Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 09–08

ALGEBRAIC THEORY OF MULTIPLE ORTHOGONAL POLYNOMIALS

A. BRANQUINHO, L. COTRIM AND A. FOULQUIÉ MORENO

Abstract: In this work we present an algebraic theory of multiple orthogonal polynomials. Our departure point is the three term recurrence relation, with matrix coefficients, satisfied by a sequence of vector multiple orthogonal polynomials. We give some characterizations of multiple orthogonal polynomials including recurrence relations, a Favard type theorem and a Christoffel-Darboux type formulas. An reinterpretation of the problems of Hermite-Padé approximation is presented.

Keywords: Multiple orthogonal polynomials, Hermite-Padé approximants, block tridiagonal operator, Favard type theorem.
AMS Subject Classification (2000): Primary 33C45; Secondary 39B42.

1. Introduction

As it is known, there has recently reemerged an interest in the extension of the notion of orthogonality known as multiple orthogonality. Multiple orthogonal polynomials are a generalization of orthogonal polynomials in the sense that they satisfy orthogonality conditions with respect to a number of measures. For a detailed study of multiple orthogonal polynomials we refer for example Aptekarev [1], Nikishin and Sorokin [18, chapter 4] and W.V. Assche [21]. Such polynomials arise in a natural way in the study of simultaneous rational approximation, in particular in the study of Hermite-Padé approximation of a system of $d \in \mathbb{Z}^+$ Markov functions (see [18, 19]) that goes back to the nineteenth century. Since then, some aspects related with the theory of multiple orthogonal polynomials have been investigated such as, existence and uniqueness, recurrence relations, normality of indices, etc. In this way, multiple orthogonal polynomials are intimately related to Hermite-Padé approximation. In the literature we can find a lot of examples of multiple orthogonal polynomials (see [1, 2, 3, 6, 13, 15, 22]). There are two types of multiple orthogonal polynomials, type I and II. K. Douak and P. Maroni [9], P. Maroni [17], V. Kaliaguine [14] and J. Van Iseghem [23]

Received February 5, 2009.
The work of the first author was supported by CMUC/FCT. The work of the second author was partially supported by ”Acção do Prodep” reference 5.3/C/00187.010/03. The third author would like to thank UI Matemática e Aplicações from University of Aveiro.

have given some characterizations of type II multiple orthogonal polynomials with respect to the system of linear functionals and diagonal multi-index. Our purpose in this work is to present an algebraic theory of type I and II multiple orthogonal polynomials for more general families of multi-indices, called quasi-diagonal, that we introduce in [5].

Let $\vec{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ which is called a *multi-index* with length $|\vec{n}| := n_1 + \cdots + n_d$ and let $\left\{u^1, \ldots, u^d\right\}$ be a system of linear functionals $u^j : \mathbb{P} \to \mathbb{C}$ with $j = 1, 2, \ldots, d$. Let $(A_{\vec{n},1}, \ldots, A_{\vec{n},d})$ be a vector of polynomials where $\deg A_{\vec{n},j} \leq n_j - 1$, for $j = 1, \ldots, d$. We say that the vector of polynomials $(A_{\vec{n},1}, \ldots, A_{\vec{n},d})$ is type I multiple orthogonal, with respect to the system of linear functionals $\left\{u^1, \ldots, u^d\right\}$ and *multi-index* $\vec{n}$, if

$$\sum_{j=1}^{d} u^j(x^m A_{\vec{n},j}(x)) = 0, \quad m = 0, 1, \ldots, |\vec{n}| - 2. \tag{1}$$

In the particular case that the system of linear functionals proceeds from positive Borel measures, μ_j, $j = 1, \ldots, d$, we have $u^j(x^k) = \int_I x^k d\mu_j$, $k \in \mathbb{N}$, $j = 1, \ldots, d$, and the conditions of multi-orthogonality (1) can be rewritten as

$$\sum_{j=1}^{d} \int_I x^k A_{\vec{n},j}(x) d\mu_j(x) = 0, \quad k = 0, 1, \ldots, |\vec{n}| - 2\,.$$

Let $\{P_{\vec{n}}\}$ be a sequence of polynomials where $\deg P_{\vec{n}} \leq |\vec{n}|$. We say that $\{P_{\vec{n}}\}$ *is type II multiple orthogonal with respect to the system of linear functionals* $\left\{u^1, \ldots, u^d\right\}$ *and multi-index* $\vec{n}$, if

$$u^j\left(x^m P_{\vec{n}}\right) = 0, \quad m = 0, 1, \ldots, n_j - 1, \quad j = 1, \ldots, d. \tag{2}$$

Similarly, if the system of linear functionals is a system of positive Borel measures, μ_j, $j = 1, \ldots, d$, the multi-orthogonality conditions, (2), can be rewritten as $\int_I P_{\vec{n}}(x) x^k d\mu_j(x) = 0$, $k = 0, 1, \ldots, n_j - 1$, $j = 1, \ldots, d$.

A *multi-index* $\vec{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ is said to be *normal* for the system of linear functionals $\left\{u^1, \ldots, u^d\right\}$, if for any non trivial solution $(A_{\vec{n},1}, \ldots, A_{\vec{n},d})$ of (1) (respectively, non trivial solution $P_{\vec{n}}$ of (2)), $\deg A_{\vec{n},j} = n_j - 1$ (respectively, $\deg P_{\vec{n}} = |\vec{n}|$). When all the multi-indices of a given family are normal, we say that *the system of linear functionals* $\left\{u^1, \ldots, u^d\right\}$ *is regular* for this given family.

This work is organized as follows. In section 2 we show which families of quasi-diagonal multi-indices, $\mathcal{J}$, will be considered in this work, and we

$n = \|\vec{n}\|$	$\vec{n} = (n_1, \ldots, n_d)$
0	$(0, \ldots, 0)$
1	$(1, 0, \ldots, 0)$
$\vdots$	$\vdots$
i	$(k_i^1, \ldots, k_i^d)$
$\vdots$	$\vdots$
$sd-1$	$(s, \ldots, s, s-1)$

Table 1. Pattern blocks

present a characterization for type I and II multiple orthogonal polynomials with respect to the regular system of linear functionals $\{u^1, \ldots, u^d\}$ and family of quasi-diagonal multi-index $\mathcal{J}$. In section 3 we present an algebraic theory and some definitions which enables us to operate with the new objects presented in this work, and give a necessary and sufficient condition for the type I and II regularity for a vector of linear functionals and quasi-diagonal multi-index $\mathcal{J}$. In section 4 we present a matrix interpretation of the type I and II multi-orthogonality conditions with respect to the regular system of linear functionals $\{u^1, \ldots, u^d\}$ and family of quasi-diagonal multi-indices, $\mathcal{J}$. In [10, 11, 12, 14] we can find a characterization of orthogonal polynomials in terms of matrix three term recurrence relations. In this section we prove a Christoffel-Darboux formula verified by the sequences of type I and II multiple orthogonal polynomials and give a reproducing kernel property. Some Christoffel-Darboux formulas for multiple orthogonal polynomials can be found in [7, 8, 20]. In section 5 we present a reinterpretation of type I and II Hermite-Padé approximation in terms of matrix functions, and an important result of bi-orthogonality with respect to the matrix generating functions of the moments associated with the system of linear functionals (see for instance [4, 16, 19]).

2. Quasi-diagonal multi-indices

We begin introducing the multi-indices being considered in this work. For that, we consider blocks with sd elements of $\mathbb{Z}_+^d$ in the Table 1. The multi-indices $\left(k_i^1, \ldots, k_i^d\right)$ where $i = 0, 1, \ldots, sd-1$ are defined by the following conditions:

- $k_{i+1}^j \geq k_i^j, \;\; i = 0, 1, \ldots, sd-2, \;\; j = 1, \ldots, d;$
- $k_i^{j+1} \leq k_i^j, \;\; i = 0, 1, \ldots, sd-1, \;\; j = 1, \ldots, d-1;$

- $\displaystyle\sum_{j=1}^{d} k_i^j = i, \;\; i = 0, 1, \ldots, sd-1, \;\; j = 1, \ldots, d;$
- $k_{sd-1}^j = \begin{cases} s, & j = 1, 2, \ldots, d-1 \\ s-1, & j = d. \end{cases}$

Now, we identify as $\mathcal{J}_0$, the set whose elements are the ones of any of the blocks presented in the Table 1, i.e, $\mathcal{J}_0 = \{(0, \ldots, 0), (1, 0, \ldots, 0), \ldots, (s, \ldots, s, s-1)\}$. From $\mathcal{J}_0$ we generate a sequence of sets which we denote by $\mathcal{J}_n$, $n \in \mathbb{N}$, according to

$$\mathcal{J}_n = \mathcal{J}_0 + n\{(s, \ldots, s)\}, \; n \in \mathbb{N}\,.$$

In this way we obtain a set of multi-indices, $\mathcal{J}$, given by $\mathcal{J} = \{\mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_n, \ldots\}$, $n \in \mathbb{N}$. Remark that for $s = 1$ we have that $\mathcal{J}_0$ is given by, $\mathcal{J}_0 = \{(0, \ldots, 0), (1, 0, \ldots, 0), (1, 1, \ldots, 0), \ldots, (1, \ldots, 1, 0)\}$, and the multi-indices are called *diagonal*.

In this work we restrict ourselves to the families of quasi-diagonal multi-indices, $\mathcal{J}$. We identify the vectors $\vec{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ with $n \in \mathbb{Z}_0^+$ because in our sets of quasi-diagonal multi-indices, $\mathcal{J}$, there is an one-to-one correspondence, $\mathbf{i}$, between the sets $\mathbb{Z}_+^d$ and $\mathbb{Z}_0^+$ given by, $\mathbf{i}(\vec{n}) = |\vec{n}| = n$.

Algorithm (Construction of linear functionals)**.** *Let us consider the system of linear functionals $\{u^1, \ldots, u^d\}$ and family of quasi-diagonal multi-indices given in Table 1, $\mathcal{J} = \{\mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_n, \ldots\}$, $n \in \mathbb{N}$.*

Let $v^1 = u^1$, $v^i = x^{k_{i-1}^j} u^j$, $i = 2, \ldots, sd-1$ where j is uniquely defined by the condition $k_i^j = k_{i-1}^j + 1$ and $v^{sd} = x^{s-1} u^d$. Hence, we have

$$v^i \in \{x^k u^j : k = 0, 1, \ldots, s-1, \; j = 1, 2, \ldots, d\}, \; i = 1, 2, \ldots, sd$$

Theorem 1 (type I multi-orthogonality)**.** *The vector $(A_{n,1}, \ldots, A_{n,d})$ where the polynomials $A_{n,j}$ have degree $n_j - 1$, for $j = 1, \ldots, d$ and $n \in \mathbb{N}$, is type I multiple orthogonal with respect to the regular system of linear functionals $\{u^1, \ldots, u^d\}$ and family of quasi-diagonal multi-indices $\mathcal{J}$ if, and only if,*

$$(\sum_{j=1}^{d} \sum_{k=0}^{s-1} (A_{n,j}^k(x^s) x^k u^j))(x^m) = 0, \;\; m = 0, 1, \ldots, |\vec{n}| - 2\,,$$

$$(\sum_{j=1}^{d} \sum_{k=0}^{s-1} (A_{n,j}^k(x^s) x^k u^j))(x^{|\vec{n}|-1}) \neq 0\,,$$

where

$$A_{n,j}(x) = \sum_{k=0}^{s-1} x^k A_{n,j}^k(x^s)\,, \quad \textit{for} \quad j = 1, 2, \ldots, m\,. \tag{3}$$

Proof: By (1) we have

$$\sum_{j=1}^{d} u^j(x^m A_{n,j}(x)) = 0, \;\; m = 0, 1, \ldots, |\vec{n}| - 2\,.$$

Now considering the representation for the polynomials $A_{n,j}$ we have for all $m = 0, 1, \ldots, |\vec{n}| - 2$,

$$\sum_{j=1}^{d} u^j(x^m A_{n,j}(x)) = (\sum_{j=1}^{d}\sum_{k=0}^{s-1}(A_{n,j}^k(x^s)x^k u^j))(x^m)\,.$$

Now, let us suppose that,

$$(\sum_{j=1}^{d}\sum_{k=0}^{s-1}(A_{n,j}^k(x^s)x^k u^j))(x^{|\vec{n}|-1}) = 0\,.$$

Then, the vector of polynomials $(A_{n,1}, \ldots, A_{n,d})$ verify the same multi-orthogonality conditions of the vector of polynomials $(A_{n+1,1}, \ldots, A_{n+1,d})$ and taking into account that $n_j \leq (n+1)_j$, $j = 1, \ldots, d$ (because of the increasing struture of the quasi-diagonal indices), we get a contradiction with the normality of the multi-indices.

Reciprocally, if

$$(\sum_{j=1}^{d}\sum_{k=0}^{s-1}(A_{n,j}^k(x^s)x^k u^j))(x^m) = 0, \;\; m = 0, 1, \ldots, |\vec{n}| - 2\,,$$

and as $\deg A_{n,j} = n_j - 1$, for $j = 1, \ldots, d$ and $n \in \mathbb{N}$, by the normality of the multi-indices, we conclude that the vector of polynomials $(A_{n,1}, \ldots, A_{n,d})$ is type I multiple orthogonal with respect to the system of linear functionals $\{u^1, \ldots, u^d\}$ and quasi-diagonal multi-index n. ■

Theorem 2 (type II multi-orthogonality, cf. [5])**.** *The sequence of monic polynomials* $\{B_n\}$ *where* $n = sdr + k$, $k = 0, 1, \ldots, sd - 1$ *and* $r = 0, 1, \ldots,$ *is type II multiple orthogonal with respect to the regular system of linear functionals* $\{u^1, \ldots, u^d\}$ *and quasi-diagonal multi-index* $\mathcal{J}$ *if, and only if,*

$$v^j((x^s)^m B_{sdr+i}) = 0\,, \;\; m = 0, 1, \ldots, r-1, \; j = 1, \ldots, sd$$

$$v^{\alpha}((x^s)^r B_{sdr+i}) = 0\,,\ \alpha = 1, \ldots, i\,,\quad v^{i+1}((x^s)^r B_{sdr+i}) \neq 0\,,$$

where the linear functionals v^j, $j = 1, \ldots, sd$ are defined by the algorithm.

3. Matrix interpretation of multi-orthogonality

Now, we present an algebraic theory of multiple orthogonal polynomials, with respect to a family of quasi-diagonal multi-indices. Let us consider the family of vectors of polynomials

$$\mathbb{P}^{sd} = \left\{ [P_1 \cdots P_{sd}]^T : P_j \in \mathbb{P} \right\},$$

and $\mathcal{M}_{sd\times sd}$ the set of $sd \times sd$ matrices with entries in $\mathbb{C}$. Let $\{\mathcal{P}_j\}$ be a sequence of vectors of polynomials given by

$$\mathcal{P}_j = \left[x^{jsd} \cdots x^{(j+1)sd-1} \right]^T,\ j \in \mathbb{N}\,. \tag{4}$$

Let $\{B_n\}$ be a sequence of polynomials, $\deg B_n = n$, $n \in \mathbb{N}$. We define the *associated vector polynomial sequence* $\{\mathcal{B}_n\}$ by

$$\mathcal{B}_n = \left[B_{nsd} \cdots B_{(n+1)sd-1} \right]^T,\ n \in \mathbb{N}\,.$$

It is easy to see that

$$\mathcal{B}_n = \sum_{j=0}^{n} B_j^n \, \mathcal{P}_j,\ \ B_j^n \in \mathcal{M}_{sd\times sd}\,,$$

where the matrix coefficients B_j^n, $j = 0, 1, \ldots, n$ are uniquely determined.

Taking into account (4) we have that $\mathcal{P}_j = (x^{sd})^j \mathcal{P}_0$, $j \in \mathbb{N}$. Therefore, $\mathcal{B}_n = V_n(x^{sd})\mathcal{P}_0$, where V_n is a matrix polynomial of degree n and dimension sd, given by

$$V_n(x) = \sum_{j=0}^{n} B_j^n x^j,\ \ B_j^n \in \mathcal{M}_{sd\times sd}\,.$$

Now, we present a *vector of functionals* acting on $\mathbb{P}^{sd}$ over $\mathcal{M}_{sd\times sd}$.

Definition 1. Let $v^j : \mathbb{P} \to \mathbb{C}$ with $j = 1, \ldots, sd$ be linear functionals. We define the *vector of functionals* $\mathcal{U} = \left[v^1 \cdots v^{sd} \right]^T$ acting in $\mathbb{P}^{sd}$ over $\mathcal{M}_{sd\times sd}$, by

$$\mathcal{U}(\mathcal{P}) := (\mathcal{U}.\mathcal{P}^T)^T = \begin{bmatrix} v^1(P_1) & \cdots & v^{sd}(P_1) \\ \vdots & \ddots & \vdots \\ v^1(P_{sd}) & \cdots & v^{sd}(P_{sd}) \end{bmatrix},$$

where "." means the symbolic product of the vectors $\mathcal{U}$ and $\mathcal{P}^T$.

Let $\widehat{A} = \sum_{k=0}^{l} A_k\, x^k$, where $A_k \in \mathcal{M}_{sd\times sd}$ and $\mathcal{U}$ a vector of linear functionals. We define the vector of linear functionals, *left multiplication of* $\mathcal{U}$ *by* $\widehat{A}$, and denote it by $\widehat{A}\mathcal{U}$, to the application of $\mathbb{P}^{sd}$ in $\mathcal{M}_{sd\times sd}$, defined by

$$(\widehat{A}\mathcal{U})(\mathcal{P}) := (\widehat{A}\mathcal{U}.\mathcal{P}^T)^T = \sum_{k=0}^{l}(x^k\,\mathcal{U})(\mathcal{P})\,(A_k)^T\,.$$

Now, we present the *duality theory*. We denote by $\mathbb{P}^*$ the *dual space* of $\mathbb{P}$, i.e., the vector space of linear applications defined on $\mathbb{P}$ over $\mathbb{C}$. As an example we have the *Dirac linear functional* on $c \in \mathbb{C}$, δ_c, defined by $\delta_c\left(p\left(x\right)\right) = p\left(c\right),\ \forall p \in \mathbb{P}$.

Let $\{B_n\}$ be a sequence of monic polynomials. The *sequence of linear functionals* $\{L_m\}$, *where* $L_m \in \mathbb{P}^*$ is its *dual sequence*, if $L_m(B_n) = \delta_{m,n},\ \ m, n \in \mathbb{N}$, and $\delta_{n,m}$ is the *Kronecker delta*.

If $v \in \mathbb{P}^*$ we have that $v = \sum_{i=0}^{\infty} \alpha_i L_i$ where $\alpha_i = v(B_i)$, $i \in \mathbb{N}$. In this way, if $v \in \mathbb{P}^*$ satisfies $v(B_i) = 0$ for $i \geq l$ then $v = \sum_{i=0}^{l-1} \alpha_i L_i$.

Let $\{L_m\}$ be a sequence of linear functionals where $L_m \in \mathbb{P}^*$. The vector sequence of linear functionals $\{\mathcal{L}_m\}$ given by

$$\mathcal{L}_m = \left[L_{msd} \cdots L_{(m+1)sd-1}\right]^T,\ \ m \in \mathbb{N}\,,$$

is called *vector sequence of linear functionals* associated with $\{L_m\}$. Taking into account the Definition 1, we have

$$\mathcal{L}_m(\mathcal{B}_n) = \begin{bmatrix} L_{msd}(B_{nsd}) & \cdots & L_{(m+1)sd-1}(B_{nsd}) \\ \vdots & \ddots & \vdots \\ L_{msd}(B_{(n+1)sd-1}) & \cdots & L_{(m+1)sd-1}(B_{(n+1)sd-1}) \end{bmatrix} = I_{sd\times sd}\delta_{m,n}\,.$$

Let $\{\mathcal{B}_n\}$ be a vector sequence of polynomials. We say that the *vector sequence of linear functionals* $\{\mathcal{L}_m\}$ is its *dual vector sequence*, if

$$\mathcal{L}_m(\mathcal{B}_n) = I_{sd\times sd}\,\delta_{m,n},\ \ n \in \mathbb{N}\,.$$

If $\mathcal{V}$ is a vector sequence of linear functionals there is an unique sequence $(\lambda_n) \subset \mathcal{M}_{sd\times sd}$, such that, $\mathcal{V} = \sum_{n=0}^{\infty} \lambda_n \mathcal{L}_n$, where $(\lambda_n)^T = \mathcal{V}(\mathcal{B}_n)\,,\ n \in \mathbb{N}$. In this way, if $\mathcal{V}(\mathcal{B}_n) = 0_{sd\times sd}$ for $n = l, l+1, \ldots,$ then $\mathcal{V} = \sum_{n=0}^{l-1} \lambda_n \mathcal{L}_n$.

Example . Let $\{\boldsymbol{\delta}_m\}$ be a vector sequence of linear functionals, where

$$\boldsymbol{\delta}_m = \left[(-1)^{smd}\frac{\delta_0^{(smd)}}{(smd)!}\ \ \cdots\ \ (-1)^{sd(m+1)-1}\frac{\delta_0^{(sd(m+1)-1)}}{(sd(m+1)-1)!}\right]^T,\ \ m \in \mathbb{N}\,,$$

in which $\{\delta_0^{(k)}\}$ is the *Dirac delta sequence.* Taking into account the definition of derivative of a linear functional, we have $(-1)^n/n!\,\delta_0^{(n)}(x^m) = \delta_{m,n},\ \ m,n \in \mathbb{N}$. By using the Definition 1 we have $\boldsymbol{\delta}_m(\mathcal{P}_j) = I_{sd\times sd}\,\delta_{m,j},\ m,j \in \mathbb{N}$. Then the vector sequence of linear functionals $\{\boldsymbol{\delta}_m\}$ is the dual sequence of $\{\mathcal{P}_j\}$.

Theorem 3. *The vector of polynomials $(A_{n,1},\ldots,A_{n,d})$ where $\deg A_{n,j} = n_j-1$, is type I multiple orthogonal with respect to the regular system of linear functionals $\{u^1,\ldots,u^d\}$ and family of quasi-diagonal multi-indices $\mathcal{J}$ if, and only if,*

$$\begin{array}{l} i)\ \ ((G_n(x^s))^T\mathcal{U})(\mathcal{P}_j) = 0_{sd\times sd},\ \ j=0,1,\ldots,n-1 \\ ii)\ \ ((G_n(x^s))^T\mathcal{U})(\mathcal{P}_n) = S_n, \end{array} \tag{5}$$

where $\mathcal{U} = \begin{bmatrix} v^1 & \cdots & v^{sd}\end{bmatrix}^T$ with v^r, $r=1,\ldots,sd$ defined by the algorithm, S_n is a regular lower triangular $sd\times sd$ matrix and

$$G_n(x^s) = \begin{bmatrix} \tilde{A}^1_{nsd+1}(x^s) & \cdots & \tilde{A}^1_{(n+1)sd}(x^s) \\ \vdots & \ddots & \vdots \\ \tilde{A}^{sd}_{nsd+1}(x^s) & \cdots & \tilde{A}^{sd}_{(n+1)sd}(x^s)\end{bmatrix},\ n\in\mathbb{N}\,.$$

where $\tilde{A}^r_{nsd+k} = A^i_{nsd+k,l}$ in case that $v^r = x^iu^l$ and $A^i_{nsd+k,l}$ are given by (3), $G_n(x) = \sum_{k=0}^n G^n_k x^k$, $G^n_k \in \mathcal{M}_{sd\times sd}$, and G^n_n is a regular upper triangular matrix.

Proof: We have,

$$((G_n(x^s))^T\mathcal{U})(\mathcal{P}_j) = \begin{bmatrix} B_{1,1} & \cdots & B_{1,sd} \\ \vdots & \ddots & \vdots \\ B_{sd,1} & \cdots & B_{sd,sd}\end{bmatrix},$$

where for $r,k = 1,\ldots,sd$ we have

$$B_{r,k} = (\sum_{r=1}^{sd}(\tilde{A}^r_{sdn+k}(x^s)v^r))(x^{jsd+r-1})\,.$$

Using the one to one correspondence between the polynomials $\tilde{A}^r_{nsd+k}$ and $A^i_{nsd+k,l}$ it holds

$$B_{r,k} = (\sum_{l=1}^{d}\sum_{i=0}^{s-1}(A^i_{sdn+k,l}(x^s)x^iu^l))(x^{jsd+r-1})\,.$$

Using the multi-orthogonality conditions of the Theorem 1, we get (5), and reciprocally.
Let us prove that G_n^n is a regular upper triangular matrix. In fact, notice that $v^1 = u^1$, so we have $\tilde{A}^1_{nsd+k} = A^0_{nsd+k,1}$, for $k = 1, \ldots, sd$. Taking into account that $\deg A_{nsd+1,1} = ns$, we have $\deg \tilde{A}^1_{sdn+1} = n$, and so $(G_n^n)_{1,1} \neq 0$. For the same reason, $\deg A^i_{sdn+1,1} < n$ for $i = 1, \ldots, s-1$. We also have that $\deg A_{nsd+1,l} = ns - 1$, for $l = 2, \ldots, d$ and so $\deg A^i_{sdn+1,l} < n$ for $i = 0, \ldots, s-1$ and $l = 2, \ldots, d$. Now, because of the one to one correspondende between the polynomials $\tilde{A}^r_{nsd+k}$ and $A^i_{nsd+k,l}$, we get $\deg \tilde{A}^r_{sdn+1} < n$ for $r = 2, \ldots, sd$, hence $(G_n^n)_{r,1} = 0$ for $r = 2, \ldots, sd$.
Let us now suppose that, for $i \in \{2, \ldots, sd-1\}$

$$(G_n^n)_{r,i-1} \neq 0\,,\ r = 1, \ldots, i-1\,,\quad (G_n^n)_{r,i-1} = 0\,,\ r = i, \ldots, sd$$

and $v^i = x^{k^j_{i-1}} u^j$ where j is uniquelly defined by the condition $k^j_i = k^j_{i-1} + 1$. This implies that $\deg A_{nsd+i,j} = k^j_{i-1} + ns$. So it holds that

$$A_{nsd+i,j} = a_{nsd+i,j}(x^s)^n x^{k^j_{i-1}} + \cdots \quad \text{and so} \quad A^{k^j_{i-1}}_{nsd+i,j}(x^s) = a_{nsd+i,j}(x^s)^n + \cdots .$$

We know that $A^{k^j_{i-1}}_{nsd+i,j} = \tilde{A}^i_{nsd+i}$, and so $(G_n^n)_{i,i} = a_{nsd+i,j} \neq 0$.
Notice that because of the increasing structure of the quase-diagonal multi-indeces the degree of $\tilde{A}^r_{nsd+i-1}$ is equal to the degree of $\tilde{A}^r_{nsd+i}$ for $r \neq i$, so this implies

$$(G_n^n)_{r,i} \neq 0\,,\ r = 1, \ldots, i\,,\quad (G_n^n)_{r,i} = 0\,,\ r = i+1, \ldots, sd$$

and G_n^n is a regular upper triangular matrix. ■

Definition 2. Let $\mathcal{U} = \begin{bmatrix} v^1 & \cdots & v^{sd} \end{bmatrix}^T$ be a vector of linear functionals and consider a sequence of matrix polynomials $\{G_n\}$. We say that $\{G_n\}$ is *type I multiple orthogonal* with respect to the vector of linear functionals $\mathcal{U}$ if

$$\begin{array}{ll} i) & ((G_n(x^s))^T \mathcal{U})(\mathcal{P}_j) = 0_{sd\times sd},\ \ j = 0, 1, \ldots, n-1 \\ ii) & ((G_n(x^s))^T \mathcal{U})(\mathcal{P}_n) = S_n, \end{array} \tag{6}$$

where S_n is a regular $sd \times sd$ matrix.

Theorem 4. *Let $\{G_n\}$ be a sequence of type I multiple orthogonal matrix polynomials with respect to the vector of linear functionals $\mathcal{U}$. Let us consider $\mathcal{E}_n(x) = G_n(x)\mathcal{F}_n$, $n \in \mathbb{N}$, where $\mathcal{F}_n$ are regular $sd \times sd$ matrices. Then, the sequence of matrix polynomials $\{\mathcal{E}_n\}$ is also type I multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$.*

Proof: Let $\{G_n\}$ be a type I multiple orthogonal matrix polynomials sequence, with respect to the vector of linear functionals $\mathcal{U}$, i.e.,

$$((G_n(x^s))^T\mathcal{U})(\mathcal{P}_j) = S_n\,\delta_{k,n}, \;\; k = 0, 1, \ldots, n, \;\; n \in \mathbb{N}\,,$$

where S_n is a regular $sd \times sd$ matrix. From,

$$((G_n(x^s))^T\mathcal{U})(\mathcal{P}_j) = \mathcal{P}_j\,\mathcal{U}^T(G_n(x^s)\mathcal{F}_n)(\mathcal{F}_n)^{-1} = (\mathcal{E}_n^T(x^s)\mathcal{U})(\mathcal{P}_j)(\mathcal{F}_n)^{-1}\,,$$

we have $(\mathcal{E}_n^T(x^s)\mathcal{U})(\mathcal{P}_j)(\mathcal{F}_n)^{-1} = S_n\,\delta_{k,n}$, $k = 0, 1, \ldots, n$, $n \in \mathbb{N}$, and so $(\mathcal{E}_n^T(x^s)\,\mathcal{U})(\mathcal{P}_j) = S_n\,\mathcal{F}_n\,\delta_{k,n}$, $k = 0, 1, \ldots, n$, $n \in \mathbb{N}$, where $S_n\,\mathcal{F}_n$ is a regular $sd \times sd$ matrix. Thus, the sequence of matrix polynomials $\{\mathcal{E}_n\}$ is also type I multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$. ∎

The same can be said for the type II multiple orthogonal polynomials.

Theorem 5 (type II vector multi-orthogonality, cf. [5])**.** *A sequence of monic polynomials $\{B_m\}$, is type II multiple orthogonal with respect to the regular system of linear functionals $\{u^1, \ldots, u^d\}$ and family of quasi-diagonal multi-indices $\mathcal{J}$ if, and only if, its associated vector polynomials sequence, $\{\mathcal{B}_m\}$, verifies:*

$$\begin{aligned} &i)\;\; ((x^s)^k\mathcal{U})(\mathcal{B}_m) = 0_{sd\times sd}, \;\; k = 0, 1, \ldots, m-1 \\ &ii)\;\; ((x^s)^m\mathcal{U})(\mathcal{B}_m) = \Delta_m, \end{aligned}$$

where $\mathcal{U} = \begin{bmatrix} v^1 & \cdots & v^{sd} \end{bmatrix}^T$, v^j, $j = 1, \ldots, sd$ are defined by the algorithm, and Δ_m is a regular upper triangular $sd \times sd$ matrix.

Now, we introduce the notions of *moments* and *block Hankel matrices* associated with the vector of linear functionals $\mathcal{U}$. We define *the moments of order $j \in \mathbb{N}$ associated with the vector of linear functionals* $(x^s)^k\mathcal{U}$, by

$$U_j^k := ((x^s)^k\mathcal{U})(\mathcal{P}_j) = \begin{bmatrix} v^1\big(x^{jsd+ks}\big) & \cdots & v^{sd}\big(x^{jsd+ks}\big) \\ \vdots & \ddots & \vdots \\ v^1\big(x^{(j+1)sd+ks-1}\big) & \cdots & v^{sd}\big(x^{(j+1)sd+ks-1}\big) \end{bmatrix}.$$

The *Hankel matrices* for $\mathcal{U}$ is defined by

$$\mathcal{H}_m = \begin{bmatrix} U_0^0 & \cdots & U_0^m \\ \vdots & \ddots & \vdots \\ U_m^0 & \cdots & U_m^m \end{bmatrix}, \;\; m \in \mathbb{N}\,.$$

$\mathcal{U}$ is called *regular* if $\det \mathcal{H}_m \neq 0$, $m \in \mathbb{N}$.

Now, we give the existence and uniqueness of a sequence of type I matrix multiple orthogonal polynomials with respect to a vector of linear functionals, $\mathcal{U}$.

Theorem 6. *Let $\mathcal{U}$ be a vector of linear functionals. Then, $\mathcal{U}$ is regular if, and only if, given a sequence of regular matrices, (S_n), there is an unique sequence of matrix polynomials $\{G_n\}$ with $G_n(x)=\sum_{k=0}^{n} G_k^n x^k$, $G_k^n \in \mathcal{M}_{sd\times sd}$ where G_n^n is regular, such that*

$$\begin{aligned} &i)\ \ ((G_n(x^s))^T\mathcal{U})(\mathcal{P}_j)=0_{sd\times sd},\ \ j=0,1,\ldots,n-1\\ &ii)\ \ ((G_n(x^s))^T\mathcal{U})(\mathcal{P}_n)=S_n\,,\end{aligned}$$

i.e, the sequence of matrix polynomials $\{G_n\}$, is type I multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$.

Proof: Let $\{G_n\}$ be a sequence of matrix polynomials where G_n is for each $n\in\mathbb{N}$, $G_n(x)=\sum_{k=0}^{n} G_k^n x^k$, $G_k^n\in\mathcal{M}_{sd\times sd}$. Applying $\mathcal{U}$, we have

$$((G_n(x^s))^T\mathcal{U})(\mathcal{P}_j)=\sum_{k=0}^{n}((x^s)^k\mathcal{U})(\mathcal{P}_j)G_k^n=\sum_{k=0}^{n}U_j^k G_k^n\,.$$

In matrix form we have,

$$\begin{bmatrix} U_0^0 & \cdots & U_0^n\\ \vdots & \ddots & \vdots\\ U_n^0 & \cdots & U_n^n\end{bmatrix}\begin{bmatrix} G_0^n\\ \vdots\\ G_n^n\end{bmatrix}=\begin{bmatrix} ((G_n(x^s))^t\mathcal{U})(\mathcal{P}_0)\\ \vdots\\ ((G_n(x^s))^t\mathcal{U})(\mathcal{P}_n)\end{bmatrix}.$$

By the multi-orthogonality conditions (6), the sequence of matrix polynomials $\{G_n\}$, is type I multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$ if,

$$\begin{bmatrix} U_0^0 & \cdots & U_0^n\\ \vdots & \ddots & \vdots\\ U_{n-1}^0 & \cdots & U_{n-1}^n\\ U_n^0 & \cdots & U_n^n\end{bmatrix}\begin{bmatrix} G_0^n\\ \vdots\\ G_{n-1}^n\\ G_n^n\end{bmatrix}=\begin{bmatrix} 0_{sd\times sd}\\ \vdots\\ 0_{sd\times sd}\\ S_n\end{bmatrix},\tag{7}$$

where S_n is a regular $sd\times sd$ matrix. Using the regularity of the vector of linear functionals $\mathcal{U}$, we have

$$\begin{bmatrix} G_0^n\\ \vdots\\ G_{n-1}^n\\ G_n^n\end{bmatrix}=\begin{bmatrix} U_0^0 & \cdots & U_0^n\\ \vdots & \ddots & \vdots\\ U_{n-1}^0 & \cdots & U_{n-1}^n\\ U_n^0 & \cdots & U_n^n\end{bmatrix}^{-1}\begin{bmatrix} 0_{sd\times sd}\\ \vdots\\ 0_{sd\times sd}\\ S_n\end{bmatrix},\tag{8}$$

and so we uniquelly obtain the sequence of matrix polynomials $\{G_n\}$. Taking $m = 0$ in (7), we have $U_0^0\, G_0^0 = S_0$. Using the regularity of the matrices U_0^0 and S_0 we have that G_0^0 is a regular matrix. Similarly, taking $m = 1$ in (7), we have

$$\begin{cases} U_0^0\, G_0^1 + U_0^1\, G_1^1 = 0_{sd\times sd} \\ U_1^0\, G_0^1 + U_1^1\, G_1^1 = S_1\,, \end{cases} \qquad \text{and so,}\ \ (U_1^1 - U_1^0(U_0^0)^{-1}U_0^1)G_1^1 = S_1\,.$$

Using the regularity of $\mathcal{U}$ and the structure by blocks, we have

$$\det(U_1^1 - U_1^0(U_0^0)^{-1}U_0^1) \neq 0\,,$$

and so G_1^1 is a regular matrix. Using the same argument we can conclude that G_n^n is, for each $n \in \mathbb{N}$, a regular matrix. Reciprocally, and in the same way, if G_n^n, $n \in \mathbb{N}$, is regular we obtain the regularity of the vector of linear functionals $\mathcal{U}$. ∎

Theorem 7 (cf. [5]). *Let $\mathcal{U}$ be a vector of linear functionals. Then $\mathcal{U}$ is regular if, and only if, given a sequence of regular $sd \times sd$ matrices, (Δ_m), there is a unique polynomial vector sequence $\{\mathcal{B}_m\}$ where $\mathcal{B}_m = [B_{m,1}\ \cdots\ B_{m,sd}]^T$, $m \in \mathbb{N}$, such that $\mathcal{B}_m = \sum_{j=0}^{m} B_j^m \mathcal{P}_j$ where $B_j^m \in \mathcal{M}_{sd\times sd}$ and B_m^m is a regular matrix that verifies*

$$\begin{aligned} &i)\ \ ((x^s)^k\mathcal{U})(\mathcal{B}_m) = 0_{sd\times sd}, \ \ k = 0, 1, \ldots, m-1 \\ &ii)\ \ ((x^s)^m\mathcal{U})(\mathcal{B}_m) = \Delta_m\,, \end{aligned} \tag{9}$$

i.e, $\{\mathcal{B}_m\}$ is type II multiple orthogonal polynomial sequence, with respect to the vector of linear functionals $\mathcal{U}$.
Moreover,

$$\mathcal{B}_m = \begin{bmatrix} 0_{sd\times sd} & \cdots & 0_{sd\times sd} & \Delta_m \end{bmatrix} \begin{bmatrix} U_0^0 & \cdots & U_0^m \\ \vdots & \ddots & \vdots \\ U_m^0 & \cdots & U_m^m \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{P}_0 \\ \vdots \\ \mathcal{P}_m \end{bmatrix}. \tag{10}$$

Theorem 8. *Let $\mathcal{U}$ be a regular vector of linear functionals, $\{G_n\}$ and $\{\mathcal{B}_m\}$ be defined by (8) and (10), respectively. Then, $\{\mathcal{B}_m\}$ and $\{G_n\}$ are biorthogonal with respect to $\mathcal{U}$, i.e.*

$$((G_n(x^s))^T\mathcal{U})(\mathcal{B}_m) = I_{sd\times sd}\,\delta_{n,m}, \ \ n, m \in \mathbb{N}$$

if, and only if, $S_m = (B_m^m)^{-1}$ and $\Delta_m = (G_m^m)^{-1}$, and so, the dual sequence $\{\mathcal{L}_n\}$ associated with $\{\mathcal{B}_m\}$ is given by, $\mathcal{L}_n = (G_n(x^s))^T\mathcal{U}$, $n \in \mathbb{N}$.

Proof: There is an unique sequence of matrices $(B_j^m) \subset \mathcal{M}_{sd\times sd}$, such that, $\mathcal{B}_m = \sum_{j=0}^m B_j^m \mathcal{P}_j$, where B_m^m is a regular matrix. Hence,

$$((G_n(x^s))^T\mathcal{U})(\mathcal{B}_m) = ((G_n(x^s))^T\mathcal{U})(\sum_{j=0}^m B_j^m \mathcal{P}_j) = \sum_{j=0}^m B_j^m ((G_n(x^s))^T\mathcal{U})(\mathcal{P}_j).$$

Using (6)

$$((G_n(x^s))^T\mathcal{U})(\mathcal{B}_m) = \begin{cases} B_m^m S_m, & m = n \\ 0_{sd\times sd}, & m < n\,. \end{cases}$$

Thus, $((G_m(x^s))^T\mathcal{U})(\mathcal{B}_m) = I_{sd\times sd}$ if, and only if, $B_m^m S_m = I_{sd\times sd}$, i.e., $S_m = (B_m^m)^{-1}$. Now, let us consider,

$$((G_n(x^s))^T\mathcal{U})(\mathcal{B}_m) = ((\sum_{j=0}^n G_j^n (x^s)^j)^T\mathcal{U})(\mathcal{B}_m) = \sum_{j=0}^n ((x^s)^j\mathcal{U})(\mathcal{B}_m) G_j^n\,.$$

As before, using (9)

$$((G_n(x^s))^t\mathcal{U})(\mathcal{B}_m) = \begin{cases} \Delta_m G_m^m, & m = n \\ 0_{sd\times sd}, & m > n\,. \end{cases}$$

So, $((G_m(x^s))^T\mathcal{U})(\mathcal{B}_m) = I_{sd\times sd}$ if, and only if, $\Delta_m G_m^m = I_{sd\times sd}$, i.e., $\Delta_m = (G_m^m)^{-1}$, as we wanted to show. ∎

4. A characterization for multiple orthogonal polynomials

Theorem 9. *Let $\{B_m\}$ be a sequence of monic polynomials, $\{\mathcal{B}_m\}$ its associated sequence of vector polynomials, and a regular system of linear functionals $\{u^1, \ldots, u^d\}$. Then, the following conditions are equivalent:*

a) $\{B_m\}$ is the type II multiple orthogonal with respect to the system of linear functionals $\{u^1, \ldots, u^d\}$ and family of quasi-diagonal multi-indices $\mathcal{J}$.

b) $\{\mathcal{B}_m\}$ is the type II multiple orthogonal with respect to the vector of linear functionals, $\mathcal{U} = [v^1 \ \ldots \ v^{sd}]^T$, where v^j, $j = 1, \ldots, sd$ are defined in terms of the system of linear functionals $\{u^1, \ldots, u^d\}$ by the algorithm.

c) There are sequences of $sd \times sd$ matrices $(\alpha_m^{s,d})$, $(\beta_m^{s,d})$ and $(\gamma_m^{s,d})$, $m \in \mathbb{N}$, with $\gamma_m^{s,d}$ regular upper triangular matrix such that $\{\mathcal{B}_m\}$ is defined by the three-term recurrence relation

$$x^s\mathcal{B}_m(x) = \alpha_m^{s,d}\,\mathcal{B}_{m+1}(x) + \beta_m^{s,d}\,\mathcal{B}_m(x) + \gamma_m^{s,d}\,\mathcal{B}_{m-1}(x), \quad m = 0, 1, \ldots \tag{11}$$

with $\mathcal{B}_{-1} = 0_{d\times 1}$ *and* $\mathcal{B}_0$ *given.*

d) There are sequences of $sd\times sd$ *matrices* $(\gamma^{s,d}_{n+1})$, $(\beta^{s,d}_{n})$ *and* $(\alpha^{s,d}_{n-1})$, $m\in\mathbb{N}$, *with* $\gamma^{s,d}_{n+1}$ *regular upper triangular matrix such that the dual sequence of* $\{\mathcal{B}_m\}$, $\{\mathcal{L}_n\}$, *is defined by the three-term recurrence relation*

$$x^s\mathcal{L}_n = (\gamma^{s,d}_{n+1})^T\mathcal{L}_{n+1} + (\beta^{s,d}_{n})^T\mathcal{L}_n + (\alpha^{s,d}_{n-1})^T\mathcal{L}_{n-1},\ n=1,2,\ldots \tag{12}$$

with $\mathcal{L}_1 = (\gamma^{s,d}_{1})^{-T}(x^sI_{sd\times sd} - (\beta^{s,d}_0)^T)(\mathcal{U}(\mathcal{B}_0))^{-T}\mathcal{U}$, $\mathcal{L}_0 = (\mathcal{U}(\mathcal{B}_0))^{-T}\mathcal{U}$, *and,* $\mathcal{L}_n = (G_n(x^s))^T\mathcal{U}$ *where* G_n *is for each* $n\in\mathbb{N}$ *a matrix polynomial with* $G_n(x) = \sum_{k=0}^{n} G^n_k x^k$, $G^n_k\in\mathcal{M}_{sd\times sd}$, *and* G^n_n *a regular upper triangular matrix.*

e) The sequence of matrix polynomials $\{G_n\}$ *is type I multiple orthogonal with respect to the vector of linear functionals* $\mathcal{U}$.

f) There are sequences of $sd\times sd$ *matrices* (l^n_{n+1}), (l^n_n) *and* (l^n_{n-1}), $n\in\mathbb{N}$, *such that* $G_n(x^s)$ *is defined by the three-term recurrence relation*

$$x^sG_n(x^s) = G_{n+1}(x^s)(l^n_{n+1})^T + G_n(x^s)(l^n_n)^T + G_{n-1}(x^s)(l^n_{n-1})^T\,, \tag{13}$$

for $n=1,2,\ldots$, *with* $G_0(x^s) = (\mathcal{U}(\mathcal{B}_0))^{-1}$, $G_1(x^s) = (\mathcal{U}(\mathcal{B}_0))^{-1}(x^sI_{sd\times sd} - (l^0_0)^T)((l^0_1)^{-1})^T$.

Proof: $a)\Leftrightarrow b)$. cf. Theorem 5. $b)\Leftrightarrow c)$. cf. [5].
$c)\Rightarrow d)$. Let

$$x^s\mathcal{L}_n = \sum_{j=0}^{n+1}\lambda^n_j\mathcal{L}_j \text{ where } (\lambda^n_j)^T = (x^s\mathcal{L}_n)(\mathcal{B}_j){=}\mathcal{L}_n(x^s\mathcal{B}_j),\ \ j\in\mathbb{N}\,.$$

Applying the vector of linear functionals $\mathcal{L}_n$ to both members of the three-term recurrence relation, we have

$$\begin{aligned}\mathcal{L}_n(x^s\mathcal{B}_j) &= \alpha^{s,d}_j\,\mathcal{L}_n(\mathcal{B}_{j+1}) + \beta^{s,d}_j\,\mathcal{L}_n(\mathcal{B}_j) + \gamma^{s,d}_j\,\mathcal{L}_n(\mathcal{B}_{j-1})\\ &= \begin{cases}\alpha^{s,d}_{n-1}, & j=n-1\\ \beta^{s,d}_n, & j=n\\ \gamma^{s,d}_{n+1}, & j=n+1\\ 0_{sd\times sd}, & j\neq n-1,n,n+1,\end{cases}\end{aligned}$$

i.e., $\lambda^n_{n-1} = (\alpha^{s,d}_{n-1})^T$, $\lambda^n_n = (\beta^{s,d}_n)^T$ and $\lambda^n_{n+1} = (\gamma_{n+1})^T$, obtaining the three-term recurrence relation for the vector sequence of linear functionals $\{\mathcal{L}_n\}$.

By induction, we prove that $\mathcal{L}_n = (G_n(x^s))^T\mathcal{U}$, $n\in\mathbb{N}$. For $n=0$, we have that, $\mathcal{L}_0 = ((\mathcal{U}(\mathcal{B}_0))^{-1})^T\mathcal{U}$. Now, let us suppose that the property is valid for $k=1,\ldots,p$, i.e., $\mathcal{L}_k = (G_k(x^s))^T\mathcal{U}$ with $\deg G_k = k$, $k=1,\ldots,p$ and we verify that it is also valid for $k = p+1$, i.e., $\mathcal{L}_{p+1} = (G_{p+1}(x^s))^T\mathcal{U}$, $p \in \mathbb{N}$. Considering the three-term recurrence relation (12) and taking into account the hypothesis of induction, we have

$$\mathcal{L}_{p+1} = ((\gamma_{p+1}^{s,d})^T)^{-1}\left[(x^s I_{sd\times sd} - (\beta_p^{s,d})^T)(G_p(x^s))^T - (\alpha_{p-1}^{s,d})^T(G_{p-1}(x^s))^T\right]\mathcal{U}\,.$$

Thus, $\mathcal{L}_{p+1} = (G_{p+1}(x^s))^T\mathcal{U}$, $p \in \mathbb{N}$, i.e, if the condition is true for $k = 1,\dots,p$, it is also for $p+1$, showing our purpose. It easily holds that G_n^n is a regular upper triangular matrix.

$d) \Rightarrow c)$. Let $x^s\mathcal{B}_m = \sum_{k=0}^{m+1}\eta_k^m\mathcal{B}_k$ where $\eta_k^m \in \mathcal{M}_{sd\times sd}$. Applying the vector of linear functionals $\mathcal{L}_k$ to both members of this representation we have $\eta_k^m = (x^s\mathcal{L}_k)(\mathcal{B}_m)$. Now, applying our hypothesis, we have

$$\begin{aligned}\eta_k^m &= \mathcal{L}_{k+1}(\mathcal{B}_m)\,\gamma_{k+1}^{s,d} + \mathcal{L}_{k-1}(\mathcal{B}_m)\,\beta_k^{s,d} + \mathcal{L}_{k-1}(\mathcal{B}_m)\,\alpha_{k-1}^{s,d}\\ &= \begin{cases}\alpha_m^{s,d}, & k = m+1\\ \beta_m^{s,d}, & k = m\\ \gamma_m^{s,d}, & k = m-1\\ 0_{sd\times sd}, & k \neq m-1, m, m+1,\end{cases}\end{aligned}$$

and so, we get (11).

$d) \Rightarrow e)$. By Theorem 8 we know that $\mathcal{L}_n = (G_n(x^s))^T\mathcal{U}$, $n \in \mathbb{N}$, is the general term of the dual sequence of $\{\mathcal{B}_m\}$ type II multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$. In this way, using the regularity of the vector of linear functionals $\mathcal{U}$, we can identify in an unique way the sequence of matrix polynomials $\{G_n\}$, of $c)$, with the type I multiple orthogonal sequence with respect to the vector of linear functionals $\mathcal{U}$.

$e) \Rightarrow f)$. Being $\{G_n\}$ a sequence of matrix polynomials, for each $n \in \mathbb{N}$, there is an unique sequence $(l_k^n) \subset \mathcal{M}_{sd\times sd}$, such that,

$$x^s(G_n(x^s))^T = \sum_{k=0}^{n+1} l_k^n (G_k(x^s))^T\,.$$

Applying $\mathcal{U}(\mathcal{P}_j)$, $j = 0,1,\dots$ to both members of the previous identity, we have

$$((G_n(x^s))^T\mathcal{U})(x^s\mathcal{P}_j) = \sum_{k=0}^{n+1}((G_k(x^s))^T\mathcal{U})(\mathcal{P}_j)(l_k^n)^T\,.$$

Being,

$$x^s\mathcal{P}_j = \vartheta_1^j\mathcal{P}_{j+1} + \vartheta_2^j\mathcal{P}_j,\ \ \vartheta_i^j \in \mathcal{M}_{sd\times sd}, \tag{14}$$

we have

$$((G_n(x^s))^T\mathcal{U})(\vartheta_1^j\mathcal{P}_{j+1}+\vartheta_2^j\mathcal{P}_j)=\sum_{k=0}^{n+1}((G_k(x^s))^T\mathcal{U})(\mathcal{P}_j)(l_k^n)^T\,.$$

As the sequence of matrix polynomials $\{G_n\}$ is type I multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$, we can verify that:
For $j=0$: $0_{sd\times sd}=((G_0(x^s))^T\mathcal{U})(\mathcal{P}_0)(l_0^n)^T$, $n=2,3,\ldots$, hence $l_0^n=0_{sd\times sd}$ as $((G_0(x^s))^T\mathcal{U})(\mathcal{P}_0)$ is a regular matrix.
For $j=1$: $0_{sd\times sd}=((G_1(x^s))^T\mathcal{U})(\mathcal{P}_1)(l_1^n)^T$, $n=3,4,\ldots$, and so $l_1^n=0_{sd\times sd}$, as $((G_1(x^s))^T\mathcal{U})(\mathcal{P}_1)$ is a regular matrix. Continuing this procedure, we get for $j=n-2$: $0_{sd\times sd}=((G_{n-2}(x^s))^T\mathcal{U})(\mathcal{P}_{n-2})(l_{n-2}^n)^T$. But, $((G_{n-2}(x^s))^T\mathcal{U})(\mathcal{P}_{n-2})$ a regular matrix, so $l_{n-2}^n=0_{sd\times sd}$. Hence, we get (13).
Next, we determine the matrix coefficients l_{n-1}^n, l_n^n and l_{n+1}^n. Applying $\mathcal{U}(\mathcal{P}_{n-1})$ to both members of (13) and taking into account that the sequence of matrix polynomials $\{G_n\}$ is type I multiple orthogonal with respect to the vector of linear functionals $\mathcal{U}$, we have

$$((G_n(x^s))^T\mathcal{U})(x^s\mathcal{P}_{n-1})=l_{n-1}^nS_{n-1}\,.$$

Using (14), we have $\vartheta_1^{n-1}S_n=l_{n-1}^nS_{n-1}$, i.e. $l_{n-1}^n=\vartheta_1^{n-1}S_n(S_{n-1})^{-1}$. Similarly applying $\mathcal{U}(\mathcal{P}_n)$ to both members of (13), we have

$$((G_n(x^s))^T\mathcal{U})(x^s\mathcal{P}_n)=l_n^nS_n+l_{n-1}^n((G_{n-1}(x^s))^T\mathcal{U})(\mathcal{P}_n)\,.$$

Using (14), we have

$$\begin{aligned}l_n^n=&\left[\vartheta_1^n((G_n(x^s))^T\mathcal{U})(\mathcal{P}_{n+1})+\vartheta_2^nS_n\right](S_n)^{-1}\\&-\left[l_{n-1}^n((G_{n-1}(x^s))^T\mathcal{U})(\mathcal{P}_n)\right](S_n)^{-1}\,.\end{aligned}$$

Finally, applying $\mathcal{U}(\mathcal{P}_{n+1})$ to both members of (13), we have

$$\begin{aligned}((G_n(x^s))^T\mathcal{U})(x^s\mathcal{P}_{n+1})=&\,l_{n+1}^nS_{n+1}+l_n^n((G_n(x^s))^T\mathcal{U})(\mathcal{P}_{n+1})\\&+l_{n-1}^n((G_{n-1}(x^s))^T\mathcal{U})(\mathcal{P}_{n+1})\,.\end{aligned}$$

Using (14), we have

$$\begin{aligned}l_{n+1}^n=&\left[\vartheta_1^{n+1}((G_n(x^s))^T\mathcal{U})(\mathcal{P}_{n+2})+\vartheta_2^{n+1}((G_n(x^s))^T\mathcal{U})(\mathcal{P}_{n+1})\right](S_{n+1})^{-1}\\&-\left[l_n^n((G_n(x^s))^T\mathcal{U})(\mathcal{P}_{n+1})+l_{n-1}^n((G_{n-1}(x^s))^T\mathcal{U})(\mathcal{P}_{n+1})\right](S_{n+1})^{-1}\,.\end{aligned}$$

$f)\Rightarrow d)$. From (13) we have

$$x^s\mathcal{L}_n=l_{n+1}^n\mathcal{L}_{n+1}+l_n^n\mathcal{L}_n+l_{n-1}^n\mathcal{L}_{n-1}\ \text{ with }\ \mathcal{L}_n=(G_n(x^s))^T\mathcal{U},\ n\in\mathbb{N}\,.$$

Hence, $l^n_{n+1} = (\gamma^{s,d}_{n+1})^T$, $l^n_n = (\beta^{s,d}_n)^T$ and $l^n_{n-1} = (\alpha^{s,d}_{n-1})^T$. ∎

Theorem 10 (Christoffel-Darboux type)**.** *Let $\mathcal{U}$ be a regular vector linear functional. Let $\{G_n\}$ and $\{\mathcal{B}_m\}$ be, respectively, a sequence of matrix polynomials with $\deg G_n = n$, for all $n \in \mathbb{N}$, and $\mathcal{B}_m(x) = V_m(x^{sd})\mathcal{P}_0(x)$, where V_m is a matrix polynomial with $\deg V_m = m$, for all $m \in \mathbb{N}$. Then the following conditions are equivalent:*

a) $\{G_n\}$ and $\{\mathcal{B}_m\}$ are type I and II sequences of polynomials, multiple orthogonal with respect to $\mathcal{U}$.

b) There exists sequences of $sd \times sd$ matrices, $(\alpha^{s,d}_m)$ and $(\gamma^{s,d}_m)$, with $\gamma^{s,d}_m$ regular, such that

$$(x-z)\sum_{k=0}^{m} G_k(z)V_k(x^d) = G_m(z)\alpha^{s,d}_m V_{m+1}(x^d) - G_{m+1}(z)\gamma^{s,d}_{m+1}V_m(x^d)\,. \tag{15}$$

c) There exists sequences of $sd \times sd$ matrices, $(\alpha^{s,d}_m)$ and $(\gamma^{s,d}_m)$, with $\gamma^{s,d}_m$ regular, such that

$$G_{m+1}(x)\gamma^{s,d}_{m+1}V_m(x^d) = G_m(x)\alpha^{s,d}_m V_{m+1}(x^d)\,, \tag{16}$$

$$\sum_{k=0}^{m} G_k(x)V_k(x^d) = G'_{m+1}(x)\gamma^{s,d}_{m+1}V_m(x^d) - G'_m(x)\alpha^{s,d}_m V_{m+1}(x^d)\,. \tag{17}$$

Proof: First we prove that $a)$ implies $b)$. In fact, taking into account the Theorem 9, we can that the polynomials $\{V_m\}$ and $\{G_m\}$ verify the three-term recurrence relations with $sd \times sd$ matrix coefficients given for all $m = 1, 2\ldots$ by,

$$xV_m(x^d) = \alpha^{s,d}_m V_{m+1}(x^d) + \beta^{s,d}_m V_m(x^d) + \gamma^{s,d}_m V_{m-1}(x^d)\,, \tag{18}$$

$$zG_m(z) = G_{m+1}(z)\gamma^{s,d}_{m+1} + G_m(z)\beta^{s,d}_m + G_{m-1}(z)\alpha^{s,d}_{m-1}\,. \tag{19}$$

Subtracting the result of multiplying on the left both members of (18) by $G_m(z)$, and on the right members of (19) by $V_m(x^d)$, we have

$$\begin{aligned}(x-z)G_m(z)V_m(x^d) = &\left[G_m(z)\alpha^{s,d}_m V_{m+1}(x^d) - G_{m-1}(z)\alpha^{s,d}_{m-1}V_m(x^d)\right] \\ &- \left[G_{m+1}(z)\gamma^{s,d}_{m+1}V_m(x^d) - G_m(z)\gamma^{s,d}_m V_{m-1}(x^d)\right],\end{aligned}$$

and so we have (15).
Now we prove that $b)$ implies $c)$. In order to obtain (16) we substitute z by x

in (15). To obtain (17) we add $G_{m+1}(x)\gamma_{m+1}^{s,d}V_m(x^d) - G_m(x)\alpha_m^{s,d}V_{m+1}(x^d)$, to (15), i.e.

$$\sum_{k=0}^{m} G_k(z)V_k(x^d) = -\frac{G_m(z)-G_m(x)}{z-x}\alpha_m^{s,d}V_{m+1}(x^d) \\ + \frac{G_{m+1}(z)-G_{m+1}(x)}{z-x}\gamma_{m+1}^{s,d}V_m(x^d) \, .$$

By letting $z \to x$, we get (17).
Finally we show that c) implies a). From (17)

$$G_m(x)V_m(x^d) + G'_m(x)\gamma_m^{s,d}V_{m-1}(x^d) - G'_{m-1}(x)\alpha_{m-1}^{s,d}V_m(x^d) \\ = G'_{m+1}(x)\gamma_{m+1}^{s,d}V_m(x^d) - G'_m(x)\alpha_m^{s,d}V_{m+1}(x^d) \, .$$

Now, taking into account (17), we get

$$G_m(x)V_m(x^d) + G'_m(x)G_m^{-1}(x)G_{m-1}(x)\alpha_{m-1}^{s,d}V_m(x^d) - G'_{m-1}(x)\alpha_{m-1}^{s,d}V_m(x^d) \\ = G'_{m+1}(x)\gamma_{m+1}^{s,d}V_m(x^d) - G'_m(x)G_m^{-1}(x)G_{m+1}(x)\gamma_{m+1}^{s,d}V_m(x^d) \, .$$

Because, for only a finite number of $x \in \mathbb{C}$, $\det G_m(x) = 0$ or $\det V_m(x^d) = 0$, multiply, in the last equation, on the left by $G_m^{-1}(x)$, on the right by $V_m^{-1}(x^d)$, and taking into account $(G_m^{-1}(x))' = -G_m^{-1}(x)G'_m(x)G_m^{-1}(x)$, to get

$$(G_m^{-1}(x))'(G_{m+1}(x)\gamma_{m+1}^{s,d} + G_{m-1}(x)\alpha_{m-1}^{s,d}) \\ + G_m^{-1}(x)(G_{m+1}(x)\gamma_{m+1}^{s,d} + G_{m-1}(x)\alpha_{m-1}^{s,d})' = I_{sd\times sd} \, .$$

Integrating on x we obtain that $\{G_n\}$ verifies a three term recurrence relation of type (19), and by Theorem 9, the result follows. ■

Let $\mathcal{U}$ be a regular vector of linear functionals, $\{G_m\}$ and $\{\mathcal{B}_m\}$, respectively the type I and II multiple orthogonal polynomials sequences with respect to $\mathcal{U}$. We denote the *polynomial kernel* by,

$$\mathbf{K}_m(z,x) = \sum_{k=0}^{m-1} G_k(z)V_k(x^d) \, ,$$

where V_m is a matrix polynomial of degree m given by $\mathcal{B}_m(x) = V_m(x^{sd})\mathcal{P}_0(x)$.

Theorem 11 (reproducing kernel property)**.** *Let $\mathcal{U}$ be a regular vector of linear functionals, $\{G_m\}$ and $\{\mathcal{B}_m\}$ be, respectively, the type I and II multiple* *orthogonal polynomials sequences with respect to* $\mathcal{U}$*. Then, given a vector polynomial* $\pi \in \mathbb{P}^{sd}$*, we have*

$$\pi(x) = ((\mathbf{K}_{r+1}(z^s, x^s))^T \mathcal{U}_z)(\pi(z))\mathcal{P}_0(x)\,.$$

Proof: As $\pi \in \mathbb{P}^{sd}$ we have

$$\pi(x) = \sum_{j=0}^{r} \alpha_j^r \mathcal{B}_j(x) \text{ with } \alpha_j^r = \mathcal{L}_j(\pi(z)) = ((G_j(z^s))^T \mathcal{U}_z)(\pi(z))\,.$$

Hence, $\pi(x) = \sum_{j=0}^{r}((G_j(z^s))^T \mathcal{U}_z)(\pi(z))\mathcal{B}_j(x)$. Using $\mathcal{B}_j(x) = V_j(x^{sd})\mathcal{P}_0(x)$ the above equality can be written by,

$$\begin{aligned}
\pi(x) &= \sum_{j=0}^{r}((G_j(z^s))^T \mathcal{U}_z)(\pi(z)) V_j(x^{sd})\mathcal{P}_0(x) \\
&= \sum_{j=0}^{r}((V_j(x^{sd}))^T (G_j(z^s))^T \mathcal{U}_z)(\pi(z))\mathcal{P}_0(x) \\
&= ((\sum_{j=0}^{r} G_j(z^s) V_j(x^{sd}))^T \mathcal{U}_z)(\pi(z))\mathcal{P}_0(x)\,,
\end{aligned}$$

as we wanted to show. ■

5. Hermite-Padé approximation

Now, we present a reinterpretation of type I Hermite-Padé approximation in terms of the matrix functions.

Definition 3. Let $\{G_n\}$ be a sequence of matrix polynomials with $sd \times sd$ matrix coefficients and $\mathcal{U}$ a regular vector of linear functionals. To the sequence of polynomials $\{G_{n-1}^{(1)}\}$ defined by

$$G_{n-1}^{(1)}(z) := \left(\frac{(G_n(z))^T - (G_n(x^s))^T}{z - x^s} \mathcal{U}_x \right) (\mathcal{P}_0(x))\,,$$

where $\mathcal{U}_x$ represents the action of $\mathcal{U}$ over the variable x, we called the *associated sequence of polynomials* of $\{G_n\}$ and $\mathcal{U}$.

Theorem 12. *Let* $\{G_n\}$ *and* $\{\mathcal{B}_m\}$ *be, respectively, the type I and II sequences of polynomials, multiple orthogonal with respect to the regular vector linear functional* $\mathcal{U}$*. Then, the associated sequence of matrix polynomials of* $\{G_n\}$ *and* $\mathcal{U}$*,* $\{G_{n-1}^{(1)}\}$*, verifies the same three-term recurrence relation as* $\{G_n\}$*, i.e*

$$zG_{n-1}^{(1)}(z) = G_n^{(1)}(z)\gamma_{n+1}^{s,d} + G_{n-1}^{(1)}(z)\beta_n^{s,d} + G_{n-2}^{(1)}(z)\alpha_{n-1}^{s,d}\,,\ n = 1, 2, \ldots$$

with $G_{-1}^{(1)}(z) = 0_{sd\times sd}$ *and* $G_0^{(1)}(z)$ *given.*

Proof: We know that $\{G_n\}$ is defined by the three-term recurrence relation

$$x^s G_n(x^s) = G_{n+1}(x^s)\gamma_{n+1}^{s,d} + G_n(x^s)\beta_n^{s,d} + G_{n-1}\alpha_{n-1}^{s,d}\,,\ n = 1, 2, \dots \tag{20}$$

with $G_0(x^s) = (\mathcal{U}(\mathcal{B}_0))^{-1}\,,\ G_1(x^s) = (\mathcal{U}(\mathcal{B}_0))^{-1}(x^s I_{sd\times sd} - \beta_0^{s,d})(\gamma_1^{s,d})^{-1}\,.$
Hence, taking transpose operator on (20) and multiplying both members by $\mathcal{U}$, we have

$$\begin{aligned} x^s(G_n(x^s))^T\mathcal{U} = {} & (\gamma_{n+1}^{s,d})^T(G_{n+1}(x^s))^T\mathcal{U} \\ & + (\beta_n^{s,d})^T(G_n(x^s))^T\mathcal{U} + (\alpha_{n-1}^{s,d})^T(G_{n-1}(x^s))^T\mathcal{U}. \end{aligned} \tag{21}$$

Substituting x by z in (21) and subtracting from (21), we have

$$\begin{aligned} & (z^s - x^s)\frac{(G_n(x^s))^T}{z^s - x^s}\mathcal{U} + z^s\frac{(G_n(z^s))^T - (G_n(x^s))^T}{z^s - x^s}\mathcal{U} \\ = {} & (\gamma_{n+1}^{s,d})^T\frac{(G_{n+1}(z^s))^T - (G_{n+1}(x^s))^T}{z^s - x^s}\mathcal{U} + (\beta_n^{s,d})^T\frac{(G_n(z^s))^T - (G_n(x^s))^T}{z^s - x^s}\mathcal{U} \\ & + (\alpha_{n-1}^{s,d})^T\frac{(G_{n-1}(z^s))^T - (G_{n-1}(x^s))^T}{z^s - x^s}\mathcal{U}\,. \end{aligned}$$

Applying both members over the vector polynomial $\mathcal{P}_0$, we have

$$\begin{aligned} & ((G_n(x^s))^T\mathcal{U})(\mathcal{P}_0) + z^s\left(\frac{(G_n(z^s))^T - (G_n(x^s))^T}{z^s - x^s}\mathcal{U}\right)(\mathcal{P}_0) \\ & = \left(\frac{(G_{n+1}(z^s))^T - (G_{n+1}(x^s))^T}{z^s - x^s}\mathcal{U}\right)(\mathcal{P}_0)\gamma_{n+1}^{s,d} \\ & \quad + \left(\frac{(G_n(z^s))^T - (G_n(x^s))^T}{z^s - x^s}\mathcal{U}\right)(\mathcal{P}_0)\beta_n^{s,d} \\ & \quad + \left(\frac{(G_{n-1}(z^s))^T - (G_{n-1}(x^s))^T}{z^s - x^s}\mathcal{U}\right)(\mathcal{P}_0)\alpha_{n-1}^{s,d}\,. \end{aligned}$$

Being $\{\mathcal{L}_n\}$ the dual sequence of $\{\mathcal{B}_m\}$, i.e., $\mathcal{L}_n(\mathcal{B}_m) = I_{sd\times sd}\,\delta_{n,m}$, $n, m \in \mathbb{N}$, we have $\mathcal{L}_n(\mathcal{B}_0) = 0_{sd\times sd}$, $n = 1, 2, \dots$. Taking into account that

$$\mathcal{B}_0 = B_0^0\,\mathcal{P}_0,\ B_0^0 \in \mathcal{M}_{sd\times sd} \text{ and } \mathcal{L}_n = (G_n(x^s))^T\mathcal{U}\,,$$

we have $((G_n(x))^T\mathcal{U})(\mathcal{P}_0) = 0_{sd\times sd}$, $n = 1, 2, \dots$.
Hence, we get after substituting z^s by z the desired three term recurrence relation for $\{G_n^{(1)}\}$. ∎

Lemma 1. *Let $\mathcal{U}$ be a regular vector of linear functionals and $\{G_n\}$ a sequence of matrix polynomials with $sd \times sd$ matrix coefficients. If $\{G_n\}$*

is a type I multiple orthogonal sequence with respect to $\mathcal{U}$, *then for* $m = 0, 1, \ldots, n-2$ *and* $l = 0, 1, \ldots, d-1$, *we have* $((G_n(x^s))^T\mathcal{U})((x^s)^l\mathcal{P}_m) = 0_{sd\times sd}$.

Proof: The vector of polynomials $(x^s)^l\mathcal{P}_m,\ \ m = 0, 1, \ldots$ e $l = 0, 1, \ldots, d-1$, can be written by $(x^s)^l\mathcal{P}_m = \vartheta_1^m\mathcal{P}_m + \vartheta_2^m\mathcal{P}_{m+1},\ \ \vartheta_i^m \in \mathcal{M}_{sd\times sd}$. Applying $(G_n(x^s))^T\mathcal{U}$ to both members of the previous equality, we have

$$((G_n(x^s))^T\mathcal{U})((x^s)^l\mathcal{P}_m) = \vartheta_1^m((G_n(x^s))^T\mathcal{U})(\mathcal{P}_m) + \vartheta_2^m((G_n(x^s))^T\mathcal{U})(\mathcal{P}_{m+1})\,.$$

Being $\{G_n\}$ the type I multiple orthogonal polynomial sequence with respect to $\mathcal{U}$, we have

$$((G_n(x^s))^T\mathcal{U})((x^s)^l\mathcal{P}_m) = 0_{sd\times sd}, \quad m = 0, 1, \ldots, n-2\,,$$

as we wanted to show. ■

Let $\mathcal{U}$ be a vector of linear functionals. We define the *matrix generating function associated* with $\mathcal{U}$, $\mathcal{F}$, by

$$\mathcal{F}(z) := \mathcal{U}_x(\frac{\mathcal{P}_0(x)}{z-x^s}) = \begin{bmatrix} v_x^1(\frac{1}{z-x^s}) & \cdots & v_x^{sd}(\frac{1}{z-x^s}) \\ \vdots & \ddots & \vdots \\ v_x^1(\frac{x^{sd-1}}{z-x^s}) & \cdots & v_x^{sd}(\frac{x^{sd-1}}{z-x^s}) \end{bmatrix}. \tag{22}$$

Being,

$$\frac{1}{z-x^s} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{x^s}{z}\right)^k \quad \text{for} \quad |x^s| < |z|\,, \tag{23}$$

we have $\mathcal{F}(z) = \sum_{k=0}^{\infty}\frac{((x^s)^k\mathcal{U}_x)(\mathcal{P}_0(x))}{z^{k+1}}$.

Theorem 13. *Let* $\mathcal{U}$ *be a regular vector of linear functionals,* $\{G_n\}$ *a matrix polynomials sequence with* $sd\times sd$ *matrix coefficients,* $\{G_{n-1}^{(1)}\}$, *its associated sequence of polynomials, and* $\mathcal{F}$ *be the matrix generating function associated with* $\mathcal{U}$, *defined by* (22). *Then,* $\{G_n\}$ *is type I multiple orthogonal polynomial with respect to the vector of the linear functions* $\mathcal{U}$ *if, and only if,*

$$\mathcal{F}(z)G_n(z) - G_{n-1}^{(1)}(z) = \sum_{m=n-1}^{\infty}\sum_{l=0}^{d-1}\frac{((G_n(x^s))^T\mathcal{U}_x)((x^s)^l\mathcal{P}_m(x))}{z^{dm+l+1}}\,. \tag{24}$$

Proof: Taking into account the Definition 3, we have

$$\begin{aligned} G_{n-1}^{(1)}(z) &= \left(\frac{(G_n(z))^T - (G_n(x^s))^T}{z-x^s}\mathcal{U}_x\right)(\mathcal{P}_0(x)) \\ &= ((G_n(z))^T\mathcal{U}_x)(\frac{\mathcal{P}_0(x)}{z-x^s}) - ((G_n(x^s))^T\mathcal{U}_x)(\frac{\mathcal{P}_0(x)}{z-x^s}) \end{aligned}$$

$$= \mathcal{F}(z)G_n(z) - ((G_n(x^s))^T\mathcal{U}_x)(\frac{\mathcal{P}_0(x)}{z-x^s}),$$

and by (23) we have

$$G^{(1)}_{n-1}(z) = \mathcal{F}(z)G_n(z) - ((G_n(x^s))^T\mathcal{U}_x)(\frac{1}{z}\sum_{k=0}^{\infty}(\frac{x^s}{z})^k\mathcal{P}_0(x))$$
$$= \mathcal{F}(z)G_n(z) - \sum_{k=0}^{\infty}\frac{((G_n(x^s))^T\mathcal{U}_x)((x^s)^k\mathcal{P}_0(x))}{z^{k+1}}$$
$$= \mathcal{F}(z)G_n(z) - \sum_{m=0}^{\infty}\sum_{l=0}^{d-1}\frac{((G_n(x^s))^T\mathcal{U}_x)((x^s)^l\mathcal{P}_m(x))}{z^{dm+l+1}}.$$

Hence, we get (24) if, and only if, $\{G_n\}$ is type I multiple orthogonal polynomial with respect to $\mathcal{U}$. ■

Theorem 14. *Let $\mathcal{U}$ be a regular vector of linear functionals, and $\mathcal{F}$ be its matrix generating function. Then, the following conditions are equivalent:*

a) The sequences $\{G_n\}$ and $\{\mathcal{B}_m\}$ are bi-orthogonal with respect to $\mathcal{U}$, i.e.

$$((G_n(x^s))^T\mathcal{U}_x)(\mathcal{B}_m) = I_{sd\times sd}\,\delta_{n,m}, \quad n,m\in\mathbb{N}.$$

b) The sequences $\{G_n\}$ and $\{V_m\}$, where $\{V_m\}$ is defined by $\mathcal{B}_m(z) = V_m(z^{sd})\mathcal{P}_0$ for $m\in\mathbb{N}$, are bi-orthogonal with respect to $\mathcal{F}$, i.e.,

$$\frac{1}{2\pi i}\int_C V_m(z^d)\mathcal{F}(z)G_n(z)dz = I_{sd\times sd}\,\delta_{n,m}, \quad n,m\in\mathbb{N}.$$

Proof: Taking into account that

$$V_m(z^d)\mathcal{F}(z)G_n(z) = (\frac{V_m(z^d)\mathcal{P}_0(x)}{z-x^s})\mathcal{U}_x^T G_n(z) = ((G_n(z))^T\mathcal{U}_x)(\frac{V_m(z^d)\mathcal{P}_0(x)}{z-x^s}),$$

we have

$$\frac{1}{2\pi i}\int_C V_m(z^d)\mathcal{F}(z)G_n(z)dz = \frac{1}{2\pi i}\int_C((G_n(z))^T\mathcal{U}_x)(\frac{V_m(z^d)\mathcal{P}_0(x)}{z-x^s})dz.$$

Being G_n, V_n and $\mathcal{P}_0$ analitic functions, by the Cauchy integral formula, we have

$$\frac{1}{2\pi i}\int_C((G_n(z))^T\mathcal{U}_x)(\frac{V_m(z^d)\mathcal{P}_0(x)}{z-x^s})dz = ((G_n(x^s))^T\mathcal{U}_x)(V_m(x^{sd})\mathcal{P}_0(x)),$$

and so, we have for all $n,m\in\mathbb{N}$

$$\frac{1}{2\pi i}\int_C V_m(z^d)\mathcal{F}(z)G_n(z)dz = ((G_n(x^s))^T\mathcal{U}_x)(\mathcal{B}_m(x)) = I_{sd\times sd}\,\delta_{n,m}.$$

From this the result follows. ■

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A. Branquinho
CMUC, Department of Mathematics, University of Coimbra, Largo D. Dinis, 3001-454 Coimbra, Portugal.

E-mail address: ajplb@mat.uc.pt

L. Cotrim
School of Technology and Management, Polytechnic Institute of Leiria, Campus 2 - Morro do Lena - Alto do Vieiro, 2411 - 901 LEIRIA - PORTUGAL.

E-mail address: lmsc@estg.ipleiria.pt

A. Foulquié Moreno
Departamento de Matemática, Universidade de Aveiro, Campus de Santiago 3810, Aveiro, Portugal.

E-mail address: foulquie@ua.pt