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THE SEMICONTINUOUS QUASI-UNIFORMITY OF A FRAME, REVISITED

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ABSTRACT: In this note we present a new treatment of the pointfree version of the semicontinuous quasi-uniformity based on the new tool of the ring of arbitrary (not necessarily continuous) real-valued functions made available recently by J. Gutiérrez García, T. Kubiak and J. Picado [Localic real functions: a general setting, *Journal of Pure and Applied Algebra* 213 (2009) 1064-1074]. The purpose is to show how the basic facts about the semicontinuous quasi-uniformity can be easily presented and proved with that tool at hand.

KEYWORDS: Frame, quasi-uniform frame, quasi-uniform biframe, quasi-metric quasi-uniformity, totally bounded quasi-uniformity, semicontinuous real function, biframe of reals, countably compact frame.

AMS SUBJECT CLASSIFICATION (2000): 06D22, 54C30, 54E05, 54E15, 54E55.

1. Introduction

Let X be a locale with corresponding frame $L = \mathcal{O}(X)$. The lattice of sublocales of X (that is, the subobject lattice of X in the category of locales) may be described in several equivalent ways. Here we use the following one [18]:

> a subset S of L is a *sublocale* of X if, whenever $A \subseteq S$, $a \in L$ and $b \in S$, then $\bigwedge A \in S$ and $a \to b \in S$.

Any intersection of sublocales is again a sublocale, so that the set of all sublocales is a complete lattice under inclusion. In fact, it is a co-frame. We make it into a frame $\mathcal{S}(L)$ by considering the dual ordering $S_1 \leq S_2$ iff $S_2 \subseteq S_1$. Among the important examples of sublocales are the *closed sublocales*

$$\mathfrak{c}(a) = \uparrow a = \{b \in L : a \leq b\}$$

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and the *open sublocales*

$$\mathfrak{o}(a) = \{a \to b : b \in L\}$$

for every $a \in L$ (which are complements of each other). The map $a \mapsto \mathfrak{c}(a)$ is a frame embedding $L \hookrightarrow \mathcal{S}(L)$. The subframe of $\mathcal{S}(L)$ consisting of all closed sublocales will be denoted by $\mathfrak{c}L$. It is isomorphic to L. Denoting by $\mathfrak{o}L$ the subframe of $\mathcal{S}(L)$ generated by all $\mathfrak{o}(a)$, $a \in L$, the triple $(\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$ constitutes a biframe.

It is well-known that a quasi-uniformity $\mathcal{E}$ on a set X may be described in several equivalent ways, most notably as a collection of ordered pairs of covers of X (the *paircover* approach) and as a collection of relations on X (the *entourage* approach). Associated with any quasi-uniformity $\mathcal{E}$ on X there is the bitopological space $(X, \mathfrak{T}_{\mathcal{E}}, \mathfrak{T}_{\mathcal{E}^{-1}})$ induced by $\mathcal{E}$.

In the pointfree setting, the theory of quasi-uniformities was first exploited using the paircover approach [8, 9]; the Weil entourages of [15, 16, 17] provided then the direct analogue of entourages. The former is defined as a structure $\mathcal{U}$ on a biframe (L_0, L_1, L_2) and the latter directly as a structure $\mathcal{E}$ on a frame L which establishes two subframes $L_1(\mathcal{E})$ and $L_2(\mathcal{E})$ of L such that the triple $(L, L_1(\mathcal{E}), L_2(\mathcal{E}))$ is a biframe (this is the pointfree version of the bitopological space $(X, \mathfrak{T}_{\mathcal{E}}, \mathfrak{T}_{\mathcal{E}^{-1}})$ above). The two approaches are equivalent [15, 16].

While the approach via paircovers is most convenient for calculations (the entourage approach asks for a good knowledge of the construction of binary coproducts of frames), the entourage approach allows to formulate the theory directly on frames, in a way very similar to the spatial setting [4, 5, 17]. For instance, given a frame L, there exists a (entourage) transitive quasi-uniformity $\mathcal{E}$ on the sublocale frame $\mathcal{S}(L)$ which is *compatible* with L, that is, $L_1(\mathcal{E}) = \mathfrak{c}L$ (which means that $L_1(\mathcal{E})$ is an isomorphic copy of the given frame L inside $\mathcal{S}(L)$) [4, 5]. This is the pointfree analogue of the well-known classical fact that for every topological space $(X, \mathfrak{T})$ there exists a transitive quasi-uniformity $\mathcal{E}$ on X, compatible with $(X, \mathfrak{T})$, that is, which induces as its first topology $\mathfrak{T}_{\mathcal{E}}$ the given topology $\mathfrak{T}$.

The semicontinuous quasi-uniformity $\mathcal{USC}(L)$ of L is a nice example of a transitive compatible quasi-uniformity [5, 6]. The purpose of this paper is to show how the basic facts about $\mathcal{USC}(L)$ can be nicely presented with the help of the ring of arbitrary (not necessarily continuous) real-valued functions made available recently by J. Gutiérrez García, T. Kubiak and J. Picado

[12]. To keep the background at the minimum possible we use the paircover approach [8, 10] to quasi-uniformities.

2. Background

For general information on locales and frames we refer to [13] and [18]. A *biframe* [2] is a triple $L = (L_0, L_1, L_2)$ in which L_0 is a frame, L_1 and L_2 are subframes of L_0 and $L_1 \cup L_2$ generates L_0 (by joins of finite meets). A *biframe map* $h : L \to M$ is a frame homomorphism from L_0 to M_0 such that the image of L_i under h is contained in M_i for $i = 1, 2$. Biframes and biframe maps are the objects and arrows of the category BiFrm. For more details on biframes consult [2].

Let $L = (L_0, L_1, L_2)$ be a biframe. A subset C of $L_1 \times L_2$ is a *paircover* [8, 10] of L if $\bigvee\{c_1 \wedge c_2 \mid (c_1, c_2) \in C\} = 1$. A paircover C of L is *strong* if, for any $(c_1, c_2) \in C$, $c_1 \vee c_2 = 0$ whenever $c_1 \wedge c_2 = 0$ (that is, $(c_1, c_2) = (0, 0)$ whenever $c_1 \wedge c_2 = 0$).

For any paircovers C and D of L we write $C \leq D$ (and say that C *refines* D) if for any $(c_1, c_2) \in C$ there is $(d_1, d_2) \in D$ with $c_1 \leq d_1$ and $c_2 \leq d_2$. Further $C \wedge D = \{(c_1 \wedge d_1, c_2 \wedge d_2) \mid (c_1, c_2) \in C, (d_1, d_2) \in D\}$. It is obvious that $C \wedge D$ is a paircover of L. For $a \in L_0$ and C, D paircovers of L, let

$$st_1(a, C) = \bigvee\{c_1 \mid (c_1, c_2) \in C \text{ and } c_2 \wedge a \neq 0\},$$

$$st_2(a, C) = \bigvee\{c_2 \mid (c_1, c_2) \in C \text{ and } c_1 \wedge a \neq 0\}$$

and

$$C \cdot D = \{(st_1(d_1, C), st_2(d_2, C)) \mid (d_1, d_2) \in D\}.$$

The particular case $C \cdot C$ is usually denoted by C^*. The paircover C is said to *star-refines* D if $C^* \leq D$.

The following lemma is easy to prove [8].

Lemma 2.1. *For any paircovers C, D of (L_0, L_1, L_2) and any $a, b \in L_0$ we have:*

(1) *$a \leq st_i(a, C)$ $(i = 1, 2)$.*
(2) *$a \leq b \Rightarrow st_i(a, C) \leq st_i(b, C)$ $(i = 1, 2)$.*
(3) *If $D^* \leq C$ then $st_i(st_i(a, D), D) \leq st_i(a, C)$ $(i = 1, 2)$.*
(4) *For any biframe map $h : (L_0, L_1, L_2) \to (M_0, M_1, M_2)$, $st_i(h(a), h[C]) \leq h(st_i(a, C))$ $(i = 1, 2)$, where $h[C] = \{(h(c_1), h(c_2)) \mid (c_1, c_2) \in C\}$.*

A non-empty family $\mathcal{U}$ of paircovers of $L = (L_0, L_1, L_2)$ is a *quasi-uniformity* on L if:

(U1) The family of strong members of $\mathcal{U}$ is a filter-base for $\mathcal{U}$ with respect to $\wedge$ and $\leq$.
(U2) For any $C \in \mathcal{U}$ there is $D \in \mathcal{U}$ such that $D^* \leq C$.
(U3) For each $a \in L_i$, $a = \bigvee\{b \in L_i \mid st_i(b, C) \leq a$ for some $C \in \mathcal{U}\}$, $(i = 1, 2)$.

The pair $(L, \mathcal{U})$ is called a *quasi-uniform biframe* [10]. $\mathcal{B} \subseteq \mathcal{U}$ is a *base* for $\mathcal{U}$ if, for each $C \in \mathcal{U}$, there is $B \in \mathcal{B}$ such that $B \leq C$.

Let $(L, \mathcal{U})$ and $(M, \mathcal{V})$ be quasi-uniform biframes. A biframe map $h : L \to M$ is *uniform* if for every $C \in \mathcal{U}$, $h[C] \in \mathcal{V}$. Quasi-uniform biframes and uniform maps constitute a category that we denote by $\mathsf{QUBiFrm}$.

The *biframe of reals* is the triple $(\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R}))$ where $\mathfrak{L}(\mathbb{R})$ is the *frame of reals* [1] defined by generators $(p, q) \in \mathbb{Q} \times \mathbb{Q}$ and relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,
(R3) $(p, q) = \bigvee\{(r, s) : p < r < s < q\}$,
(R4) $\bigvee_{p,q \in \mathbb{Q}}(p, q) = 1$.

We shall use also the following notation:

$$(p, -) = \bigvee_{q \in \mathbb{Q}} (p, q) \quad \text{and } (-, q) = \bigvee_{p \in \mathbb{Q}} (p, q);$$

note that $(p, -) \wedge (-, q) = (p, q)$.

Equivalently, $\mathfrak{L}(\mathbb{R})$ may be defined by taking $(p, -)$ and $(-, q)$ as primitive notions, with relations

(S1) $(p, -) \wedge (-, q) = 0$ whenever $p \geq q$,
(S2) $(p, -) \vee (-, q) = 1$ whenever $p < q$,
(S3) $(p, -) = \bigvee_{r > p}(r, -)$,
(S4) $(-, q) = \bigvee_{s < q}(-, s)$,
(S5) $\bigvee_{p \in \mathbb{Q}}(p, -) = 1$,
(S6) $\bigvee_{q \in \mathbb{Q}}(-, q) = 1$.

Then $\mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_l(\mathbb{R})$ are just the following subframes of $\mathfrak{L}(\mathbb{R})$:

$$\begin{aligned} \mathfrak{L}_u(\mathbb{R}) &= \langle \{(p, -) : p \in \mathbb{Q},\ (p, -) \text{ satisfy (R3) and (R5) for all } p \in \mathbb{Q}\} \rangle, \\ \mathfrak{L}_l(\mathbb{R}) &= \langle \{(-, q) : q \in \mathbb{Q},\ (-, q) \text{ satisfy (R4) and (R6) for all } q \in \mathbb{Q}\} \rangle. \end{aligned}$$

In general topology one sometimes deals with arbitrary (not necessarily continuous) real-valued functions on a topological space X. This is also possible in the pointfree setting with the approach recently introduced in [12] (which extends the approach to pointfree continuous real functions of Banaschewski [1]). Let L be a frame. A *real-valued function* on L is a frame homomorphism $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$. It is

(1) *lower semicontinuous* if $f(\mathfrak{L}_u(\mathbb{R})) \subseteq \mathfrak{c}L$,
(2) *upper semicontinuous* if $f(\mathfrak{L}_l(\mathbb{R})) \subseteq \mathfrak{c}L$,
(3) *continuous* if $f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L$.

The set $\mathrm{F}(L)$ of all real-valued functions on L is partially ordered by

$$\begin{aligned} f \le g \qquad &\Leftrightarrow \qquad f(p,\text{—}) \le g(p,\text{—}) \quad \text{for every } p \in \mathbb{Q} \\ &\Leftrightarrow \qquad g(\text{—},q) \le f(\text{—},q) \quad \text{for every } q \in \mathbb{Q}. \end{aligned}$$

We denote by $\mathrm{LSC}(L), \mathrm{USC}(L)$ and $\mathrm{C}(L)$ the collections of all lower semicontinuous, upper semicontinuous, and continuous members of $\mathrm{F}(L)$. Of course, one has

$$\mathrm{C}(L) = \mathrm{LSC}(L) \cap \mathrm{USC}(L).$$

Note that $\mathrm{USC}(L) \simeq \mathsf{BiFrm}\left((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), (\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)\right)$.

A nice way of constructing real functions is with the help of the so called scales [12]. A collection of sublocales $\{S_r : r \in \mathbb{Q}\} \subseteq \mathcal{S}(L)$ is a *scale* on $\mathcal{S}(L)$ if $S_r \vee S_s^* = 1$ whenever $r < s$ and $\bigvee\{S_r : r \in \mathbb{Q}\} = 1 = \bigvee\{S_r^* : r \in \mathbb{Q}\}$ (here S^* denotes the pseudocomplement of S). For each scale $\{S_r : r \in \mathbb{Q}\}$ in $\mathcal{S}(L)$ the function f defined by

$$f(p,\text{—}) = \bigvee_{r>p} S_r \quad \text{and} \quad f(\text{—},q) = \bigvee_{r<q} {S_r}^* \quad (p,q \in \mathbb{Q}) \tag{2.1}$$

belongs to $\mathrm{F}(L)$. If, moreover, each S_r is an open sublocale then $f \in \mathrm{USC}(L)$.

For instance, given a complemented sublocale S of L, with complement $\neg S$, the *characteristic map* $\chi_S : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ is defined by

$$\chi_S(p,\text{—}) = \begin{cases} 1 & \text{if } p < 0, \\ \neg S & \text{if } 0 \le p < 1, \\ 0 & \text{if } p \ge 1, \end{cases} \quad \text{and} \quad \chi_S(\text{—},q) = \begin{cases} 0 & \text{if } q \le 0, \\ S & \text{if } 0 < q \le 1, \\ 1 & \text{if } q > 1, \end{cases}$$

for each $p, q \in \mathbb{Q}$ [12]. Then, as in the classical context, we have:

(a) $\chi_S \in \mathrm{LSC}(L)$ if and only if S is open,
(b) $\chi_S \in \mathrm{USC}(L)$ if and only if S is closed,

(c) $\chi_S \in \mathrm{C}(L)$ if and only if S is clopen.

For any $f \in \mathrm{F}(L)$ the *upper regularization* $f^- \in \mathrm{USC}(L)$ of f is defined by

$$f^-(p,-) = \bigvee_{q>p} \neg\, \overline{f(-,q)} \quad \text{and} \quad f^-(-,p) = \bigvee_{q<p} \overline{f(-,q)}$$

(see [11] and [12] for more information). Of course, when $f \in \mathrm{USC}(L)$ then $f^- = f$. Thus, for any $f \in \mathrm{USC}(L)$, we have

$$f(p,-) = \bigvee_{q>p} \neg\, f(-,q) \in \mathfrak{o}L \quad \text{and} \quad f(-,p) = \bigvee_{q<p} f(-,q) \in \mathfrak{c}L. \tag{2.2}$$

3. The semicontinuous quasi-uniformity $\mathcal{USC}(L)$

For each $n \in \mathbb{N}$,

$$Q_n = \Big\{((-,q),(p,-)) \mid p,q \in \mathbb{Q},\, 0 < q-p < \frac{1}{n}\Big\}$$

is a strong paircover of the biframe $(\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R}))$. These paircovers satisfy the following (easy to check) properties:

Lemma 3.1. (1) *For every $n \in \mathbb{N}$ and $p,q \in \mathbb{Q}$ with $p<q$, $\frac{1}{q-p} < n$, we have:*
   (a) $st_1((-,p),Q_n) \le (-,q)$.
   (b) $st_2((q,-),Q_n) \le (p,-)$.
(2) *For every $p_i, q_i \in \mathbb{Q}$ with $p_i < q_i$, we have:*
   (a) $st_1(\bigvee_{i\in I}(p_i,q_i),Q_n) = st_1(\bigvee_{i\in I}(-,q_i),Q_n)$.
   (b) $st_2(\bigvee_{i\in I}(p_i,q_i),Q_n) = st_2(\bigvee_{i\in I}(p_i,-),Q_n)$.
(3) *For each $n \in \mathbb{N}$, $Q_{n+1} \subseteq Q_n$ (thus $Q_{n+1} \le Q_n$).* ■

Moreover:

Proposition 3.2. *For every $n \in \mathbb{N}$ and $p \in \mathbb{Q}$, we have:*

(1) $Q^*_{3n} \le Q_n$.
(2) $(-,p) = \bigvee\{(-,q) \in \mathfrak{L}_l(\mathbb{R}) \mid st_1((-,q),Q_n) \le (-,p)$ *for some* $n \in \mathbb{N}\}$.
(3) $(p,-) = \bigvee\{(q,-) \in \mathfrak{L}_u(\mathbb{R}) \mid st_2((q,-),Q_n) \le (p,-)$ *for some* $n \in \mathbb{N}\}$.

Proof: (1) Let $((-,q),(p,-)) \in Q_{3n}$. We have to show that there is

$$((-,\tilde{q}),(\tilde{p},-)) \in Q_n$$

such that $st_1((-,q),Q_{3n}) \leq (-,\tilde{q})$ and $st_2((p,-),Q_{3n}) \leq (\tilde{p},-)$. But

$$\begin{aligned} st_1((-,q),Q_{3n}) &= \bigvee\{(-,d_1) \mid ((-,d_1),(d_2,-)) \in Q_{3n}, (d_2,-) \wedge (-,q) \neq 0\} \\ &\leq (-,q+\frac{1}{3n}) \end{aligned}$$

since $(d_2,-) \wedge (-,q) \neq 0 \Leftrightarrow d_2 < q$ and $0 < d_1 - d_2 < \frac{1}{3n}$ (which implies $d_1 < d_2 + \frac{1}{3n} < q + \frac{1}{3n}$). Similarly,

$$\begin{aligned} st_2((p,-),Q_{3n}) &= \bigvee\{(d_2,-) \mid ((-,d_1),(d_2,-)) \in Q_{3n}, (-,d_1) \wedge (p,-) \neq 0\} \\ &\leq (p-\frac{1}{3n},-). \end{aligned}$$

It suffices then to take $\tilde{q} = q + \frac{1}{3n}$ and $\tilde{p} = p - \frac{1}{3n}$. Indeed, $((-,q+\frac{1}{3n}),(p-\frac{1}{3n},-)) \in Q_n$, since $0 < q + \frac{1}{3n} - p + \frac{1}{3n} < \frac{1}{3n} + \frac{1}{3n} + \frac{1}{3n} = \frac{1}{n}$.
(2) By Lemma 3.1(1), for every $q < p$ there is some $n \in \mathbb{N}$ such that $st_1((-,q),Q_n) \leq (-,p)$. Thus, by Lemma 2.1(1),

$$\begin{aligned} (-,p) &= \bigvee_{q<p}(-,q) \leq \bigvee\{(-,q) \mid st_1((-,q),Q_n) \leq (-,p) \text{ for some } n \in \mathbb{N}\} \\ &\leq (-,p). \end{aligned}$$

(3) may be proved similarly. ∎

In conclusion, the strong paircovers Q_n ($n \in \mathbb{N}$), generate a quasi-uniformity $\mathcal{Q}$ on the biframe of reals $(\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R}))$.

Corollary 3.3. *The pair* $((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), \mathcal{Q})$ *is a quasi-uniform biframe.* ∎

We refer to it as the *quasi-metric quasi-uniformity* of the reals.

Now let $f \in \mathrm{USC}(L)$. Then (recall (2.2))

$$f(p,-) = \bigvee_{q>p} \neg f(-,q) \in \mathfrak{o}L \quad \text{and} \quad f(-,p) = \bigvee_{q<p} f(-,q) \in \mathfrak{c}L$$

so $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ is a biframe map

$$f : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L).$$

Clearly, for each $n \in \mathbb{N}$,

$$C_{f,n} = \{(f(\text{—},q), f(p,\text{—})) \mid p,q \in \mathbb{Q},\ f(p,q) \neq 0,\ 0 < q-p < \frac{1}{n}\}$$

is a strong paircover of the sublocale lattice $(\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$. Further, we have [6]:

Lemma 3.4. (1) *For any* $f_1, \dots, f_k \in \mathrm{USC}(L)$*,* $n_1, \dots, n_k \in \mathbb{N}$ *and* $S \in \mathcal{S}(L)$*:*

(a) $st_1(S, \bigwedge_{i=1}^k C_{f_i,n_i}) \in \mathfrak{c}L$.

(b) $st_2(S, \bigwedge_{i=1}^k C_{f_i,n_i}) \in \mathfrak{o}L$.

(2) *For any* $a \in L$ *and* $n \in \mathbb{N}$*:*

(a) $st_1(\mathfrak{c}(a), C_{\chi_{\mathfrak{c}(a)},n}) = \mathfrak{c}(a)$.

(b) $st_2(\mathfrak{o}(a), C_{\chi_{\mathfrak{c}(a)},n}) = \mathfrak{o}(a)$. ■

We have finally the required result that extends Proposition 1.1 of [14] (also Theorem 3.1 of [3]).

Proposition 3.5. $\{C_{f,n} \mid f \in \mathrm{USC}(L), n \in \mathbb{N}\}$ *is a subbase for a quasi-uniformity* $\mathcal{USC}(L)$ *on the biframe* $(\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$*.*

Proof: For each $(f(-,q), f(p,-)) \in C_{f,3n}$ we have

$$st_1(f(-,q), C_{f,3n}) \leq f(-, q + \frac{1}{3n})$$

and

$$st_2(f(p,-), C_{f,3n}) \leq f(p - \frac{1}{3n}, -)$$

(the proof goes as in Proposition 3.2). Since $f(p - \frac{1}{3n}, q + \frac{1}{3n}) \geq f(p,q) \neq 0$, this shows that $C^*_{f,3n} \leq C_{f,n}$.

Conditions (U1) and (U3) follow immediately from Lemma 3.4. ■

$\mathcal{USC}(L)$ is called the *semicontinuous quasi-uniformity* on L. This can be immediately generalized to any collection $\mathcal{C}$ containing all characteristic functions χ_S for a closed sublocale S:

Corollary 3.6. *Let* $\mathcal{C}$ *be a collection of upper semicontinuous real functions, containing all upper characteristic functions* $\chi_{\mathfrak{c}(a)}$ $(a \in L)$*. Then* $\{C_{f,n} \mid f \in \mathcal{C}, n \in \mathbb{N}\}$ *is a subbase for a quasi-uniformity* $\mathcal{U}_{\mathcal{C}}$ *on the biframe* $(\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$*.* ■

4. Properties of $\mathcal{USC}(L)$

Proposition 4.1. *$\mathcal{USC}(L)$ is the coarsest quasi-uniformity $\mathcal{U}$ on $(\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$ for which each biframe map $h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$ is a uniform homomorphism $h : ((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), \mathcal{Q}) \to ((\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L), \mathcal{U})$.*

Proof: We begin by checking that any biframe map

$$h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$$

is a uniform homomorphism

$$((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), \mathcal{Q}) \to ((\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L), \mathcal{USC}(L)),$$

that is, $h[Q_n] \in \mathcal{USC}(L)$ for every $n \in \mathbb{N}$. Obviously, the frame map $h : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ belongs to $\mathrm{USC}(L)$. It suffices then to show that $C_{h,n} \leq h[Q_n]$, which is obvious since $C_{h,n} \subseteq h[Q_n]$.

Now let $\mathcal{U}$ be a quasi-uniformity on $(\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$ for which any biframe map

$$h : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$$

is a uniform homomorphism

$$h : ((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), \mathcal{Q}) \to ((\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L), \mathcal{U}).$$

In order to show that $\mathcal{USC}(L) \subseteq \mathcal{U}$ it suffices to check that, for any $f \in \mathrm{USC}(L)$ and $n \in \mathbb{N}$, $C_{f,n} \in \mathcal{U}$. By hypothesis,

$$f[Q_n] = \{(f(-,q), f(p,-)) \mid p, q \in \mathbb{Q},\ 0 < q - p < \frac{1}{n}\} \in \mathcal{U}.$$

So there is a strong paircover $C \in \mathcal{U}$ such that $C \leq f[Q_n]$. Then $C \leq C_{f,n}$. Indeed, for any $((-,q),(p,-)) \in C$ there are $\tilde{p}, \tilde{q} \in \mathbb{Q}$ with $(-,q) \leq f(-,\tilde{q})$, $(p,-) \leq f(\tilde{p},-)$ and $0 < \tilde{q} - \tilde{p} < \frac{1}{n}$; since $(p,q) \neq 0$, then $f(\tilde{p},\tilde{q}) \neq 0$.

Hence $C_{f,n} \in \mathcal{U}$ as required. ■

For every frame L,

$$\{(\mathfrak{c}(a), 1), (1, \mathfrak{o}(a)) \mid a \in L\}$$

is a subbase for a quasi-uniformity on $(\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$ [8]. It is clearly a quasi-uniformity compatible with the given frame L since the first subframe $\mathfrak{c}L$ is an isomorphic copy of L. This is the pointfree analogue of the Császár-Pervin quasi-uniformity of a set X. We refer to it as the *Frith quasi-uniformity* and denote it by $\mathcal{F}$.

Since

$$C_{\chi_{\mathfrak{c}(a)},n} = \{(\chi_{\mathfrak{c}(a)}(-,q), \chi_{\mathfrak{c}(a)}(p,-)) \mid p,q \in \mathbb{Q}, 0 < q-p < \frac{1}{n}, \chi_{\mathfrak{c}(a)}(p,q) \neq 0\},$$

then it is straightforward to check the following.

Lemma 4.2. *For each characteristic function $\chi_{\mathfrak{c}(a)}$, $a \in L$,*

$$C_{\chi_{\mathfrak{c}(a)},n} = \{(\mathfrak{c}(a), 1), (1, \mathfrak{o}(a))\}. \quad \blacksquare$$

Therefore, for $\mathcal{C} = \{\chi_{\mathfrak{c}(a)} \mid a \in L\}$, $\mathcal{U}_\mathcal{C}$ and $\mathcal{F}$ have a common subbase and we have:

Corollary 4.3. *Let $\mathcal{C} = \{\chi_{\mathfrak{c}(a)} \mid a \in L\}$. Then $\mathcal{U}_\mathcal{C}$=$\mathcal{F}$.* $\blacksquare$

A real-valued function $f \in \mathrm{F}(L)$ is *bounded* [12] if there exist some $p < q$ in $\mathbb{Q}$ for which $f(p,q) = 1$. More generally, f is *upper bounded* if $f(-,q) = 1$ for some $q \in \mathbb{Q}$. Since every upper characteristic function $\chi_{\mathfrak{c}(a)}$ is bounded, the previous corollary leads immediately to the following result, which is the pointfree extension of Proposition 2.10 of [7].

Proposition 4.4. *Let $\mathcal{C}$ be the collection of all bounded upper semicontinuous real functions on L. Then $\{C_{f,n} \mid f \in \mathcal{C}, n \in \mathbb{N}\}$ is a subbase for $\mathcal{F}$.* $\blacksquare$

Proposition 4.5. *Let $h : (\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L) \to (\mathcal{S}(M), \mathfrak{c}M, \mathfrak{o}M)$ be a biframe map. Then h is a uniform homomorphism*

$$((\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L), \mathcal{USC}(L)) \to ((\mathcal{S}(M), \mathfrak{c}M, \mathfrak{o}M), \mathcal{USC}(M)).$$

Proof: Let $C_{f,n} \in \mathcal{USC}(L)$, for some $f \in \mathrm{USC}(L)$ and $n \in \mathbb{N}$. Evidently, $hf \in \mathrm{USC}(M)$ and

$$\begin{aligned} h[C_{f,n}] &= \{(hf(-,q), hf(p,-)) \mid p,q \in \mathbb{Q}, 0 < q-p < \frac{1}{n}, f(p,q) \neq 0\} \\ &\geq C_{hf,n} \in \mathcal{SC}(M) \end{aligned}$$

because $hf(p,q) \neq 0 \Rightarrow f(p,q) \neq 0$. $\blacksquare$

We say that a quasi-uniform biframe $(L, \mathcal{U})$ is *totally bounded* if $\mathcal{U}$ has a base of finite paircovers.

Lemma 4.6. *If $((\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L), \mathcal{U})$ is a totally bounded quasi-uniform biframe then every uniform homomorphism*

$$h : ((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), \mathcal{Q}) \to ((\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L), \mathcal{U})$$

is bounded.

Proof: Let $h : ((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), \mathcal{Q}) \to ((\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L), \mathcal{U})$ be a uniform homomorphism. For each $n \in \mathbb{N}$, $h[Q_n] \in \mathcal{U}$, so there exists a finite paircover

$$C = \{(\mathfrak{c}(a_1), \mathfrak{o}(b_1)), \cdots, (\mathfrak{c}(a_k), \mathfrak{o}(b_k))\}$$

of $\mathcal{S}(L)$ such that $C \leq h[Q_n]$. Therefore, for each $i \in \{1, \cdots, k\}$, $\mathfrak{c}(a_i) \leq h(-, q_i)$ and $\mathfrak{o}(b_i) \leq h(p_i, -)$ for some $p_i, q_i \in \mathbb{Q}$ with $0 < q_i - p_i < \frac{1}{n}$. Hence $1 = \bigvee_{i=1}^{k} \mathfrak{c}(a_i) \wedge \mathfrak{o}(b_i) \leq \bigvee_{i=1}^{k} h(p_i, q_i)$. Let $q = \max_{i=1,\dots,k} q_i$ and $p = \min\limits_{i=1,\dots,k} p_i$. Immediately, $h(p, q) = 1$ and h is bounded. ∎

Proposition 4.7. *Let $((\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L), \mathcal{U})$ be a totally bounded quasi-uniform frame. Then there exists a collection $\mathcal{C}$ of bounded $f \in \mathrm{USC}(L)$ such that $\{C_{f,n} \mid f \in \mathcal{C}, n \in \mathbb{N}\}$ is a subbase for $\mathcal{U}$.*

Proof: Let $((\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L), \mathcal{U})$ be a totally bounded quasi-uniform frame. Every uniform homomorphism

$$h : ((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), \mathcal{Q}) \to ((\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L), \mathcal{U}),$$

which is bounded by Lemma 4.6, is upper semicontinuous. Let $\mathcal{C}$ be the collection of every such maps. Since $\mathcal{C}$ contains all characteristic functions $\chi_{\mathfrak{c}(a)}$ $(a \in L)$, then, by Corollary 3.6, $\{C_{h,n} \mid h \in \mathcal{C}, n \in \mathbb{N}\}$ is a subbase for a quasi-uniformity $\mathcal{U}_{\mathcal{C}}$ on $(\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$. Since h is uniform,

$$h[Q_n] = \{(h(-, q), h(p, -)) \mid p, q \in \mathbb{Q},\, 0 < q - p < \frac{1}{n}\} \in \mathcal{U}.$$

So there is a strong paircover $C \in \mathcal{U}$ such that $C \leq h[Q_n]$. Then $C \leq C_{h,n}$ (the proof is similar to the proof at the end of 4.1 that $C \leq C_{f,n}$). Hence $\{C_{h,n} \mid h \in \mathcal{C}, n \in \mathbb{N}\}$ is also a subbase for $\mathcal{U}$. ∎

Theorem 4.8. *Let L be a frame. Then $\mathcal{USC}(L)$ is totally bounded if and only if every $f \in \mathsf{USC}(L)$ is bounded.*

Proof: Assume that $\mathcal{USC}(L)$ is totally bounded and let $f \in \mathrm{USC}(L)$. Then we have a biframe map $f : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \to (\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$ which, by Proposition 4.1, is uniform. Then, by Lemma 4.6, f is bounded.

Conversely, let $\mathcal{C} = \mathrm{USC}(L) = \{\text{bounded } f \in \mathrm{USC}(L)\}$. Then $\mathcal{U}_{\mathcal{C}} = \mathcal{USC}(L)$ coincides by Proposition 4.4 with $\mathcal{F}$. Since $\mathcal{F}$ is totally bounded, then $\mathcal{USC}(L)$ is totally bounded. ∎

Recall that a frame is *countably compact* if each countable cover has a finite subcover.

Theorem 4.9. *Let L be a frame. Then every $f \in \mathsf{USC}(L)$ is upper bounded if and only if L is countably compact.*

Proof: Let $A = \{a_i \mid i \in \mathbb{N}\}$ be a countable cover of L. For each $q \in \mathbb{Q}$ let $m(q) = \min\{n \in \mathbb{N}_0 \mid n \geq q\}$. Further, let $a_0 = 0$ and define, for each $r \in \mathbb{Q}$,

$$S_r = \mathfrak{o}\Big(\bigvee_{i=0}^{m(r)} a_i\Big).$$

This is clearly a scale of open sublocales so, by (2.1), the function f defined by

$$f(p,-) = \bigvee_{r>p} \mathfrak{o}\Big(\bigvee_{i=0}^{m(r)} a_i\Big) \quad \text{and} \quad f(-,q) = \bigvee_{r<q} \mathfrak{c}\Big(\bigvee_{i=0}^{m(r)} a_i\Big) \quad (p, q \in \mathbb{Q})$$

is in USC(L). By hypothesis, f is bounded. Consequently, there is some $q \in \mathbb{Q}$ for which $f(-,q) = 1$. This means precisely that

$$1 = \bigvee_{r<q} \mathfrak{c}\Big(\bigvee_{i=0}^{m(r)} a_i\Big) = \mathfrak{c}\Big(\bigvee_{r<q} \bigvee_{i=0}^{m(r)} a_i\Big) = \mathfrak{c}\Big(\bigvee_{i=0}^{m(q)} a_i\Big),$$

that is, $\bigvee_{i=0}^{m(q)} a_i = 1$. Hence $\{a_1, \ldots, a_{m(q)}\}$ is a finite subcover of A. This shows that L is countably compact.

Conversely, let L be countably compact and let $f \in \mathsf{USC}(L)$. Then $\{f(-,q) \mid q \in \mathbb{Q}\}$ is a countable cover of $\mathfrak{c}L \cong L$. By hypothesis, there exist $q_1, \ldots, q_k \in \mathbb{Q}$ such that $\bigvee_{i=1}^{k} f(-,q_i) = 1$, that is, $f(-, \bigvee_{i=1}^{k} q_i) = 1$, which shows that f is upper bounded. ■

This is the pointfree counterpart of Lemma 3.2 of [3]. Our last result extends Corollary 3.3 of [3]. It asserts that every frame L with a unique compatible quasi-uniform structure is countably compact.

Corollary 4.10. *If $(\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$ has a unique quasi-uniform structure then L is countably compact.*

Proof: If $\mathcal{U}$ is the unique quasi-uniform structure on $(\mathcal{S}(L), \mathfrak{c}L, \mathfrak{o}L)$ then $\mathcal{U}$ coincides with $\mathcal{F}$ which is totally bounded. But also $\mathcal{U} = \mathcal{USC}(L)$ so, by the theorems above, L is countably compact. ■

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