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CONVEXITY OF THE KREIN SPACE TRACIAL NUMERICAL RANGE AND MORSE THEORY

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ABSTRACT: In this paper we present a Krein space convexity theorem on the tracial-numerical range of a matrix. This theorem is the analogue of Westwick's theorem. The proof is an application of Morse theory.

KEYWORDS: Numerical range, indefinite inner product, J-Hermitian matrix, non-interlacing.

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1. Introduction

Let J denote a Hermitian involutive matrix, that is, $J^* = J$, $J^2 = I_n$, with *signature* $(r, n-r)$, $0 < r \leq n$ (i.e. with r positive and $n-r$ negative eigenvalues). We consider $\mathbf{C}^n$ endowed with the indefinite inner product induced by J:

$$[x, y] = y^* J x, \qquad x, y \in \mathbf{C}^n.$$

Let M_n be the algebra of $n \times n$ complex matrices. A matrix $H \in M_n$ is J-Hermitian if $H^\# = H$, where $H^\# = JH^*J$ is the *J-adjoint* of H. A matrix $U \in M_n$ is *J-unitary* if $U^\# U = UU^\# = I_n$, the identity matrix of size n. The J-unitary matrices form a locally compact connected group denoted by $G = \mathcal{U}(r, n-r)$ and called the *J-unitary group*. For $C, T \in M_n$, the *J-tracial numerical range* of T is denoted and defined as

$$W_C^J(T) = \{\mathrm{Tr}(CUTU^{-1}) : U \in \mathcal{U}(r, n-r)\}. \tag{1.1}$$

This set is a connected set in the Gaussian plane $\mathbf{C}$ and satisfies the symmetry property $W_C^J(T) = W_T^J(C)$ (see [1], [2], [4] for other properties). In the case $r = n$, the matrix J reduces to the identity matrix and the group $\mathcal{U}(n, 0) = \mathcal{U}(n)$ is the *unitary group*, which is a compact group. In this case $W_C^J(T)$ is simply denoted by $W_C(T)$ and called the *C-numerical range of T*. Hausdorff proved the convexity of $W_C(T)$ in the case C is a rank one orthogonal projection. Using Morse theory, Westwick [8] proved the

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convexity of $W_C(T)$ for C Hermitian and in [5] Poon obtained an alternative proof by majorization techniques. In [7], Tam generalized Westwick's result for certain compact Lie groups. In this paper we obtain a non-compact group analogue of Westwick's convexity theorem applying Morse theory (cf. Lemma 4.1). The eigenvalues of a J-unitarily diagonalizable J-Hermitian matrix T are real and will be denoted as:

$$\sigma_+(T) = \{\lambda \in \mathbf{C} : T\xi = \lambda\xi \,\text{for some}\, \xi \in \mathbf{C}^n, [\xi, \xi] > 0\},$$

$$\sigma_-(T) = \{\lambda \in \mathbf{C} : T\xi = \lambda\xi \,\text{for some}\, \xi \in \mathbf{C}^n, [\xi, \xi] < 0\}.$$

The eigenvalues are said to be *noninterlacing* if

$$\max \sigma_-(T) < \min \sigma_+(T) \quad \text{or} \quad \max \sigma_+(T)) < \min \sigma_-(T). \tag{1.2}$$

Replacing $<$ in (1.2) by $\leq$ the eigenvalues are said to be *weakly noninterlacing*. Noninterlacing property plays a key role in the investigation of numerical ranges associated with non-compact groups.

In the sequel we consider $J = I_r \oplus (-I_{n-r})$. Our main result is the following.

Theorem 1.1. *Let C be a J-unitarily diagonalizable J-Hermitian matrix with noninterlacing eigenvalues and let $T \in M_n$ be a matrix such that for some $\theta \in \mathbf{R}$*

$$T_\theta = \cos\theta(T + T^\#)/2 + \sin\theta(T - T^\#)/(2i)$$

is a J-unitarily diagonalizable J-Hermitian matrix with noninterlacing eigenvalues:

$$\sigma_+(C) = \{c_1 \geq \cdots \geq c_r\}, \sigma_-(C) = \{c_{r+1} \geq \cdots \geq c_n\},$$

$$\sigma_+(T_\theta) = \{\lambda_1 \geq \cdots \geq \lambda_r\}, \sigma_-(T_\theta) = \{\lambda_{r+1} \geq \cdots \geq \lambda_n\},$$

$$c_n > c_1 \text{ or } c_r > c_{r+1}, \quad \lambda_n > \lambda_1 \text{ or } \lambda_r > \lambda_{r+1}.$$

Then $W_C^J(T)$ is a closed convex set in $\mathbf{C}$ contained in a closed cone $\{z_0 + r e^{i\eta} : r \geq 0, \theta_1 \leq \eta \leq \theta_2\}$ with $0 \leq \theta_2 - \theta_1 < \pi$.

Denote by $C^\infty(G)$ the commutative algebra of all C^∞-differentiable complex-valued functions on $G = \mathcal{U}(r, n-r)$. We consider the even dimensional coset space $M = G/D$, where

$$D = \{\text{diag}(d_1, \ldots, d_n) : d_1, \ldots, d_n \in \mathbf{C}, |d_1| = \cdots = |d_n| = 1\}.$$

Let the real valued function f_0 on M be defined by

$$f_0(g) = \sum_{h,j=1}^{n} c_h \lambda_j |u_{hj}|^2 \epsilon_h \epsilon_j = \mathrm{Tr}\,(C g T_\theta g^{-1}), \tag{1.3}$$

where $\epsilon_h = 1$ for $1 \leq h \leq r$ and $\epsilon_h = -1$ for $r+1 \leq h \leq n$.

This paper is organized as follows. The critical points and the Morse indices of the function f_0 are investigated in Section 2. Global properties of f_0 and of $W_C^J(T)$ are studied in Section 3. The proof of Theorem 1.1 is presented in Section 4 using the results in Sections 2, 3.

2. Local properties of the function f_0

To treat local properties of the function f_0, we introduce some notation and prerequisites. The Lie algebra of $G = \mathcal{U}(r, n-r)$ is given as

$$\mathfrak{G} = \{X \in M_n : X^{\#} = -X\}. \tag{2.1}$$

For $1 \leq k_0 < l_0 \leq r$ or $r+1 \leq k_0 < l_0 \leq n$, let

$$X_{k_0,l_0} = (\delta_{kk_0}\delta_{ll_0} - \delta_{kl_0}\delta_{lk_0}), \quad Y_{k_0,l_0} = i(\delta_{kk_0}\delta_{ll_0} + \delta_{kl_0}\delta_{lk_0}),$$

where δ_{kl} is the Kronecker symbol. For $1 \leq k_0 \leq r,\ r+1 \leq l_0 \leq n$, let

$$V_{k_0,l_0} = (\delta_{kk_0}\delta_{ll_0} + \delta_{kl_0}\delta_{lk_0}), \quad W_{k_0,l_0} = i(\delta_{kk_0}\delta_{ll_0} - \delta_{kl_0}\delta_{lk_0}).$$

Throughout, we consider the basis of the Lie algebra $\mathfrak{G}$ constituted by the matrices X_{k_0,l_0}, Y_{k_0,l_0}, V_{k_0,l_0}, W_{k_0,l_0} and the diagonal matrices $D_1 = \mathrm{diag}(i, 0, \ldots, 0), \cdots, D_n = \mathrm{diag}(0, \ldots, 0, i)$, which generate the Lie algebra $\mathfrak{D}$. We shall use a canonical coordinate system of second kind

$$\exp(t_1 X_1)\exp(t_2 X_2)\cdots\exp(t_{n^2} X_{n^2})$$

in a neighborhood of the identity I_n, where $\{X_{n^2-n+1}, \ldots, X_{n^2}\}$ is the above basis of $\mathfrak{D}$, and $\{X_1, X_2, \ldots, X_{n^2-n}\}$ is the remaining system of vectors of the basis of $\mathfrak{G}$.

For $f \in C^\infty(G)$, consider a representation α of $\mathfrak{G}$ on $C^\infty(G)$ given by

$$\alpha(X)(f)(g) = -\lim_{t\to 0} \frac{f(\mathrm{e}^{tX} g) - f(g)}{t} = \lim_{t\to 0} \frac{f(\mathrm{e}^{-tX} g) - f(g)}{t}. \tag{2.2}$$

The function space $C^\infty(G/D)$ is identified with the space

$$\{f \in C^\infty(G) : f(gd) = f(g)\,\mathrm{for}\, d \in D\}. \tag{2.3}$$

For S_n the symmetric group of degree n, let

$$S_{r,n-r} = \{\sigma_1\sigma_2 : \sigma_1(j) = j\,(r+1 \le j \le n), \sigma_2(j) = j\,(1 \le j \le r)\}.$$

Proposition 2.1. *Let $H = \mathrm{diag}\,(\lambda_1, \cdots, \lambda_n)$ and $C = \mathrm{diag}\,(c_1, \cdots, c_n)$ be such that $c_h \neq c_j$, $\lambda_h \neq \lambda_j$ for $1 \le h \neq j \le n$. A point $g = (u_{hj})$ of G is a critical point of f_0 if and only if (u_{hj}) is a permutation matrix associated with a permutation in $S_{r,n-r}$.*

Proof. We take an arbitrary critical point g. A straightforward computation shows that

$$-\alpha(X)(f_0)(g) = \mathrm{Tr}\,(X[gHg^{-1}, C]) = 0,$$

where $[X, Y] = XY - YX$ stands for the commutator. Thus, gHg^{-1} commutes with the diagonal matrix C, whose diagonal entries are pairwise distinct, and so

$$gHg^{-1} = \mathrm{diag}\,(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}),$$

for some $\sigma \in S_n$. For $g = (u_{hj})$, we have $u_{hj}\lambda_j = \lambda_{\sigma(h)}u_{hj}$ and so $u_{hj} = 0$ unless $\lambda_j = \lambda_{\sigma(h)}$. Therefore, $u_{hj} = \eta_j\delta_{j\sigma(h)}$, where $|\eta_j| = 1$. We notice that $gHg^{-1} = dgHg^{-1}d^{-1}$ for any $d \in D$. Moreover, σ belongs to $S_{r,n-r}$ because $g^{-1} = Jg^*J$. The converse is clear. □

Proposition 2.2. *Let $H = \mathrm{diag}\,(\lambda_1, \cdots, \lambda_n)$ and $C = \mathrm{diag}\,(c_1, \cdots, c_n)$ be such that $c_h \neq c_j$, $\lambda_h \neq \lambda_j$ for $1 \le h \neq j \le n$. The numbers of positive and negative eigenvalues of the Hessian matrix of f_0 at each critical point are even.*

Proof. We analyze the Hessian matrix of the function f_0 at a critical point. For $f \in C^\infty(G)$, the representation α of $\mathfrak{G}$ satisfies

$$\alpha(X)\alpha(Y)(f)(g) = \frac{d}{dt}\frac{d}{ds}\, f(\mathrm{e}^{-sY}\mathrm{e}^{-tX}g)\Big|_{s,t=0}. \tag{2.4}$$

For $X = -X^\#$, consider the expansion of $f(\mathrm{e}^{-tX}g)$ in powers of t

$$f_0(\mathrm{e}^{-tX}g) = \mathrm{Tr}(CgHg^{-1}) - t\,\mathrm{Tr}(C[X, gHg^{-1}]) + \frac{t^2}{2}\mathrm{Tr}(C[X, [X, gHg^{-1}]]) + \mathcal{O}(t^3).$$

Assuming g is critical, we get

$$f_0(\mathrm{e}^{-tX}g) = \mathrm{Tr}(CgHg^{-1}) + \frac{t^2}{2}\mathrm{Tr}(C[X, [X, gHg^{-1}]]) + \mathcal{O}(t^3).$$

Let $\mathcal{M}(X,Y) = \alpha(X)\alpha(Y)(f_0)(g) = \mathrm{Tr}(C[Y,[X,gHg^{-1}]])$. Having in mind Jacobi identity and performing some computations, we find

$$\alpha(X)\alpha(Y)(f_0)(g) = \mathrm{Tr}(C[Y,[X,gHg^{-1}]]) = \mathrm{Tr}([C,Y][X,gHg^{-1}]), \tag{2.5}$$

and so $\mathcal{M}(X,Y) = \mathcal{M}(Y,X)$. For $X = (x_{kl})$, $Y = (y_{kl})$, we have $[C,Y] = (y_{hj}(c_h - c_j))$ and $[X,gHg^{-1}] = (x_{hj}(\lambda_{\sigma(j)} - \lambda_{\sigma(h)}))$. By straightforward computations we obtain

$$\begin{aligned}\mathcal{M}(X,Y) &= \mathrm{Tr}(C[Y,[X,gHg^{-1}]]) = \mathrm{Tr}\left((y_{hj}(c_h - c_j))\left(x_{hj}(\lambda_{\sigma(j)} - \lambda_{\sigma(h)})\right)\right) \\ &= \sum_{k,l=1}^{n} y_{kl}x_{lk}(c_k - c_l)(\lambda_{\sigma(k)} - \lambda_{\sigma(l)}).\end{aligned}$$

The matrix representation of $\mathcal{M}(X,Y)$, relative to the fixed basis in $\mathfrak{G}/\mathfrak{D}$, is diagonal. In fact, for $1 \leq k_0 < l_0 \leq r$ or $r+1 \leq k_0 < l_0 \leq n$, an easy computation leads to

$$\mathcal{M}(X_{k_0l_0}, X_{k_0'l_0'}) = \mathcal{M}(Y_{k_0l_0}, Y_{k_0'l_0'}) = -2\delta_{k_0k_0'}\delta_{l_0l_0'}(c_{k_0} - c_{l_0})(\lambda_{\sigma(k_0)} - \lambda_{\sigma(l_0)}).$$

For $1 \leq k_0, k_0' \leq r,\ r+1 \leq l_0, l_0' \leq n$, we analogously have

$$\mathcal{M}(V_{k_0l_0}, V_{k_0'l_0'}) = \mathcal{M}(W_{k_0l_0}, W_{k_0'l_0'}) = 2\delta_{k_0k_0'}\delta_{l_0l_0'}(c_{k_0} - c_{l_0})(\lambda_{\sigma(k_0)} - \lambda_{\sigma(l_0)}).$$

The off-diagonal entries of $\mathcal{M}(X,Y)$ vanish. Thus, the $(n^2-n) \times (n^2-n)$-Hessian matrix of the function f_0 at every critical point is nonsingular. Moreover, the numbers of positive and negative eigenvalues of the Hessian matrix are even. □

We recall, in passing, that the number of negative eigenvalues of the Hessian is called the *Morse index.*

3. Global properties of the function f_0 and $W_C^J(H)$

Proposition 3.1. *Let C and H be J-unitarily diagonalizable J-Hermitian matrices with respective eigenvalues $\sigma_+(C) = \{c_1, \ldots, c_r\}$, $\sigma_-(C) = \{c_{r+1}, \ldots, c_n\}$, $\sigma_+(H) = \{\lambda_1, \ldots, \lambda_r\}$, $\sigma_-(H) = \{\lambda_{r+1}, \ldots, \lambda_n\}$ satisfying the conditions*

$$c_{r+1} \geq \cdots \geq c_n > c_1 \geq \cdots \geq c_r, \tag{3.1}$$

$$\lambda_1 \geq \cdots \geq \lambda_r > \lambda_{r+1} \geq \cdots \geq \lambda_n. \tag{3.2}$$

Then the superlevel set $\{U \in \mathcal{U}(r, n-r) : a \leq f_0(U)\}$ is compact for an arbitrary real number a.

Proof. Analogously to (1.3) we define a function g_0 for the diagonal J-Hermitian matrices $C' = \mathrm{diag}(c'_1, \dots, c'_n)$, $H' = \mathrm{diag}(\lambda'_1, \dots, \lambda'_n)$ with eigenvalues $c'_1 = \cdots = c'_r = c_1, c'_{r+1} = \cdots = c'_n = c_n$ and $\lambda'_1 = \cdots = \lambda'_r = \lambda_r$, $\lambda'_{r+1} = \cdots = \lambda'_n = \lambda_{r+1}$. We show that the functions f_0, g_0 satisfy the inequality $f_0(g) \leq g_0(g)$ for $g \in G$. To prove this inequality, we may assume that $c_n > 0 > c_1$, $\lambda_r > 0 > \lambda_{r+1}$ by adding suitable scalar operators to C and H. Observing that $c_1\lambda_r - c_h\lambda_j = (c_1 - c_h)\lambda_r - c_h(\lambda_j - \lambda_r)$ $(1 \leq h, j \leq r)$, the function $g_0(U) - f_0(U)$ can be expressed as a linear combination of $|u_{h,j}|^2$ $(1 \leq h, j \leq n)$ with nonnegative coefficients. Henceforth the inclusion holds

$$\{[U] \in M : f_0([U]) \geq a\} \subset \{[U] \in M : g_0(U) \geq a\}, \tag{3.3}$$

and the left-hand side of (3.3) is a closed subset of the set in the right-hand side. It can be easily checked that

$$g_0(U) = rc_1\lambda_r + (n-r)c_n\lambda_{r+1} - \frac{c_n - c_1}{4}(\lambda_r - \lambda_{r+1}) \left(\sum_{h,j=1}^{n} |u_{h,j}|^2 - n \right),$$

so the right-hand side of (3.3) is compact for an arbitrary a and so is the left-hand side. □

Proposition 3.2. *Let C be a J-unitarily diagonalizable J-Hermitian matrix with noninterlacing eigenvalues $\sigma_+(C) = \{c_1 \geq \cdots \geq c_r\}$, $\sigma_-(C) = \{c_{r+1} \geq \ldots \geq c_n\}$, $c_n > c_1$. Let H be a J-Hermitian matrix such that $W_C^J(H) = (-\infty, a_0]$. Then the matrix H is J-unitarily diagonalizable and its eigenvalues are weakly noninterlacing.*

Proof. We assume that the maximum value a_0 is attained at $\mathrm{Tr}(CgHg^{-1})$. This assumption implies that $\mathrm{Tr}(X[gHg^{-1}, C]) = 0$ for an arbitrary $X \in \mathfrak{G}$, so that $[gHg^{-1}, C] = 0$. The condition $c_n > c_1$ implies that $gHg^{-1} = H_1 \oplus H_2$, where H_1, H_2 are Hermitian block matrices of sizes r and $n-r$, respectively. Thus, gHg^{-1} is J-unitarily diagonalizable as $VgHg^{-1}V^{-1} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, where V is a J-unitary matrix of the form $V = V_1 \oplus V_2$, the blocks V_1, V_2 being unitary of sizes r and $n-r$, respectively. Weak noninterlacing property of the eigenvalues follows from Theorem 1.1 (iii) in [1]. □

Proposition 3.3. *Let $C = \mathrm{diag}(c_1, \dots, c_n)$ with $\sigma_+(C) = \{c_1 \geq \cdots \geq c_r\}$, $\sigma_-(C) = \{c_{r+1} \geq \cdots \geq c_r\}$, $c_n > c_1$, and let $H = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ with*

$\sigma_+(H) = \{\lambda_1 \geq \cdots \geq \lambda_r\}$, $\sigma_-(H) = \{\lambda_{r+1} \geq \cdots \geq \lambda_n\}$, $\lambda_r \geq \lambda_{r+1}$, so that

$$\left\{\mathrm{Tr}(CgHg^{-1}) : g \in \mathcal{U}(r, n-r)\right\} = \left(-\infty, \sum_{j=1}^{n} c_j\lambda_j\right].$$

Then, for K J-Hermitian, the set

$$\mathcal{W} = \left\{\mathrm{Tr}(CgKg^{-1}) : g \in \mathcal{U}_{r,n-r}, \mathrm{Tr}(CgHg^{-1}) = \sum_{j=1}^{n} c_j\lambda_j\right\}$$

is connected.

Proof. Let $H = (\lambda'_1 I_{n_1}) \oplus \cdots \oplus (\lambda'_s I_{n_s})$, where the λ'_j are all distinct. Consider $C = C_1 \oplus \cdots \oplus C_s$, $J = J_1 \oplus \cdots \oplus J_s$, and let K be the J-Hermitian block matrix

$$K = \begin{pmatrix} K_{11} & \dots & K_{1s} \\ \dots & \dots & \dots \\ K_{s1} & \dots & K_{ss} \end{pmatrix},$$

where C_j, J_j, $K_{jj} \in M_{n_j}$, $j = 1, \dots, s$. If $\mathrm{Tr}(CgHg^{-1}) = \sum_{j=1}^{n} c_j\lambda_j$, then $g = WV$, where $W, V \in \mathcal{U}(r, n-r)$, $WCW^{-1} = C$ and $VHV^{-1} = H$. Obviously, $V = V_1 \oplus \cdots \oplus V_s$, $V_j \in M_{n_j}$, and $V_j J_j V_j^* = J_j$. Moreover,

$$VKV^{-1} = \begin{pmatrix} V_1K_{11}V_1^{-1} & \dots & V_1K_{1s}V_s^{-1} \\ \dots & \dots & \dots \\ V_sK_{s1}V_1^{-1} & \dots & V_sK_{ss}V_s^{-1} \end{pmatrix}$$

and

$$\mathrm{Tr}(CgKg^{-1}) = \sum_{j=1}^{n} \mathrm{Tr}(C_jV_jK_{jj}V_j^{-1}).$$

Thus

$$\mathcal{W} = W_C^{J_1}(K_{11}) + W_{C_2}^{J_2}(K_{22}) + \cdots + W_{C_s}^{J_s}(K_{ss}). \tag{3.4}$$

Since each summand in the Minkowski sum in the right hand side of (3.4) is connected, the result follows. $\square$

4. Proof of Theorem 1.1

Lemma 4.1. (cf. [3, 6]). *Let M be an m-dimensional connected C^∞-manifold ($m \geq 2$) and let f be a real-valued C^∞-function on M. Assume that the function f satisfies the following conditions:*

- (i) *The function f attains a minimum value;*
- (ii) *The set $\{x \in M : a \leq f(x) \leq b\}$ is compact for every $-\infty < a < b < +\infty$;*
- (iii) *The number of critical points $x_1, x_2, \ldots, x_n$ of f is finite, and the critical values $f(x_1), f(x_2), \ldots, f(x_n)$ are all distinct;*
- (iv) *The Hessian matrix of f at each critical point x_j is non-singular;*
- (v) *The number of the negative eigenvalues of the Hessian matrix of f at each critical point x_j is neither 1 nor $m-1$.*

Then the level set $\{x \in M : f(x) = a\}$ is connected for every $a \in \mathbf{R}$.

Some critical values of the function f_0 given by (1.3) at distinct critical points may coincide. Then we apply a perturbative method. We choose a vector $(y_1, \ldots, y_n)$ satisfying $\sum_{k=1}^n y_k(c_{\sigma(k)} - c_{\tau(k)}) \neq 0$ for any distinct pair $\sigma, \tau \in \mathcal{S}_{r,n-r}$. We can find $t_0 \in \mathbf{R}$ with sufficiently small modulus such that $\lambda_k + y_k t_0$ satisfy the condition (iii) of Lemma 4.1.

Now, we prove Theorem 1.1. First we show the simple connectedness of $W_C^J(T)$ under the assumptions of the theorem. Consider the case $c_n > c_1$, $\lambda_r > \lambda_{r+1}$. The remaining cases reduce to this one by replacing C by $-C$ and/or T_θ by $T_{\theta+\pi}$. By a rotation in the Gaussian plane $\mathbf{C}$, we assume that $\theta = 0$ and so $H = T_0 = (T + T^\#)/2$ is a J-unitarily diagonalizable J-Hermitian matrix with noninterlacing eigenvalues $\lambda_r > \lambda_{r+1}$. Define the following sequences of J-Hermitian matrices

$$C_m = \mathrm{diag}(c_1^{(m)}, \ldots, c_n^{(m)}), \quad H_m = \mathrm{diag}(\lambda_1^{(m)}, \ldots, \lambda_n^{(m)}),$$

where $c_n^{(m)} = c_n > c_1 = c_1^{(m)}$, $\lambda_r^{(m)} = \lambda_r > \lambda_{r+1} = \lambda_{r+1}^{(m)}$ for $m = 1, 2, 3, \ldots$. Suppose that $|c_h^{(m)} - c_h| \to 0, \quad |\lambda_h^{(m)} - \lambda_h| \to 0$ as $m \to \infty$; $c_h^{(m)} \neq c_j^{(m)}$, $\lambda_h^{(m)} \neq \lambda_j^{(m)}$ for each m, $1 \leq h < j \leq n$ and assume that the critical values $\sum_{h=1}^n c_h^{(m)} \lambda_{\sigma(h)}^{(m)}$ for any distinct permutations $\sigma \in \mathcal{S}_{r,n-r}$ are distinct. By Proposition 3.1 the set

$$\{g \in U(r, n-r) : \mathrm{Tr}(C_m g H_m g^{-1}) \geq b\} \cup \{g \in U(r, n-r) : \mathrm{Tr}(C g H g^{-1}) \geq b\}$$

is contained in a compact set B for any $b \in \mathbf{R}$. The pairs $\{C_m, H_m\}$ satisfy the conditions of Propositions 2.1 and 2.2. It follows that these pairs satisfy the conditions of Lemma 4.1. By Lemma 4.1, each set $\{g \in U(r, n-r) : \mathrm{Tr}(C_m g H_m g^{-1}) = a - 1/2^m\}$ is a compact connected set. Hence for $T_m = H_m + i(T - T^{\#})/(2i) \in M_n$ and $a \in \mathbf{R}$, $\mathcal{I}_m = \{z \in W^J_{C_m}(T_m) : \Re(z) = a - 1/2^m\}$ is a closed interval. For every $a \leq \sum_{h=1}^n c_h \lambda_h$, the set $\{z \in W^J_C(T) : \Re(z) = a\}$ is compact. Let

$$b_0 = \min\{y \in \mathbf{R} : a + iy \in W^J_C(T)\}, \quad c_0 = \max\{y \in \mathbf{R} : a + iy \in W^J_C(T)\},$$

$$\begin{aligned} b_m &= \min\{y \in \mathbf{R} : a - \frac{1}{2^m} + iy \in W^J_{C_m}(T_m)\}, \\ c_m &= \max\{y \in \mathbf{R} : a - \frac{1}{2^m} + iy \in W^J_{C_m}(T_m)\}. \end{aligned}$$

Let us take an arbitrary $0 < t_0 < 1$ and choose a point U_m in B satisfying

$$\mathrm{Tr}(C_m U_m T_m U_m^{-1}) = a - \frac{1}{2^m} + i(t_0 b_m + (1 - t_0) c_m)$$

for each m. Since the set B is compact, we consider a subsequence U_{m_k} of U_m converging to a point U_0 of B. Then as $k \to \infty$

$$\mathrm{Tr}(C_{m_k} U_{m_k} T_{m_k} U_{m_k}^{-1}) \to \mathrm{Tr}(C U_0 T U_0^{-1}) = a + i(t_0 b_0 + (1 - t_0) c_0).$$

Hence the set

$$\{z \in W^J_C(T) : \Re(z) = a\}, \tag{4.1}$$

is also a closed interval. We set $a_0 = \sum_{h=1}^n c_h \lambda_h$. Since the set (4.1) depends continuously on a, then

$$W^J_C(T) = \{x + iy : -\infty < x \leq a_0, \phi_1(x) \leq y \leq \phi_2(x)\}$$

for some continuous real-valued functions $\phi_1(x) \leq \phi_2(x)$ defined on the half-line $(-\infty, a_0]$, and so it is simply connected.

Now we prove that the boundary of $W^J_C(T)$, $\partial W^J_C(T)$, is convex. The set $W^J_C(T)$ has a point z_0 with $\Re(z_0) = a_0$ and the line $\Re(z) = a_0$ is a support line of $W^J_C(T)$. If $\partial W^J_C(T)$ is not convex, there exists a tangent line at some point $z_1 \in \partial W^J_C(T)$ expressed as $\Re(z \mathrm{e}^{-i\theta_1}) = b_0$ for some $\theta_1, b_0 \in \mathbf{R}$ satisfying $\Re(z_2 \mathrm{e}^{-i\theta_1}) < b_0$, $\Re(z_3 \mathrm{e}^{-i\theta_1}) > b_0$, where $z_2, z_3 \in W^J_C(T)$. Then we find a point z_4 on the arc of $\partial W^J_C(T)$ joining z_0, z_1 and a support line ℓ_θ passing through z_4, $\Re(z \mathrm{e}^{-i\theta_0}) = a_\theta$, such that $W^J_C(T) \cap \ell_{\theta_0}$ is not connected. However the last condition of ℓ_θ contradicts Proposition 3.3. So $\partial W^J_C(T)$ is convex.

We take a point $x_0 < a_0$ at which ϕ_1, ϕ_2 are differentiable. Then we have $\phi_2'(x_0) \leq \phi_1'(x_0)$ and the cone $\{x + iy \in \mathbf{C} : \phi_1'(x_0)(x - x_0) + \phi_1(x_0) \leq y \leq \phi_2'(x_0)(x - x_0) + \phi_2(x_0)\}$ satisfies the last assertion of Theorem 1.1. This completes the proof of the theorem. $\square$

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