_j^{k-1},$$

$$\mathcal{U}((x^s)^k \mathcal{B}_m) = \sum_{j=0}^{m-1} (\alpha_m^{s,d} B_j^{m+1} + \beta_m^{s,d} B_j^m + \gamma_m^{s,d} B_j^{m-1}) \mathcal{U}_j^{k-1} + (\alpha_m^{s,d} B_m^{m+1} + \beta_m^{s,d} B_m^m) \mathcal{U}_m^{k-1} + \alpha_m^{s,d} B_{m+1}^{m+1} \mathcal{U}_{m+1}^{k-1}.$$

Taking into account that,

$$\mathcal{U}((x^s)^k \mathcal{B}_m) = \mathcal{U}((x^s)^k \sum_{j=0}^{m} B_j^m \mathcal{P}_j) = B_m^m \mathcal{U}_m^k + \sum_{j=0}^{m-1} B_j^m \mathcal{U}_j^k,$$

we have

$$\mathcal{U}_m^k = (B_m^m)^{-1} \sum_{j=0}^{m-1} (\alpha_m^{s,d} B_j^{m+1} + \beta_m^{s,d} B_j^m + \gamma_m^{s,d} B_j^{m-1}) \mathcal{U}_j^{k-1} + (B_m^m)^{-1} ((\alpha_m^{s,d} B_m^{m+1} + \beta_m^{s,d} B_m^m) \mathcal{U}_m^{k-1} + \alpha_m^{s,d} B_{m+1}^{m+1} \mathcal{U}_{m+1}^{k-1} - \sum_{j=0}^{m-1} B_j^m \mathcal{U}_j^k).$$

For $m = k$ we have

$$\mathcal{U}((x^s)^k \mathcal{B}_k) = \gamma_k^{s,d} \gamma_{k-1}^{s,d} \cdots \gamma_1^{s,d} B_0^0 \mathcal{U}_0^0 ,$$

and so,

$$\mathcal{U}_k^k = (B_k^k)^{-1} (\gamma_k^{s,d} \gamma_{k-1}^{s,d} \cdots \gamma_1^{s,d} B_0^0 \mathcal{U}_0^0 - \sum_{j=0}^{k-1} B_j^k \mathcal{U}_j^k) .$$

For $m > k$ we have $\mathcal{U}((x^s)^k \mathcal{B}_m) = 0_{sd \times sd}$, i.e.,

$$\mathcal{U}_m^k = \sum_{j=0}^{m-1} -(B_m^m)^{-1} B_j^m \mathcal{U}_j^k .$$

Therefore, the moments associated to the vector of linear functionals $\mathcal{U}$ are uniquely determined from (16) and considering the fact that B_m^m is regular we obtain the regularity of the vector of linear functionals $\mathcal{U}$. Hence, this result is proved. ■

Note that, in matrix notation the three-term recurrence relation of the previous Theorem, (15), is written by

$$J \begin{bmatrix} \mathcal{B}_0 \\ \vdots \\ \mathcal{B}_m \\ \vdots \end{bmatrix} = x^s \begin{bmatrix} \mathcal{B}_0 \\ \vdots \\ \mathcal{B}_m \\ \vdots \end{bmatrix}, \tag{17}$$

where the tridiagonal matrix by blocks

$$J = \begin{bmatrix} \beta_0^{s,d} & \alpha_0^{s,d} & 0_{sd\times sd} & & & \\ \gamma_1^{s,d} & \beta_1^{s,d} & \alpha_1^{s,d} & 0_{sd\times sd} & & \\ 0_{sd\times sd} & \gamma_2^{s,d} & \beta_2^{s,d} & \alpha_2^{s,d} & 0_{sd\times sd} & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{18}$$

is designated by *block Jacobi matrix*.

4. Type II Hermite-Padé approximation

Definition 11. Let $\mathcal{U}$ be a vector of linear functionals. We define the *matrix generating function associated* to $\mathcal{U}$, $\mathcal{F}$, by

$$\mathcal{F}(z) := \mathcal{U}_x\left(\frac{\mathcal{P}_0(x)}{z-x^s}\right) = \begin{bmatrix} v_x^1\left(\frac{1}{z-x^s}\right) & \cdots & v_x^{sd}\left(\frac{1}{z-x^s}\right) \\ \vdots & \ddots & \vdots \\ v_x^1\left(\frac{x^{sd-1}}{z-x^s}\right) & \cdots & v_x^{sd}\left(\frac{x^{sd-1}}{z-x^s}\right) \end{bmatrix}. \tag{19}$$

Being,

$$\frac{1}{z-x^s} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{x^s}{z}\right)^k \quad \text{for} \quad |x^s| < |z|, \tag{20}$$

we have $\mathcal{F}(z) = \displaystyle\sum_{k=0}^{\infty} \frac{((x^s)^k \mathcal{U}_x)(\mathcal{P}_0(x))}{z^{k+1}}$.

Theorem 7. *Let $\mathcal{U}$ be a regular vector of linear functionals, $\{\mathcal{B}_m\}$ a vector type II multiple orthogonal polynomials sequence, with respect to $\mathcal{U}$, and $\mathcal{R}$ the resolvent function associated to the linear operator defined by the block*

Jacobi matrix, J, given in (18), i.e.,

$$\mathcal{R}(z)=\sum_{n=0}^{\infty}\frac{e_0^tJ^ne_0}{z^{n+1}}\,,\ \textit{where } e_0=[I_{sd\times sd}\,0_{sd\times sd}\ \cdots]^T\,.$$

Then, $\mathcal{R}(z)=B_0^0\,\mathcal{F}(z)(\mathcal{U}(\mathcal{P}_0))^{-1}(B_0^0)^{-1}$, where B_0^0 is the matrix coefficient in $\mathcal{B}_0=B_0^0\,\mathcal{P}_0$.

Proof: In order to determine the value of $e_0^tJ^ne_0$, $n\in\mathbb{N}$, we consider the matrix identity (17), from which we can obtain,

$$J^n\begin{bmatrix}\mathcal{B}_0(x)\\ \vdots\\ \mathcal{B}_m(x)\\ \vdots\end{bmatrix}=(x^s)^n\begin{bmatrix}\mathcal{B}_0(x)\\ \vdots\\ \mathcal{B}_m(x)\\ \vdots\end{bmatrix},\quad n\in\mathbb{N}. \tag{21}$$

Let $(x^s)^n\mathcal{B}_m(x)=\displaystyle\sum_{j=m-n}^{m+n}\eta_{j,n}^m\mathcal{B}_j(x)$, $\eta_{j,n}^m\in\mathcal{M}_{sd\times sd}$. In particular, for $m=0$ we have, $(x^s)^n\mathcal{B}_0(x)=\displaystyle\sum_{j=0}^{n}\eta_{j,n}^0\mathcal{B}_j(x)\,.$

By (21), $e_0^tJ^ne_0$, $n\in\mathbb{N}$, it is given by $\eta_{0,n}^0$. Applying the vector of linear functionals $\mathcal{U}$ to both members of the previous matrix indentity, we have

$$\eta_{0,n}^0=((x^s)^n\mathcal{U})(\mathcal{B}_0)(\mathcal{U}(\mathcal{B}_0))^{-1}\,.$$

Using $\mathcal{B}_0=B_0^0\,\mathcal{P}_0$, we have $\eta_{0,n}^0=B_0^0((x^s)^n\mathcal{U})(\mathcal{P}_0)(\mathcal{U}(\mathcal{P}_0))^{-1}(B_0^0)^{-1}$. Hence,

$$\mathcal{R}(z)=B_0^0\left\{\sum_{n=0}^{\infty}\frac{((x^s)^n\mathcal{U})(\mathcal{P}_0)(\mathcal{U}(\mathcal{P}_0))^{-1}}{z^{n+1}}\right\}(B_0^0)^{-1}\,,$$

as we want to show. ■

Now, we present a reinterpretation of type II Hermite-Padé approximation in terms of the matrix functions.

Definition 12. Let $\{\mathcal{B}_m\}$ be a vector sequence of polynomials and $\mathcal{U}$ a regular vector of linear functionals. To the sequence of polynomials $\{\mathcal{B}_{m-1}^{(1)}\}$ given by

$$\mathcal{B}_{m-1}^{(1)}(z):=\mathcal{U}_x\Big(\frac{V_m(z^d)-V_m(x^{sd})}{z-x^s}\,\mathcal{P}_0(x)\Big)\,,$$

where $\mathcal{U}_x$ represents the action of $\mathcal{U}$ over the variable x, we designate *sequence of polynomials associated to* $\{\mathcal{B}_m\}$ and to $\mathcal{U}$.

Theorem 8. *Let $\mathcal{U}$ be a regular vector of linear functionals, $\{\mathcal{B}_m\}$ a vector sequence of polynomials, $\{\mathcal{B}^{(1)}_{m-1}\}$ the sequence of associated polynomials and $\mathcal{F}$ the matrix generating function defined in (19). Then, $\{\mathcal{B}_m\}$ is the type II multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$ if, and only if,*

$$V_m(z^d)\mathcal{F}(z) - \mathcal{B}^{(1)}_{m-1}(z) = \sum_{k=m}^{\infty} \frac{((x^s)^k \mathcal{U}_x)(\mathcal{B}_m(x))}{z^{k+1}} \, .$$

Proof: Taking into account the Definition 12, we have

$$\mathcal{B}^{(1)}_{m-1}(z) = \mathcal{U}_x(\frac{V_m(z^d) - V_m(x^{sd})}{z - x^s}\, \mathcal{P}_0(x)) = V_m(z^d)\mathcal{F}(z) - \mathcal{U}_x(\frac{V_m(x^{sd})}{z - x^s}\, \mathcal{P}_0(x)) \, ,$$

i.e., $V_m(z^d)\mathcal{F}(z) - \mathcal{B}^{(1)}_{m-1}(z) = \mathcal{U}_x(\frac{V_m(x^{sd})}{z - x^s}\, \mathcal{P}_0(x)) \, .$

Taking into account (20) we have

$$V_m(z^d)\mathcal{F}(z) - \mathcal{B}^{(1)}_{m-1}(z) = \sum_{k=0}^{\infty} \frac{((x^s)^k \mathcal{U}_x)(\mathcal{B}_m(x))}{z^{k+1}} \, .$$

Hence, we get the desired result. ■

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