Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 09–34

ON THE DESCRIPTION OF FRAME ASYMMETRIC UNIFORMITIES VIA PAIRCOVERS

JORGE PICADO

Abstract: A quasi-uniformity on a frame may be equivalently described either in terms of paircovers or in terms of entourages. The former is defined as a structure $\mathcal{U}$ on a biframe (L_0, L_1, L_2) and the latter directly as a structure $\mathcal{E}$ on a frame L which induces two subframes $L_1(\mathcal{E})$ and $L_2(\mathcal{E})$ of L such that the triple $(L, L_1(\mathcal{E}), L_2(\mathcal{E}))$ is a biframe (this is the pointfree analogue of the bitopological space $(X, \mathfrak{T}(\mathcal{E}), \mathfrak{T}(\mathcal{E}^{-1}))$ induced by any quasi-uniformity $\mathcal{E}$ on the set X). While the approach via paircovers is most convenient for calculations, it does not faithfully reflect the spatial original notion since it is not formulated directly on frames.

Here, it will be shown that it is possible to describe frame quasi-uniformities by defining the paircovering structure $\mathcal{U}$ directly on a frame L without requiring the prior knowledge of the underlying biframe. It turns out that the biframe structure $(L, L_1(\mathcal{U}), L_2(\mathcal{U}))$ appears only *a posteriori*, induced by the structure in a very natural way. In addition, we indicate how to do the same for strong relations on biframes and the corresponding quasi-proximal frames.

Keywords: Frame, biframe, entourage, paircover, quasi-uniform frame, quasi-proximal frame, strong inclusion, quasi-uniformity, quasi-proximity.

AMS Subject Classification (2000): 06D22, 54E05, 54E15, 54E55.

1. The problem

The study of quasi-uniformities began in 1948 with Nachbin's investigations on uniform preordered spaces (see [13, 14]). A quasi-uniformity $\mathcal{E}$ on a set X may be described in several equivalent ways, most notably as a collection of relations on X (the *entourage* approach [5]) and as a collection of ordered pairs of covers of X (the *paircover* approach of [10]). Associated with any quasi-uniformity $\mathcal{E}$ on X there is the well-known bitopological space $(X, \mathfrak{T}(\mathcal{E}), \mathfrak{T}(\mathcal{E}^{-1}))$ induced by $\mathcal{E}$. In the pointfree setting, the theory of quasi-uniformities was first exploited by J. Frith using the paircover

Received September 16, 2009.
Support from the Centre for Mathematics of the University of Coimbra (CMUC/FCT) and the Ministry of Education and Science of Spain and FEDER under grant MTM2009-12872-C02-02 is gratefully acknowledged.

approach [6, 7]; the Weil entourages of [15, 16, 17] provided then the direct analogue of entourages. The former is defined as a structure $\mathcal{C}$ on a biframe (L_0, L_1, L_2) and the latter directly as a structure $\mathcal{E}$ on a frame L which establishes two subframes $L_1(\mathcal{E})$ and $L_2(\mathcal{E})$ of L such that the triple $(L, L_1(\mathcal{E}), L_2(\mathcal{E}))$ is a biframe (this is the pointfree version of the bitopological space $(X, \mathfrak{T}(\mathcal{E}), \mathfrak{T}(\mathcal{E}^{-1}))$ above). The categories defined by the two approaches are isomorphic [15, 16].

The motivation for this paper originates in the talk presented by J. Frith and A. Schauerte at the III Workshop on Aspects of Contemporary Topology (Antwerp, December 2007) and their subsequent paper [9]. Frith's treatment of the category of quasi-uniform frames made use of the approach to quasi-uniform spaces via conjugate pairs of covers due to Gantner and Steinlage [10] (the concept of a biframe, which was crucial for Frith's study had been introduced before by Banaschewski, Brümmer and Hardie [1]).

Let (L_0, L_1, L_2) be a biframe. A subset C of $L_1 \times L_2$ is a *paircover* [6] of (L_0, L_1, L_2) if $\bigvee\{c_1 \wedge c_2 \mid (c_1, c_2) \in C\} = 1$. A paircover C of (L_0, L_1, L_2) is *strong* if, for any $(c_1, c_2) \in C$, $c_1 \vee c_2 = 0$ whenever $c_1 \wedge c_2 = 0$ (that is, $(c_1, c_2) = (0, 0)$ whenever $c_1 \wedge c_2 = 0$). For any paircovers C and D of (L_0, L_1, L_2) one writes $C \leq D$ (and say that C *refines* D) if for any $(c_1, c_2) \in C$ there is $(d_1, d_2) \in D$ with $c_1 \leq d_1$ and $c_2 \leq d_2$. Further $C \wedge D = \{(c_1 \wedge d_1, c_2 \wedge d_2) \mid (c_1, c_2) \in C, (d_1, d_2) \in D\}$. It is obvious that $C \wedge D$ is a paircover of (L_0, L_1, L_2). For $a \in L_0$ and C a paircover of (L_0, L_1, L_2), let

$$st_1(a, C) = \bigvee\{c_1 \mid (c_1, c_2) \in C \text{ and } c_2 \wedge a \neq 0\},$$

$$st_2(a, C) = \bigvee\{c_2 \mid (c_1, c_2) \in C \text{ and } c_1 \wedge a \neq 0\}$$

and

$$C^* = \{(st_1(c_1, C), st_2(c_2, C)) \mid (c_1, c_2) \in C\}.$$

A non-empty family $\mathcal{C}$ of paircovers of (L_0, L_1, L_2) is a *quasi-uniformity* [6] on (L_0, L_1, L_2) if:

(C1) For any $C \in \mathcal{C}$ and any paircover D with $C \leq D$, then $D \in \mathcal{C}$.
(C2) For any $C, D \in \mathcal{C}$ there exists a strong $E \in \mathcal{C}$ such that $E \leq C \wedge D$.
(C3) For any $C \in \mathcal{C}$ there is a $D \in \mathcal{C}$ such that $D^* \leq C$.
(C4) For each $a \in L_i$, $a = \bigvee\{b \in L_i \mid st_i(b, C) \leq a$ for some $C \in \mathcal{C}\}$ $(i = 1, 2)$.

Axioms (C1) and (C2) assert that the family of strong members of $\mathcal{C}$ is a filter-base for $\mathcal{C}$ with respect to $\wedge$ and $\leq$.

$((L_0, L_1, L_2), \mathcal{C})$ is called a *quasi-uniform biframe* [9] (*quasi-uniform frame* in the original [6]). A subset $\mathcal{B}$ of $\mathcal{C}$ is a *base* for $\mathcal{C}$ if, for each $C \in \mathcal{C}$, there is a $B \in \mathcal{B}$ such that $B \leq C$.

Let $((L_0, L_1, L_2), \mathcal{C})$ and $((M_0, M_1, M_2), \mathcal{D})$ be quasi-uniform biframes. A biframe homomorphism $h : (L_0, L_1, L_2) \to (M_0, M_1, M_2)$ is *uniform* if for every $C \in \mathcal{C}$,

$$h[C] := \{(h(c_1), h(c_2)) \mid (c_1, c_2) \in C\} \in \mathcal{D}.$$

Quasi-uniform biframes and uniform biframe homomorphisms constitute a category that we denote by $\mathsf{QUBiFrm}$.

While the approach via paircovers is most convenient for calculations (the entourage approach asks for a good knowledge of the construction of binary coproducts of frames), in the entourage approach the quasi-uniformity is defined directly on a frame; do not require – as the (paircovering) quasi-uniformities – prior knowledge of the underlying biframe. This makes the analogy with the spatial case evident [17, 3] and clarifies the discussion in D. Doitchinov paper [2].

This paper is concerned with the description of frame quasi-uniformities via paircovers: is it possible to define a (paircovering) quasi-uniformity directly on a frame, without requiring prior knowledge of the underlying biframe? This is the question that we address in this paper.

After this introductory section we shall give, in the second section, the solution to the problem just described: the category QUFrm of quasi-uniform frames and uniform homomorphisms. Then, in Section 3, with a number of lemmas we prepare the ground for the proof, in Section 4, that our QUFrm is indeed a solution for the problem. Finally, in Section 5 we briefly present the dual adjunction between QUFrm and the category of quasi-uniform spaces and, in Section 6, we show how to introduce quasi-proximities directly on a frame.

For general information on locales and frames we refer to [12] and [18]. We recall that a *biframe* [1] is a triple (L_0, L_1, L_2) in which L_0 is a frame, L_1 and L_2 are subframes of L_0 and $L_1 \cup L_2$ generates L_0 (by joins of finite meets). A *biframe homomorphism* $h : (L_0, L_1, L_2) \to (M_0, M_1, M_2)$ is a frame homomorphism from L_0 to M_0 such that the image of L_i under h is contained in M_i for $i = 1, 2$. Biframes and biframe homomorphisms are the objects and arrows of the category BiFrm. For more details on biframes consult [1] or [19]. If (L_0, L_1, L_2) is a biframe and $a \in L_i$ $(i = 1, 2)$, we denote by $a^\bullet$ the element $\bigvee\{b \in L_j \mid a \wedge b = 0\}$ $(j \in \{1, 2\},\ j \neq i)$ [19]. This is the analogue on biframes of the *pseudocomplement* $a^* = \bigvee\{b \in L \mid a \wedge b = 0\}$ of any element a of a frame L.

2. The solution

Let L be a frame. In analogy with [6] we call $C \subseteq L \times L$ a *paircover* of L if $\bigvee\{c_1 \wedge c_2 \mid (c_1, c_2) \in C\} = 1$. A paircover C of L is *strong* if, for any $(c_1, c_2) \in C$, $c_1 \vee c_2 = 0$ whenever $c_1 \wedge c_2 = 0$. For any $C, D \subseteq L \times L$ we write $C \leq D$ (and say that C *refines* D) if for any $(c_1, c_2) \in C$ there is $(d_1, d_2) \in D$ with $c_1 \leq d_1$ and $c_2 \leq d_2$. Further, for any paircovers C and D, $C \wedge D = \{(c_1 \wedge d_1, c_2 \wedge d_2) \mid (c_1, c_2) \in C, (d_1, d_2) \in D\}$ is a paircover. The set of all paircovers of L has also non-empty arbitrary joins given just by union. For $a \in L$ and $C, D \subseteq L \times L$, we set

$$st_1(a, C) = \bigvee\{c_1 \mid (c_1, c_2) \in C \text{ and } c_2 \wedge a \neq 0\},$$

$$st_2(a, C) = \bigvee\{c_2 \mid (c_1, c_2) \in C \text{ and } c_1 \wedge a \neq 0\},$$

$$C^{-1} = \{(c_2, c_1) \mid (c_1, c_2) \in C\}$$

and

$$st(D, C) = \{(st_1(d_1, C), st_2(d_2, C)) \mid (d_1, d_2) \in D\}.$$

The particular case $st(C, C)$ is denoted by C^*. We say that C *star-refines* D if $C^* \leq D$.

Given a non-empty family $\mathcal{U}$ of paircovers of L, we write $a \overset{\mathcal{U}}{\lhd_i} b$ $(i = 1, 2)$ whenever $st_i(a, U) \leq b$ for some $U \in \mathcal{U}$, and define

$$L_i(\mathcal{U}) = \{a \in L \mid a = \bigvee\{b \in L \mid b \overset{\mathcal{U}}{\lhd_i} a\}\} \quad (i = 1, 2).$$

Definition 2.1. We say that a non-empty family $\mathcal{U}$ of paircovers of L is a *quasi-uniformity* on L if:

(U1) For any $U \in \mathcal{U}$ and any paircover V with $U \leq V$, then $V \in \mathcal{U}$.
(U2) For any $U, V \in \mathcal{U}$ there exists a strong $W \in \mathcal{U}$ such that $W \leq U \wedge V$.
(U3) For any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V^* \leq U$.
(U4) $(L, L_1(\mathcal{U}), L_2(\mathcal{U}))$ is a biframe.

In the following section we will show that Axiom (U4) may be replaced by a condition similar to (C4) containing no reference to biframes.

The pair $(L,\mathcal{U})$ is called a *quasi-uniform frame.* $\mathcal{B} \subseteq \mathcal{U}$ is a *base* for $\mathcal{U}$ if, for each $U \in \mathcal{U}$, there is a $B \in \mathcal{B}$ such that $B \leq U$.

Let $(L,\mathcal{U})$ and $(M,\mathcal{V})$ be quasi-uniform frames. A frame homomorphism $h : L \to M$ is *uniform* if $h[U] \in \mathcal{V}$ for every $U \in \mathcal{U}$. Quasi-uniform frames and uniform homomorphisms constitute a category that we denote by QUFrm. Proving that this category is isomorphic to QUBFrm is the main aim of this paper.

3. Lemmas for the proof

Lemma 3.1. *Let $U, V, W \subseteq L \times L$ and $a, b \in L$. Then, for $i, j = 1, 2$, we have:*

(1) *If $a \leq b$ then $st_i(a,U) \leq st_i(b,U)$.*
(2) *If $U \leq V$ then $st_i(a,U) \leq st_i(a,V)$.*
(3) *$a \wedge st_1(b,U) = 0$ iff $b \wedge st_2(a,U) = 0$.*
(4) *If U is a paircover then $a \leq st_i(a,U)$ and $U \leq U^*$.*
(5) *If U is a paircover then $st_i(st_i(a,U),U) \leq st_i(a,U^*)$.*
(6) *$st_i(a,U^{-1}) = st_j(a,U)$ $(j \neq i)$.*
(7) *If U is a paircover then $V \leq st(V,U)$.*
(8) *If U and V are paircovers and V is strong then $V \leq st(U,V)$.*
(9) *If $V \leq W$ then $st(U,V) \leq st(U,W)$ and $st(V,U) \leq st(W,U)$.*
(10) *If U is a paircover then $st(st(V,U),U) \leq st(V,U^*)$.*
(11) *For any frame homomorphism $h : L \to M$, $st_i(h(a),h[U]) \leq h(st_i(a,U))$.*
(12) *For any frame homomorphism $h : L \to M$, $h[U]^* \leq h[U^*]$.*

Proof: (1), (2) and (3) are trivial.

(4) For each $a \in L$ we have $a = a \wedge 1 = a \wedge \bigvee\{u_1 \wedge u_2 \mid (u_1,u_2) \in U\} = \bigvee\{a \wedge u_1 \wedge u_2 \mid (u_1,u_2) \in U, a \wedge u_1 \wedge u_2 \neq 0\} \leq \bigvee\{u_1 \mid (u_1,u_2) \in U, a \wedge u_2 \neq 0\}$. Thus $a \leq st_1(a,U)$. The case $i = 2$ is similar. Hence $U \leq U^*$.

(5) It is an immediate consequence of properties (2) and (3).

(6) Trivial.

(7) Follows immediately from property (4).

(8) Let $(v_1,v_2) \in V$. If $v_1 \wedge v_2 = 0$ then $(v_1,v_2) = (0,0)$ and so there is obviously some $(u_1,u_2) \in U$ such that $v_i \leq st_i(u_i,V)$ $(i = 1,2)$. Otherwise

$v_1 \wedge v_2 \neq 0$. Then, since U is a paircover, there is some $(u_1, u_2) \in U$ such that $u_1 \wedge u_2 \wedge v_1 \wedge v_2 \neq 0$. Immediately $v_i \leq st_i(u_i, V)$ $(i = 1, 2)$.

(9) The first assertion follows from property (2) and the second from (1).

(10) It follows from property (5).

(11) It is obvious, since $h(a) \neq 0$ implies $a \neq 0$ for every frame homomorphism h.

(12) It is an immediate consequence of (11). ∎

Lemma 3.2. *Let $\mathcal{U}$ be a base for a filter of paircovers of L. Then, for $i = 1, 2$, the relations $\overset{\mathcal{U}}{\lhd}_i$ are sublattices of $L \times L$, stronger than $\leq$, satisfying the following properties:*

1. *For any $a, b, c, d \in L$, $a \leq b \overset{\mathcal{U}}{\lhd}_i c \leq d$ implies $a \overset{\mathcal{U}}{\lhd}_i d$.*
2. *For any $a, b \in L$, $a \overset{\mathcal{U}}{\lhd}_i b$ implies $a \prec b$ (that is, $a^* \vee b = 1$).*
3. *$L_i(\mathcal{U})$ is a subframe of L.*

Proof: The fact that each $\overset{\mathcal{U}}{\lhd}_i$ is stronger than $\leq$ follows from 3.1(4). Clearly $0 \overset{\mathcal{U}}{\lhd}_i 0$ and $1 \overset{\mathcal{U}}{\lhd}_i 1$. If $st_i(a_1, U_1) \leq b_1$ and $st_i(a_2, U_2) \leq b_2$ with $U_1, U_2 \in \mathcal{U}$ then, immediately, $st_i(a_1 \wedge a_2, U_1 \wedge U_2) \leq st_i(a_1, U_1) \wedge st_i(a_2, U_2) \leq b_1 \wedge b_2$. Since $\mathcal{U}$ is a filter basis, there exists some $V \in \mathcal{U}$ such that $V \leq U_1 \wedge U_2$. Hence, using property 3.1(2), we may conclude that $a_1 \wedge a_2 \overset{\mathcal{U}}{\lhd}_i b_1 \wedge b_2$. On the other hand, as can be easily checked, $st_i(a_1 \vee a_2, U_1 \wedge U_2) \leq st_i(a_1, U_1) \vee st_i(a_2, U_2) \leq b_1 \vee b_2$. Thus $a_1 \vee a_2 \overset{\mathcal{U}}{\lhd}_i b_1 \vee b_2$.

(1) It is obvious from Lemma 3.1(1).

(2) Let $i, j \in \{1, 2\}$ with $i \neq j$. Assume $a \overset{\mathcal{U}}{\lhd}_i b$, that is, $st_i(a, U) \leq b$ for some $U \in \mathcal{U}$. Let $u_1 \wedge u_2$ with $(u_1, u_2) \in U$. If $u_j \wedge a = 0$ then $u_1 \wedge u_2 \leq u_j \leq a^*$; otherwise, $u_1 \wedge u_2 \leq u_i \leq st_i(a, U) \leq b$. Hence $1 = \bigvee\{u_1 \wedge u_2 \mid (u_1, u_2) \in U\} \leq a^* \vee b$.

(3) It is an immediate consequence of the fact that each $\overset{\mathcal{U}}{\lhd}_i$ is a sublattice of $L \times L$, stronger than $\leq$, and assertion (1). ∎

Let $\mathcal{U}$ be a quasi-uniformity on L. By property (7) of Lemma 3.1, the relation $\overset{\mathcal{U}}{\Subset}$ on $\mathcal{P}(L \times L)$ defined by

$$C \overset{\mathcal{U}}{\Subset} D \equiv st(C, U) \leq D \text{ for some } U \in \mathcal{U}$$

is stronger than $\leq$. We shall also need the following (interior) operator on $\mathcal{P}(L \times L)$:

$$\mathrm{int}(C) = \bigcup \{D \subseteq L \times L \mid D \overset{\mathcal{U}}{\Subset} C\}.$$

Lemma 3.3. *Let $\mathcal{U}$ be a quasi-uniformity on L. For each $U \in \mathcal{U}$ we have:*

(1) *$int(U) \leq U \leq int(U^*)$.*

(2) *For every $a \in L$, $st_i(a, int(U)) \in L_i(\mathcal{U})$ $(i = 1, 2)$.*

Proof: (1) The inequality $\mathrm{int}(U) \leq U$ is trivial. The other follows from the obvious fact that for every $U \in \mathcal{U}$, $U \overset{\mathcal{U}}{\Subset} U^*$.

(2) We only prove the case $i = 1$ (the case $i = 2$ may be proved in a similar way). We need to show that

$$st_1(a, \mathrm{int}(U)) \leq \bigvee \{y \in L \mid y \overset{\mathcal{U}}{\lhd}_1 st_1(a, \mathrm{int}(U))\}.$$

By definition, $st_1(a, \mathrm{int}(U)) = \bigvee\{d_1 \mid (d_1, d_2) \in \mathrm{int}(U), d_2 \wedge a \neq 0\}$. So, let $(d_1, d_2) \in \mathrm{int}(U)$ such that $d_2 \wedge a \neq 0$. Then $(d_1, d_2) \in D \subseteq L \times L$ and there exists $V \in \mathcal{U}$ such that $st(D, V) \subseteq U$. We need to show that $d_1 \overset{\mathcal{U}}{\lhd}_1 st_1(a, \mathrm{int}(U))$. To see this consider $W \in \mathcal{U}$ such that $W^* \leq V$. It suffices then to prove that $st_1(d_1, W) \leq st_1(a, \mathrm{int}(U))$.

By properties (9) and (10) of Lemma 3.1 we have

$$st(st(D, W), W) \leq st(D, W^*) \leq st(D, V) \leq U,$$

which shows that $st(D, W) \overset{\mathcal{U}}{\Subset} U$. Thus $st(D, W) \subseteq \mathrm{int}(U)$. Therefore we only need to check that $st_1(d_1, W) \leq st_1(a, st(D, W))$, which is easy since $(st_1(d_1, W), st_2(d_2, W)) \in st(D, W)$ and $st_2(d_2, W) \wedge a \geq d_2 \wedge a \neq 0$. ■

Remark 3.4. Let $\overline{\mathcal{U}}$ be the filter of paircovers of L generated by $\{U \wedge U^{-1} \mid U \in \mathcal{U}\}$. Then, by Lemma 3.1(6), $st_1(a, U \wedge U^{-1}) = st_2(a, (U \wedge U^{-1})^{-1})$. Since $(U \wedge U^{-1})^{-1} = U \wedge U^{-1}$, then $\overset{\overline{\mathcal{U}}}{\lhd}_1 = \overset{\overline{\mathcal{U}}}{\lhd}_2$. Below we denote this relation on L just by $\overset{\overline{\mathcal{U}}}{\lhd}$.

Proposition 3.5. *Let $\mathcal{U}$ be a non-empty family of paircovers of L satisfying axioms* (U1), (U2) *and* (U3). *Then $\mathcal{U}$ satisfies* (U4) *if and only if*

$$(\mathrm{U4}') \text{ For each } a \in L,\ a = \bigvee\{b \in L \mid b \overset{\overline{\mathcal{U}}}{\lhd} a\}.$$

Proof: "$\Rightarrow$": For each $a \in L$ we may write $a = \bigvee_{i \in I}(a_i^1 \wedge a_i^2)$ for some

$$\{a_i^1 \mid i \in I\} \subseteq L_1(\mathcal{U}) \quad \text{and} \quad \{a_i^2 \mid i \in I\} \subseteq L_2(\mathcal{U}).$$

Taking into account that, for any $i \in I$,

$$a_i^1 = \{b \in L \mid b \overset{\mathcal{U}}{\lhd}_1 a_i^1\} \quad \text{and} \quad a_i^2 = \{b \in L \mid b \overset{\mathcal{U}}{\lhd}_2 a_i^2\},$$

it suffices to show that $b_1 \wedge b_2 \overset{\overline{\mathcal{U}}}{\lhd} a_1 \wedge a_2$ whenever $b_1 \overset{\mathcal{U}}{\lhd}_1 a_1$ and $b_2 \overset{\mathcal{U}}{\lhd}_2 a_2$. This property is an immediate consequence of Remark 3.4 and properties (1) and (2) of Lemma 3.1.

"$\Leftarrow$": By Lemma 3.2, each $L_i(\mathcal{U})$ $(i = 1, 2)$ is a subframe of L. It remains to show that each $a \in L$ is a join of finite meets in $L_1(\mathcal{U}) \cup L_2(\mathcal{U})$.

Let $a \in L$. Then $a = \bigvee S$ where $S = \{b \in L \mid b \overset{\overline{\mathcal{U}}}{\lhd} a\}$. For each $b \in S$ there exists $\overline{U}_b \in \overline{\mathcal{U}}$ and $U_b \in \mathcal{U}$ such that $st_1(b, \overline{U}_b) \leq a$ and $U_b \wedge U_b^{-1} \leq \overline{U}_b$. Consider $V_b \in \mathcal{U}$ such that $V_b^* \leq \overline{U}_b$. Then $\overline{V}_b = V_b \wedge V_b^{-1} \in \overline{\mathcal{U}}$ and $\overline{V}_b^* \leq \overline{U}_b$. Therefore $\mathrm{int}(\overline{V}_b^*) \leq \mathrm{int}(\overline{U}_b)$. Thus

$$\begin{aligned}
a = \bigvee S &\leq \bigvee_{b \in S}(st_1(b, \overline{V}_b) \wedge st_2(b, \overline{V}_b)) \\
&\leq \bigvee_{b \in S}(st_1(b, \mathrm{int}(\overline{V}_b^*)) \wedge st_2(b, \mathrm{int}(\overline{V}_b^*))) \\
&\leq \bigvee_{b \in S}(st_1(b, \mathrm{int}(\overline{U}_b)) \wedge st_2(b, \mathrm{int}(\overline{U}_b))) \\
&\leq st_1(b, \overline{U}_b) \\
&\leq a.
\end{aligned}$$

Hence

$$a = \bigvee_{b \in S}(st_1(b, \mathrm{int}(\overline{U}_b)) \wedge st_2(b, \mathrm{int}(\overline{U}_b)))$$

and, by Lemma 3.3, $st_i(b, \mathrm{int}(\overline{U}_b)) \in L_i(\mathcal{U})$ $(i = 1, 2)$. ■

Remark 3.6. Note that our Definition 2.1 of a quasi-uniformity contains, of course, the particular case of uniform structures defined by covers: it is one that has a base consisting of pairs of covers, both coordinates of which are the same (i.e., of the form (U, U) where U is a cover of L); certainly such paircovers are strong, moreover $(U, U)^* = (U^*, U^*)$, where U^* is the usual star of the cover U, and $\overset{\overline{\mathcal{U}}}{\triangleleft}$ is the usual $\overset{\mathcal{U}}{\triangleleft}$.

4. The proof

Proposition 4.1. *Let $((L_0, L_1, L_2), \mathcal{C})$ be an object of* $\mathsf{QUBiFrm}$*. Then $(L_0, \mathcal{C})$ is a quasi-uniform frame.*

Proof: Every element of $\mathcal{C}$, being a paircover of (L_0, L_1, L_2), is of course a paircover of L_0. In addition, $\mathcal{C}$ satisfies axioms (U1), (U2) and (U3) trivially so it suffices to check (U4). By Lemma 3.2, each $L_i(\mathcal{C})$ is a subframe of L_0 so it suffices to show that $L_i \subseteq L_i(\mathcal{C})$ for $i = 1, 2$. Let $a \in L_i$. Then $a = \bigvee\{b \in L_i \mid st_i(b, C) \leq a$ for some $C \in \mathcal{C}\} \leq \bigvee\{b \in L_0 \mid b \overset{\mathcal{C}}{\triangleleft}_i a$ for some $C \in \mathcal{C}\} \leq a$. ■

Concerning maps, the following is obvious:

Proposition 4.2. *Let $h : ((L_0, L_1, L_2), \mathcal{C}) \to ((M_0, M_1, M_2), \mathcal{D})$ be a morphism of* $\mathsf{QUBiFrm}$*. Then $h : (L_0, \mathcal{C}) \to (M_0, \mathcal{D}) \in \mathsf{QUFrm}$.* ■

Propositions 4.1 and 4.2 establish a functor $\Phi : \mathsf{QUBiFrm} \to \mathsf{QUFrm}$.

Let $U \subseteq L \times L$ and $a \in L$. In the following, we denote the element

$$\bigvee\{u_1 \wedge u_2 \mid (u_1, u_2) \in U, u_1 \wedge u_2 \wedge a \neq 0\}$$

by $st(a, U)$. It is obvious that for every $a \in L$ and every paircover U of L,

$$a \leq st(a, U) \leq st_1(a, U) \wedge st_2(a, U).$$

Given a paircover U of a frame L, we say that an element a of L is *U-small* if $a \leq st(b, U)$ whenever $a \wedge b \neq 0$. Note that, for any $(u_1, u_2) \in U$, $u_1 \wedge u_2$ is U-small. We define also

$$C_U = \{(st_1(a, \mathrm{int}(U)), st_2(a, \mathrm{int}(U))) \mid a \text{ is an } U\text{-small member of } L\}.$$

Lemma 4.3. *Let $(L,\mathcal{U}) \in$* QUFrm*. For each $a \in L$ and $U, V \in \mathcal{U}$ we have:*

(1) *Each C_U is a strong paircover of the biframe $(L, L_1(\mathcal{U}), L_2(\mathcal{U}))$.*
(2) *$C_{U \wedge V} \leq C_U \wedge C_V$.*

(3) $st_i(a, C_U) \leq st_i(a, U^{**})$ $(i = 1, 2)$.
(4) $st_i(a, U) \leq st_i(a, C_{U^*})$ $(i = 1, 2)$.
(5) $(C_U)^* \leq C_{U^{***}}$.

Proof: (1) By Lemma 3.3(2) each C_U is a subset of $L_1(\mathcal{U}) \times L_2(\mathcal{U})$. It is a paircover since $\bigvee\{st_1(a, \mathrm{int}(U)) \wedge st_2(a, \mathrm{int}(U)) \mid a \text{ is } U\text{-small}\} \geq \bigvee\{a \in L \mid a \text{ is } U\text{-small}\} \geq \bigvee\{u_1 \wedge u_2 \mid (u_1, u_2) \in U\} = 1$. Finally, it is strong: if $st_1(a, \mathrm{int}(U)) \vee st_2(a, \mathrm{int}(U)) \neq 0$ then $a \neq 0$ so $st_1(a, \mathrm{int}(U)) \wedge st_2(a, \mathrm{int}(U)) \geq a \neq 0$.

(2) Trivial.

(3) Fix $i \in \{1, 2\}$ and let $j \in \{1, 2\}$ with $j \neq i$. By definition,

$$st_i(a, C_U) = \bigvee\{st_i(b, \mathrm{int}(U)) \mid b \text{ is } U\text{-small}, st_j(b, \mathrm{int}(U)) \wedge a \neq 0\}.$$

By Lemma 3.1(3), $st_j(b, \mathrm{int}(U)) \wedge a \neq 0$ is equivalent to $st_i(a, \mathrm{int}(U)) \wedge b \neq 0$ and, since b is U-small, this implies that $b \leq st(st_i(a, \mathrm{int}(U)), U) \leq st_i(st_i(a, U), U) \leq st_i(a, U^*)$ (using property (4) of Lemma 3.1). Hence

$$st_i(b, \mathrm{int}(U)) \leq st_i(st_i(a, U^*), \mathrm{int}(U)) \leq st_i(st_i(a, U^*), U^*) \leq st_i(a, U^{**}).$$

(4) We have $st_1(a, U) = \bigvee\{u_1 \mid (u_1, u_2) \in U, u_2 \wedge a \neq 0\}$ and for each such u_1, since U^* is a paircover, we may write $u_1 = \bigvee\{u_1 \wedge d_1 \wedge d_2 \mid (d_1, d_2) \in U^*, u_1 \wedge d_1 \wedge d_2 \neq 0\}$. But, for each $(d_1, d_2) \in U^*$, $u_1 \wedge d_1 \wedge d_2$ is U^*-small. Indeed, if $y \wedge u_1 \wedge d_1 \wedge d_2 \neq 0$ then $u_1 \wedge d_1 \wedge d_2 \leq st(y, U^*)$ (since $u_1 \wedge d_1 \wedge d_2 \leq d_1 \wedge d_2$, $(d_1, d_2) \in U^*$ and $d_1 \wedge d_2 \wedge y \geq y \wedge u_1 \wedge d_1 \wedge d_2 \neq 0$). Therefore

$$(st_1(u_1 \wedge d_1 \wedge d_2, \mathrm{int}(U^*)), st_2(u_1 \wedge d_1 \wedge d_2, \mathrm{int}(U^*))) \in C_{U^*}.$$

It only remains to prove that $a \wedge st_2(u_1 \wedge d_1 \wedge d_2, \mathrm{int}(U^*)) \neq 0$, which is easy:

$$\begin{aligned}
&a \wedge st_2(u_1 \wedge d_1 \wedge d_2, \mathrm{int}(U^*)) = \\
&\quad = \bigvee\{a \wedge c_2 \mid (c_1, c_2) \in \mathrm{int}(U^*), c_1 \wedge a \wedge st_2(u_1 \wedge d_1 \wedge d_2, \mathrm{int}(U^*)) \neq 0\} \\
&\quad \geq \bigvee\{a \wedge c_2 \mid (c_1, c_2) \in U, c_1 \wedge a \wedge st_2(u_1 \wedge d_1 \wedge d_2, \mathrm{int}(U^*)) \neq 0\} \\
&\quad \geq a \wedge u_2 \neq 0.
\end{aligned}$$

(5) Let a be an U-small element of L and consider

$$(st_1(st_1(a, \mathrm{int}(U)), C_U), st_2(st_2(a, \mathrm{int}(U)), C_U)) \in C_U^*.$$

For $i = 1, 2$ and $j \in \{1, 2\}, j \neq i$, we have

$$st_i(st_i(a, \mathrm{int}(U)), C_U) = \bigvee\{st_i(b, \mathrm{int}(U)) \mid b \text{ is } U\text{-small}, st_i(a, \mathrm{int}(U)) \wedge st_j(b, \mathrm{int}(U)) \neq 0\}. \quad (*)$$

But, by Lemma 3.1(3),

$$st_i(a, \mathrm{int}(U)) \wedge st_j(b, \mathrm{int}(U)) \neq 0 \Leftrightarrow b \wedge st_i(st_i(a, \mathrm{int}(U)), \mathrm{int}(U)) \neq 0.$$

Thus, by the U-smallness of b,

$$b \leq st(st_i(st_i(a, \mathrm{int}(U)), \mathrm{int}(U)), U) \leq st_i(st_i(st_i(a, U), U), U).$$

Hence, each element in the join $(*)$ satisfies

$$st_i(b, \mathrm{int}(U)) \leq st_i(st_i(st_i(st_i(a, U), U), U), U).$$

Finally, applying property (5) of Lemma 3.1 twice and Lemma 3.3(1) we get

$$st_i(b, \mathrm{int}(U)) \leq st_i(a, U^{**}) \leq st_i(a, \mathrm{int}(U^{***}))$$

(and note that, of course, a is U^{***}-small). ■

The following three results establish a functor $\Psi : \mathsf{QUFrm} \to \mathsf{QUBiFrm}$.

Proposition 4.4. *Let* $(L, \mathcal{U})$ *be an object of* QUFrm*. Then* $\{C_U \mid U \in \mathcal{U}\}$ *is a base for a quasi-uniformity* $\mathcal{C}_{\mathcal{U}}$ *on the biframe* $(L, L_1(\mathcal{U}), L_2(\mathcal{U}))$.

Proof: By Lemma 4.3, each C_U is a strong paircover of the biframe $(L, L_1(\mathcal{U}), L_2(\mathcal{U}))$. Let us check that $\mathcal{C}_{\mathcal{U}}$ satisfies axioms (C1), (C2), (C3) and (C4):

(C1) Trivial.

(C2) Lemma 4.3(2).

(C3) Let $C \in \mathcal{C}_{\mathcal{U}}$. Then there exists $U \in \mathcal{U}$ such that $C_U \leq C$. Take $V \in \mathcal{U}$ satisfying $V^{***} \leq U$. By Lemma 4.3(5), $(C_V)^* \leq C_{V^{***}} \leq C_U \leq C$.

(C4) Let $a \in L_i(\mathcal{U})$ $(i = 1, 2)$. We need to prove that

$$a = \bigvee\{b \in L_i(\mathcal{U}) \mid st_i(b, C) \leq a \text{ for some } C \in \mathcal{C}_{\mathcal{U}}\}.$$

By hypothesis, $a = \bigvee\{b \in L \mid b \overset{\mathcal{U}}{\triangleleft}_i a\}$. Therefore it is sufficient to show that $b \overset{\mathcal{U}}{\triangleleft}_i a$ implies the existence of $b' \in L_i(\mathcal{U})$ such that $b \leq b' \leq st_i(b', C_U) \leq a$ for some $U \in \mathcal{U}$.

Let $b \overset{\mathcal{U}}{\lhd}_1 a$. Then there is $U \in \mathcal{U}$ satisfying $st_1(b, U) \leq a$. Let $V \in \mathcal{U}$ such that $V^{**} \leq U$ and consider also $W \in \mathcal{U}$ such that $W^{****} \leq V$. By Lemma 3.3, $b \leq st_1(b, \mathrm{int}(W)) \in L_1(\mathcal{U})$. Let us show that $st_1(b, \mathrm{int}(W))$ is the required $b' \in L_1(\mathcal{U})$, by checking that $st_1(st_1(b, \mathrm{int}(W)), C_{W^*}) \leq a$:

By Lemma 4.3(4) we have

$$st_1(st_1(b, \mathrm{int}(W)), C_{W^*}) \leq st_1(st_1(b, W), C_{W^*}) \leq st_1(st_1(b, C_{W^*}), C_{W^*}).$$

Then, by Lemma 3.1(5) and Lemma 4.3(5),

$$st_1(st_1(b, \mathrm{int}(W)), C_{W^*}) \leq st_1(b, (C_{W^*})^*) \leq st_1(b, C_{W^{****}}) \leq st_1(b, C_V).$$

Finally, using Lemma 4.3(3), we may conclude that

$$st_1(st_1(b, \mathrm{int}(W)), C_{W^*}) \leq st_1(b, V^{**}) \leq st_1(b, U) \leq a.$$

The proof for $i = 2$ is similar. ∎

Lemma 4.5. *Let $h : (L,\mathcal{U}) \to (M, \mathcal{V})$ be a morphism of* QUFrm*, $a, b \in L$ and $U \in \mathcal{U}$. Then:*

(1) *If $b \overset{\mathcal{U}}{\lhd}_i a$ then $h(b) \overset{\mathcal{V}}{\lhd}_i h(a)$, for $i = 1, 2$.*

(2) $C_{h[U]} \leq h[C_{U^{**}}]$.

Proof: (1) Let $b \overset{\mathcal{U}}{\lhd}_i a$. Then $st_i(b, U) \leq a$ for some $U \in \mathcal{U}$. Consider $V = h[U] \in \mathcal{V}$. Using Lemma 3.1(11) we may conclude that $st_i(h(b), V) \leq h(st_i(b, U)) \leq h(a)$ and thus $h(b) \overset{\mathcal{V}}{\lhd}_i h(a)$.

(2) We need to show that for each $h[U]$-small element b of M there exists an U^{**}-small element $a \in L$ such that

$$st_i(b, \mathrm{int}(h[U])) \leq h(st_i(a, \mathrm{int}(U^{**}))) \quad (i = 1, 2).$$

So, let $b \neq 0$ be $h[U]$-small. Then, since $h[U]$ is a paircover of M, there exists $(u_1, u_2) \in U$ for which $b \wedge h(u_1) \wedge h(u_2) \neq 0$. Then, by the $h[U]$-smallness of b, $b \leq st(h(u_1) \wedge h(u_2), h[U])$. Let us denote $u_1 \wedge u_2$ by a. We have $b \leq st_i(h(a), h[U])$. Then

$$st_i(b, \mathrm{int}(h[U])) \leq st_i(b, h[U]) \leq st_i(st_i(h(a), h[U]), h[U]) \leq st_i(h(a), h[U]^*).$$

Using 3.1(12) and (11) we get

$$st_i(b, \mathrm{int}(h[U])) \leq st_i(h(a), h[U^*]) \leq h(st_i(a, U^*)).$$

Hence, by Lemma 3.3, $st_i(b, \mathrm{int}(h[U])) \leq h(st_i(a, \mathrm{int}(U^{**})))$. Finally, a is U^{**}-small because $a = u_1 \wedge u_2$ and $(u_1, u_2) \in U \leq U^{**}$. ∎

Proposition 4.6. *Let* $h : (L, \mathcal{U}) \to (M, \mathcal{V})$ *be a morphism of* QUFrm*. Then* $h : ((L, L_1(\mathcal{U}), L_2(\mathcal{U})), \mathcal{C}_{\mathcal{U}}) \to ((M, M_1(\mathcal{V}), M_2(\mathcal{V})), \mathcal{C}_{\mathcal{V}}) \in \mathsf{QUBiFrm}$*.*

Proof: First we check that the frame homomorphism $h : (L, \mathcal{U}) \to (M, \mathcal{V})$ is also a biframe homomorphism from $(L, L_1(\mathcal{U}), L_2(\mathcal{U}))$ to $(M, M_1(\mathcal{V}), M_2(\mathcal{V}))$. Let $a \in L_i(\mathcal{U})$ $(i = 1, 2)$. We need to show that $h(a) \in M_i(\mathcal{V})$. Since $a = \bigvee\{b \in L \mid b \overset{\mathcal{U}}{\lhd_i} a\}$, we may conclude by property (1) of the previous lemma that

$$h(a) = \bigvee\{h(b) \mid b \in L, b \overset{\mathcal{U}}{\lhd_i} a\} \leq \bigvee\{m \in M \mid m \overset{\mathcal{V}}{\lhd_i} h(a)\} \leq h(a)$$

which ensures that $h(a) = \bigvee\{m \in M \mid m \overset{\mathcal{V}}{\lhd_i} h(a)\}$ — meaning that $h(a) \in M_i(\mathcal{V})$.

Finally, it remains to show that $h[C] \in \mathcal{C}_{\mathcal{V}}$ for every $C \in \mathcal{C}_{\mathcal{U}}$. Let $C \in \mathcal{C}_{\mathcal{U}}$ and $U \in \mathcal{U}$ such that $C_U \leq C$. Consider $V \in \mathcal{U}$ satisfying $V^{**} \leq U$. By Lemma 4.5(2), $C_{h[V]} \leq h[C_{V^{**}}] \leq h[C_U] \leq h[C]$. Since $h[V] \in \mathcal{V}$, then $h[C] \in \mathcal{C}_{\mathcal{V}}$. ■

In order now to prove the isomorphism $\mathsf{QUFrm} \cong \mathsf{QUBiFrm}$ we only need the following:

Lemma 4.7. *Let* $(L, \mathcal{U})$ *be a quasi-uniform frame. For any paircover* U *in* $\mathcal{U}$ *we have:*

1. *If* U *is strong then* $U \leq C_{U^*}$*.*
2. $C_U \leq U^{**}$*.*

Proof: (1) Let $(u_1, u_2) \in U$. Since U is strong we may assume that $u_1 \wedge u_2 \neq 0$. Since U^* is a paircover then $u_1 \wedge u_2 = \bigvee\{u_1 \wedge u_2 \wedge v_1 \wedge v_2 \mid (v_1, v_2) \in U^*\}$. Therefore, there exists $(v_1, v_2) \in U^*$ for which $u_1 \wedge u_2 \wedge v_1 \wedge v_2 \neq 0$. This implies immediately that

$$u_1 \leq st_1(v_1 \wedge v_2, U) \leq st_1(v_1 \wedge v_2, \mathrm{int}(U^*))$$

and

$$u_2 \leq st_2(v_1 \wedge v_2, U) \leq st_2(v_1 \wedge v_2, \mathrm{int}(U^*)).$$

But $v_1 \wedge v_2$ is U^*-small (because $(v_1, v_2) \in U^*$) hence

$$(st_1(v_1 \wedge v_2, \mathrm{int}(U^*)), st_2(v_1 \wedge v_2, \mathrm{int}(U^*))) \in C_{U^*}$$

and it is proved.

(2) Let $a \neq 0$ be an U-small element of L. Since U is a paircover, there exists $(u_1, u_2) \in U$ such that $a \wedge u_1 \wedge u_2 \neq 0$. Then, by the U-smallness of a, $a \leq st(u_1 \wedge u_2, U) \leq st_1(u_1, U) \wedge st_2(u_2, U)$. Consequently, $st_i(a, \mathrm{int}(U)) \leq st_i(st_i(u_i, U), U) \leq st_i(u_i, U^*)$, for $i = 1, 2$. Of course, there exists $(v_1, v_2) \in U^*$ for which $u_i \leq v_i$ $(i = 1, 2)$. Hence $st_i(a, \mathrm{int}(U)) \leq st_i(v_i, U^*)$, which guarantees that $C_U \leq U^{**}$. ■

Now we are ready for the main theorem.

Theorem 4.8. *The functors* Φ *and* Ψ *establish an isomorphism between the concrete categories* QUFrm *and* $\mathsf{QUBiFrm}$.

Proof: Let us show that $\Psi\Phi = \mathrm{Id}_{\mathsf{QUBiFrm}}$ and $\Phi\Psi = \mathrm{Id}_{\mathsf{QUFrm}}$. For morphisms there is nothing to prove (after Propositions 4.2 and 4.6). With respect to objects we have

$$\Psi\Phi(((L_0, L_1, L_2), \mathcal{D})) = \Psi((L_0, \mathcal{D})) = ((L_0, (L_0)_1(\mathcal{C}), (L_0)_2(\mathcal{C})), \mathcal{C}_{\mathcal{D}})$$

and

$$\Phi\Psi((L, \mathcal{U})) = \Phi((L, L_1(\mathcal{U}), L_2(\mathcal{U})), \mathcal{C}_{\mathcal{U}}) = (L, \mathcal{C}_{\mathcal{U}}),$$

so we need to prove that (a) $(L_0)_i(\mathcal{C}) = L_i$ $(i = 1, 2)$, (b) $\mathcal{C}_{\mathcal{D}} = \mathcal{D}$ and (c) $\mathcal{C}_{\mathcal{U}} = \mathcal{U}$.

(a) By hypothesis, $((L_0, L_1, L_2), \mathcal{D})$ is an object of $\mathsf{QUBiFrm}$ so $L_i \subseteq (L_0)_i(\mathcal{C})$ $(i = 1, 2)$. The reverse inclusion may be proved in the same way as we proved, in Proposition 4.4, that axiom (C3) is satisfied.

(b) Let $D \in \mathcal{D}$ and consider $E \in \mathcal{D}$ such that $E^{**} \leq D$. By Lemma 4.7, $C_E \leq E^{**} \leq D$ thus $D \in \mathcal{C}_{\mathcal{D}}$. Conversely, let $U \in \mathcal{C}_{\mathcal{D}}$. Then there exists $D \in \mathcal{D}$ such that $C_D \leq U$. Take $E \in \mathcal{D}$, strong, such that $E^* \leq D$. Then, using Lemma 4.7, we get $E \leq C_{E^*} \leq C_D \leq U$ and $U \in \mathcal{D}$, as required.

(c) It is an immediate consequence of Lemma 4.7, as in (b). ■

Remark 4.9. We have now a formulation for the theory of (covering) quasi-uniformities very similar to the spatial one: the structure $\mathcal{U}$ is defined on a frame L and induces a bi-structure (specifically, a biframe) $(L, L_1(\mathcal{U}), L_2(\mathcal{U}))$ in such a way that, for instance: every frame L is *quasi-uniformizable* (in the sense that there exists a quasi-uniform frame $(M, \mathcal{U})$ such that the first induced subframe $M_1(\mathcal{U})$ is isomorphic to the given L); for every quasi-uniform frame the induced biframe is always regular (in fact completely regular, assuming the axiom of countable dependent choice); every completely regular biframe (L_0, L_1, L_2) is *uniformizable* (in the sense that there exists a quasi-uniformity on the total part L_0 of the biframe whose induced biframe coincides with the given biframe); for every compact regular biframe there is a unique quasi-uniformity on the total part of the biframe whose induced biframe coincides with the given biframe; etc.

5. The adjunction $\mathsf{QUnif} \rightleftarrows \mathsf{QUFrm}$

Let QUnif denote the category of quasi-uniform spaces and uniformly continuous maps [5]. Here, in the description of its objects we follow the covering approach of Gantner and Steinlage [10] that uses conjugate pairs of covers of a set X.

The expected open and spectrum functors establishing a dual adjunction between QUnif and QUFrm are easy to describe now.

Let (X, μ) be a quasi-uniform space. To explain the notation and terminology, recall that μ induces two topologies $\mathfrak{T}_1(\mu)$ and $\mathfrak{T}_2(\mu)$ on X in the following manner:

- $A \subseteq X$ is $\mathfrak{T}_1(\mu)$-open if for every $a \in A$ there exists $\mathcal{U}_a \in \mu$ such that $st_1(a, \mathcal{U}_a) \subseteq A$;
- similarly, $A \subseteq X$ is $\mathfrak{T}_2(\mu)$-open if for every $a \in A$ there exists $\mathcal{U}_a \in \mu$ such that $st_2(a, \mathcal{U}_a) \subseteq A$.

Let $\mathcal{U} \in \mu$. We say that $\mathcal{U}$ is an *open* paircover of (X, μ) if for each $(U_1, U_2) \in \mathcal{U}$, U_1 is $\mathfrak{T}_1(\mu)$-open and U_2 is $\mathfrak{T}_2(\mu)$-open.

Set $\Omega(X, \mu) = (\mathfrak{T}_1(\mu) \vee \mathfrak{T}_2(\mu), \mathcal{C}_\mu)$ where $\mathcal{C}_\mu$ is the set of all open paircovers of (X, μ). It is not hard to check that $\Omega(X, \mu) \in \mathsf{QUFrm}$. Moreover, if $f : (X, \mu) \to (Y, \nu)$ is uniformly continuous then $\Omega(f) : \Omega(Y, \nu) \to \Omega(X, \mu)$, defined by $\Omega(f)(B) = f^{-1}(B)$ for any $B \in \mathfrak{T}_1(\nu) \vee \mathfrak{T}_2(\nu)$, is a uniform frame homomorphism. Thus, Ω is a contravariant functor from QUnif into QUFrm.

For each frame L, we consider its *spectrum*, that is, the topological space

$$(\mathsf{Pt}(L), \{\Sigma_a \mid a \in L\}),$$

where $\mathsf{Pt}(L)$ is the set of all frame homomorphisms $p : L \to \{0, 1\}$ and $\Sigma_a = \{p \in \mathsf{Pt}(L) \mid p(a) = 1\}$. Let $(L, \mathcal{U})$ be a quasi-uniform frame. For each $U \in \mathcal{U}$, let $\Sigma_U = \{(\Sigma_{u_1}, \Sigma_{u_2}) \mid (u_1, u_2) \in U\}$ and let $\Sigma_{\mathcal{U}}$ be the filter of paircovers of $\mathsf{Pt}(L)$ generated by $\{\Sigma_U \mid U \in \mathcal{U}\}$.

Proposition 5.1. *Let* $(L, \mathcal{U}) \in \mathsf{QUFrm}$. *Then* $\Sigma((L, \mathcal{U}) = (\mathsf{Pt}(L), \Sigma_{\mathcal{U}})$ *is a quasi-uniform space.*

Proof: (1) Let $U \in \mathcal{U}$. Then $\bigcup\{\Sigma_{u_1} \cap \Sigma_{u_2} \mid (u_1, u_2) \in U\} = \bigcup\{\Sigma_{u_1 \wedge u_2} \mid (u_1, u_2) \in U\} = \Sigma_{\bigvee\{u_1 \wedge u_2 | (u_1,u_2) \in U\}} = \Sigma_1 = \mathsf{Pt}(L)$.
(2) Let $U, V \in \mathcal{U}$. Trivially $U \leq V$ implies $\Sigma_U \leq \Sigma_V$ and $\Sigma_{U \wedge V} = \Sigma_U \wedge \Sigma_V$.
(3) Let $U^* \leq V$. Then $(\Sigma_U)^* \leq \Sigma_V$. Indeed: for each $(u_1, u_2) \in U$ there exists $(v_1, v_2) \in V$ satisfying $st_1(u_1, U) \leq v_1$ and $st_2(u_2, U) \leq v_2$; then

$$\begin{aligned} st_1(\Sigma_{u_1}, \Sigma_U) &= \bigcup\{\Sigma_{u'_1} \mid \Sigma_{u'_2} \cap \Sigma_{u_1} \neq \varnothing, (u'_1, u'_2) \in U\} \\ &= \bigcup\{\Sigma_{u'_1} \mid \Sigma_{u'_2 \wedge u_1} \neq \varnothing, (u'_1, u'_2) \in U\} \\ &\subseteq \bigcup\{\Sigma_{u'_1} \mid u'_2 \wedge u_1 \neq 0\} \\ &= \Sigma_{st_1(u_1,U)} \subseteq \Sigma_{v_1}. \end{aligned}$$

Similarly, $st_2(\Sigma_{u_2}, \Sigma_U) \subseteq \Sigma_{v_2}$. ■

Concerning morphisms, given a map $h : (L,\mathcal{U}) \to (M, \nu)$ of QUFrm, let $\Sigma(h) : \Sigma(M, \nu) \to \Sigma(L,\mathcal{U})$ be defined by $\Sigma(h)(p) = ph$. It is a straightforward exercise to check that $\Sigma(h) \in \mathsf{QUnif}$. We have therefore a contravariant functor $\Sigma : \mathsf{QUFrm} \to \mathsf{QUnif}$.

Remark 5.2. It is also a straightforward exercise to check that the topologies $\mathfrak{T}_1(\Sigma_\mathcal{U})$ and $\mathfrak{T}_2(\Sigma_\mathcal{U})$ induced by the quasi-uniformity $\Sigma_\mathcal{U}$ on $\mathsf{Pt}(L)$ coincide with the spectral topologies of the spectrums of $L_1(\mathcal{U})$ and $L_2(\mathcal{U})$ respectively.

Theorem 5.3. *The two above contravariant functors Ω and Σ define a dual adjunction, with adjoint units $\eta_{(X,\mu)} : (X, \mu) \to \Sigma\Omega(X, \mu)$ and $\xi_{(L,\mathcal{U})} : (L,\mathcal{U}) \to \Omega\Sigma(L,\mathcal{U})$ given by $\eta_{(X,\mu)}(a)(U) = 1$ iff $a \in U$ and $\xi_{(L,\mathcal{U})}(a) = \Sigma_a$.*

Proof: The checking that each $\eta_{(X,\mu)}$ is uniformly continuous and that each $\xi_{(L,\mathcal{U})}$ is a uniform frame homomorphism is left to the reader. They define natural transformations $\eta : \mathrm{Id}_{\mathsf{Unif}} \stackrel{\cdot}{\to} \Sigma\Omega$ and $\xi : \mathrm{Id}_{\mathsf{QUFrm}} \stackrel{\cdot}{\to} \Omega\Sigma$, that is, the following diagrams commute, for every $f : (X, \mu) \to (Y, \nu)$ and every $h : (L,\mathcal{U}) \to (M, \mathcal{V})$.

$$\begin{tikzcd} (X,\mu) \arrow[r, "\eta_{(X,\mu)}"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)"] & \Sigma\Omega(X,\mu) \arrow[d, "\Sigma\Omega(f)"] \\ (Y,\nu) \arrow[r, "\eta_{(Y,\nu)}"'] & \Sigma\Omega(Y,\nu) \end{tikzcd} \qquad \begin{tikzcd} (L,\mathcal{U}) \arrow[r, "\xi_{(L,\mathcal{U})}"] \arrow[d, "h"'] \arrow[dr, phantom, "(2)"] & \Omega\Sigma(L,\mathcal{U}) \arrow[d, "\Omega\Sigma(h)"] \\ (M,\mathcal{V}) \arrow[r, "\xi_{(M,\mathcal{V})}"'] & \Omega\Sigma(M,\mathcal{V}) \end{tikzcd}$$

Indeed, for each $f : (X, \mu) \to (Y, \nu)$ in QUnif and every $x \in X$,

$$\Sigma\Omega(f)(\eta_{(X,\mu)}(x)) = \eta_{(X,\mu)}(x)\Omega(f)$$

is the map $F : \Omega(Y, \nu) \to \{0, 1\}$ given by $F(B) = 1$ iff $x \in f^{-1}(B)$. Since $x \in f^{-1}(B)$ iff $f(x) \in B$, this is precisely the map $\eta_{(Y,\nu)}(f(x))$ and diagram (1) commutes.

On the other hand, for each $h : (L,\mathcal{U}) \to (M, V)$ in QUFrm and every $a \in L$,

$$\Omega\Sigma(h)(\xi_{(L,\mathcal{U})}(a)) = \Omega\Sigma(h)(\Sigma_a) = (\Sigma(h))^{-1}(\Sigma_a) = \{p \in \mathsf{Pt}(M) \mid \Sigma(h)(p) \in \Sigma_a\}.$$

Since $\Sigma(h)(p) \in \Sigma_a$ iff $ph \in \Sigma_a$ iff $p(h(a)) = 1$ iff $p \in \Sigma_{h(a)}$, then diagram (2) also commutes.

Finally, $\eta : \mathrm{Id}_{\mathsf{Unif}} \stackrel{\cdot}{\to} \Sigma\Omega$ and $\xi : \mathrm{Id}_{\mathsf{QUFrm}} \stackrel{\cdot}{\to} \Omega\Sigma$ satisfy the triangular identities (a) $\Omega(\eta_{(X,\mu)}) \cdot \xi_{\Omega(X,\mu)} = 1$ and (b) $\Sigma(\xi_{(L,\mathcal{U})}) \cdot \eta_{\Sigma(L,\mathcal{U})} = 1$ for every $(X, \mu) \in \mathsf{QUnif}$ and every $(L,\mathcal{U}) \in \mathsf{QUFrm}$:
(a) For each $A \in \mathfrak{T}_1(\mu) \vee \mathfrak{T}_2(\mu)$, $(\Omega(\eta_{(X,\mu)}) \cdot \xi_{\Omega(X,\mu)})(A) = \Omega(\eta_{(X,\mu)})(\Sigma_A) = \eta^{-1}_{(X,\mu)}(\Sigma_A)$ and $x \in \eta^{-1}_{(X,\mu)}(\Sigma_A)$ iff $\eta_{(X,\mu)}(x) \in \Sigma_A$ iff $\eta_{(X,\mu)}(x)(A) = 1$ iff $x \in A$.
(b) For each $p : L \to \{0, 1\}$ in $\mathsf{Pt}(L)$, $(\Sigma(\xi_{(L,\mathcal{U})}) \cdot \eta_{\Sigma(L,\mathcal{U})})(p) = q \cdot \xi_{(L,\mathcal{U})}$ where $q : \Omega\Sigma(L,\mathcal{U}) \to \{0, 1\}$ maps A into 1 iff $p \in A$. But $(q \cdot \xi_{(L,\mathcal{U})})(a) = q(\Sigma_a)$ is equal to 1 iff $p \in \Sigma_a$, that is, iff $p(a) = 1$. Hence $q \cdot \xi_{(L,\mathcal{U})} = p$ as required. ■

6. Quasi-proximities

The quasi-proximal frames (proximal biframes in [9]) introduced by Frith [6] in terms of the so-called strong inclusions of biframes, have the same drawback of quasi-uniform biframes: the structure is defined on a biframe (L_0, L_1, L_2) rather than directly on a frame L. In this final section we briefly describe how this can be overcome.

Recall that a *strong inclusion* [19] on a biframe (L_0, L_1, L_2) is a pair $(\lhd_1, \lhd_2)$ of relations on L_1 and L_2 respectively satisfying the following conditions, for $i = 1, 2$:

(S1) $\lhd_i$ is a sublattice of $L_i \times L_i$.
(S2) $a \le b \lhd_i c \le d$ implies that $a \lhd_i d$.
(S3) $a \lhd_i b$ implies that $a^\bullet \vee b = 1$ (usually denoted by $a \prec_i b$).
(S4) $a \lhd_i b$ implies that there exists $c \in L_i$ with $a \lhd_i c \lhd_i b$.
(S5) If $a \lhd_i b$ then $b^\bullet \lhd_j a^\bullet$ for $j \in \{1, 2\}$ and $j \neq i$.
(S6) For every $a \in L_i$, $a = \bigvee\{b \in L_i \mid b \lhd_i a\}$.

A triple $((L_0, L_1, L_2), \lhd_1, \lhd_2)$ where $(\lhd_1, \lhd_2)$ is a strong inclusion on the biframe (L_0, L_1, L_2) is called a *proximal biframe* [9]. Given proximal biframes

$$((L_0, L_1, L_2), \lhd_1^L, \lhd_2^L) \quad \text{and} \quad ((M_0, M_1, M_2), \lhd_1^M, \lhd_2^M),$$

a biframe homomorphism $h : (L_0, L_1, L_2) \to (M_0, M_1, M_2)$ is a *proximal biframe homomorphism* if $a \lhd_i^L b$ implies $h(a) \lhd_i^M h(b)$ for $i = 1, 2$ and every $a, b \in L_0$. We denote the category of proximal biframes and proximal biframe homomorphisms by $\mathsf{ProxBiFrm}$.

Let L be a frame, $\lhd$ a binary relation in L and

$$L_\lhd = \{a \in L \mid a = \bigvee\{b \in L \mid b \lhd a\}\}.$$

Lemma 6.1. *If $\lhd$ is a sublattice of $L \times L$, stronger than $\leq$, satisfying*

$$a \leq b \lhd c \leq d \Rightarrow a \lhd d \tag{$*$}$$

then $L_\lhd$ is a subframe of L.

Proof: Since $0 \lhd 0$ and $1 \lhd 1$, then $0, 1 \in L_\lhd$. Since $\lhd$ is stronger than the partial order of L, we have always $\bigvee\{b \in L \mid b \lhd a\} \leq a$. Let $a, b \in L_\lhd$. Then, since $\lhd$ is closed under finite meets, we have

$$\begin{aligned} a \wedge b &= \bigvee\{a' \in L \mid a' \lhd a\} \wedge \bigvee\{b' \in L \mid b' \lhd b\} \\ &= \bigvee\{a' \wedge b' \mid a', b' \in L, a' \lhd a, b' \lhd b\} \\ &\leq \bigvee\{c \in L \mid c \lhd a \wedge b\} \end{aligned}$$

which shows that $a \wedge b \in L_\lhd$.

Finally, let $a_i \in L_\lhd$ $(i \in I)$. Then $\bigvee_{i \in I} a_i = \bigvee_{i \in I} \bigvee\{b \in L \mid b \lhd a_i\}$. For each such b, $b \lhd a_i \leq \bigvee_{i \in I} a_i$. Consequently, by condition $(*)$, $\bigvee_{i \in I} a_i \leq \bigvee\{b \in L \mid b \lhd \bigvee_{i \in I} a_i\}$. Hence $\bigvee_{i \in I} a_i \in L_\lhd$. ■

Since in the sequel we shall need to refer to pseudocomplements relatively to different pairs of subframes, we will adopt the following notation. Given a pair of subframes L_1, L_2 of a frame L and $a \in L_i$ $(i = 1, 2)$, we denote by $a^\bullet[L_j]$ $(j \in \{1, 2\}, j \neq i)$ the element $\bigvee\{b \in L_j \mid b \wedge a = 0\}$.

Definition 6.2. A pair $(\lhd_1, \lhd_2)$ of relations in L, stronger than $\leq$, will be called a *strong bi-inclusion* on L if for $i, j = 1, 2$ we have:

(SB1) $\lhd_i$ is a sublattice of $L \times L$.

(SB2) $a \leq b \lhd_i c \leq d$ implies that $a \lhd_i d$.

(SB3) $a \lhd_i b$ implies that $a \prec b$ (i.e. $a^* \vee b = 1$).
(SB4) For every $a, b \in L_{\lhd_i}$, $a \lhd_i b$ implies $a^\bullet[L_{\lhd_j}] \vee b = 1$ ($j \neq i$).
(SB5) If $a \lhd_i b$ then there exists $c \in L_{\lhd_i}$ such that $a \lhd_i c \lhd_i b$.
(SB6) For every $a, b \in L_{\lhd_i}$, $a \lhd_i b$ implies $b^\bullet[L_{\lhd_j}] \lhd_j a^\bullet[L_{\lhd_j}]$ ($j \neq i$).
(SB7) $(L, L_{\lhd_1}, L_{\lhd_2})$ is a biframe.

The triple $(L, \lhd_1, \lhd_2)$ will be called a *quasi-proximal frame*. Given quasi-proximal frames $(L, \lhd_1^L, \lhd_2^L)$ and $(M, \lhd_1^M, \lhd_2^M)$, a *quasi-proximal map*

$$h : (L, \lhd_1^L, \lhd_2^L) \to (M, \lhd_1^M, \lhd_2^M)$$

is a frame homomorphism $h : L \to M$ such that $a \lhd_i^L b \Rightarrow h(a) \lhd_i^M h(b)$ for every $a, b \in L$. The corresponding category QPFrm is isomorphic to the category ProxBiFrm of proximal biframes, as we shall prove next. This is proved in a similar way as in Section 4 we proved the isomorphism QUFrm $\cong$ QUBiFrm, using new functors Ψ and Φ defined below.

Given a proximal biframe $((L_0, L_1, L_2), \lhd_1, \lhd_2)$, let

$$\Phi((L_0, L_1, L_2), \lhd_1, \lhd_2) = (L_0, \overline{\lhd_1}, \overline{\lhd_2})$$

where, for any $a, b \in L_0$,

$$a \,\overline{\lhd_i}\, b \equiv \exists\, c, d \in L_i : a \leq c \lhd_i d \leq b.$$

Proposition 6.3. *For each proximal biframe* $((L_0, L_1, L_2), \lhd_1, \lhd_2)$, $\Phi((L_0, L_1, L_2), \lhd_1, \lhd_2)$ *is a quasi-proximal frame.*

Proof: The fact that each $\overline{\lhd_i}$ is stronger than $\leq$ is obvious.
(SB1) $0\overline{\lhd_i}0$ and $1\overline{\lhd_i}1$ are trivial. Let $a_1, a_2\overline{\lhd_i}b$. Then $a_1 \leq c_1 \lhd_i d_1 \leq b$ and $a_2 \leq c_2 \lhd_i d_2 \leq b$ for some $c_1, c_2, d_1, d_2 \in L_i$. Consequently, $a_1 \leq c_1 \lhd_i d_1 \vee d_2 \leq b$ and $a_2 \leq c_2 \lhd_i d_1 \vee d_2 \leq b$ with $c_1, c_2, d_1 \vee d_2 \in L_i$. By hypothesis, $c_1 \wedge c_2 \lhd_i d_1 \vee d_2$ and $c_1 \wedge c_2 \in L_i$. Thus $a_1 \wedge a_2\overline{\lhd_i}b$. Similarly, $a_1 \vee a_2 \leq c_1 \vee c_2 \lhd_i d_1 \vee d_2 \leq b$ and $a_1 \vee a_2\overline{\lhd_i}b$.
(SB2) Trivial.
(SB3) Let $a\overline{\lhd_i}b$, that is $a \leq c \lhd_i d \leq b$ for some $c, d \in L_i$. Since $c \lhd_i d$ implies $c \prec_i d$, we have that $c^\bullet[L_j] \vee d = 1$. Therefore $c^* \vee d \geq c^\bullet[L_j] \vee d = 1$ and then $a^* \vee b = 1$.
(SB4) First note that $L_i \subseteq (L_0)_{\overline{\lhd_i}}$. Indeed, for each $a \in L_i$, since $\lhd_i \subseteq \overline{\lhd_i}$, we have $a = \bigvee\{b \in L_i \mid b \lhd_i a\} \leq \bigvee\{b \in L_0 \mid b\overline{\lhd_i}a\} \leq a$. Now, let $a, b \in (L_0)_{\overline{\lhd_i}}$ and $a \leq c \lhd_i d \leq b$ with $c, d \in L_i$. By hypothesis, $c \prec_i d$ (with respect to the subframes L_1 and L_2) that is, $c^\bullet[L_j] \vee d = 1$. From the inclusion $L_i \subseteq (L_0)_{\overline{\lhd_i}}$

and the fact that $a \leq c$ it follows that

$$a^\bullet[(L_0)_{\overline{\lhd_j}}] = \bigvee\{a' \in (L_0)_{\overline{\lhd_j}} \mid a' \wedge a = 0\} \geq \bigvee\{c' \in L_j \mid c' \wedge c = 0\} = c^\bullet[L_j].$$

Thus $a^\bullet[(L_0)_{\overline{\lhd_j}}] \vee b = 1$.
(SB5) It follows immediately from condition (S4) and the fact proved earlier that $L_i \subseteq (L_0)_{\overline{\lhd_i}}$.
(SB6) Let $a, b \in (L_0)_{\overline{\lhd_i}}$ with $a\overline{\lhd_i}b$, that is, $a \leq c \lhd_i d \leq b$ for some $c, d \in L_i$. Then, by hypothesis, $d^\bullet[L_j] \lhd_j c^\bullet[L_j]$ and, of course, $d^\bullet[L_j], c^\bullet[L_j] \in L_j$. It suffices then to show that $b^\bullet[(L_0)_{\overline{\lhd_j}}] \leq d^\bullet[L_j]$ and $c^\bullet[L_j] \leq a^\bullet[(L_0)_{\overline{\lhd_j}}]$. The latter was already proved in (SB4) above and the former can be proved in a similar way.
(SB7) By Lemma 6.1, each $(L_0)_{\overline{\lhd_i}}$ is a subframe of L_0. Since (L_0, L_1, L_2) is a biframe and $L_i \subseteq (L_0)_{\overline{\lhd_i}}$ $(i = 1, 2)$, then immediately $(L_0, (L_0)_{\overline{\lhd_1}}, (L_0)_{\overline{\lhd_2}})$ is also a biframe. ∎

Given a quasi-proximal frame $(L, \lhd_1, \lhd_2)$, let

$$\Psi(L, \lhd_1, \lhd_2) = ((L, L_{\lhd_1}, L_{\lhd_2}), \lhd_{1|_{L_{\lhd_1}}}, \lhd_{2|_{L_{\lhd_2}}}).$$

Proposition 6.4. *For each quasi-proximal frame* $(L, \lhd_1, \lhd_2)$*,* $\Psi(L, \lhd_1, \lhd_2)$ *is a proximal biframe.*

Proof: By hypothesis, $(L, L_{\lhd_1}, L_{\lhd_2})$ is a biframe and $\Psi(L, \lhd_1, \lhd_2)$ satisfies conditions (S1)-(S5) trivially. It remains to check (S6):

For every $a \in L_{\lhd_i}$, $a = \bigvee\{b \in L \mid b \lhd_i a\}$. But by condition (SB5) there is some $c \in L_{\lhd_i}$ satisfying $b \lhd_i c \lhd_i a$. Therefore $a = \bigvee\{b \in L_{\lhd_i} \mid b \lhd_i a\}$, as required. ∎

Concerning morphisms, the next result allows us to define $\Phi(h) = h$ for every $h \in \mathsf{ProxBiFrm}$ and $\Psi(h) = h$ for every $h \in \mathsf{QPFrm}$.

Proposition 6.5. (1) *Let* $h : ((L_0, L_1, L_2), \lhd_1^L, \lhd_2^L) \to ((M_0, M_1, M_2), \lhd_1^M, \lhd_2^M)$ *be a proximal biframe homomorphism. Then*

$$h : \Phi((L_0, L_1, L_2), \lhd_1^L, \lhd_2^L) \to \Phi((M_0, M_1, M_2), \lhd_1^M, \lhd_2^M) \in \mathsf{QPFrm}.$$

(2) *Let* $h : (L, \lhd_1^L, \lhd_2^L) \to (M, \lhd_1^M, \lhd_2^M)$ *be a quasi-proximal map. Then*

$$h : \Psi(L, \lhd_1^L, \lhd_2^L) \to \Psi(M, \lhd_1^M, \lhd_2^M) \in \mathsf{ProxBiFrm}.$$

Proof: (1) We need to check that $a\overline{\lhd_i^L}b \Rightarrow h(a)\overline{\lhd_i^M}h(b)$ for $i = 1, 2$ and every $a, b \in L_i$. Let $a\overline{\lhd_i^L}b$, that is $a \leq c \lhd_i^L d \leq b$ for some $c, d \in L_i$. Then, by hypothesis, $h(c), h(d) \in M_i$ and $h(a) \leq h(c) \lhd_i^M h(d) \leq h(b)$, which shows that $h(a)\overline{\lhd_i^M}h(b)$.

(2) It suffices to check that h is a biframe map

$$(L, L_{\lhd_1}, L_{\lhd_2}) \to (M, M_{\lhd_1}, M_{\lhd_2})$$

(the rest is obvious). Consider $a \in L_{\lhd_i}$. Since $a = \bigvee\{b \in L \mid b \lhd_i^L a\}$ and $b \lhd_i^L a$ implies $h(b) \lhd_i^M h(a)$, then

$$h(a) = \bigvee\{h(b) \mid b \in L, b \lhd_i^L a\} \leq \bigvee\{c \in M \mid c \lhd_i^M h(a)\} \leq h(a).$$

Hence $h(a) \in M_{\lhd_i}$. ■

Theorem 6.6. *The functors* Ψ *and* Φ *establish an isomorphism between the concrete categories* QPFrm *and* $\mathsf{ProxBiFrm}$.

Proof: It is sufficient to show that (a) $\Phi\Psi = \mathrm{Id}_{\mathsf{QPFrm}}$ and (b) $\Psi\Phi = \mathrm{Id}_{\mathsf{ProxBiFrm}}$ on objects.

(a) We need to show that $\overline{\lhd_{|_{L_{\lhd_i}}}} = \lhd_i$. Consider $a, b \in L$ with $a \lhd_i b$. By condition (SB5), there is $c, d \in L_{\lhd_i}$ such that $a \lhd_i c \lhd_i d \lhd_i b$. Since $c \lhd_{i|_{L_{\lhd_i}}} d$ then, immediately, $a\overline{\lhd_{|_{L_{\lhd_i}}}}b$. On the other hand, if $a, b \in L$ are such that $a\overline{\lhd_{|_{L_{\lhd_i}}}}b$ then there exists a pair c, d of elements of $L_{\lhd_i}$ satisfying $a \leq c \lhd_i d \leq b$. Thus $a \lhd_i b$.

(b) It suffices to check that $(L_0)_{\overline{\lhd_i}} = L_i$ for $i = 1, 2$. Let $a \in L_i$. Then, since $\lhd_i \subseteq \overline{\lhd_i}$, we have $a = \bigvee\{b \in L_i \mid b \lhd_i a\} \leq \bigvee\{b \in L_0 \mid b\overline{\lhd_i}a\} \leq a$. Conversely, if $a \in (L_0)_{\overline{\lhd_i}}$, meaning that $a \in L_0$ and $a = \bigvee\{b \in L_0 \mid b\overline{\lhd_i}a\}$, then for each such b there is $c_b, d_b \in L_i$ satisfying $b \leq c_b \lhd_i d_b \leq a$. Consequently, $a \leq \bigvee\{c \in L_i \mid c \lhd_i a\} \leq a$ and, therefore, $a \in L_i$. ■

Remarks 6.7. (1) Our Definition 6.2 of a quasi-proximal frame contains, of course, the symmetric case of proximal frames [6] defined by strong inclusions: it is a frame equipped with a strong bi-inclusion $(\lhd_1, \lhd_2)$, both coordinates of which are the same.

(2) In closing we note that it can be shown, in analogy with the spatial case or the symmetric case, that the category QPFrm is isomorphic to the full subcategory of QUFrm of all *totally bounded* quasi-uniform frames (that is, the quasi-uniform frames $(L, \mathcal{U})$ for which the paircovering structure $\mathcal{U}$ has a base of finite paircovers). We omit the details.

It is now felt that, with the reformulation of the covering approach here presented (Definition 2.1), paircovers are the most convenient tool to deal with quasi-uniformities on frames, better than the entourages of [17], and will be certainly used "ten-tenths of the time" [11] in asymmetric pointfree topology.

References

[1] B. Banaschewski, G. C. L. Brümmer and K. A. Hardie, Biframes and bispaces, *Quaest. Math.* 6 (1983) 13–25.

[2] D. Doitchinov, Some reflections on quasi-uniform frames, in: *Topology with Applications* (Szekszàrd, Hungary, 1993), vol. 4, Bolyai Society, Math. Studies, 1995, pp. 151–158.

[3] M. J. Ferreira and J. Picado, Functorial quasi-uniformities on frames, *Appl. Categ. Structures* 13 (2005) 281–303.

[4] P. Fletcher, W. Hunsaker and W. Lindgren, Frame quasi-uniformities, *Monatsh. Math.* 117 (1994) 223–236.

[5] P. Fletcher and W. Lindgren, *Quasi-uniform Spaces*, Marcel Dekker, New York, 1982.

[6] J. Frith, *Structured frames*, PhD thesis, University of Cape Town, 1987.

[7] J. Frith, The category of quasi-uniform frames, *Research Reports*, Department of Mathematics, University of Cape Town, 1991.

[8] J. Frith, W. Hunsaker and J. Walters-Wayland, The Samuel compactification of a quasi-uniform frame, *Topology Proc.* 23 (1998) 115–126.

[9] J. Frith and A. Schauerte, The Samuel compactification for quasi-uniform biframes, *Topol. Appl.* 156 (2009) 2116–2122.

[10] T. E. Gantner and R. C. Steinlage, Characterizations of quasi-uniformities, *J. London Math. Soc. (2)* 5 (1972) 48–52.

[11] J. R. Isbell, *Uniform Spaces*, Math. Surveys, vol. 12, Amer. Math. Soc., 1964.

[12] P. T. Johnstone, *Stone Spaces*, Cambridge Studies in Advanced Math., vol. 3, Cambridge Univ. Press, Cambridge, 1982.

[13] L. Nachbin, Sur les espaces uniformes ordonnés, *C. R. Acad. Sci. Paris* 226 (1948) 774–775.

[14] L. Nachbin, *Topology and Order*, D. van Nostrand, Princeton, 1965.

[15] J. Picado, Weil uniformities for frames, *Comment. Math. Univ. Carolin.* 36 (1995) 357–370.

[16] J. Picado, Frame quasi-uniformities by entourages, in: *Symposium on Categorical Topology (Univ. of Cape Town 1994)*, Department of Mathematics, Univ. of Cape Town, 1999, pp. 161–175.

[17] J. Picado, Structured frames by Weil entourages, *Appl. Categ. Structures* 8 (2000) 351–366.

[18] J. Picado and A. Pultr, *Locales mostly treated in a covariant way*, Textos de Matemática, vol. 41, University of Coimbra, 2008.

[19] A. Schauerte, Biframe compactifications, *Comment. Math. Univ. Carolin.* 34 (1993) 567–574.

Jorge Picado
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: picado@mat.uc.pt