

ON THE UNIFORMIZATION OF L -VALUED FRAMES

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ABSTRACT: This note discusses the appropriate way of uniformizing the notion of an L -valued frame introduced by A. Pultr and S. Rodabaugh in [*Lattice-valued frames, functor categories, and classes of sober spaces*, Chapter 6 of *Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets*, Kluwer, 2003]. It covers the case of a completely distributive lattice L (which is, in a certain sense, the most general one) and studies the corresponding category of uniform L -valued frames.

KEYWORDS: L -valued frames; L -Frm; L -topological spaces; L -Top; uniform L -valued frames; iota functors; upper/lower forgetful functors.

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1. Introduction

L -valued frames are structures of increasing interest for fuzzy topology (see [14], [8], [15], [16], [17], [6], [7]). They were introduced by Pultr and Rodabaugh in [14] for the case of a complete chain L and were recently extended for the more general case of a completely distributive lattice L by Gutiérrez García, Höhle and de Prada Vicente [7]. This paper is a continuation of our previous paper [6], and has its roots in the convenience of finding (after [7]) an appropriate notion of a uniform L -valued frame. There are two obvious candidates for it: the most direct one, provided by the direct approach of uniformizing the L -topologies as a frame, and the one suggested by [14], based on the concept of an L -valued frame. In the previous [6] we chose the latter (that would eventually provide a nice categorical picture of the categories at hand) since the former reveals to make no sense after the following observation of Pultr and Rodabaugh in [14]:

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(...) we envisage potential applications of the notion of an L -valued frame in the linear case. One of the questions of interest in fuzzy topology concerns well-founded definitions of the structures of the uniformity type. The case of a complete chain L , important historically and still very important for applications, does not allow the direct uniformization of an L -topology τ as a frame: a uniformity on τ induces a uniformity on L (as observed by Banaschewski); and since the only linearly ordered frame admitting a uniformity is the two point Boolean algebra $2 = \{0 < 1\}$ we would be left with the crisp case. One can, however, think of definitions based on the concept of an L -valued frame which would be more satisfactory.

However, as we now have found with the following example, this observation is not true in general (it holds only for the stratified case): Let L be a non-uniformizable frame, for example $L = [0, 1]$ or any linearly ordered frame different from $2 = \{0 < 1\}$. Let $1_\emptyset$ and 1_X denote, respectively, the bottom and top elements in L^X . Clearly enough $\tau = \{1_\emptyset, 1_X\}$ is a uniformizable L -topology on X . If the observation above would be true, then there would exist a uniformity on the frame L , a contradiction.

So, this places again the former alternative as the most natural candidate for a good definition of a uniform L -valued frame and prompts for its study. This is the problem that we address in this paper, having in mind the treatment of the case of a completely distributive lattice L , following the lines of [7]. As it is shown in [7], the case of a completely distributive lattice cannot, in a certain sense, be weakened.

2. Preliminaries and notation

For standard notions and facts from category theory used here we refer to [1]. As a general reference to frames we suggest [11] or [13].

2.1. Uniform frames. A *frame* is a complete lattice A satisfying the distributive law

$$\forall a \in A, \forall S \subseteq A, \quad a \wedge (\bigvee S) = \bigvee \{a \wedge b \mid b \in S\}.$$

Given two frames A and B , a *frame homomorphism* $h : A \rightarrow B$ is a mapping preserving all joins and finite meets. The category of frames and frame homomorphism will be denoted by $\mathbf{Frm}$.

Let A be a frame. A set $U \subseteq A$ is a *cover* of A if $\bigvee U = 1$. The set of covers of A can be preordered: a cover U *refines* a cover V , written $U \leq V$, if for each $a \in U$ there is a $b \in V$ with $a \leq b$.

For $a \in A$, the element $st(a, U) = \bigvee \{b \in U \mid b \wedge a \neq 0\}$ is called the *star of a in U* . Further, for a family $\mathcal{U}$ of covers of A , we write

$$a \overset{\mathcal{U}}{\triangleleft} b \text{ if there exists } U \in \mathcal{U} \text{ such that } st(a, U) \leq b.$$

Remark 2.1. The following useful properties are easy to check:

- (i) $a \overset{\mathcal{U}}{\triangleleft} b \implies a \leq b$,
- (ii) $a \leq b \overset{\mathcal{U}}{\triangleleft} c \leq d \implies a \overset{\mathcal{U}}{\triangleleft} d$,
- (iii) $a \overset{\mathcal{U}}{\triangleleft} b, c \overset{\mathcal{U}}{\triangleleft} d \implies a \wedge c \overset{\mathcal{U}}{\triangleleft} b \wedge d$.

A family $\mathcal{U}$ of covers of A is a *uniformity basis* [12] provided that:

- (U1) $\mathcal{U}$ is a filter basis of the preordered set $(Cov(A), \leq)$ of all covers of A .
- (U2) Every $U \in \mathcal{U}$ has a star-refinement, i.e., for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ with

$$st(V) := \{st(a, V) \mid a \in V\} \leq U.$$

- (U3) For every $a \in A$, $a = \bigvee \{b \in A \mid b \overset{\mathcal{U}}{\triangleleft} a\}$.

Also, we understand by a *uniformity subbasis* a family of covers of A such that finite meets of elements of the family constitute a uniformity basis.

A *uniformity* on A is a filter $\mathcal{U}$ of covers of A generated by some uniformity basis. The pair $(A, \mathcal{U})$ is then called a *uniform frame*. Let $(A, \mathcal{U})$ and $(B, \mathcal{V})$ be uniform frames. A frame homomorphism $h : A \rightarrow B$ is a *uniform homomorphism* if, for every $U \in \mathcal{U}$, $h[U] = \{h(a) \mid a \in U\} \in \mathcal{V}$. We denote by $\mathbf{UFrm}$ the category of uniform frames and uniform homomorphisms.

2.2. The iota functor $\iota_L^{\mathbf{T}} : L\text{-Top} \rightarrow \mathbf{Top}$. Let L denote a complete lattice. An element $p \in L$ is called *prime* if for each $a, b \in L$ with $a \wedge b \leq p$ either $a \leq p$ or $b \leq p$. As in [4], we denote by $\mathbf{PRIME} L$ the set of all prime elements of L and $\mathbf{Spec} L = \mathbf{PRIME} L \setminus \{1\}$.

Following [7], for each $\alpha \in L$ we denote by $\uparrow\alpha$ the set $\{\beta \in L : \alpha \triangleleft \beta\}$, where $\triangleleft$ is the opposite relation of the way-below relation in the lattice L^{op} :

$$\alpha \triangleleft \beta \equiv \text{for all } S \subseteq L \text{ such that } \bigwedge S \leq \alpha \text{ there exist } \gamma_1, \dots, \gamma_n \in S : \bigwedge_{i=1}^n \gamma_i \leq \beta.$$

An *L-valued topological space* [2, 5] (shortly, an *L-topological space*) is a pair (X, τ) consisting of a set X and a subset τ of L^X (the *L-valued topology* or *L-topology* on the set X), containing $1_\emptyset$ and 1_X and closed under finite meets and arbitrary joins (where meets and joins in L^X are defined pointwisely).

Given two *L-valued topological spaces* $(X, \tau_1), (Y, \tau_2)$ a map $f : X \rightarrow Y$ is an *L-continuous map* if the correspondence $b \mapsto f^\leftarrow(b) = b \circ f$ maps τ_2 into τ_1 . The resulting category will be denoted by *L-Top*.

Of course, when $L = 2$, an *L-topological space* is precisely a topological space and there is an isomorphism between *Top* and *L-Top*, via the characteristic functor (the one associating to each subset its characteristic function and leaving morphisms unchanged). If L is a frame then the *L-topologies*, being subframes of the frame L^X , are frames as well.

The well-known iota functor ι_L , originally introduced by Lowen [10] for $L = [0, 1]$ and later on extended by Kubiak [9] to an arbitrary complete lattice, was the original motivation to define chain-valued frames and the corresponding category in [14]. It constructs at the fibre level, for each *L-topology*, a traditional topology with subbasis the family of all level sets for all members of the *L-topology*:

For each set X and each $\alpha \in L$, the α -level mapping $\iota_\alpha : L^X \rightarrow 2^X$ is defined by

$$\iota_\alpha(a) = [a \not\leq \alpha] := \{x \in X \mid a(x) \not\leq \alpha\}, \quad \text{for each } a \in L^X.$$

Now, given an *L-topology* τ on X , we consider the associated crisp topology

$$\iota_L^\top(\tau) = \langle \{\iota_\alpha(\tau) \mid \alpha \in L\} \rangle = \langle \{\iota_\alpha(a) \mid a \in \tau, \alpha \in L\} \rangle.$$

Remark 2.2. Notice that when L is a completely distributive lattice, the collection $\{\iota_p(a) \mid a \in \tau, p \in \text{Spec } L\}$ is also a subbase of the topology $\iota_L^\top(\tau)$.

The correspondence $(X, \tau) \mapsto (X, \iota_L^\top(\tau))$ defines a functor $\iota_L^\top : L\text{-Top} \rightarrow \text{Top}$: for each *L-continuous map* $f : (X, \tau_1) \rightarrow (Y, \tau_2)$, $f : (X, \iota_L^\top(\tau_1)) \rightarrow (Y, \iota_L^\top(\tau_2))$ is continuous, since $b \circ f \in \tau_1$ and $f^{-1}[\iota_\alpha(b)] = \iota_\alpha(b \circ f)$ for every $b \in \tau_2$.

Note that whenever L is a frame the mapping $\iota_p : \tau \rightarrow \iota_L^\top(\tau)$ is a frame homomorphism for each $p \in \text{Spec } L$ (see Corollary 3.5 below; this is not true in general if p fails to be prime). Consequently we can consider the system of frame homomorphisms

$$(\iota_p : \tau \rightarrow \iota_L^\top(\tau) \mid p \in \text{Spec } L). \quad (2.2.1)$$

2.3. L -valued frames. From now on let L denote a completely distributive lattice. Recall that any completely distributive lattice is a spatial frame (i.e. a frame isomorphic to the lattice of open sets of some topological space) and therefore in any completely distributive lattice each element is a meet of primes.

An L -valued frame (shortly, an L -frame) is a system

$$\mathfrak{A} = (\varphi_p^\mathfrak{A} : A^u \rightarrow A^l \mid p \in \text{Spec } L)$$

of frame homomorphisms (A^u is the *upper frame* and A^l is the *lower frame*) satisfying the following conditions:

- (F0) For every $p \in \text{Spec } L$, $\varphi_p^\mathfrak{A} = \bigvee \{\varphi_q^\mathfrak{A} \mid q \in \uparrow p \cap \text{Spec } L\}$.
- (F1) $A^l = \langle \bigcup_{p \in \text{Spec } L} \varphi_p^\mathfrak{A}(A^u) \rangle$. (collectionwise extremally epimorphic)
- (F2) If $a \neq b$ then $\varphi_p^\mathfrak{A}(a) \neq \varphi_p^\mathfrak{A}(b)$ for some $p \in \text{Spec } L$. (collectionwise monomorphic)

L -frames were introduced by Pultr and Rodabaugh [14] for the case of a complete chain L and extended by Gutiérrez García, Höhle and de Prada Vicente [7] for the case of a completely distributive lattice.

Given two L -frames $\mathfrak{A}$ and $\mathfrak{B}$ an L -frame homomorphism $h : \mathfrak{A} \rightarrow \mathfrak{B}$ is an ordered pair of frame morphisms

$$(h^u : A^u \rightarrow B^u, h^l : A^l \rightarrow B^l)$$

satisfying

$$\forall p \in \text{Spec } L, \quad h^l \circ \varphi_p^\mathfrak{A} = \varphi_p^\mathfrak{B} \circ h^u.$$

The resulting category, with composition and identities component-wise in $\mathbf{Frm}$, is denoted by $L\text{-Frm}$.

Note that for each L -topological space (X, τ) with L completely distributive, the system (2.2.1) of frame homomorphisms defines an L -frame [7].

This was the motivating example for the notion of an L -frame.

We recall also the *upper*, resp. *lower*, forgetful functors

$$\mathcal{U}^u : L\text{-Frm} \rightarrow \mathbf{Frm}, \quad \mathcal{U}^l : L\text{-Frm} \rightarrow \mathbf{Frm}$$

defined by sending $(\varphi_p^{\mathfrak{A}} : A^u \rightarrow A^l \mid p \in \text{Spec } L)$ to A^u , resp. A^l , and (h^u, h^l) to h^u , resp. h^l .

3. An extension of the iota functor

Let $L\text{-UTop}$ denote the category whose objects are of the form $(X, \tau_X, \mathcal{U}_X)$ where τ_X is an L -topology on X and $\mathcal{U}_X$ is a uniformity on the frame τ_X , and morphisms $f : (X, \tau_X, \mathcal{U}_X) \rightarrow (Y, \tau_Y, \mathcal{U}_Y)$ are the L -continuous functions $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ for which $f^{\leftarrow} : (\tau_Y, \mathcal{U}_Y) \rightarrow (\tau_X, \mathcal{U}_X)$ is a uniform homomorphism.

In the particular case $L = 2$ the objects of 2-UTop are topological spaces $(X, \mathcal{T}_X)$ endowed with a uniformity on the spatial frame $\mathcal{T}_X$.

One can then try to extend the iota functor for $L\text{-UTop}$:

$$\iota_L^{\text{TUF}} : L\text{-UTop} \rightarrow 2\text{-UTop}.$$

We will do that by defining how it acts on the additional structure (the uniformity on the L -topology) since in the L -topology it will act precisely as the iota functor already defined in Subsection (2.2).

For each $p \in \text{Spec } L$ and each $\mathcal{A} \subseteq L^X$, let $\iota_p[\mathcal{A}] = \{\iota_p(a) : a \in \mathcal{A}\} \subseteq 2^X$.

We state without proof some basic facts satisfied by the maps $\{\iota_p \mid p \in \text{Spec } L\}$:

Lemma 3.1. *Let $\mathcal{A} \subseteq L^X$, $f : X \rightarrow Y$, $a \in L^X$, $b \in L^Y$ and $p, q \in \text{Spec } L$. Then:*

- (1) $\iota_p(\bigvee \mathcal{A}) = \bigcup_{a \in \mathcal{A}} \iota_p(a)$.
- (2) $\iota_p(\bigwedge \mathcal{A}) = \bigcap_{a \in \mathcal{A}} \iota_p(a)$ whenever $\mathcal{A}$ is finite.
- (3) $p \leq q \Rightarrow \iota_q(a) \subseteq \iota_p(a)$.
- (4) $f^{-1}(\iota_p(b)) = \iota_p(f^{\leftarrow}(b))$.

We have now the following, which is easy to check (cf. [6]):

Proposition 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be covers of the frame L^X and $p \in \text{Spec } L$. Then:*

- (1) $\iota_p[\mathcal{A}]$ is a cover of X .
- (2) If $\mathcal{A} \leq \mathcal{B}$ then $\iota_p[\mathcal{A}] \leq \iota_p[\mathcal{B}]$. Hence $\iota_p[\mathcal{A} \wedge \mathcal{B}] \leq \iota_p[\mathcal{A}]$ and $\iota_p[\mathcal{A} \wedge \mathcal{B}] \leq \iota_p[\mathcal{B}]$.
- (3) $\text{st}(\iota_p(a), \iota_p[\mathcal{A}]) \subseteq \iota_p[\text{st}(a, \mathcal{A})]$.
- (4) If $\text{st}(\mathcal{A}) \leq \mathcal{B}$ then $\text{st}(\iota_p[\mathcal{A}]) \leq \iota_p[\mathcal{B}]$.

Corollary 3.3. *Let $(X, \tau_X, \mathcal{U}_X) \in L\text{-UTop}$. The family*

$$\{\iota_p[U] \mid U \in \mathcal{U}_X \text{ and } p \in \text{Spec } L\}$$

generates (as a subbase) a uniformity $\iota_L^{\text{UF}}(\mathcal{U}_X)$ on the frame $\iota_L^T(\tau_X)$ whose base is

$$\{\iota_p[U] \wedge \iota_q[U] \mid U \in \mathcal{U}_X \text{ and } p, q \in \text{Spec } L\}.$$

Proof: It follows from Lemma 3.1 and the properties in Proposition 3.2. Regarding condition (U3) notice that, by Proposition 3.2(3), $b \stackrel{\mathcal{U}_X}{\triangleleft} a$ implies that $\iota_p(b) \stackrel{\iota_L^{\text{UF}}(\mathcal{U}_X)}{\triangleleft} \iota_p(a)$, thus we have $\iota_p(a) = \bigcup \{\iota_p(b) \mid \iota_p(b) \stackrel{\iota_L^{\text{UF}}(\mathcal{U}_X)}{\triangleleft} \iota_p(a)\}$ for every $p \in \text{Spec } L$ and every $a \in \tau_X$, which suffices to conclude (U3). Indeed: for any element H of $\iota_L^T(\tau_X)$, there are $a_i, b_i \in \tau_X$ and $p_i, q_i \in \text{Spec } L$ such that (cf. Remark 2.2)

$$\begin{aligned} H &= \bigcup_{i \in I} (\iota_{p_i}(a_i) \cap \iota_{q_i}(b_i)) \\ &= \bigcup_{i \in I} \left(\bigcup \{ \iota_{p_i}(c) \mid \iota_{p_i}(c) \stackrel{\iota_L^{\text{UF}}(\mathcal{U}_X)}{\triangleleft} \iota_{p_i}(a_i) \} \cap \bigcup \{ \iota_{q_i}(d) \mid \iota_{q_i}(d) \stackrel{\iota_L^{\text{UF}}(\mathcal{U}_X)}{\triangleleft} \iota_{q_i}(b_i) \} \right) \\ &\subseteq \bigcup_{i \in I} \bigcup \{ \iota_{p_i}(c) \cap \iota_{q_i}(d) \mid \iota_{p_i}(c) \cap \iota_{q_i}(d) \stackrel{\iota_L^{\text{UF}}(\mathcal{U}_X)}{\triangleleft} \iota_{p_i}(a_i) \cap \iota_{q_i}(b_i) \} \\ &\subseteq \bigcup \{ G \in \iota_L^T(\tau_X) \mid G \stackrel{\iota_L^{\text{UF}}(\mathcal{U}_X)}{\triangleleft} H \} \subseteq H, \end{aligned}$$

where the last inclusions follow from Remark 2.1. $\square$

The following is obvious:

Corollary 3.4. *If $f : (X, \tau_X, \mathcal{U}_X) \rightarrow (Y, \tau_Y, \mathcal{U}_Y)$ is a morphism of $L\text{-UTop}$ then $f : (X, \iota_L^T(\tau_X), \iota_L^{\text{UF}}(\mathcal{U}_X)) \rightarrow (Y, \iota_L^T(\tau_Y), \iota_L^{\text{UF}}(\mathcal{U}_Y))$ is a morphism of 2-UTop . $\square$*

Consequently, we have a functor $\iota_L^{\text{TUF}} : L\text{-UTop} \rightarrow 2\text{-UTop}$ given on objects by

$$\iota_L^{\text{TUF}}(X, \tau_X, \mathcal{U}_X) = (X, \iota_L^T(\tau_X), \iota_L^{\text{UF}}(\mathcal{U}_X)) \quad \text{for each } (X, \tau_X, \mathcal{U}_X) \in L\text{-UTop}$$

and, on morphisms, for each $f : (X, \tau_X, \mathcal{U}_X) \rightarrow (Y, \tau_Y, \mathcal{U}_Y)$, $\iota_L^{\text{TUF}}(f) = f$.

Hence the diagram

$$\begin{array}{ccc} L\text{-}\mathbf{UTop} & \xrightarrow{\iota_L^{\text{UF}}} & 2\text{-}\mathbf{UTop} \\ \downarrow F_L^{\text{UF}} & & \downarrow F^{\text{UF}} \\ L\text{-}\mathbf{Top} & \xrightarrow{\iota_L^{\text{T}}} & \mathbf{Top} \end{array}$$

commutes (where F_L^{UF} and F^{UF} denote, respectively, the forgetful functors).

Corollary 3.5. *For each $(X, \tau_X, \mathcal{U}_X) \in L\text{-}\mathbf{UTop}$ and each $p \in \text{Spec } L$ the mapping $\iota_p : (\tau_X, \mathcal{U}_X) \rightarrow (\iota_L^{\text{T}}(\tau_X), \iota_L^{\text{UF}}(\mathcal{U}_X))$ is a uniform homomorphism.*

Proof: By Lemma 3.1(1) and (2), each ι_p is a frame homomorphism. Clearly, it is moreover uniform, that is, for every $U \in \mathcal{U}_X$, $\iota_p[U] \in \iota_L^{\text{UF}}(\mathcal{U}_X)$. $\square$

4. L -valued uniform frames

Let us consider the following system of uniform homomorphisms:

$$(\iota_p : (\tau_X, \mathcal{U}_X) \rightarrow (\iota_L^{\text{T}}(\tau_X), \iota_L^{\text{UF}}(\mathcal{U}_X)) \mid p \in \text{Spec } L).$$

Remarks 4.1. (1) As it was shown in [7], for each $p \in \text{Spec } L$,

$$\iota_p = \bigvee \{ \iota_q \mid q \in \uparrow p \cap \text{Spec } L \}. \quad (4.1.1)$$

In particular, the assignment $p \mapsto \iota_p$ is antitone.

(2) We have

$$(\iota_L^{\text{T}}(\tau_X), \iota_L^{\text{UF}}(\mathcal{U}_X)) = \left\langle \bigvee_{p \in \text{Spec } L} \iota_p(\tau_X, \mathcal{U}_X) \right\rangle, \quad (4.1.2)$$

where, by the previous expression we mean that the topology $\iota_L^{\text{T}}(\tau_X)$ is generated by the subbase $\{ \iota_p(a) \mid a \in \tau_X \text{ and } p \in \text{Spec } L \}$ and the uniformity $\iota_L^{\text{UF}}(\mathcal{U}_X)$ is generated by the subbase $\{ \iota_p[U] \mid U \in \mathcal{U}_X \text{ and } p \in \text{Spec } L \}$.

(3) For each pair of distinct $a, b \in \tau_X$ there exists $x \in X$ such that $a(x) \neq b(x)$, hence there exists $p \in \text{Spec } L$ such that either $a(x) \leq p$ and $b(x) \not\leq p$ or $a(x) \not\leq p$ and $b(x) \leq p$ and so $[a \not\leq p] \neq [b \not\leq p]$. It follows that

$$\text{if } a \neq b \in \tau_X \text{ then } \iota_p(a) \neq \iota_p(b) \text{ for some } p \in \text{Spec } L. \quad (4.1.3)$$

(4) As a consequence of the previous comments, the system of uniform homomorphisms

$$(\iota_p : (\tau_X, \mathcal{U}_X) \rightarrow (\iota_L^{\text{T}}(\tau_X), \iota_L^{\text{UF}}(\mathcal{U}_X)) \mid p \in \text{Spec } L)$$

satisfies conditions (4.1.1), (4.1.2) and (4.1.3) above.

Proposition 4.2. *Let $(A, \mathcal{U})$ be a uniform frame and $(X, \mathcal{T}_X, \mathcal{U}_X) \in 2\text{-UTop}$. Let $(\varphi_p : (A, \mathcal{U}) \rightarrow (\mathcal{T}_X, \mathcal{U}_X) \mid p \in \text{Spec } L)$ be a system of uniform homomorphisms satisfying:*

$$\bullet \text{ for each } p \in \text{Spec } L, \varphi_p = \bigvee \{\varphi_q \mid q \in \uparrow p \cap \text{Spec } L\}. \quad (4.2.1)$$

$$\bullet (\mathcal{T}_X, \mathcal{U}_X) = \langle \bigvee_{p \in \text{Spec } L} \varphi_p((A, \mathcal{U})) \rangle. \quad (4.2.2)$$

$$\bullet \text{ if } a \neq b \in A \text{ then } \varphi_p(a) \neq \varphi_p(b) \text{ for some } p \in \text{Spec } L. \quad (4.2.3)$$

Then there is a uniform frame $(B, \mathcal{V})$ and a uniform isomorphism $h : (A, \mathcal{U}) \rightarrow (B, \mathcal{V})$ satisfying the following:

- (1) B is an L -topology on X (hence $(X, B, \mathcal{V}) \in L\text{-UTop}$).
- (2) $\iota_L^{\text{UF}}(X, B, \mathcal{V}) = (X, \mathcal{T}_X, \mathcal{U}_X)$.
- (3) For each $p \in \text{Spec } L$, $\iota_p \circ h = \varphi_p$.

Proof: The proof follows the lines of Proposition 2.5 in [7].

Given $a \in A$, let $h(a) \in L^X$ be the L -valued function induced by the family $\{\varphi_p(a) \mid p \in \text{Spec } L\}$, that is,

$$h(a)(x) = \bigwedge \{p \in \text{Spec } L \mid x \notin \varphi_p(a)\} \quad \text{for every } x \in X.$$

Take $B = \{h(a) \mid a \in A\}$ and $\mathcal{V} = \{h[U] : U \in \mathcal{U}\}$. It is easy to check (see [7, Proposition 2.5]) that condition (4.2.1) implies that $h(a)(x) \leq p \iff x \notin \varphi_p(a)$ and so $\iota_p(h(a)) = [h(a) \not\leq p] = \varphi_p(a)$ for every $a \in A$ and $p \in \text{Spec } L$, i.e., $\iota_p \circ h = \varphi_p$ for every $p \in \text{Spec } L$.

We now check that h is injective: Given $a \neq b \in A$, by (4.2.3) we have $\varphi_p(a) \neq \varphi_p(b)$ for some $p \in \text{Spec } L$. We can assume, without loss of generality, that there exists $x \in \varphi_p(a)$ such that $x \notin \varphi_p(b)$. Then $h(b)(x) \leq p$ and $h(a)(x) \not\leq p$ which implies $h(b) \neq h(a)$.

Moreover, h is a frame homomorphism. Indeed:

Let $a, b \in A$, $x \in X$ and $p \in \text{Spec } L$. We have

$$\begin{aligned} h(a \wedge b)(x) \leq p &\iff x \notin \varphi_p(a \wedge b) = \varphi_p(a) \cap \varphi_p(b) \\ &\iff x \notin \varphi_p(a) \text{ or } x \notin \varphi_p(b) \\ &\iff h(a)(x) \leq p \text{ or } h(b)(x) \leq p \\ &\iff h(a)(x) \wedge h(b)(x) \leq p \end{aligned}$$

and therefore, since L is a spatial frame, we conclude that $h(a \wedge b) = h(a) \wedge h(b)$. Further, $h(1) = 1$, since $\varphi_p(1) = X$ for every $p \in \text{Spec } L$. Thus h preserves finite meets.

On the other hand, given $\{a_i\}_{i \in I} \subseteq A$, $x \in X$ and $p \in \text{Spec } L$ we have

$$\begin{aligned} h(\bigvee_{i \in I} a_i)(x) \leq p &\iff x \notin \varphi_p(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} \varphi_p(a_i) \\ &\iff x \notin \varphi_p(a_i) \text{ for all } i \in I \\ &\iff h(a_i)(x) \leq p \text{ for all } i \in I \\ &\iff \bigvee_{i \in I} h(a_i)(x) \leq p \end{aligned}$$

and thus, once again because L is a spatial frame, we may conclude that $h(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} h(a_i)$.

Hence $B = h(A)$ is a subframe of L^X , i.e. an L -topology on X , and therefore, $h : A \rightarrow B$ is a frame isomorphism. The latter fact makes straightforward the proof that $\mathcal{V} = \{h[U] : U \in \mathcal{U}\}$ is a uniformity on B and that $h : (A, \mathcal{U}) \rightarrow (B, \mathcal{V})$ is a uniform isomorphism, since h and h^{-1} are trivially uniform.

Finally, it follows from (4.2.2) that

$$\begin{aligned} \iota_L^\top(B) &= \langle \{\iota_p(h(a)) \mid a \in A, p \in \text{Spec } L\} \rangle \\ &= \langle \{\varphi_p(a) \mid a \in A, p \in \text{Spec } L\} \rangle = \mathcal{T}_X \end{aligned}$$

and

$$\begin{aligned} \iota_L^{\text{UF}}(\mathcal{V}) &= \langle \{\iota_p[h(U)] \mid U \in \mathcal{U}, p \in \text{Spec } L\} \rangle \\ &= \langle \{\varphi_p[U] \mid U \in \mathcal{U}, p \in \text{Spec } L\} \rangle = \mathcal{U}_X, \end{aligned}$$

i.e., $\iota_L^{\text{TUF}}(X, B, \mathcal{V}) = (X, \mathcal{T}_X, \mathcal{U}_X)$. $\square$

Proposition 4.2 leads immediately to the following definition:

Definition 4.3. An L -valued uniform frame $\mathfrak{A}$ is a system

$$\mathfrak{A} = (\varphi_p^\mathfrak{A} : (A^u, \mathcal{U}^u) \rightarrow (A^l, \mathcal{U}^l) \mid p \in \text{Spec } L)$$

of uniform homomorphisms ($(A^u, \mathcal{U}^u)$ is the *upper uniform frame* and $(A^l, \mathcal{U}^l)$ is the *lower uniform frame*) satisfying the following conditions:

(UF0) For every $p \in \text{Spec } L$, $\varphi_p^\mathfrak{A} = \bigvee \{\varphi_q^\mathfrak{A} \mid q \in \uparrow p \cap \text{Spec } L\}$.

(UF1) $(A^l, \mathcal{U}^l) = \langle \bigvee_{p \in \text{Spec } L} \varphi_p^\mathfrak{A}((A^u, \mathcal{U}^u)) \rangle$.

(UF2) If $a \neq b$ then $\varphi_p^\mathfrak{A}(a) \neq \varphi_p^\mathfrak{A}(b)$ for some $p \in \text{Spec } L$.

An L -uniform homomorphism $h : \mathfrak{A} \rightarrow \mathfrak{B}$ is an ordered pair of uniform homomorphisms

$$(h^u : (A^u, \mathcal{U}^u) \rightarrow (B^u, \mathcal{V}^u), h^l : (A^l, \mathcal{U}^l) \rightarrow (B^l, \mathcal{V}^l))$$

satisfying

$$\forall p \in \text{Spec } L, \quad h^l \circ \varphi_p^{\mathfrak{A}} = \varphi_p^{\mathfrak{B}} \circ h^u.$$

The resulting category, with composition and identities component-wise in $\mathbf{UFrm}$, is denoted by $L\text{-UFrm}$.

Remarks 4.4. (1) Note that any L -valued uniform frame is, in particular, an L -valued frame.

(2) For $L = 2$, we have exactly one uniform homomorphism $\varphi_0^{\mathfrak{A}}$ which automatically satisfies (UF0): conditions (UF1) and (UF2) imply that $\varphi_0^{\mathfrak{A}}$ is an isomorphism $(A^u, \mathcal{U}^u) \rightarrow (A^l, \mathcal{U}^l)$. Thus a 2-uniform frame is a pair of isomorphic uniform frames. Further, each 2-uniform homomorphism is a pair of uniform homomorphisms (h^u, h^l) such that each factors through the other via isomorphisms. Each 2-uniform frame is a functor $\mathbf{2} \rightarrow \mathbf{UFrm}$ where $\mathbf{2}$ denotes the category

$$\begin{array}{ccc} \textcircled{\bullet}^u & \xrightarrow{\quad} & \textcircled{\bullet}^l \end{array}$$

It is then easy to conclude that the category $\mathbf{2}\text{-UFrm}$ of 2-valued uniform frames is the category $\mathbf{UFrm}^{\mathbf{2}}$ of functors $\mathbf{2} \rightarrow \mathbf{UFrm}$ and natural transformations between such functors. Consequently, $\mathbf{2}\text{-UFrm}$ is equivalent to the category $\mathbf{UFrm}$, since $\mathbf{UFrm}^{\mathbf{2}}$ is a category equivalent to $\mathbf{UFrm}$ via functors $F : \mathbf{UFrm}^{\mathbf{2}} \rightarrow \mathbf{UFrm}$, with $F((A^u, \mathcal{U}^u), (A^l, \mathcal{U}^l)) = (A^u, \mathcal{U}^u)$ and $F(h^u, h^l) = h^u$, and $G : \mathbf{UFrm} \rightarrow \mathbf{UFrm}^{\mathbf{2}}$, with $G(A, \mathcal{U}) = ((A, \mathcal{U}), (A, \mathcal{U}))$ and $G(h) = (h, h)$.

(3) For a general completely distributive L , $L\text{-UFrm}$ is a full subcategory of the category $\mathbf{UFrm}^{\mathfrak{L}}$ where $\mathfrak{L}$ denotes the category

$$\begin{array}{ccc} \textcircled{\bullet}^u & \xrightarrow[p \in \text{Spec } L]{\quad} & \textcircled{\bullet}^l \end{array}$$

(the morphisms between u and l are indexed by $\text{Spec } L$).

Theorem 4.5. *The category $L\text{-UFrm}$ is complete and cocomplete.*

Proof: Let UF_0 and UF_1 denote the categories defined by relaxing the axioms for objects in the definition of $L\text{-UFrm}$ as follows: in UF_0 the objects are systems of uniform homomorphisms for which only (UF0) is required and in UF_1 the objects are systems of uniform homomorphisms for which only (UF0) and (UF1) are required.

Claim 1. *The category UF_0 is complete and cocomplete.*

Proof of Claim 1. Given any family of objects $(\mathfrak{A}_i)_{i \in I}$, with $\mathfrak{A}_i = (\varphi_p^{\mathfrak{A}_i} : (A_i^u, \mathcal{U}_i^u) \rightarrow (A_i^l, \mathcal{U}_i^l) \mid p \in \text{Spec } L)$, in UF_0 , consider the uniform frame coproduct [18] $(A^u, \mathcal{U}^u)$ of the $(A_i^u, \mathcal{U}_i^u)$ (with injections u_i^u) and the uniform frame coproduct $(A^l, \mathcal{U}^l)$ of the $(A_i^l, \mathcal{U}_i^l)$ (with injections u_i^l). Then, for each $p \in \text{Spec } L$, there exists a unique $\varphi_p^{\mathfrak{A}} : (A^u, \mathcal{U}^u) \rightarrow (A^l, \mathcal{U}^l)$ such that $\varphi_p^{\mathfrak{A}} \circ u_i^u = u_i^l \circ \varphi_p^{\mathfrak{A}_i}$ for every $i \in I$:

$$\begin{array}{ccc} (A_i^u, \mathcal{U}_i^u) & \xrightarrow{u_i^u} & (A^u, \mathcal{U}^u) \\ \downarrow \varphi_p^{\mathfrak{A}_i} & & \downarrow \varphi_p^{\mathfrak{A}} \\ (A_i^l, \mathcal{U}_i^l) & \xrightarrow{u_i^l} & (A^l, \mathcal{U}^l) \end{array}$$

Further, $\mathfrak{A} = (\varphi_p^{\mathfrak{A}} : (A^u, \mathcal{U}^u) \rightarrow (A^l, \mathcal{U}^l))_{p \in \text{Spec } L}$ is in UF_0 . Indeed, for each i ,

$$\begin{aligned} \varphi_p^{\mathfrak{A}} \circ u_i^u &= u_i^l \circ \varphi_p^{\mathfrak{A}_i} = u_i^l \circ \bigvee \{ \varphi_q^{\mathfrak{A}_i} \mid q \in \uparrow p \cap \text{Spec } L \} \\ &= \bigvee \{ u_i^l \circ \varphi_q^{\mathfrak{A}_i} \mid q \in \uparrow p \cap \text{Spec } L \} \\ &= \bigvee \{ \varphi_q^{\mathfrak{A}} \circ u_i^u \mid q \in \uparrow p \cap \text{Spec } L \} \\ &= \left(\bigvee \{ \varphi_q^{\mathfrak{A}} \mid q \in \uparrow p \cap \text{Spec } L \} \right) \circ u_i^u, \end{aligned}$$

from which it readily follows that $\varphi_p^{\mathfrak{A}} = \bigvee \{ \varphi_q^{\mathfrak{A}} \mid q \in \uparrow p \cap \text{Spec } L \}$.

Finally, $(\mathfrak{A}, (u_i^u, u_i^l)_{i \in I}) = ((\varphi_p^{\mathfrak{A}} : (A^u, \mathcal{U}^u) \rightarrow (A^l, \mathcal{U}^l))_{p \in \text{Spec } L}, (u_i^u, u_i^l)_{i \in I})$ is the coproduct in UF_0 of the system of all $\mathfrak{A}_i$. Indeed, given any $\mathfrak{B} = (\varphi_p^{\mathfrak{B}} : (B^u, \mathcal{V}^u) \rightarrow (B^l, \mathcal{V}^l))_{p \in \text{Spec } L} \in \text{UF}_0$ and any collection of L -uniform homomorphisms $\{h_i = (h_i^u, h_i^l) : \mathfrak{A}_i \rightarrow \mathfrak{B} \mid i \in I\}$, since $(A^u, \mathcal{U}^u)$ is the coproduct in UFrm of the $(A_i^u, \mathcal{U}_i^u)$, there exists a unique uniform homomorphism $h^u : (A^u, \mathcal{U}^u) \rightarrow (B^u, \mathcal{V}^u)$ such that $h^u \circ u_i^u = h_i^u$ for every $i \in I$. Similarly,

there exists a unique uniform homomorphism $h^l : (A^l, \mathcal{U}^l) \rightarrow (B^l, \mathcal{V}^l)$ such that $h^l \circ u_i^l = h_i^l$ for every $i \in I$.

$$\begin{array}{ccccc}
 & & h_i^u & & \\
 & \nearrow & & \searrow & \\
 (A_i^u, \mathcal{U}_i^u) & \xrightarrow{u_i^u} & (A^u, \mathcal{U}^u) & \dashrightarrow^{h^u} & (B^u, \mathcal{V}^u) \\
 \downarrow \varphi_p^{\mathfrak{A}_i} & & \downarrow \varphi_p^{\mathfrak{A}} & & \downarrow \varphi_p^{\mathfrak{B}} \\
 (A_i^l, \mathcal{U}_i^l) & \xrightarrow{u_i^l} & (A^l, \mathcal{U}^l) & \dashrightarrow^{h^l} & (B^l, \mathcal{V}^l) \\
 & \searrow & & \nearrow & \\
 & & h_i^l & &
 \end{array}$$

Since, for each $i \in I$ and $p \in \text{Spec } L$,

$$\varphi_p^{\mathfrak{B}} \circ h^u \circ u_i^u = \varphi_p^{\mathfrak{B}} \circ h_i^u = h_i^l \circ \varphi_p^{\mathfrak{A}_i} = h^l \circ u_i^l \circ \varphi_p^{\mathfrak{A}_i} = h^l \circ \varphi_p^{\mathfrak{A}} \circ u_i^u,$$

then $\varphi_p^{\mathfrak{B}} \circ h^u = h^l \circ \varphi_p^{\mathfrak{A}}$, and the pair (h^u, h^l) is an L -uniform homomorphism $\mathfrak{A} \rightarrow \mathfrak{B}$. It is clearly the unique L -uniform homomorphism $\mathfrak{A} \rightarrow \mathfrak{B}$ such that $(h^u, h^l) \circ (u_i^u, u_i^l) = (h_i^u, h_i^l)$ for every $i \in I$.

In a similar way, one can construct the coequalizers, products and equalizers of $\mathbf{UF}_0$ from the corresponding constructions in $\mathbf{UFrm}$. $\square$

Claim 2. $\mathbf{UF}_1$ is mono-coreflective in $\mathbf{UF}_0$. Consequently, $\mathbf{UF}_1$ inherits colimits from $\mathbf{UF}_0$ and the limits of $\mathbf{UF}_1$ are the coreflections of limits of $\mathbf{UF}_0$, and hence $\mathbf{UF}_1$ is also complete and cocomplete.

Proof of Claim 2. Let $\mathfrak{B} = (\varphi_p^{\mathfrak{B}} : (B^u, \mathcal{V}^u) \rightarrow (B^l, \mathcal{V}^l))_{p \in \text{Spec } L} \in \mathbf{UF}_0$ be given and consider $\mathfrak{A} = (\varphi_p^{\mathfrak{A}} : (A^u, \mathcal{U}^u) \rightarrow (A^l, \mathcal{U}^l))_{p \in \text{Spec } L}$ where $(A^u, \mathcal{U}^u) = (B^u, \mathcal{V}^u)$, A^l is the subframe of B^l generated by $\bigcup_{p \in \text{Spec } L} \varphi_p^{\mathfrak{B}}[B^u]$, $\mathcal{U}^l$ is the uniformity contained in $\mathcal{V}^l$ generated by $\{\varphi_p^{\mathfrak{B}}[V] \mid p \in \text{Spec } L, V \in \mathcal{V}^u\}$ and $\varphi_p^{\mathfrak{A}} = \varphi_p^{\mathfrak{B}}|_{\text{cod}=\varphi_p^{\mathfrak{B}}[B^u]}$. It is immediate that $\mathfrak{A}$ satisfies (UF0) and (UF1). Further, define $h : \mathfrak{A} \rightarrow \mathfrak{B}$ by $h^u = \text{id}_{B^u}$ and $h^l : A^l \hookrightarrow B^l$. It is straightforward to check that h is a monomorphism in $\mathbf{UF}_0$ satisfying the required universal property. $\square$

Claim 3. $\mathbf{UF}_1$ is an $(E, \mathcal{M})$ -category, for E the class RegEpi of regular epimorphisms and $\mathcal{M}$ the class Mono-source of mono-sources.

Proof of Claim 3. By Proposition 15.13 of [1], it suffices to check that each source in $\mathbf{UF}_1$ has a (RegEpi, Mono-source)-factorization.

So let $(h_i : \mathfrak{A} \rightarrow \mathfrak{A}_i)_{i \in I}$ be a mono-source in $\mathbf{UF}_1$ and consider the product $((B^u, \mathcal{V}^u), p_i^u)$ (resp. $((B^l, \mathcal{V}^l), p_i^l)$) of the domains $(A_i^u, \mathcal{U}_i^u)$ (resp. codomains $(A_i^l, \mathcal{U}_i^l)$) of the $\varphi_p^{\mathfrak{A}_i}$. Then there exists, for each $p \in \text{Spec } L$, a unique $\varphi_p^{\mathfrak{B}} : (B^u, \mathcal{V}^u) \rightarrow (B^l, \mathcal{V}^l)$ such that $p_i^l \circ \varphi_p^{\mathfrak{B}} = \varphi_p^{\mathfrak{A}_i} \circ p_i^u$ for every $i \in I$.

The function $f^u : (A^u, \mathcal{U}^u) \rightarrow (B^u, \mathcal{V}^u)$ (resp. $f^l : (A^l, \mathcal{U}^l) \rightarrow (B^l, \mathcal{V}^l)$) defined by $p_i^u \circ f^u = h_i^u$ for each $i \in I$ (resp. $p_i^l \circ f^l = h_i^l$ for each $i \in I$) has a factorization

$$(A^u, \mathcal{U}^u) \xrightarrow{e^u} (C^u, \mathcal{W}^u) \xrightarrow{m^u} (B^u, \mathcal{V}^u)$$

(resp. $(A^l, \mathcal{U}^l) \xrightarrow{e^l} (C^l, \mathcal{W}^l) \xrightarrow{m^l} (B^l, \mathcal{V}^l)$) with e^u and e^l surjections and m^u and m^l injective functions.

Since, for each $c \in C^u$, $\varphi_p^{\mathfrak{B}}(m^u(c)) = \varphi_p^{\mathfrak{B}}(m^u(e^u(a))) = \varphi_p^{\mathfrak{B}}(f^u(a)) = f^l(\varphi_p^{\mathfrak{A}}(a)) = m^l(e^l(\varphi_p^{\mathfrak{A}}(a)))$ for some $a \in A^u$, we may define

$$\varphi_p^{\mathfrak{C}} : (C^u, \mathcal{W}^u) \rightarrow (C^l, \mathcal{W}^l)$$

by $\varphi_p^{\mathfrak{C}}(c) = (m^l)^{-1}(\varphi_p^{\mathfrak{B}}(m^u(c))) = e^l(\varphi_p^{\mathfrak{A}}(a))$ (easily seen to be well defined).

Finally, if we define $m_i^u : C^u \rightarrow A_i^u$ by $m_i^u = p_i^u \circ m^u$ and $m_i^l : C^l \rightarrow A_i^l$ by $m_i^l = p_i^l \circ m^l(a)$ we have the following diagram:

$$\begin{array}{ccccc}
 & & h_i^u & & \\
 & \nearrow & & \searrow & \\
 (A^u, \mathcal{U}^u) & \xrightarrow{e^u} & (C^u, \mathcal{W}^u) & \xrightarrow{m_i^u} & (A_i^u, \mathcal{U}_i^u) \\
 & \searrow f^u & \downarrow m^u & \nearrow p_i^u & \\
 & & (B^u, \mathcal{V}^u) & & \\
 & & \downarrow \varphi_p^{\mathfrak{B}} & & \\
 & & (B^l, \mathcal{V}^l) & & \\
 & \nearrow f^l & \uparrow m^l & \searrow p_i^l & \\
 (A^l, \mathcal{U}^l) & \xrightarrow{e^l} & (C^l, \mathcal{W}^l) & \xrightarrow{m_i^l} & (A_i^l, \mathcal{U}_i^l) \\
 & \searrow & & \nearrow & \\
 & & h_i^l & &
 \end{array}$$

$\varphi_p^{\mathfrak{A}}$ (left vertical), $\varphi_p^{\mathfrak{A}_i}$ (right vertical), $\varphi_p^{\mathfrak{C}}$ (bottom horizontal), $\varphi_p^{\mathfrak{B}}$ (middle vertical), m^u (top middle vertical), m^l (bottom middle vertical), p_i^u (top right diagonal), p_i^l (bottom right diagonal), f^u (top left diagonal), f^l (bottom left diagonal).

It follows that $(h_i^u, h_i^l) = (m_i^u \circ e^u, m_i^l \circ e^l)$ is a (RegEpi, Mono-source)-factorization of $(h_i)_{i \in I} = (h_i^u, h_i^l)_{i \in I}$ in $\mathbf{UF}_1$. $\square$

Claim 4. $L\text{-UFrm}$ is a full subcategory of $\mathbf{UF}_1$ closed under the formation of mono-sources in $\mathbf{UF}_1$.

Proof of Claim 4. First, notice that for any mono-source $(h_i : \mathfrak{A} \rightarrow \mathfrak{A}_i)_{i \in I}$ in $\mathbf{UF}_1$, $(h_i^u)_{i \in I}$ is a mono-source in $\mathbf{UFrm}$ and, therefore, if $h_i^u(a) = h_i^u(b)$ for every $i \in I$ then $a = b$. Thus, if each $\mathfrak{A}_i$ belongs to $L\text{-UFrm}$ and $a, b \in A^u$ then, for each $p \in \text{Spec } L$,

$$\begin{aligned} \varphi_p^{\mathfrak{A}}(a) = \varphi_p^{\mathfrak{A}}(b) &\implies h_i^l(\varphi_p^{\mathfrak{A}}(a)) = h_i^l(\varphi_p^{\mathfrak{A}}(b)), \text{ for all } i \in I \\ &\iff \varphi_p^{\mathfrak{A}_i}(h_i^u(a)) = \varphi_p^{\mathfrak{A}_i}(h_i^u(b)), \text{ for all } i \in I \\ &\implies h_i^u(a) = h_i^u(b), \text{ for all } i \in I \\ &\implies a = b, \end{aligned}$$

which shows that $\mathfrak{A}$ is also in $L\text{-UFrm}$. Hence $L\text{-UFrm}$ is closed under the formation of mono-sources in $\mathbf{UF}_1$. $\square$

In conclusion, $L\text{-UFrm}$ is a full subcategory of an $(E, \mathcal{M})$ -category $\mathbf{UF}_1$, closed under the formation of $\mathcal{M}$ -sources in $\mathbf{UF}_1$. Hence, by Theorem 16.8 of [1], $L\text{-UFrm}$ is E -reflective in $\mathbf{UF}_1$. Consequently, $L\text{-UFrm}$ inherits limits from $\mathbf{UF}_1$ and the colimits of $L\text{-UFrm}$ are the reflections of colimits of $\mathbf{UF}_1$, and hence $L\text{-UFrm}$ is complete and cocomplete. $\square$

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