ON THE DESCRIPTION OF FRAME ASYMMETRIC UNIFORMITIES VIA PAIRCOVERS

JORGE PICADO

Abstract: A quasi-uniformity on a frame may be equivalently described either in terms of paircovers or in terms of entourages. The former is defined as a structure $\mu$ on a biframe $(L_0, L_1, L_2)$ and the latter directly as a structure $\E$ on a frame $L$ which induces two subframes $L_1(\E)$ and $L_2(\E)$ of $L$ such that the triple $(L, L_1(\E), L_2(\E))$ is a biframe (this is the pointfree analogue of the bitopological space $(X, \mathcal{F}(\E), \mathcal{F}(\E^{-1}))$ induced by any quasi-uniformity $\E$ on the set $X$). While the approach via paircovers is most convenient for calculations, it does not faithfully reflect the spatial original notion since it is not formulated directly on frames.

Here, it will be shown that it is possible to describe frame quasi-uniformities by defining the paircovering structure $\mu$ directly on a frame $L$ without requiring the prior knowledge of the underlying biframe. It turns out that the biframe structure $(L, L_1(\mu), L_2(\mu))$ appears only a posteriori, induced by the structure in a very natural way. In addition, we indicate how to do the same for strong relations on biframes and the corresponding quasi-proximal frames.

Keywords: Frame, biframe, entourage, paircover, quasi-uniform frame, quasi-proximal frame, strong inclusion, quasi-uniformity, quasi-proximity.

AMS Subject Classification (2000): 06D22, 54E05, 54E15, 54E55.

1. The problem

The study of quasi-uniformities began in 1948 with Nachbin’s investigations on uniform preordered spaces (see [13, 14]). A quasi-uniformity $\E$ on a set $X$ may be described in several equivalent ways, most notably as a collection of relations on $X$ (the entourage approach [5]) and as a collection of ordered pairs of covers of $X$ (the paircover approach of [10]). Associated with any quasi-uniformity $\E$ on $X$ there is the well-known bitopological space $(X, \mathcal{F}(\E), \mathcal{F}(\E^{-1}))$ induced by $\E$. In the pointfree setting, the theory of quasi-uniformities was first exploited by J. Frith using the paircover

Received September 16, 2009.

Support from the Centre for Mathematics of the University of Coimbra (CMUC/FCT) and the Ministry of Education and Science of Spain and FEDER under grant MTM2009-12872-C02-02 is gratefully acknowledged.
and introduced before by Banaschewski, Brümmer and Hardie [1].

The former is defined as a structure $C$ on a biframe $(L_0, L_1, L_2)$ and the latter directly as a structure $E$ on a frame $L$ which establishes two subframes $L_1(E)$ and $L_2(E)$ of $L$ such that the triple $(L, L_1(E), L_2(E))$ is a biframe (this is the pointfree version of the bitopological space $(X, \tau(E), \tau(E^{-1}))$ above). The categories defined by the two approaches are isomorphic [15, 16].

The motivation for this paper originates in the talk presented by J. Frith and A. Schauerte at the III Workshop on Aspects of Contemporary Topology (Antwerp, December 2007) and their subsequent paper [9]. Frith’s treatment of the category of quasi-uniform frames made use of the approach to quasi-uniform spaces via conjugate pairs of covers due to Gantner and Steinlage [10] (the concept of a biframe, which was crucial for Frith’s study had been introduced before by Banaschewski, Brümmer and Hardie [1]).

Let $(L_0, L_1, L_2)$ be a biframe. A subset $C$ of $L_1 \times L_2$ is a paircover [6] of $(L_0, L_1, L_2)$ if $\bigvee \{c_1 \land c_2 \mid (c_1, c_2) \in C\} = 1$. A paircover $C$ of $(L_0, L_1, L_2)$ is strong if, for any $(c_1, c_2) \in C$, $c_1 \lor c_2 = 0$ whenever $c_1 \land c_2 = 0$ (that is, $(c_1, c_2) = (0, 0)$ whenever $c_1 \land c_2 = 0$). For any paircovers $C$ and $D$ of $(L_0, L_1, L_2)$ one writes $C \leq D$ (and say that $C$ refines $D$) if for any $(c_1, c_2) \in C$ there is $(d_1, d_2) \in D$ with $c_1 \leq d_1$ and $c_2 \leq d_2$. Further $C \land D = \{(c_1 \land d_1, c_2 \land d_2) \mid (c_1, c_2) \in C, (d_1, d_2) \in D\}$. It is obvious that $C \land D$ is a paircover of $(L_0, L_1, L_2)$. For $a \in L_0$ and $C$ a paircover of $(L_0, L_1, L_2)$, let

$$st_1(a, C) = \bigvee \{c_1 \mid (c_1, c_2) \in C \land c_2 \land a \neq 0\},$$

$$st_2(a, C) = \bigvee \{c_2 \mid (c_1, c_2) \in C \land c_1 \land a \neq 0\}$$

and

$$C^* = \{(st_1(c_1, C), st_2(c_2, C)) \mid (c_1, c_2) \in C\}.$$

A non-empty family $C$ of paircovers of $(L_0, L_1, L_2)$ is a quasi-uniformity [6] on $(L_0, L_1, L_2)$ if:

(C1) For any $C \in C$ and any paircover $D$ with $C \leq D$, then $D \in C$.

(C2) For any $C, D \in C$ there exists a strong $E \in C$ such that $E \leq C \land D$.

(C3) For any $C \in C$ there is a $D \in C$ such that $D^* \leq C$.

(C4) For each $a \in L_i$, $a = \bigvee \{b \in L_i \mid st_i(b, C) \leq a$ for some $C \in C\}$ ($i = 1, 2$).
ON THE DESCRIPTION OF FRAME ASYMMETRIC UNIFORMITIES VIA PAIRCOVERS

Axioms (C1) and (C2) assert that the family of strong members of $C$ is a filter-base for $C$ with respect to $\wedge$ and $\leq$.

$((L_0, L_1, L_2), C)$ is called a quasi-uniform biframe [9] (quasi-uniform frame in the original [6]). A subset $B$ of $C$ is a base for $C$ if, for each $C \in C$, there is a $B \in B$ such that $B \leq C$.

Let $((L_0, L_1, L_2), C)$ and $((M_0, M_1, M_2), D)$ be quasi-uniform biframes. A biframe homomorphism $h : (L_0, L_1, L_2) \rightarrow (M_0, M_1, M_2)$ is uniform if for every $C \in C$,

$$h[C] := \{(h(c_1), h(c_2)) \mid (c_1, c_2) \in C\} \in D.$$ 

Quasi-uniform biframes and uniform biframe homomorphisms constitute a category that we denote by $\textbf{QUBiFrm}$.

While the approach via paircovers is most convenient for calculations (the entourage approach asks for a good knowledge of the construction of binary coproducts of frames), in the entourage approach the quasi-uniformity is defined directly on a frame; do not require – as the (paircovering) quasi-uniformities – prior knowledge of the underlying biframe. This makes the analogy with the spatial case evident [17, 3] and clarifies the discussion in D. Doitchinov paper [2].

This paper is concerned with the description of frame quasi-uniformities via paircovers: is it possible to define a (paircovering) quasi-uniformity directly on a frame, without requiring prior knowledge of the underlying biframe? This is the question that we address in this paper.

After this introductory section we shall give, in the second section, the solution to the problem just described: the category $\textbf{QUFrm}$ of quasi-uniform frames and uniform homomorphisms. Then, in Section 3, with a number of lemmas we prepare the ground for the proof, in Section 4, that our $\textbf{QUFrm}$ is indeed a solution for the problem. Finally, in Section 5 we briefly present the dual adjunction between $\textbf{QUFrm}$ and the category of quasi-uniform spaces and, in Section 6, we show how to introduce quasi-proximities directly on a frame.

For general information on locales and frames we refer to [12] and [18]. We recall that a biframe [1] is a triple $(L_0, L_1, L_2)$ in which $L_0$ is a frame, $L_1$ and $L_2$ are subframes of $L_0$ and $L_1 \cup L_2$ generates $L_0$ (by joins of finite meets). A biframe homomorphism $h : (L_0, L_1, L_2) \rightarrow (M_0, M_1, M_2)$ is a frame homomorphism from $L_0$ to $M_0$ such that the image of $L_i$ under $h$ is contained in $M_i$ for $i = 1, 2$. Biframes and biframe homomorphisms are the objects and
arrows of the category \texttt{BiFrm}. For more details on biframes consult [1] or [19]. If $(L_0, L_1, L_2)$ is a biframe and $a \in L_i$ ($i = 1, 2$), we denote by $a^*$ the element $igvee \{b \in L_j \mid a \land b = 0\}$ ($j \in \{1, 2\}, j \neq i$) [19]. This is the analogue on biframes of the pseudocomplement $a^* = \bigvee \{b \in L \mid a \land b = 0\}$ of any element $a$ of a frame $L$.

2. The solution

Let $L$ be a frame. In analogy with [6] we call $C \subseteq L \times L$ a paircover of $L$ if $\bigvee \{c_1 \land c_2 \mid (c_1, c_2) \in C\} = 1$. A paircover $C$ of $L$ is strong if, for any $(c_1, c_2) \in C$, $c_1 \lor c_2 = 0$ whenever $c_1 \land c_2 = 0$. For any $C, D \subseteq L \times L$ we write $C \leq D$ (and say that $C$ refines $D$) if for any $(c_1, c_2) \in C$ there is $(d_1, d_2) \in D$ with $c_1 \leq d_1$ and $c_2 \leq d_2$. Further, for any paircovers $C$ and $D$, $C \land D = \{(c_1 \land d_1, c_2 \land d_2) \mid (c_1, c_2) \in C, (d_1, d_2) \in D\}$ is a paircover. The set of all paircovers of $L$ has also non-empty arbitrary joins given just by union. For $a \in L$ and $C, D \subseteq L \times L$, we set

$$st_1(a, C) = \bigvee \{c_1 \mid (c_1, c_2) \in C \text{ and } c_2 \land a \neq 0\},$$

$$st_2(a, C) = \bigvee \{c_2 \mid (c_1, c_2) \in C \text{ and } c_1 \land a \neq 0\},$$

$$C^{-1} = \{(c_2, c_1) \mid (c_1, c_2) \in C\}$$

and

$$st(D, C) = \{(st_1(d_1, C), st_2(d_2, C)) \mid (d_1, d_2) \in D\}.$$ 

The particular case $st(C, C)$ is denoted by $C^*$. We say that $C$ star-refines $D$ if $C^* \leq D$.

Given a non-empty family $U$ of paircovers of $L$, we write $a \triangleleft_i b$ ($i = 1, 2$) whenever $st_i(a, U) \leq b$ for some $U \in \mathcal{U}$, and define

$$L_i(U) = \{a \in L \mid a = \bigvee \{b \in L \mid b \triangleleft_i a\}\} \quad (i = 1, 2).$$

\textbf{Definition 2.1.} We say that a non-empty family $U$ of paircovers of $L$ is a quasi-uniformity on $L$ if:

(U1) For any $U \in \mathcal{U}$ and any paircover $V$ with $U \leq V$, then $V \in \mathcal{U}$.

(U2) For any $U, V \in \mathcal{U}$ there exists a strong $W \in \mathcal{U}$ such that $W \leq U \land V$.

(U3) For any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V^* \leq U$.

(U4) $(L, L_1(U), L_2(U))$ is a biframe.
In the following section we will show that Axiom (U4) may be replaced by a condition similar to (C4) containing no reference to biframes.

The pair \((L, \mathcal{U})\) is called a \textit{quasi-uniform frame}. \(\mathcal{B} \subseteq \mathcal{U}\) is a base for \(\mathcal{U}\) if, for each \(U \in \mathcal{U}\), there is a \(B \in \mathcal{B}\) such that \(B \subseteq U\).

Let \((L, \mathcal{U})\) and \((M, \mathcal{V})\) be quasi-uniform frames. A frame homomorphism \(h : L \to M\) is \textit{uniform} if \(h[U] \in \mathcal{V}\) for every \(U \in \mathcal{U}\). Quasi-uniform frames and uniform homomorphisms constitute a category that we denote by \(\text{QUFrm}\). Proving that this category is isomorphic to \(\text{QUBFrm}\) is the main aim of this paper.

3. Lemmas for the proof

\textbf{Lemma 3.1.} Let \(U, V, W \subseteq L \times L\) and \(a, b \in L\). Then, for \(i, j = 1, 2\), we have:

1. If \(a \leq b\) then \(st_i(a, U) \leq st_i(b, U)\).
2. If \(U \leq V\) then \(st_i(a, U) \leq st_i(a, V)\).
3. \(a \wedge st_1(b, U) = 0\) iff \(b \wedge st_2(a, U) = 0\).
4. If \(U\) is a paircover then \(a \leq st_i(a, U)\) and \(U \leq U^*\).
5. If \(U\) is a paircover then \(st_i(st_i(a, U), U) \leq st_i(a, U^*)\).
6. \(st_i(a, U^{-1}) = st_j(a, U)\) \((j \neq i)\).
7. If \(U\) is a paircover then \(V \leq st(V, U)\).
8. If \(U\) and \(V\) are paircovers and \(V\) is strong then \(V \leq st(U, V)\).
9. If \(V \leq W\) then \(st(U, V) \leq st(U, W)\) and \(st(V, U) \leq st(W, U)\).
10. If \(U\) is a paircover then \(st(st(V, U), U) \leq st(V, U^*)\).
11. For any frame homomorphism \(h : L \to M\), \(st_i(h(a), h[U]) \leq h(st_i(a, U))\).
12. For any frame homomorphism \(h : L \to M\), \(h[U]^* \leq h[U^*]\).

\textit{Proof:} (1), (2) and (3) are trivial.

4. For each \(a \in L\) we have \(a = a \wedge 1 = a \wedge \bigvee \{u_1 \wedge u_2 \mid (u_1, u_2) \in U\} = \bigvee \{a \wedge u_1 \wedge u_2 \mid (u_1, u_2) \in U, a \wedge u_1 \wedge u_2 \neq 0\} \leq \bigvee \{u_1 \mid (u_1, u_2) \in U, a \wedge u_2 \neq 0\}\). Thus \(a \leq st_1(a, U)\). The case \(i = 2\) is similar. Hence \(U \leq U^*\).

5. It is an immediate consequence of properties (2) and (3).

6. Trivial.

7. Follows immediately from property (4).

8. Let \((v_1, v_2) \in V\). If \(v_1 \wedge v_2 = 0\) then \((v_1, v_2) = (0, 0)\) and so there is obviously some \((u_1, u_2) \in U\) such that \(v_i \leq st_i(u_i, V)\) \((i = 1, 2)\). Otherwise
$v_1 \land v_2 \neq 0$. Then, since $U$ is a paircover, there is some $(u_1, u_2) \in U$ such that $u_1 \land u_2 \land v_1 \land v_2 \neq 0$. Immediately $v_i \leq st_i(u_i, V) \ (i = 1, 2)$.

(9) The first assertion follows from property (2) and the second from (1).

(10) It follows from property (5).

(11) It is obvious, since $h(a) \neq 0$ implies $a \neq 0$ for every frame homomorphism $h$.

(12) It is an immediate consequence of (11).

**Lemma 3.2.** Let $U$ be a base for a filter of paircovers of $L$. Then, for $i = 1, 2$, the relations $\prec_i$ are sublattices of $L \times L$, stronger than $\leq$, satisfying the following properties:

1. For any $a, b, c, d \in L$, $a \leq b \prec_i c \leq d$ implies $a \prec_i d$.
2. For any $a, b \in L$, $a \prec_i b$ implies $a \ast \lor b = 1$.
3. $L_i(U)$ is a subframe of $L$.

**Proof:** The fact that each $\prec_i$ is stronger than $\leq$ follows from 3.1(4). Clearly $0 \prec_i 0$ and $1 \prec_i 1$. If $st_i(a_1, U_1) \leq b_1$ and $st_i(a_2, U_2) \leq b_2$ with $U_1, U_2 \in U$ then, immediately, $st_i(a_1 \land a_2, U_1 \land U_2) \leq st_i(a_1, U_1) \land st_i(a_2, U_2) \leq b_1 \land b_2$. Since $U$ is a filter basis, there exists some $V \in U$ such that $V \leq U_1 \land U_2$. Hence, using property 3.1(2), we may conclude that $a_1 \land a_2 \prec_i b_1 \land b_2$. On the other hand, as can be easily checked, $st_i(a_1 \lor a_2, U_1 \land U_2) \leq st_i(a_1, U_1) \lor st_i(a_2, U_2) \leq b_1 \lor b_2$. Thus $a_1 \lor a_2 \prec_i b_1 \lor b_2$.

(1) It is obvious from Lemma 3.1(1).

(2) Let $i, j \in \{1, 2\}$ with $i \neq j$. Assume $a \prec_i b$, that is, $st_i(a, U) \leq b$ for some $U \in U$. Let $u_1 \land u_2$ with $(u_1, u_2) \in U$. If $u_j \land a = 0$ then $u_1 \land u_2 \leq u_j \leq a^*$; otherwise, $u_1 \land u_2 \leq u_i \leq st_i(a, U) \leq b$. Hence $1 = \bigvee \{u_1 \land u_2 \mid (u_1, u_2) \in U\} \leq a^* \lor b$.

(3) It is an immediate consequence of the fact that each $\prec_i$ is a sublattice of $L \times L$, stronger than $\leq$, and assertion (1).
Let $\mathcal{U}$ be a quasi-uniformity on $L$. By property (7) of Lemma 3.1, the relation $\in \mathcal{U}$ on $\mathcal{P}(L \times L)$ defined by

$$C \in \mathcal{U} \implies D \equiv \text{st}(C, U) \leq D \text{ for some } U \in \mathcal{U}$$

is stronger than $\leq$. We shall also need the following (interior) operator on $\mathcal{P}(L \times L)$:

$$\text{int}(C) = \bigcup \{D \subseteq L \times L \mid D \in \mathcal{U}, D \subseteq C\}.$$  

**Lemma 3.3.** Let $\mathcal{U}$ be a quasi-uniformity on $L$. For each $U \in \mathcal{U}$ we have:

1. $\text{int}(U) \leq U \leq \text{int}(U^*)$.
2. For every $a \in L$, $\text{st}_i(a, \text{int}(U)) \in L_i(U)$ ($i = 1, 2$).

**Proof:** (1) The inequality $\text{int}(U) \leq U$ is trivial. The other follows from the obvious fact that for every $U \in \mathcal{U}$, $U \in \mathcal{U}$.

(2) We only prove the case $i = 1$ (the case $i = 2$ may be proved in a similar way). We need to show that

$$\text{st}_1(a, \text{int}(U)) \leq \bigvee \{y \in L \mid y \leq \text{st}_1(a, \text{int}(U))\}.$$  

By definition, $\text{st}_1(a, \text{int}(U)) = \bigvee \{d_1 \mid (d_1, d_2) \in \text{int}(U), d_2 \land a \neq 0\}$. So, let $(d_1, d_2) \in \text{int}(U)$ such that $d_2 \land a \neq 0$. Then $(d_1, d_2) \in D \subseteq L \times L$ and there exists $V \in \mathcal{U}$ such that $\text{st}(D, V) \subseteq U$. We need to show that

$$d_1 \leq \text{st}_1(a, \text{int}(U)).$$  

To see this consider $W \in \mathcal{U}$ such that $W^* \leq V$. It suffices then to prove that $\text{st}_1(d_1, W) \leq \text{st}_1(a, \text{int}(U))$.

By properties (9) and (10) of Lemma 3.1 we have

$$\text{st}(D, W) \leq \text{st}(D, W^*) \leq \text{st}(D, V) \leq U,$$  

which shows that $\text{st}(D, W) \in \mathcal{U}$. Thus $\text{st}(D, W) \subseteq \text{int}(U)$. Therefore we only need to check that $\text{st}_1(d_1, W) \leq \text{st}_1(a, \text{st}(D, W))$, which is easy since $(\text{st}_1(d_1, W), \text{st}_2(d_2, W)) \in \text{st}(D, W)$ and $\text{st}_2(d_2, W) \land a \geq d_2 \land a \neq 0$.

**Remark 3.4.** Let $\overline{\mathcal{U}}$ be the filter of paircovers of $L$ generated by $\{U \land U^{-1} \mid U \in \mathcal{U}\}$. Then, by Lemma 3.1(6), $\text{st}_1(a, U \land U^{-1}) = \text{st}_2(a, (U \land U^{-1})^{-1})$.

Since $(U \land U^{-1})^{-1} = U \land U^{-1}$, then $\triangleleft_1 = \triangleleft_2$. Below we denote this relation on $L$ just by $\triangleleft_1$.  

Proposition 3.5. Let $\mathcal{U}$ be a non-empty family of paircovers of $L$ satisfying axioms (U1), (U2) and (U3). Then $\mathcal{U}$ satisfies (U4) if and only if

$$(U4') \text{ For each } a \in L, \ a = \bigvee \{b \in L \mid b \triangleleft a\}.$$ 

Proof: “⇒”: For each $a \in L$ we may write $a = \bigvee_{i \in I} (a^1_i \land a^2_i)$ for some

$$\{a^1_i \mid i \in I\} \subseteq L_1(\mathcal{U}) \quad \text{and} \quad \{a^2_i \mid i \in I\} \subseteq L_2(\mathcal{U}).$$

Taking into account that, for any $i \in I$,

$$a^1_i = \{b \in L \mid b \triangleleft^1 a^1_i\} \quad \text{and} \quad a^2_i = \{b \in L \mid b \triangleleft^2 a^2_i\},$$

it suffices to show that $b_1 \land b_2 \triangleleft^1 a_1 \land a_2$ whenever $b_1 \triangleleft^1 a_1$ and $b_2 \triangleleft^2 a_2$. This property is an immediate consequence of Remark 3.4 and properties (1) and (2) of Lemma 3.1.

“⇐”: By Lemma 3.2, each $L_i(\mathcal{U})$ ($i = 1, 2$) is a subframe of $L$. It remains to show that each $a \in L$ is a join of finite meets in $L_1(\mathcal{U}) \cup L_2(\mathcal{U})$.

Let $a \in L$. Then $a = \bigvee S$ where $S = \{b \in L \mid b \triangleleft a\}$. For each $b \in S$ there exists $\overline{U}_b \in \overline{\mathcal{U}}$ and $\bar{U}_b \in \mathcal{U}$ such that $st_1(b, \overline{U}_b) \leq a$ and $\bar{U}_b \land \bar{U}_b^{-1} \leq \overline{U}_b$. Consider $V_b \in \mathcal{U}$ such that $V^*_b \leq \overline{U}_b$. Then $\overline{V}_b = V_b \land V_b^{-1} \in \overline{\mathcal{U}}$ and $\overline{V}^*_b \leq \overline{U}_b$. Therefore $\text{int}(\overline{V}_b) \leq \text{int}(\overline{U}_b)$. Thus

$$a = \bigvee S \leq \bigvee_{b \in S} (st_1(b, \overline{V}_b) \land st_2(b, \overline{V}_b))$$

$$\leq \bigvee_{b \in S} (st_1(b, \text{int}(\overline{V}_b^*)) \land st_2(b, \text{int}(\overline{V}_b^*)))$$

$$\leq \bigvee_{b \in S} (st_1(b, \text{int}(\overline{U}_b)) \land st_2(b, \text{int}(\overline{U}_b)))$$

$$\leq \bigvee_{b \in S} (st_1(b, \overline{U}_b))$$

$$\leq a.$$

Hence

$$a = \bigvee_{b \in S} (st_1(b, \text{int}(\overline{U}_b)) \land st_2(b, \text{int}(\overline{U}_b)))$$

and, by Lemma 3.3, $st_i(b, \text{int}(\overline{U}_b)) \in L_i(\mathcal{U})$ ($i = 1, 2$).
Remark 3.6. Note that our Definition 2.1 of a quasi-uniformity contains, of course, the particular case of uniform structures defined by covers: it is one that has a base consisting of pairs of covers, both coordinates of which are the same (i.e., of the form \((U,U)\) where \(U\) is a cover of \(L\)); certainly such paircovers are strong, moreover \((U,U)^* = (U^*,U^*)\), where \(U^*\) is the usual star of the cover \(U\), and \(\lhd\) is the usual \(\lhd\).

4. The proof

Proposition 4.1. Let \(((L_0,L_1,L_2),C)\) be an object of \(\text{QUBiFrm}\). Then \((L_0,C)\) is a quasi-uniform frame.

Proof: Every element of \(C\), being a paircover of \((L_0,L_1,L_2)\), is of course a paircover of \(L_0\). In addition, \(C\) satisfies axioms (U1), (U2) and (U3) trivially so it suffices to check (U4). By Lemma 3.2, each \(L_i(C)\) is a subframe of \(L_0\) so it suffices to show that \(L_i \subseteq L_i(C)\) for \(i = 1,2\). Let \(a \in L_i\). Then \(a = \bigvee \{b \in L_i \mid st_i(b,C) \leq a\}\) for some \(C \in C\}\) \(\leq \bigvee \{b \in L_0 \mid b \lhd_i a\}\) for some \(C \in C\}\) \(\leq a\).

Concerning maps, the following is obvious:

Proposition 4.2. Let \(h : ((L_0,L_1,L_2),C) \to ((M_0,M_1,M_2),D)\) be a morphism of \(\text{QUBiFrm}\). Then \(h : (L_0,C) \to (M_0,D) \in \text{QUFrm}\).

Propositions 4.1 and 4.2 establish a functor \(\Phi : \text{QUBiFrm} \to \text{QUFrm}\).

Let \(U \subseteq L \times L\) and \(a \in L\). In the following, we denote the element \(\bigvee \{u_1 \land u_2 \mid (u_1,u_2) \in U, u_1 \land u_2 \land a \neq 0\}\) by \(st(a,U)\). It is obvious that for every \(a \in L\) and every paircover \(U\) of \(L\),

\[a \leq st(a,U) \leq st_1(a,U) \land st_2(a,U).\]

Given a paircover \(U\) of a frame \(L\), we say that an element \(a\) of \(L\) is \(U\)-small if \(a \leq st(b,U)\) whenever \(a \land b \neq 0\). Note that, for any \((u_1,u_2) \in U\), \(u_1 \land u_2\) is \(U\)-small. We define also

\[C_U = \{(st_1(a,\text{int}(U)),st_2(a,\text{int}(U))) \mid a \text{ is an } U\text{-small member of } L\}.\]

Lemma 4.3. Let \((L,U) \in \text{QUFrm}\). For each \(a \in L\) and \(U,V \in U\) we have:

1. Each \(C_U\) is a strong paircover of the biframe \((L,L_1(U),L_2(U))\).
2. \(C_{U \land V} \leq C_U \land C_V\).
(3) \( st_i(a, C_U) \leq st_i(a, U^*) \) \((i = 1, 2)\).
(4) \( st_i(a, U) \leq st_i(a, C_{U^*}) \) \((i = 1, 2)\).
(5) \((C_U)^{+} \leq C_{U^{**}}\).

Proof: (1) By Lemma 3.3(2) each \( C_U \) is a subset of \( L_1(U) \times L_2(U) \). It is a paircover since \( \bigvee \{st_1(a, \text{int}(U)) \land st_2(a, \text{int}(U)) \mid a \) is \( U \)-small\} \( \geq \bigvee \{a \in L \mid a \) is \( U \)-small\} \( \geq \bigvee \{u_1 \land u_2 \mid (u_1, u_2) \in U\} = 1 \). Finally, it is strong: if \( st_1(a, \text{int}(U)) \lor st_2(a, \text{int}(U)) \neq 0 \) then \( a \neq 0 \) so \( st_1(a, \text{int}(U)) \land st_2(a, \text{int}(U)) \geq a \neq 0 \).

(2) Trivial.

(3) Fix \( i \in \{1, 2\} \) and let \( j \in \{1, 2\} \) with \( j \neq i \). By definition,
\[
st_i(a, C_U) = \bigvee \{st_i(b, \text{int}(U)) \mid b \) is \( U \)-small, \( st_j(b, \text{int}(U)) \land a \neq 0\}.
\]

By Lemma 3.1(3), \( st_j(b, \text{int}(U)) \land a \neq 0 \) is equivalent to \( st_i(a, \text{int}(U)) \land b \neq 0 \) and, since \( b \) is \( U \)-small, this implies that \( b \leq st(st_i(a, \text{int}(U)), U) \leq st_i(a, U^*) \) (using property (4) of Lemma 3.1). Hence
\[
st_i(b, \text{int}(U)) \leq st_i(st_i(a, U^*), \text{int}(U)) \leq st_i(st_i(a, U^*), U^*) \leq st_i(a, U^{**}).
\]

(4) We have \( st_1(a, U) = \bigvee \{u_1 \mid (u_1, u_2) \in U, u_2 \land a \neq 0\} \) and for each such \( u_1 \), since \( U^* \) is a paircover, we may write \( u_1 = \bigvee \{u_1 \land d_1 \land d_2 \mid (d_1, d_2) \in U^*, u_1 \land d_1 \land d_2 \neq 0\} \). But, for each \( (d_1, d_2) \in U^* \), \( u_1 \land d_1 \land d_2 \) is \( U^* \)-small. Indeed, if \( y \land u_1 \land d_1 \land d_2 \neq 0 \) then \( u_1 \land d_1 \land d_2 \leq st(y, U^*) \) (since \( u_1 \land d_1 \land d_2 \leq d_1 \land d_2 \), \( (d_1, d_2) \in U^* \) and \( d_1 \land d_2 \land y \geq y \land u_1 \land d_1 \land d_2 \neq 0 \)). Therefore
\[
(st_1(u_1 \land d_1 \land d_2, \text{int}(U^*)), st_2(u_1 \land d_1 \land d_2, \text{int}(U^*))) \in C_{U^*}.
\]

It only remains to prove that \( a \land st_2(u_1 \land d_1 \land d_2, \text{int}(U^*)) \neq 0 \), which is easy:
\[
a \land st_2(u_1 \land d_1 \land d_2, \text{int}(U^*)) = \bigvee \{a \land c_2 \mid (c_1, c_2) \in \text{int}(U^*), c_1 \land a \land st_2(u_1 \land d_1 \land d_2, \text{int}(U^*)) \neq 0\}
\geq \bigvee \{a \land c_2 \mid (c_1, c_2) \in U, c_1 \land a \land st_2(u_1 \land d_1 \land d_2, \text{int}(U^*)) \neq 0\}
\geq a \land u_2 \neq 0.
\]

(5) Let \( a \) be an \( U \)-small element of \( L \) and consider
\[
(st_1(st_1(a, \text{int}(U)), C_U), st_2(st_2(a, \text{int}(U)), C_U)) \in C_{U^*}.
\]
For $i = 1, 2$ and $j \in \{1, 2\}, j \neq i$, we have

$$\text{st}_i(\text{st}_i(a, \text{int}(U)), C_U) = \bigvee \{\text{st}_i(b, \text{int}(U)) | b \text{ is } U\text{-small}, \text{st}_i(a, \text{int}(U)) \land \text{st}_j(b, \text{int}(U)) \neq 0\}. \quad (*)$$

But, by Lemma 3.1(3),

$$\text{st}_i(a, \text{int}(U)) \land \text{st}_j(b, \text{int}(U)) \neq 0 \iff b \land \text{st}_i(a, \text{int}(U)), \text{int}(U) \neq 0.$$

Thus, by the $U$-smallness of $b$,

$$b \leq \text{st}(\text{st}_i(st_i(a, \text{int}(U)), \text{int}(U)), U) \leq \text{st}_i(st_i(a, U), U).$$

Hence, each element in the join $(*)$ satisfies

$$\text{st}_i(b, \text{int}(U)) \leq \text{st}_i(st_i(st_i(st_i(st_i(a, U), U), U), U)).$$

Finally, applying property (5) of Lemma 3.1 twice and Lemma 3.3(1) we get

$$\text{st}_i(b, \text{int}(U)) \leq \text{st}_i(a, U^{**}) \leq \text{st}_i(a, \text{int}(U^{**}))$$

(and note that, of course, $a$ is $U^{***}$-small).

The following three results establish a functor $\Psi : \text{QUFrm} \to \text{QUBiFrm}$.

**Proposition 4.4.** Let $(L, U)$ be an object of $\text{QUFrm}$. Then $\{C_U | U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{C}_U$ on the biframe $(L, L_1(U), L_2(U))$.

**Proof:** By Lemma 4.3, each $C_U$ is a strong paircover of the biframe $(L, L_1(U), L_2(U))$. Let us check that $\mathcal{C}_U$ satisfies axioms (C1), (C2), (C3) and (C4):

(C1) Trivial.

(C2) Lemma 4.3(2).

(C3) Let $C \in \mathcal{C}_U$. Then there exists $U \in \mathcal{U}$ such that $C_U \leq C$. Take $V \in \mathcal{U}$ satisfying $V^{***} \leq U$. By Lemma 4.3(5), $(C_V)^* \leq C_V^{***} \leq C_U \leq C$.

(C4) Let $a \in L_i(U) \; (i = 1, 2)$. We need to prove that

$$a = \bigvee \{b \in L_i(U) | \text{st}_i(b, C) \leq a \text{ for some } C \in \mathcal{C}_U\}.$$

By hypothesis, $a = \bigvee \{b \in L | b \leq a\}$. Therefore it is sufficient to show that $b \leq a$ implies the existence of $b' \in L_i(U)$ such that $b \leq b' \leq \text{st}_i(b', C_U) \leq a$ for some $U \in \mathcal{U}$. 

Let \( b \not\leq_U a \). Then there is \( U \in \mathcal{U} \) satisfying \( \text{st}_1(b, U) \leq a \). Let \( V \in \mathcal{U} \) such that \( V^{**} \leq U \) and consider also \( W \in \mathcal{U} \) such that \( W^{****} \leq V \). By Lemma 3.3, \( b \leq \text{st}_1(b, \text{int}(W)) \in L_1(\mathcal{U}) \). Let us show that \( \text{st}_1(b, \text{int}(W)) \) is the required \( b' \in L_1(\mathcal{U}) \), by checking that \( \text{st}_1(\text{st}_1(b, \text{int}(W)), C_{W^*}) \leq a \):

By Lemma 4.3(4) we have

\[
\text{st}_1(\text{st}_1(b, \text{int}(W)), C_{W^*}) \leq \text{st}_1(\text{st}_1(b, W), C_{W^*}) \leq \text{st}_1(\text{st}_1(b, C_{W^*}), C_{W^*}).
\]

Then, by Lemma 3.1(5) and Lemma 4.3(5),

\[
\text{st}_1(\text{st}_1(b, \text{int}(W)), C_{W^*}) \leq \text{st}_1(\text{st}_1(b, (C_{W^*})^*) \leq \text{st}_1(b, C_{W^{****}}) \leq \text{st}_1(b, C_V).
\]

Finally, using Lemma 4.3(3), we may conclude that

\[
\text{st}_1(\text{st}_1(b, \text{int}(W)), C_{W^*}) \leq \text{st}_1(b, V^{**}) \leq \text{st}_1(b, U) \leq a.
\]

The proof for \( i = 2 \) is similar.

**Lemma 4.5.** Let \( h : (L, \mathcal{U}) \to (M, \mathcal{V}) \) be a morphism of \( \text{QUFrm} \), \( a, b \in L \) and \( U \in \mathcal{U} \). Then:

1. If \( b \not\leq_U a \) then \( h(b) \not\leq_V h(a) \), for \( i = 1, 2 \).
2. \( C_{h[U]} \leq h[C_{U^{**}}] \).

**Proof:**

1. Let \( b \not\leq_U a \). Then \( \text{st}_1(b, U) \leq a \) for some \( U \in \mathcal{U} \). Consider \( V = h[U] \in \mathcal{V} \). Using Lemma 3.1(11) we may conclude that \( \text{st}_1(h(b), V) \leq h(\text{st}_1(b, U)) \leq h(a) \) and thus \( h(b) \not\leq_V h(a) \).

2. We need to show that for each \( h[U] \)-small element \( b \) of \( M \) there exists an \( U^{**} \)-small element \( a \in L \) such that

\[
\text{st}_i(b, \text{int}(h[U])) \leq h(\text{st}_i(a, \text{int}(U^{**}))) \quad (i = 1, 2).
\]

So, let \( b \neq 0 \) be \( h[U] \)-small. Then, since \( h[U] \) is a paircover of \( M \), there exists \((u_1, u_2) \in U \) for which \( b \wedge h(u_1) \wedge h(u_2) \neq 0 \). Then, by the \( h[U] \)-smallness of \( b \), \( b \leq \text{st}(h(u_1) \wedge h(u_2), h[U]) \). Let us denote \( u_1 \wedge u_2 \) by \( a \). We have \( b \leq \text{st}_1(h(a), h[U]) \).

Then

\[
\text{st}_i(b, \text{int}(h[U])) \leq \text{st}_i(b, h[U]) \leq \text{st}_i(\text{st}_i(h(a), h[U]), h[U]) \leq \text{st}_i(h(a), h[U]^*).
\]

Using 3.1(12) and (11) we get

\[
\text{st}_i(b, \text{int}(h[U])) \leq \text{st}_i(h(a), h[U^*]) \leq h(\text{st}_i(a, U^*)).
\]

Hence, by Lemma 3.3, \( \text{st}_i(b, \text{int}(h[U])) \leq h(\text{st}_i(a, \text{int}(U^{**}))) \). Finally, \( a \) is \( U^{**} \)-small because \( a = u_1 \wedge u_2 \) and \((u_1, u_2) \in U \leq U^{**} \).
**Proposition 4.6.** Let \( h : (L, \mathcal{U}) \to (M, \mathcal{V}) \) be a morphism of \( \text{QUFrm} \). Then \( h : ((L, L_1(\mathcal{U}), L_2(\mathcal{U})), \mathcal{C}_\mathcal{U}) \to ((M, M_1(\mathcal{V}), M_2(\mathcal{V})), \mathcal{C}_\mathcal{V}) \in \text{QUBiFrm} \).

**Proof:** First we check that the frame homomorphism \( h : (L, \mathcal{U}) \to (M, \mathcal{V}) \) is also a biframe homomorphism from \((L, L_1(\mathcal{U}), L_2(\mathcal{U}))\) to \((M, M_1(\mathcal{V}), M_2(\mathcal{V}))\). Let \( a \in L_i(\mathcal{U}) \) \((i = 1, 2)\). We need to show that \( h(a) \in M_i(\mathcal{V}) \). Since 

\[
\forall \{b \in L \mid b \preceq_i a\}, \text{we may conclude by property (1) of the previous lemma that}
\]

\[
h(a) = \bigvee \{h(b) \mid b \in L, b \preceq_i a\} \leq \bigvee \{m \in M \mid m \preceq_i h(a)\} \leq h(a)
\]

which ensures that \( h(a) = \bigvee \{m \in M \mid m \preceq_i h(a)\} \) — meaning that \( h(a) \in M_i(\mathcal{V}) \).

Finally, it remains to show that \( h[C] \in \mathcal{C}_\mathcal{V} \) for every \( C \in \mathcal{C}_\mathcal{U} \). Let \( C \in \mathcal{C}_\mathcal{U} \) and \( U \in \mathcal{U} \) such that \( C_U \leq C \). Consider \( V \in \mathcal{U} \) satisfying \( V^{**} \leq U \). By Lemma 4.5(2), \( C_{h[V]} \leq h[C_{V^{**}}} \leq h[C_U] \leq h[C] \). Since \( h[V] \in \mathcal{V} \), then \( h[C] \in \mathcal{C}_\mathcal{V} \).

In order now to prove the isomorphism \( \text{QUFrm} \cong \text{QUBiFrm} \) we only need the following:

**Lemma 4.7.** Let \((L, \mathcal{U})\) be a quasi-uniform frame. For any paircover \( U \) in \( \mathcal{U} \) we have:

1. If \( U \) is strong then \( U \leq C_U^{**} \).
2. \( C_U \leq U^{**} \).

**Proof:** (1) Let \((u_1, u_2) \in U \). Since \( U \) is strong we may assume that \( u_1 \land u_2 \neq 0 \). Since \( U^* \) is a paircover then \( u_1 \land u_2 = \bigvee \{u_1 \land u_2 \land v_1 \land v_2 \mid (v_1, v_2) \in U^*\} \). Therefore, there exists \((v_1, v_2) \in U^*\) for which \( u_1 \land u_2 \land v_1 \land v_2 \neq 0 \). This implies immediately that

\[
u_1 \leq st_1(v_1 \land v_2, U) \leq st_1(v_1 \land v_2, \text{int}(U^*))
\]

and

\[
u_2 \leq st_2(v_1 \land v_2, U) \leq st_2(v_1 \land v_2, \text{int}(U^*)).
\]

But \( v_1 \land v_2 \) is \( U^* \)-small (because \((v_1, v_2) \in U^*\) hence

\[
(st_1(v_1 \land v_2, \text{int}(U^*))), st_2(v_1 \land v_2, \text{int}(U^*)) \in C_U^*.
\]

and it is proved.
(2) Let \( a \neq 0 \) be an \( U \)-small element of \( L \). Since \( U \) is a paircover, there exists \((u_1, u_2) \in U\) such that \( a \land u_1 \land u_2 \neq 0 \). Then, by the \( U \)-smallness of \( a \), \( a \leq \text{st}(u_1 \land u_2, U) \leq \text{st}_1(u_1, U) \land \text{st}_2(u_2, U) \). Consequently, \( \text{st}_i(a, \text{int}(U)) \leq \text{st}_i(u_i, U) \), for \( i = 1, 2 \). Of course, there exists \((v_1, v_2) \in U^*\) for which \( u_i \leq v_i \) \((i = 1, 2)\). Hence \( \text{st}_i(a, \text{int}(U)) \leq \text{st}_i(v_i, U^*) \), which guarantees that \( C_U \leq U^* \).

Now we are ready for the main theorem.

**Theorem 4.8.** The functors \( \Phi \) and \( \Psi \) establish an isomorphism between the concrete categories \( \text{QUFrm} \) and \( \text{QUBiFrm} \).

**Proof:** Let us show that \( \Psi \Phi = \text{Id}_{\text{QUBiFrm}} \) and \( \Phi \Psi = \text{Id}_{\text{QUFrm}} \). For morphisms there is nothing to prove (after Propositions 4.2 and 4.6). With respect to objects we have

\[
\Psi \Phi(((L_0, L_1, L_2), D)) = \Psi((L_0, D)) = ((L_0, (L_0)_1(C), (L_0)_2(C), C_D)
\]

and

\[
\Phi \Psi((L, U)) = \Phi((L, L_1(U), L_2(U)), C_U) = (L, C_{\text{int}}(U)),
\]

so we need to prove that (a) \((L_0)_i(C) = L_i \) \((i = 1, 2)\), (b) \( C_D = D \) and (c) \( C_U = U \).

(a) By hypothesis, \(((L_0, L_1, L_2), D)\) is an object of \( \text{QUBiFrm} \) so \( L_i \subseteq (L_0)_i(C) \) \((i = 1, 2)\). The reverse inclusion may be proved in the same way as we proved, in Proposition 4.4, that axiom (C3) is satisfied.

(b) Let \( D \in D \) and consider \( E \in D \) such that \( E^{**} \leq D \). By Lemma 4.7, \( C_E \leq E^{**} \leq D \) thus \( D \in C_D \). Conversely, let \( U \in C_D \). Then there exists \( D \in D \) such that \( C_D \leq U \). Take \( E \in D \), strong, such that \( E^* \leq D \). Then, using Lemma 4.7, we get \( E \leq C_{E^*} \leq C_D \leq U \) and \( U \in D \), as required.

(c) It is an immediate consequence of Lemma 4.7, as in (b).

**Remark 4.9.** We have now a formulation for the theory of (covering) quasi-uniformities very similar to the spatial one: the structure \( U \) is defined on a frame \( L \) and induces a bi-structure (specifically, a biframe) \((L, L_1(U), L_2(U))\) in such a way that, for instance: every frame \( L \) is quasi-uniformizable (in the sense that there exists a quasi-uniform frame \((M, U)\) such that the first induced subframe \( M_1(U) \) is isomorphic to the given \( L \)); for every quasi-uniform frame the induced biframe is always regular (in fact completely regular, assuming the axiom of countable dependent choice); every completely regular
biframe \((L_0, L_1, L_2)\) is uniformizable (in the sense that there exists a quasi-uniformity on the total part \(L_0\) of the biframe whose induced biframe coincides with the given biframe); for every compact regular biframe there is a unique quasi-uniformity on the total part of the biframe whose induced biframe coincides with the given biframe; etc.

5. The adjunction \(\text{QUnif} \rightleftarrows \text{QUFrm}\)

Let \(\text{QUnif}\) denote the category of quasi-uniform spaces and uniformly continuous maps [5]. Here, in the description of its objects we follow the covering approach of Gantner and Steinlage [10] that uses conjugate pairs of covers of a set \(X\).

The expected open and spectrum functors establishing a dual adjunction between \(\text{QUnif}\) and \(\text{QUFrm}\) are easy to describe now.

Let \((X, \mu)\) be a quasi-uniform space. To explain the notation and terminology, recall that \(\mu\) induces two topologies \(\mathcal{T}_1(\mu)\) and \(\mathcal{T}_2(\mu)\) on \(X\) in the following manner:

- \(A \subseteq X\) is \(\mathcal{T}_1(\mu)\)-open if for every \(a \in A\) there exists \(U_a \in \mu\) such that \(st_1(a, U_a) \subseteq A\);
- similarly, \(A \subseteq X\) is \(\mathcal{T}_2(\mu)\)-open if for every \(a \in A\) there exists \(U_a \in \mu\) such that \(st_2(a, U_a) \subseteq A\).

Let \(U \in \mu\). We say that \(U\) is an open paircover of \((X, \mu)\) if for each \((U_1, U_2) \in U\), \(U_1\) is \(\mathcal{T}_1(\mu)\)-open and \(U_2\) is \(\mathcal{T}_2(\mu)\)-open.

Set \(\Omega(X, \mu) = (\mathcal{T}_1(\mu) \lor \mathcal{T}_2(\mu), \mathcal{C}_\mu)\) where \(\mathcal{C}_\mu\) is the set of all open paircovers of \((X, \mu)\). It is not hard to check that \(\Omega(X, \mu) \in \text{QUFrm}\). Moreover, if \(f : (X, \mu) \rightarrow (Y, \nu)\) is uniformly continuous then \(\Omega(f) : \Omega(Y, \nu) \rightarrow \Omega(X, \mu)\), defined by \(\Omega(f)(B) = f^{-1}(B)\) for any \(B \in \mathcal{T}_1(\nu) \lor \mathcal{T}_2(\nu)\), is a uniform frame homomorphism. Thus, \(\Omega\) is a contravariant functor from \(\text{QUnif}\) into \(\text{QUFrm}\).

For each frame \(L\), we consider its spectrum, that is, the topological space

\[(\text{Pt}(L), \{\Sigma_a \mid a \in L\}),\]

where \(\text{Pt}(L)\) is the set of all frame homomorphisms \(p : L \rightarrow \{0, 1\}\) and \(\Sigma_a = \{p \in \text{Pt}(L) \mid p(a) = 1\}\). Let \((L, U)\) be a quasi-uniform frame. For each \(U \in U\), let \(\Sigma_U = \{(\Sigma_{u_1}, \Sigma_{u_2}) \mid (u_1, u_2) \in U\}\) and let \(\Sigma_U\) be the filter of paircovers of \(\text{Pt}(L)\) generated by \(\{\Sigma_U \mid U \in U\}\).

**Proposition 5.1.** Let \((L, U) \in \text{QUFrm}\). Then \(\Sigma((L, U) = (\text{Pt}(L), \Sigma_U)\) is a quasi-uniform space.
Proof: (1) Let $U \in \mathcal{U}$. Then $\bigcup \{ \Sigma_{u_1} \cap \Sigma_{u_2} \mid (u_1, u_2) \in U \} = \bigcup \{ \Sigma_{u_1 \lor u_2} \mid (u_1, u_2) \in U \} = \Sigma_1 = \text{Pt}(L)$.

(2) Let $U, V \in \mathcal{U}$. Trivially $U \leq V$ implies $\Sigma_U \leq \Sigma_V$ and $\Sigma_{U \lor V} = \Sigma_U \land \Sigma_V$.

(3) Let $U^* \leq V$. Then $(\Sigma_U)^* \leq \Sigma_V$. Indeed: for each $(u_1, u_2) \in U$ there exists $(v_1, v_2) \in V$ satisfying $st_1(u_1, U) \leq v_1$ and $st_2(u_2, U) \leq v_2$; then

$$
st_1(\Sigma_{u_1}, \Sigma_U) = \bigcup \{ \Sigma_{u_1}' \mid \Sigma_{u_2} \cap \Sigma_{u_1} \neq \emptyset, (u_1', u_2') \in U \}
= \bigcup \{ \Sigma_{u_1}' \mid \Sigma_{u_2}' \cap \Sigma_{u_1} \neq \emptyset, (u_1', u_2') \in U \}
\subseteq \bigcup \{ \Sigma_{u_1}' \mid u_2' \cap u_1 \neq 0 \}
= \Sigma_{st_1(u_1, U)} \subseteq \Sigma_{v_1}.
$$

Similarly, $st_2(\Sigma_{u_2}, \Sigma_U) \subseteq \Sigma_{v_2}$. ■

Concerning morphisms, given a map $h : (L, \mathcal{U}) \to (M, \nu)$ of $\text{QUFrm}$, let $\Sigma(h) : \Sigma(M, \nu) \to \Sigma(L, \mathcal{U})$ be defined by $\Sigma(h)(p) = ph$. It is a straightforward exercise to check that $\Sigma(h) \in \text{QUnif}$. We have therefore a contravariant functor $\Sigma : \text{QUFrm} \to \text{QUnif}$.

Remark 5.2. It is also a straightforward exercise to check that the topologies $\mathcal{T}_1(\Sigma_{\mathcal{U}})$ and $\mathcal{T}_2(\Sigma_{\mathcal{U}})$ induced by the quasi-uniformity $\Sigma_{\mathcal{U}}$ on $\text{Pt}(L)$ coincide with the spectral topologies of the spectra of $L_1(\mathcal{U})$ and $L_2(\mathcal{U})$ respectively.

Theorem 5.3. The two above contravariant functors $\Omega$ and $\Sigma$ define a dual adjunction, with adjoint units $\eta_{(X, \mu)} : (X, \mu) \to \Sigma \Omega(X, \mu)$ and $\xi_{(L, \mathcal{U})} : (L, \mathcal{U}) \to \Omega \Sigma(L, \mathcal{U})$ given by $\eta_{(X, \mu)}(a)(U) = 1$ iff $a \in U$ and $\xi_{(L, \mathcal{U})}(a) = \Sigma a$.

Proof: The checking that each $\eta_{(X, \mu)}$ is uniformly continuous and that each $\xi_{(L, \mathcal{U})}$ is a uniform frame homomorphism is left to the reader. They define natural transformations $\eta : \text{Id}_{\text{Unif}} \Rightarrow \Sigma \Omega$ and $\xi : \text{Id}_{\text{QUFrm}} \Rightarrow \Omega \Sigma$, that is, the following diagrams commute, for every $f : (X, \mu) \to (Y, \nu)$ and every $h : (L, \mathcal{U}) \to (M, \mathcal{V})$.

$$
\begin{array}{cc}
(X, \mu) & \xrightarrow{\eta_{(X, \mu)}} & \Sigma \Omega(X, \mu) \\
(1) \downarrow f & & \downarrow \Sigma \Omega(f) \\
(Y, \nu) & \xrightarrow{\eta_{(Y, \nu)}} & \Sigma \Omega(Y, \nu)
\end{array}
\quad
\begin{array}{cc}
(L, \mathcal{U}) & \xrightarrow{\xi_{(L, \mathcal{U})}} & \Omega \Sigma(L, \mathcal{U}) \\
(2) \downarrow h & & \downarrow \Omega \Sigma(h) \\
(M, \mathcal{V}) & \xrightarrow{\xi_{(M, \mathcal{V})}} & \Omega \Sigma(M, \mathcal{V})
\end{array}
$$
Indeed, for each \( f : (X, \mu) \to (Y, \nu) \) in \( \text{QUnif} \) and every \( x \in X \),
\[
\Sigma \Omega(f)(\eta_{(X, \mu)}(x)) = \eta_{(X, \mu)}(x)\Omega(f)
\]
is the map \( F : \Omega(Y, \nu) \to \{0, 1\} \) given by \( F(B) = 1 \) iff \( x \in f^{-1}(B) \). Since \( x \in f^{-1}(B) \) iff \( f(x) \in B \), this is precisely the map \( \eta_{(Y, \nu)}(f(x)) \) and diagram (1) commutes.

On the other hand, for each \( h : (L, \mathcal{U}) \to (M, V) \in \text{QUFrm} \) and every \( a \in L \),
\[
\Omega \Sigma(h)(\xi_{(L, \mathcal{U})}(a)) = \Omega \Sigma(h)(\Sigma_a) = \Sigma(h)^{-1}(\Sigma_a) = \{ p \in \text{Pt}(M) \mid \Sigma(h)(p) \in \Sigma_a \}.
\]
Since \( \Sigma(h)(p) \in \Sigma_a \) iff \( ph \in \Sigma_a \) iff \( p(h(a)) = 1 \) iff \( p \in \Sigma_{h(a)} \), then diagram (2) also commutes.

Finally, \( \eta : \text{Id}_{\text{QUnif}} \to \Sigma \Omega \) and \( \xi : \text{Id}_{\text{QUFrm}} \to \Omega \Sigma \) satisfy the triangular identities (a) \( \Omega(\eta_{(X, \mu)}) \cdot \xi_{\Omega_{(X, \mu)}} = 1 \) and (b) \( \Sigma(\xi_{(L, \mathcal{U})}) \cdot \eta_{\Sigma_{(L, \mathcal{U})}} = 1 \) for every \( (X, \mu) \in \text{QUnif} \) and every \( (L, \mathcal{U}) \in \text{QUFrm} \):
(a) For each \( A \in \mathfrak{T}_1(\mu) \lor \mathfrak{T}_2(\mu) \), \( (\Omega(\eta_{(X, \mu)}) \cdot \xi_{\Omega_{(X, \mu)}})(A) = \Omega(\eta_{(X, \mu)})(\Sigma_A) = \eta^{-1}_{(X, \mu)}(\Sigma_A) \) and \( x \in \eta^{-1}_{(X, \mu)}(\Sigma_A) \) iff \( \eta_{(X, \mu)}(x) \in \Sigma_A \) iff \( \eta_{(X, \mu)}(x)(A) = 1 \) iff \( x \in A \).
(b) For each \( p : L \to \{0, 1\} \) in \( \text{Pt}(L) \), \( (\Sigma(\xi_{(L, \mathcal{U})}) \cdot \eta_{\Sigma_{(L, \mathcal{U})}})(p) = q \cdot \xi_{(L, \mathcal{U})} \) where \( q : \Omega \Sigma(L, \mathcal{U}) \to \{0, 1\} \) maps \( A \) into 1 iff \( p \in A \). But \( (q \cdot \xi_{(L, \mathcal{U})})(a) = q(\Sigma_a) \) is equal to 1 iff \( p \in \Sigma_a \), that is, iff \( p(a) = 1 \). Hence \( q \cdot \xi_{(L, \mathcal{U})} = p \) as required. \( \blacksquare \)

6. Quasi-proximities

The quasi-proximal frames (proximal biframes in [9]) introduced by Frith [6] in terms of the so-called strong inclusions of biframes, have the same drawback of quasi-uniform biframes: the structure is defined on a biframe \( (L_0, L_1, L_2) \) rather than directly on a frame \( L \). In this final section we briefly describe how this can be overcome.

Recall that a strong inclusion [19] on a biframe \( (L_0, L_1, L_2) \) is a pair \( (\triangleleft_1, \triangleleft_2) \) of relations on \( L_1 \) and \( L_2 \) respectively satisfying the following conditions, for \( i = 1, 2 \):

(S1) \( \triangleleft_i \) is a sublattice of \( L_i \times L_i \).
(S2) \( a \leq b \triangleleft_i c \leq d \) implies that \( a \triangleleft_i d \).
(S3) \( a \triangleleft_i b \) implies that \( a^\bullet \lor b = 1 \) (usually denoted by \( a \prec_i b \)).
(S4) \( a \triangleleft_i b \) implies that there exists \( c \in L_i \) with \( a \triangleleft_i c \triangleleft_i b \).
(S5) If \( a \triangleleft_i b \) then \( b^\bullet \triangleleft_j a^\bullet \) for \( j \in \{1, 2\} \) and \( j \neq i \).
(S6) For every \( a \in L_i \), \( a = \bigvee \{ b \in L_i \mid b \triangleleft_i a \} \).
A triple \(((L_0, L_1, L_2), \trianglelefteq_1, \trianglelefteq_2)\) where \((\trianglelefteq_1, \trianglelefteq_2)\) is a strong inclusion on the biframe \((L_0, L_1, L_2)\) is called a proximal biframe [9]. Given proximal biframes \(((L_0, L_1, L_2), \trianglelefteq^L_1, \trianglelefteq^L_2)\) and \(((M_0, M_1, M_2), \trianglelefteq^M_1, \trianglelefteq^M_2)\), a biframe homomorphism \(h : (L_0, L_1, L_2) \rightarrow (M_0, M_1, M_2)\) is a proximal biframe homomorphism if \(a \trianglelefteq^L_i b\) implies \(h(a) \trianglelefteq^M_i h(b)\) for \(i = 1, 2\) and every \(a, b \in L_0\). We denote the category of proximal biframes and proximal biframe homomorphisms by \(\text{ProxBiFrm}\).

Let \(L\) be a frame, \(<\) a binary relation in \(L\) and
\[
L_\trianglelefteq = \{a \in L \mid a = \bigvee \{b \in L \mid b < a\}\}.
\]

**Lemma 6.1.** If \(<\) is a sublattice of \(L \times L\), stronger than \(\leq\), satisfying
\[
a \leq b < c \leq d \Rightarrow a < d
\]
then \(L_\trianglelefteq\) is a subframe of \(L\).

**Proof:** Since \(0 < 0\) and \(1 < 1\), then \(0, 1 \in L_\trianglelefteq\). Since \(<\) is stronger than the partial order of \(L\), we have always \(\bigvee \{b \in L \mid b < a\} \leq a\). Let \(a, b \in L_\trianglelefteq\). Then, since \(<\) is closed under finite meets, we have
\[
a \wedge b = \bigvee \{a' \in L \mid a' < a\} \wedge \bigvee \{b' \in L \mid b' < b\} = \bigvee \{a' \wedge b' \mid a', b' \in L, a' < a, b' < b\} \leq \bigvee \{c \in L \mid c < a \wedge b\}
\]
which shows that \(a \wedge b \in L_\trianglelefteq\).

Finally, let \(a_i \in L_\trianglelefteq\) (\(i \in I\)). Then \(\bigvee_{i \in I} a_i = \bigvee_{i \in I} \bigvee \{b \in L \mid b < a_i\}\). For each such \(b, b < a_i \leq \bigvee_{i \in I} a_i\). Consequently, by condition (*), \(\bigvee_{i \in I} a_i \leq \bigvee \{b \in L \mid b < \bigvee_{i \in I} a_i\}\). Hence \(\bigvee_{i \in I} a_i \in L_\trianglelefteq\). \(\blacksquare\)

Since in the sequel we shall need to refer to pseudocomplements relatively to different pairs of subframes, we will adopt the following notation. Given a pair of subframes \(L_1, L_2\) of a frame \(L\) and \(a \in L_i\) \((i = 1, 2)\), we denote by \(a^*\left[\mathcal{L}_j\right]\) \((j \in \{1, 2\}, j \neq i)\) the element \(\bigvee \{b \in L_j \mid b \wedge a = 0\}\).

**Definition 6.2.** A pair \((\trianglelefteq_1, \trianglelefteq_2)\) of relations in \(L\), stronger than \(\leq\), will be called a strong bi-inclusion on \(L\) if for \(i, j = 1, 2\) we have:

1. \((\text{SB1})\) \(\trianglelefteq_i\) is a sublattice of \(L \times L\).
2. \((\text{SB2})\) \(a \leq b \trianglelefteq_i c \leq d\) implies that \(a \trianglelefteq_i d\).
(SB3) $a <_i b$ implies that $a < b$ (i.e. $a^* \lor b = 1$).

(SB4) For every $a, b \in L_{<i}$, $a <_i b$ implies $a^*[L_{<j}] \lor b = 1$ ($j \neq i$).

(SB5) If $a <_i b$ then there exists $c \in L_{<i}$ such that $a <_i c <_i b$.

(SB6) For every $a, b \in L_{<i}$, $a <_i b$ implies $b^*[L_{<j}] <_j a^*[L_{<j}]$ ($j \neq i$).

(SB7) $(L, L_{<1}, L_{<2})$ is a biframe.

The triple $(L, <_1, <_2)$ will be called a quasi-proximal frame. Given quasi-proximal frames $(L, <_1^L, <_2^L)$ and $(M, <_1^M, <_2^M)$, a quasi-proximal map

$$h : (L, <_1^L, <_2^L) \rightarrow (M, <_1^M, <_2^M)$$

is a frame homomorphism $h : L \rightarrow M$ such that $a <_i^L b \Rightarrow h(a) <_i^M h(b)$ for every $a, b \in L$. The corresponding category $\text{QPrFrm}$ is isomorphic to the category $\text{ProxBiFrm}$ of proximal biframes, as we shall prove next. This is proved in a similar way as in Section 4 we proved the isomorphism $\text{QUFrm} \cong \text{QUFrm}$, using new functors $\Psi$ and $\Phi$ defined below.

Given a proximal biframe $((L_0, L_1, L_2), <_1, <_2)$, let

$$\Phi((L_0, L_1, L_2), <_1, <_2) = (L_0, \overline{<_1}, \overline{<_2})$$

where, for any $a, b \in L_0$,

$$a \overline{<_i} b \equiv \exists c, d \in L_i : a \leq c <_i d \leq b.$$

**Proposition 6.3.** For each proximal biframe $((L_0, L_1, L_2), <_1, <_2)$, $\Phi((L_0, L_1, L_2), <_1, <_2)$ is a quasi-proximal frame.

**Proof:** The fact that each $\overline{<_i}$ is stronger than $\leq$ is obvious.

(SB1) $0 \overline{<_0} 0$ and $1 \overline{<_1} 1$ are trivial. Let $a_1, a_2 \overline{<_i} b$. Then $a_1 \leq c <_i d_1 \leq b$ and $a_2 \leq c <_i d_2 \leq b$ for some $c, d_1, d_2 \in L_i$. Consequently, $a_1 \leq c <_i d_1 \lor d_2 \leq b$ and $a_2 \leq c <_i d_1 \lor d_2 \leq b$ with $c, d_1, d_2 \in L_i$. By hypothesis, $c <_i d_1 \lor d_2 \leq b$ and $c \in L_i$. Thus $a_1 \lor a_2 \overline{<_i} b$. Similarly, $a_1 \lor a_2 \leq c <_i d_1 \lor d_2 \leq b$ and $a_1 \lor a_2 \overline{<_i} b$.

(SB2) Trivial.

(SB3) Let $a \overline{<_i} b$, that is $a \leq c <_i d \leq b$ for some $c, d \in L_i$. Since $c <_i d$ implies $c <_i d$, we have that $c^*[L_j] \lor d = 1$. Therefore $c^* \lor d \geq c^*[L_j] \lor d = 1$ and then $a^* \lor b = 1$.

(SB4) First note that $L_i \subseteq (L_0)_{\overline{<_i}}$. Indeed, for each $a \in L_i$, since $<_i \subseteq \overline{<_i}$, we have $a = \bigvee \{b \in L_i \mid b <_i a\} \subseteq \bigvee \{b \in L_0 \mid b\overline{<_i}a\} \leq a$. Now, let $a, b \in (L_0)_{\overline{<_i}}$ and $a \leq c <_i d \leq b$ with $c, d \in L_i$. By hypothesis, $c <_i d$ (with respect to the subframes $L_1$ and $L_2$) that is, $c^*[L_j] \lor d = 1$. From the inclusion $L_i \subseteq (L_0)_{\overline{<_i}}$
and the fact that \( a \leq c \) it follows that
\[
a^*[(L_0)_{\Xi}] = \bigvee \{ a' \in (L_0)_{\Xi} \mid a' \land a = 0 \} \geq \bigvee \{ c' \in L_j \mid c' \land c = 0 \} = c^*[L_j].
\]
Thus \( a^*[(L_0)_{\Xi}] \lor b = 1 \).

(SB5) It follows immediately from condition (S4) and the fact proved earlier that \( L_i \subseteq (L_0)_{\Xi} \).

(SB6) Let \( a, b \in (L_0)_{\Xi} \) with \( a \preceq_i b \), that is, \( a \leq c \preceq_i d \leq b \) for some \( c, d \in L_i \). Then, by hypothesis, \( d^*[L_j] \preceq_j c^*[L_j] \) and, of course, \( d^*[L_j], c^*[L_j] \in L_j \). It suffices then to show that \( b^*[(L_0)_{\Xi}] \leq d^*[L_j] \) and \( c^*[L_j] \leq a^*[(L_0)_{\Xi}] \). The latter was already proved in (SB4) above and the former can be proved in a similar way.

(SB7) By Lemma 6.1, each \((L_0)_{\Xi}\) is a subframe of \( L_0 \). Since \((L_0, L_1, L_2)\) is a biframe and \( L_i \subseteq (L_0)_{\Xi} \) \((i = 1, 2)\), then immediately \((L_0, (L_0)_{\Xi}, (L_0)_{\Xi})\) is also a biframe. 

Given a quasi-proximal frame \((L, \preceq_1, \preceq_2)\), let
\[
\Psi(L, \preceq_1, \preceq_2) = ((L, L_{\preceq_1}, L_{\preceq_2}), \preceq_{1|L_{\preceq_1}}, \preceq_{2|L_{\preceq_2}}).
\]

**Proposition 6.4.** For each quasi-proximal frame \((L, \preceq_1, \preceq_2)\), \( \Psi(L, \preceq_1, \preceq_2) \) is a proximal biframe.

**Proof:** By hypothesis, \((L, L_{\preceq_1}, L_{\preceq_2})\) is a biframe and \( \Psi(L, \preceq_1, \preceq_2) \) satisfies conditions (S1)-(S5) trivially. It remains to check (S6):

For every \( a \in L_{\preceq_i} \), \( a = \bigvee \{ b \in L \mid b \preceq_i a \} \). But by condition (SB5) there is some \( c \in L_{\preceq_i} \) satisfying \( b \preceq_i c \preceq_i a \). Therefore \( a = \bigvee \{ b \in L_{\preceq_i} \mid b \preceq_i a \} \), as required.

Concerning morphisms, the next result allows us to define \( \Phi(h) = h \) for every \( h \in \text{ProxBiFrm} \) and \( \Psi(h) = h \) for every \( h \in \text{QPFrm} \).

**Proposition 6.5.** (1) Let \( h : ((L_0, L_1, L_2), \preceq_{L_1}^L, \preceq_{L_2}^L) \to ((M_0, M_1, M_2), \preceq_{M_1}^M, \preceq_{M_2}^M) \) be a proximal biframe homomorphism. Then
\[
h : \Phi(((L_0, L_1, L_2), \preceq_{L_1}^L, \preceq_{L_2}^L) \to \Phi(((M_0, M_1, M_2), \preceq_{M_1}^M, \preceq_{M_2}^M) \in \text{QPFrm}.
\]
(2) Let \( h : (L, \preceq_{L_1}^L, \preceq_{L_2}^L) \to (M, \preceq_{M_1}^M, \preceq_{M_2}^M) \) be a quasi-proximal map. Then
\[
h : \Psi(L, \preceq_{L_1}^L, \preceq_{L_2}^L) \to \Psi(M, \preceq_{M_1}^M, \preceq_{M_2}^M) \in \text{ProxBiFrm}.
\]

**Proof:** (1) We need to check that \( a \preceq_{L_i}^L b \Rightarrow h(a) \preceq_{M_i}^M h(b) \) for \( i = 1, 2 \) and every \( a, b \in L_i \). Let \( a \preceq_{L_i}^L b \), that is \( a \preceq_i c \preceq_i d \leq b \) for some \( c, d \in L_i \). Then, by
hypothesis, \( h(c), h(d) \in M_i \) and \( h(a) \leq h(c) \triangleleft_i^M h(d) \leq h(b) \), which shows that \( h(a) \triangleleft_i^M h(b) \).

(2) It suffices to check that \( h \) is a biframe map

\[
(L, L_{\triangleleft_1}, L_{\triangleleft_2}) \to (M, M_{\triangleleft_1}, M_{\triangleleft_2})
\]

(the rest is obvious). Consider \( a \in L_{\triangleleft_i} \). Since \( a = \bigvee \{ b \in L \mid b \triangleleft_i^L a \} \) and \( b \triangleleft_i^L a \) implies \( h(b) \triangleleft_i^M h(a) \), then

\[
h(a) = \bigvee \{ h(b) \mid b \in L, b \triangleleft_i^L a \} \leq \bigvee \{ c \in M \mid c \triangleleft_i^M h(a) \} \leq h(a).
\]

Hence \( h(a) \in M_{\triangleleft_i} \).  

**Theorem 6.6.** The functors \( \Psi \) and \( \Phi \) establish an isomorphism between the concrete categories \( \text{QPFrm} \) and \( \text{ProxBiFrm} \).

**Proof:** It is sufficient to show that (a) \( \Phi \Psi = \text{Id}_{\text{QPFrm}} \) and (b) \( \Psi \Phi = \text{Id}_{\text{ProxBiFrm}} \) on objects.

(a) We need to show that \( \overline{\triangleleft_i} \vert_{L_{\triangleleft_i}} = \triangleleft_i \). Consider \( a, b \in L \) with \( a \triangleleft_i b \). By condition (SB5), there is \( c, d \in L_{\triangleleft_i} \) such that \( a \triangleleft_i c \triangleleft_i d \triangleleft_i b \). Since \( c \triangleleft_i \vert_{L_{\triangleleft_i}} d \) then, immediately, \( a \overline{\triangleleft_i} b \). On the other hand, if \( a, b \in L \) are such that \( a \overline{\triangleleft_i} b \) then there exists a pair \( c, d \) of elements of \( L_{\triangleleft_i} \) satisfying \( a \leq c \triangleleft_i d \leq b \). Thus \( a \triangleleft_i b \).

(b) It suffices to check that \( (L_0) \overline{\triangleleft_i} = L_i \) for \( i = 1, 2 \). Let \( a \in L_i \). Then, since \( \triangleleft_i \subseteq \overline{\triangleleft_i} \), we have \( a = \bigvee \{ b \in L_i \mid b \triangleleft_i a \} \leq \bigvee \{ b \in L_0 \mid b \overline{\triangleleft_i} a \} \leq a \). Conversely, if \( a \in (L_0) \overline{\triangleleft_i} \), meaning that \( a \in L_0 \) and \( a = \bigvee \{ b \in L_0 \mid b \overline{\triangleleft_i} a \} \), then for each such \( b \) there is \( c_b, d_b \in L_i \) satisfying \( b \leq c_b \triangleleft_i d_b \leq a \). Consequently, \( a \leq \bigvee \{ c \in L_i \mid c \triangleleft_i a \} \leq a \) and, therefore, \( a \in L_i \).

**Remarks 6.7.** (1) Our Definition 6.2 of a quasi-proximal frame contains, of course, the symmetric case of proximal frames [6] defined by strong inclusions: it is a frame equipped with a strong bi-inclusion \( (\triangleleft_1, \triangleleft_2) \), both coordinates of which are the same.

(2) In closing we note that it can be shown, in analogy with the spatial case or the symmetric case, that the category \( \text{QPFrm} \) is isomorphic to the full subcategory of \( \text{QUFrm} \) of all **totally bounded** quasi-uniform frames (that is, the quasi-uniform frames \( (L, U) \) for which the paircovering structure \( U \) has a base of finite paircovers). We omit the details.
It is now felt that, with the reformulation of the covering approach here presented (Definition 2.1), paircovers are the most convenient tool to deal with quasi-uniformities on frames, better than the entourages of [17], and will be certainly used “ten-tenths of the time” [11] in asymmetric pointfree topology.

References


Jorge Picado
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: picado@mat.uc.pt