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LANDAU'S NECESSARY DENSITY CONDITIONS FOR THE HANKEL TRANSFORM

LUÍS DANIEL ABREU AND AFONSO BANDEIRA

ABSTRACT: We will prove an analogue of Landau's necessary conditions [*Necessary density conditions for sampling and interpolation of certain entire functions*, Acta Math. 117 (1967).] for spaces of functions whose Hankel transform is supported in a measurable subset S of the positive semi-axis. As a special case, necessary density conditions for the existence of Fourier-Bessel frames are obtained. In the course of our proof we obtain estimates for some eigenvalues which arise in Tracy and Widom work [*Level spacing distributions and the Bessel kernel.* Comm. Math. Phys. 161 (1994), no. 2, 289–309.].

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1. Introduction

While Fourier series rely on the fact that $\{e^{ikx}\}_{k\in\mathbb{Z}}$ constitutes an orthogonal basis for $L^2(-\pi,\pi)$, *nonharmonic Fourier series* allow more general sets $\{e^{it_kx}\}_{k\in\mathbb{Z}}$. They can be nonuniform as in *Riesz basis* [19], perhaps even redundant as in *Fourier frames* [8]. On their "frequency side", nonharmonic Fourier series provide nonuniform and redundant sampling theorems in spaces of bandlimited functions. As a consequence of Landau's necessary conditions for sampling and interpolation of such functions [5, 6], we know that sampling requires $\{t_k\}_{k\in\mathbb{Z}}$ to be "denser than $\mathbb{Z}$" and that interpolation requires $\{t_k\}_{k\in\mathbb{Z}}$ to be "sparser than $\mathbb{Z}$". The set $\mathbb{Z}$ is a sequence of both sampling and interpolation for bandlimited functions (this is known as the Whittaker-Shannon-Kotel'nikov sampling theorem).

Likewise, let J_α be the Bessel function of order $\alpha > -1/2$ and $j_{n,\alpha}$ its n^{th} zero. The theory of Fourier-Bessel series is based on the fact that $\{x^{\frac{1}{2}}J_\alpha(j_{n,\alpha}x)\}_{n=0}^{\infty}$ is an orthogonal basis for $L^2[0,1]$. Thus, the study of more general sets $\{x^{\frac{1}{2}}J_\alpha(t_{n,\alpha}x)\}_{n=0}^{\infty}$ leads naturally to " nonharmonic Fourier-Bessel sets". Completeness properties of such sets have been investigated by Boas

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and Pollard [2], and some stability results concerning Riesz basis have been obtained in [12]. However, the problems that arise naturally in connection with frame theory and, in particular, questions related to frame and sampling density, have not been investigated up to the present date. We will address this question in the present paper, as a special case of a more general result, which gives Landau-type results in the context of the Hankel transform.

Let S be a measurable subset of $(0, \infty)$ and consider the space $\mathcal{B}_\alpha(S)$ of functions in $L^2(0,\infty)$ such that their Hankel transform,

$$H_\alpha\left(f\right)\left(x\right) = \int_0^\infty f(t)(xt)^{1/2} J_\alpha(xt)dt,$$

is supported in S. The special case $S = [0,1]$ is an important example of a reproducing kernel Hilbert space [4, 17]. Our investigations are also motivated by the seminal work of Tracy and Widom [16]. With a view to solving an eigenvalue problem arising in the asymptotics of certain random matrices, they have constructed a set of functions which play the role of the prolate spheroidal functions in this situation. Such functions are examples of doubly orthogonal functions in the sense of Stefan Bergman [1] and this automatically implies [15] that they solve the concentration problem

$$\lambda_k \phi_k(x) = \int_0^r \phi_k(t)\mathcal{R}_\alpha(t,x)dt,$$

where $\mathcal{R}_\alpha(t,x)$ is the reproducing kernel of $\mathcal{B}_\alpha([0,1])$. Once we know that such functions exist, it becomes natural to ask if the eigenvalues exhibit the famous "plunging phenomenon" that lies at the center of the Beurling-type density theorems revealing a "Nyquist rate", as in [6]. We will give a positive answer to this question, for the space $\mathcal{B}_\alpha(S)$, where S is a measurable subset of the positive semi-axis.

To describe our results, we require some terminology. We will impose a certain stability in the definition of sampling sequences. Specifically, we will call a sequence $\Lambda = \{t_n\}_{n=0}^\infty$ a *set of sampling* for $\mathcal{B}_\alpha(S)$ if there exists a constant A such that, for every $f \in \mathcal{B}_\alpha(S)$,

$$A\int_0^\infty \left|f(x)\right|^2 dx \leq \sum_{n=0}^\infty |f(t_n)|^2.$$

Moreover, Λ is a *set of interpolation* for $\mathcal{B}_\alpha(S)$ if, given any set of numbers $\{a_n\}_{n=0}^\infty$ with $\sum |a_n|^2 < \infty$, there exists $f \in \mathcal{B}_\alpha(S)$ such that

$$f(t_n) = a_n, \text{ for every } t_n \in \Lambda.$$

We assume that all sequences are separated, meaning that $\inf_{j\neq l} |t_j - t_l| = d > 0$.

Definition 1. *Let $n(r)$ denote the number of points of $\Lambda \subset \mathbb{R}^+$ to be found in $[0, r]$. Then the* upper density *of Λ is given by the limit*

$$D^+(\Lambda) = \lim_{r\to\infty} \sup \frac{n(r)}{r}.$$

and the lower density *of Λ is*

$$D^-(\Lambda) = \lim_{r\to\infty} \inf \frac{n(r)}{r}.$$

With these definitions, our main results read as follows.

Theorem 1. *Let $\alpha > -1/2$. If the set $\Lambda \subset \mathbb{R}^+$ is of sampling for $\mathcal{B}_\alpha(S)$, then*

$$D^-(\Lambda) \geq \frac{1}{\pi}. \tag{1}$$

Theorem 2. *Let $\alpha > -1/2$. If the set $\Lambda \subset \mathbb{R}^+$ is of interpolation for $\mathcal{B}_\alpha(S)$, then*

$$D^+(\Lambda) \leq \frac{1}{\pi}. \tag{2}$$

The above results can be seen from a *frame theory* viewpoint. A sequence of functions $\{e_j\}_{j\in I}$ is said to be a *frame* in a Hilbert space H if there exist positive constants A and B such that, for every $f \in H$,

$$A \|f\|_H^2 \leq \sum_{j\in I} |\langle f, e_j\rangle|^2 \leq B \|f\|_H^2 . \tag{3}$$

Accordingly, we say that $\{(t_n x)^{\frac{1}{2}} J_\alpha(t_n x)\}$ is a *Fourier-Bessel frame* if there exist positive constants A and B such that

$$A \int_0^1 |f(x)|^2 \, dx \leq \sum_{n=0}^\infty \left| \int_0^1 (t_n x)^{\frac{1}{2}} f(x) J_\alpha(t_n x) dx \right|^2 \leq B \int_0^1 |f(x)|^2 \, dx.$$

By choosing $S = (0, 1)$ in Theorem 1 then clearly one concludes that, if $\{(t_n x)^{\frac{1}{2}} J_\alpha(t_n x)\}$ is a Fourier Bessel frame, then $D^-(\Lambda) \geq \frac{1}{\pi}$.

Recently, Marzo [7] applied Landau's ideas to the proof of Marcinkiewicz–Zygmund inequalities in the sphere. From his work we borrow an idea to start with the estimations leading to inequality (1). It is worth noting that analogues of Landau's necessary conditions have been also studied [9, 10] using techniques from time-frequency analysis [11].

The outline of the paper is as follows. In Section 2 we recall some of the convolution structure associated with the Hankel transform. Then, Section 3 states the eigenvalue problem and contains the estimates of the its trace and norm. We prove our main result in Section 4 and in Section 5 collect some required technical lemmas.

2. Preliminaries

In this section we will use [13] and [14] as reference sources for some definitions and properties that are useful in the harmonic analysis associated with the Hankel transform. For $\alpha > -1$, the Bessel functions are defined by the power series,

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{x}{2})^{2n+\alpha}}{n!\Gamma(n+\alpha+1)}.$$

The spherical Bessel functions j_α are defined as

$$j_\alpha(x) = \Gamma(\alpha+1)\left(\frac{2}{x}\right)^\alpha J_\alpha(x).$$

They are known to satisfy $j_\alpha(0) = 1$ and $|j_\alpha(x)| \leq 1$, for all $x \in \mathbb{R}^+$. We will use some operators: the Hankel modulation:

$$(\mathrm{m}_\lambda f)(x) = j_\alpha(\lambda x) f(x),$$

the Hankel translation,

$$H_\alpha(\tau_\lambda f) = \mathrm{m}_\lambda H_\alpha f = j_\alpha(\lambda \cdot)(H_\alpha f)(\cdot),$$

and the Hankel convolution,

$$f\#g(\lambda) = \lambda^{\alpha+\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{\alpha}{2}} \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(t)\tau_\lambda g(t)dt.$$

As in the theory of Fourier transforms, the Hankel transform maps convolutions in products via the formula

$$H_\alpha(f\#g)(s) = s^{-\left(\alpha+\frac{1}{2}\right)}(2\pi)^{\frac{\alpha}{2}} H_\alpha f(s) H_\alpha g(s).$$

The following result will be required in the proof of the technical lemmas of the last section. See, for instance [13] for a proof.

Proposition 1. *If* $\operatorname{supp} g \subset [0, d]$ *and* $r > d$ *then*

$$\operatorname{supp} \tau_r g \subset [\max\{0, r-d\}, r+d].$$

3. Energy Concentration and the eigenvalue problem

We are interested in the problem of maximizing, over the functions $f \in \mathcal{B}_\alpha(S)$, the quadratic formula

$$\lambda_f = \frac{1}{\|f\|^2} \int_I |f(t)|^2 dt.$$

We can write it as

$$\lambda_f = \frac{(D_I B_S f, B_S f)}{(B_S f, B_S f)}.$$

Where D_I is the projection onto $L^2(I)$ and B_S is the projection onto $\mathcal{B}_\alpha(S)$. Since $f \in \mathcal{B}_\alpha(S)$ and the operators are self-adjoint, then

$$\lambda_f = \frac{(B_S D_I f, f)}{(f, f)}.$$

This Quadratic Form immediately leads to the eigenvalue problem

$$\lambda_k(I, S)\phi_k(x) = B_S D_I \phi_k. \tag{4}$$

Then, writing the operators explicitly and interchanging the integrals, (4) becomes

$$\lambda_k(I, S)\phi_k(x) = \int_I \phi_k(t) w_S(t, x) dt, \tag{5}$$

where, for a set X, $w_X(t, x)$ is given by

$$w_X(t, x) = \int_X J_\alpha(ty) J_\alpha(xy)(tx)^{\frac{1}{2}} y dy.$$

Multiplying both sides of (5) by $(xu)^{\frac{1}{2}} J_\alpha(xu)$, integrating in respect to dx in I and changing the order of the integrals, gives the dual problem of concentrating on S functions whose Hankel Transform are supported on I:

$$\lambda_k(I, S)\psi_k(x) = \int_S \psi_k(t) w_I(t, x) dt. \tag{6}$$

Remark 1. *Observe that, for $\beta > 0$,*

$$\begin{aligned} \lambda_k(I,S) &= \lambda_k(S,I) & (7) \\ &= \lambda_k(\beta I, \beta^{-1} S), & (8) \end{aligned}$$

since we have just proven the first identity and the second follows from a simple change of variables.

Consider the special case

$$w_{[0,1]}(t,x) = \mathcal{R}_\alpha(x,y) = \begin{cases} (xy)^{\frac{1}{2}} \frac{J_\alpha(y) x J'_\alpha(x) - J_\alpha(x) y J'_\alpha(y)}{y^2 - x^2} & \text{if } x \neq y \\ \frac{1}{2}\left(x J'_\alpha(x)^2 - x J_\alpha(x) J''_\alpha(x) - J_\alpha(x) J'_\alpha(x) \right) & \text{if } x = y \end{cases}$$

If $I = [0,r]$ then a calculation shows that

$$w_I(t,x) = r\mathcal{R}_\alpha(tr, xr).$$

Thus, if $I = [0,r]$ one can write (6) as

$$\lambda_k(r,S)\phi_k(x) = \int_S \phi_k(t) r \mathcal{R}_\alpha(tr, xr) dt, \tag{9}$$

Our results will follow from the evaluation of the following quantities (see [20]):

$$\mathrm{Trace}(r) = \sum \lambda_k(r,S) = r \int_S \mathcal{R}_\alpha(tr,tr) dt, \tag{10}$$

and

$$\mathrm{Norm}(r) = \sum \lambda_k^2(r,S) = r^2 \int_S \int_S \mathcal{R}_\alpha^2(tr,xr) dt dx. \tag{11}$$

We find it convenient to study first the eigenvalue problem

$$\lambda_k(r,[0,1])\phi_k(t) = \int_0^r \phi_k(x) \mathcal{R}_\alpha(t,x) dx.$$

We will proceed with the estimation of the trace and the norm in this simplified case (corresponding, after a change of variable, to the case when $S = [0,1]$), which we denote as $T(r)$ and $N(r)$.

3.1. Estimation of $T(r)$. The key tool will be the asymptotic formula for the Bessel function [18],

$$J_\alpha(x) = \sqrt{\frac{2}{\pi x}} \left(\sin \eta_x + \rho(x)\right), \tag{12}$$

with $\rho(x) = \mathcal{O}(x^{-1})$, where $\eta_x = x - \left(\frac{1}{2}\alpha - \frac{1}{4}\right)\pi$, from this formula its easy to derive,

$$J'_\alpha(x) = \sqrt{\frac{2}{\pi x}} \left(\cos \eta + \rho_1(x)\right), \ \rho_1(x) = \rho'(x) - \frac{1}{2x}\left(\sin \eta + \rho(x)\right) = \mathcal{O}(x^{-1}) \tag{13}$$

$$J''_\alpha(x) = \sqrt{\frac{2}{\pi x}} \left(-\sin \eta + \rho_2(x)\right) \tag{14}$$

with $\rho_2(x) = \mathcal{O}(x^{-1})$.

Lemma 1. *For $\alpha > -1/2$, $T_\alpha(r) = \frac{1}{\pi} r + \mathcal{O}(\log r)$.*

Proof: Let us estimate the trace

$$T_\alpha(r) = \int_0^r \mathcal{R}_\alpha(x,x)dx \ = \frac{1}{2}\int_0^r xJ'_\alpha(x)^2 - xJ_\alpha(x)J''_\alpha(x) - J_\alpha(x)J'_\alpha(x)dx.$$

We first prove that the trace integral converges. For small x, we have

$$J_\alpha(x) = \frac{x^\alpha}{2^\alpha \Gamma \alpha + 1} + \mathcal{O}(x^{\alpha+2}).$$

Then, for small r, a calculation shows that

$$T(r) = \frac{1}{2}\int_0^r \mathcal{O}(x^{2\alpha+1})dx.$$

So, as $\alpha > -1/2$ then $T(r)$ converges for small r. Using (12), (13) and (14) in

$$\frac{dT(r)}{dr} = \frac{1}{2}\left(rJ'_\alpha(r)^2 - rJ_\alpha(r)J''_\alpha(r) - J_\alpha(r)J'_\alpha(r)\right),$$

one obtains, after some simplification,

$$\frac{dT(r)}{dr} = \frac{1}{\pi} + \mathcal{O}\left(\frac{1}{r}\right).$$

As a result, the integral converges for all r and

$$T(r) = \frac{1}{\pi} r + \mathcal{O}(\log r).$$

■

3.2. Estimation of $N(r)$. This section contains the key step, which is the estimation of

$$N(r)=\sum_{k=0}^{\infty}\lambda_k^2(r,[0,1])=\int_0^r\int_0^r\mathcal{R}_\alpha(x,y)^2dxdy.$$

Lemma 2. *The function N satisfies the estimate*

$$N(r)\geq\frac{1}{\pi}r-K\log r-L. \tag{15}$$

for some constants K and L not depending on r.

Proof: We divide our proof in three steps. In the first one we rewrite the double integral in such a way it allows us to obtain our result from an upper estimate, as in [7, Proposition 1] and separate the resulting integral in pieces that will be estimated in steps 2 and 3.

Step 1. Since

$$N(r)=\int_0^r\int_0^\infty\mathcal{R}_\alpha^2(x,y)dxdy-\int_0^r\int_r^\infty\mathcal{R}_\alpha^2(x,y)dxdy,$$

the reproducing kernel property gives:

$$N(r)=\int_0^r\mathcal{R}_\alpha(x,x)dx-\int_0^r\int_r^\infty\mathcal{R}_\alpha^2(x,y)dxdy.$$

Thus,

$$N(r)=T(r)-\int_0^r\int_r^\infty\mathcal{R}_\alpha^2(x,y)dxdy.$$

Clearly we have $T(r)>N(r)$, we want a superior limit for

$$D_\alpha(r)=\int_0^r\int_r^\infty\mathcal{R}_\alpha^2(x,y)dxdy.$$

We separate in two summands,

$$D_\alpha(r)\leq G_\alpha(r)+F_\alpha(r)+G_\alpha(2r),$$

with

$$G_\alpha(r)=\int_0^{\frac{r}{2}}\int_r^\infty\mathcal{R}_\alpha^2(x,y)dxdy \text{ and } F_\alpha(r)=\int_{\frac{r}{2}}^r\int_r^{2r}\mathcal{R}_\alpha^2(x,y)dxdy.$$

In Step 2 we will show that $F_\alpha(r)\leq K\log r+L+\mathcal{O}(r^{-1})$, for some constants K,L and in Step 3 we will see that $G_\alpha(r)$ is bounded. Combining this with the estimate of $T(r)$, gives (15).

Step 2. In this step we will estimate F_α, essentially by reducing the problem to one of the fundamental estimates obtained by Landau [5]. A change of variables in the double integral defining F_α results in:

$$\begin{aligned} F_\alpha(r) &= \int_{\frac{1}{2}}^{1}\int_{1}^{2} r^2 R(rx,ry)^2 dxdy \\ &= \int_{\frac{1}{2}}^{1}\int_{1}^{2} r^2 xy \left(\frac{J_\alpha(ry)xJ'_\alpha(rx) - J_\alpha(rx)yJ'_\alpha(ry)}{y^2-x^2}\right)^2 dxdy. \end{aligned}$$

Now insert asymptotic formulas (12) and (12). After some computation one sees that

$$F_\alpha(r) = \int_{\frac{1}{2}}^{1}\int_{1}^{2} \left(\mathcal{L}_\alpha^r(x,y) + \mathcal{E}_\alpha^r(x,y)\right)^2 dxdy,$$

with

$$\mathcal{L}_\alpha^r(x,y) = \frac{2}{\pi}\frac{\sin\eta_{ry}x\cos\eta_{rx} - \sin\eta_{rx}y\cos\eta_{ry}}{y^2-x^2} \tag{16}$$

and $\mathcal{E}_\alpha^r(x,y) = \mathcal{O}(r^{-1})$. As a result,

$$F_\alpha(r) = \tilde{F}_\alpha(r) + \mathcal{O}(r^{-1}),$$

with

$$\tilde{F}_\alpha(r) = \int_{\frac{1}{2}}^{1}\int_{1}^{2} \left(\mathcal{L}_\alpha^r(x,y)\right)^2 dxdy.$$

Plugging (16), we get that $\tilde{F}_\alpha(r)$ is equal to,

$$\int_{\frac{1}{2}}^{1}\int_{1}^{2} \left(\frac{\sin(rx+k_\alpha)y\cos(ry+k_\alpha) - \sin(ry+k_\alpha)x\cos(rx+k_\alpha)}{x^2-y^2}\right)^2 dxdy,$$

where $k_\alpha = -\left(\frac{1}{2}\alpha - \frac{1}{4}\right)\pi$. A calculation yields,

$$\tilde{F}_\alpha(r) = \int_{\frac{1}{2}}^{1}\int_{1}^{2} \left(\frac{y\sin(r(x-y)) + (y-x)\sin(ry+k_\alpha)\cos(rx+k_\alpha)}{x^2-y^2}\right)^2 dxdy.$$

Thus, $\tilde{F}_\alpha(r)$ is equal to

$$\int_{\frac{1}{2}}^{1}\int_{1}^{2} \left(\frac{1}{x+y}\right)^2 \left(r\,\mathrm{sinc}\left(\frac{r}{\pi}(x-y)\right) - \sin(ry+k_\alpha)\cos(rx+k_\alpha)\right)^2 dxdy,$$

where we are using the usual notation $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$. From the last expression it follows that there exists a positive constant A such that

$$\tilde{F}_\alpha(r) \leq A \int_{\frac{1}{2}}^{1} \int_{1}^{2} \left(r \,\mathrm{sinc}\left(\frac{r}{\pi}(x-y) \right) \right)^2 dxdy + B.$$

Changing variables,

$$\tilde{F}_\alpha(r) \leq A\pi^2 \mathcal{S}(r) + B,$$

where $\mathcal{S}(r)$ is given by,

$$\mathcal{S}(r) = \int_{\frac{r}{2\pi}}^{\frac{r}{\pi}} \int_{\frac{r}{\pi}}^{\frac{2r}{\pi}} (\mathrm{sinc}(x-y))^2 \, dxdy.$$

As

$$\begin{aligned} \mathcal{S}(r) &\leq \int_0^{\frac{r}{\pi}} \int_{\frac{r}{\pi}}^{\infty} (\mathrm{sinc}(x-y))^2 \, dxdy + \int_0^{\frac{r}{\pi}} \int_{\mathbb{R}^-} (\mathrm{sinc}(x-y))^2 \, dxdy \\ &= \int_0^{\frac{r}{\pi}} \int_{\mathbb{R}} (\mathrm{sinc}(x-y))^2 \, dxdy - \int_0^{\frac{r}{\pi}} \int_0^{\frac{r}{\pi}} (\mathrm{sinc}(x-y))^2 \, dxdy \\ &= \frac{r}{\pi} - \int_0^{\frac{r}{\pi}} \int_0^{\frac{r}{\pi}} (\mathrm{sinc}(x-y))^2 \, dxdy, \end{aligned}$$

the last equality being true because $\int_{\mathbb{R}} \mathrm{sinc}(x-y)^2 dx = 1$. Now, Landau's inequality [5, (8)] gives

$$\int_0^{\frac{r}{\pi}} \int_0^{\frac{r}{\pi}} (\mathrm{sinc}(x-y))^2 \, dxdy \geq \frac{r}{\pi} - C\log\left(\frac{r}{\pi}\right) - B.$$

It follows that

$$\mathcal{S}(r) \leq C\log\left(\frac{r}{\pi}\right) - B.$$

Thus, for some positive constants K and L not depending on r,

$$\tilde{F}_\alpha(r) \leq K\log r + L.$$

Step 3. Let us now estimate

$$G_\alpha(r) = \int_0^{\frac{r}{2}} \int_r^{\infty} xy \left(\frac{J_\alpha(y)xJ'_\alpha(x) - J_\alpha(x)yJ'_\alpha(y)}{y^2 - x^2} \right)^2 dxdy$$

This integral is a bounded function in r. Indeed, using the triangle inequality and the estimates $|J_\alpha(x)| \leq x^{-\frac{1}{2}}$, $xy/(x+y)^2 < y/x$ and $1/(x-y)^2 \leq (x - r/2)^2$, we can write

$$G_\alpha(r) \leq A_1 + A_2,$$

with

$$A_1 = \int_0^{\frac{r}{2}} \int_r^{\infty} \frac{x}{(x-\frac{r}{2})^2} \left(J'_\alpha(x)\right)^2 dxdy$$

and

$$A_2 = \int_0^{\frac{r}{2}} \int_r^{\infty} \frac{y^3}{(x-\frac{r}{2})^2 x^2} \left(J'_\alpha(y)\right)^2 dxdy.$$

Using asymptotic formulas one can verify that both A_1 and A_2 are bounded. This completes the estimation of G_α. ■

3.3. Estimates for a finite union of intervals. We now extend the estimates to the case where S is a finite union of intervals.

Lemma 3. *let S be a finite union of non-overlapping intervals contained in $(0, \infty)$. Then*

$$\text{Trace}(r) = r\,\text{m}(S)\frac{1}{\pi} + \mathcal{O}(\log r).$$

Proof: First observe that, from minor modifications in the proof of Lemma 1, one obtains

$$\int_{xa}^{xb} \mathcal{R}_\alpha(u,u)du = x\frac{b-a}{\pi} + \mathcal{O}\left(\log x\right).$$

Then, let S be the finite union of non-overlapping intervals $[a_i, b_i]$, for $1 \leq i \leq n$. We have

$$\begin{aligned}
\text{Trace}(r) &= \sum_{i=1}^{n} \int_{a_i}^{b_i} r\mathcal{R}_\alpha(tr, tr)dt \\
&= \sum_{i=1}^{n} \int_{a_i r}^{b_i r} \mathcal{R}_\alpha(u,u)du \\
&= \sum_{i=1}^{n} r(b_i - a_i)\frac{1}{\pi} + \mathcal{O}(\log r) \\
&= r\,\text{m}(S)\frac{1}{\pi} + \mathcal{O}(\log r).
\end{aligned}$$

■

Lemma 4. *let S be a finite union of non-overlapping intervals contained in $(0, \infty)$. Then*

$$\text{Norm}(r) \geq r\,\text{m}(S)\frac{1}{\pi} - \mathcal{O}(\log r).$$

Proof: By adapting the proof of Lemma 2, one gets

$$\int_{xa}^{xb}\int_{xa}^{xb}\mathcal{R}_\alpha^2(u,v)dudv \geq x\frac{b-a}{\pi} - \mathcal{O}\left(\log x\right).$$

Again, let S be the finite union of non-overlapping $[a_i, b_i]$, for $1 \leq i \leq n$. Then

$$\begin{aligned}
\mathrm{Norm}(r) &= r^2\int_S\int_S \mathcal{R}_\alpha^2(tr,xr)dtdx \\
&\geq \sum_{i=1}^{n}\int_{a_i}^{b_i}\int_{a_i}^{b_i} r^2\mathcal{R}_\alpha^2(tr,xr)dtdx \\
&= \sum_{i=1}^{n}\int_{a_ir}^{b_ir}\int_{a_ir}^{b_ir}\mathcal{R}_\alpha^2(u,v)dudv \\
&\geq \sum_{i=1}^{n} r(b_i-a_i)\frac{1}{\pi} - \mathcal{O}(\log r) \\
&= r\,\mathrm{m}(S)\frac{1}{\pi} - \mathcal{O}(\log r).
\end{aligned}$$

■

4. Proof of the main result

Using the identities (7) and (8) one can see that $\lambda_{k-1}(r,S)$ is also the k^{th} eigenvalue of the problem of concentrating functions whose frequencies lie in $[0,1]$, on the set rS. The sampling theorem associated with the Hankel transform [4] states that $\{j_{\alpha,n}\}$ is a sequence of both sampling and interpolation, and is known to be a perturbation of the set $\{\frac{n}{\pi}\}$. Then, there exists a $\Upsilon = \mathcal{O}(1)$ such that, when S is the union of N intervals, the number of these points contained in S^+ is at most $\lfloor\frac{1}{\pi}r\,\mathrm{m}(S)\rfloor + \Upsilon N$ and their number in S^- is at least $\lfloor\frac{1}{\pi}r\,\mathrm{m}(S)\rfloor - \Upsilon N$. Now, from Lemma 5 and Lemma 6 of Section 5, there are γ_0, δ_0 such that

$$\lambda_{\lfloor\frac{1}{\pi}r\,\mathrm{m}(S)\rfloor+\Upsilon N}(r,S) \leq \gamma_0 < 1 \qquad (17)$$

$$\lambda_{\lfloor\frac{1}{\pi}r\,\mathrm{m}(S)\rfloor-\Upsilon N-1}(r,S) \geq \delta_0 > 0. \qquad (18)$$

Theorem 3 (Sampling)**.** *If $\{t_k\}$ is a set of sampling for $B_\alpha(S)$, then $[0,r]$ must contain at least $(\frac{1}{\pi}r\,\mathrm{m}(S) - A\log r - B)$ points of $\{t_k\}$, with A and B constants not depending on r.*

Proof:

Let $\{t_k\}$ is a set of sampling for $B_\alpha(S)$. By Lemma 5 there exists γ independent of r such that

$$\lambda_{n(I)+2} \leq \lambda_{n(I+)} \leq \gamma < 1.$$

And from (18)

$$\lambda_{\lfloor \frac{1}{\pi} r\, \mathrm{m}(S) \rfloor - \Upsilon N - 1}(r, S) \geq \delta_0 > 0.$$

As the number of eigenvalues between δ_0 and γ increase at most logarithmically with r we have

$$\lfloor \frac{1}{\pi} r\, \mathrm{m}(S) \rfloor - \Upsilon N - 1 - n(I) + 2 \leq A' \log r + B'.$$

Thus,

$$n(I) \geq \frac{1}{\pi} r\, \mathrm{m}(S) - A \log r - B,$$

for some A and B not depending on r. □

Theorem 4 (Interpolation). *If $\{t_k\}$ is a set of interpolation for $B_\alpha(S)$, then $[0, r]$ must not contain more than $(\frac{1}{\pi} r\, \mathrm{m}(S) - C \log r - D)$ points of $\{t_k\}$, with C and D constants not depending on r.*

Proof:

The same proof applies interchanging the roles of the results used.

Let $\{t_k\}$ is a set of interpolation for $B_\alpha(S)$. By Lemma 6 there exists δ independent of r such that

$$\lambda_{n(I)-3} \geq \lambda_{n(I-)} \geq \delta > 0.$$

And from (17)

$$\lambda_{\lfloor \frac{1}{\pi} r\, \mathrm{m}(S) \rfloor + \Upsilon N}(r, S) \leq \gamma_0 < 1.$$

As the number of eigenvalues between δ and γ_0 increase at most logarithmically with r we have

$$(n(I) - 3) - \left(\lfloor \frac{1}{\pi} r\, \mathrm{m}(S) \rfloor + \Upsilon N \right) \leq C' \log r + D'$$

Thus,

$$n(I) \leq \frac{1}{\pi} r\, \mathrm{m}(S) + C \log r + D,$$

for constants C, D not depending on r. □

Finally, with general measurable sets S as goal, we can proceed as in Landau [6, pag. 49], observing that it suffices to prove the result for compact sets and then cover a compact S set by a finite collection S_ϵ of intervals with disjoint interiors and measure arbitrary close to the measure of S. This allows to provide estimates, analogous to the ones above, to measurable sets. From the definition of lower and upper density, we conclude that:

- if $\{t_k\}$ is sampling then
$$D^-(\{t_k\}) \geq \frac{1}{\pi}\,\mathrm{m}(S).$$
- if $\{t_k\}$ is interpolation then
$$D^+(\{t_k\}) \leq \frac{1}{\pi}\,\mathrm{m}(S).$$

5. Technical lemmas

The next proposition is the analogue of Proposition 1 in [5].

Proposition 2. *Let S be bounded, and let $\{t_k\}$ be a set of interpolation for $B_\alpha(S)$. Then the points of $\{t_k\}$ are separated by at least some positive distance d, and the interpolation can be performed in a stable way.*

Proof: From
$$f(t) = \int_S H_\alpha f(x) J_\alpha(tx)(tx)^{\frac{1}{2}} dx$$
one has
$$|f(t)|^2 \leq \left(\int_0^\infty |f(x)|^2 dx\right)\left(\int_S \sup_{u\in\mathbb{R}_0^+}\{J_\alpha^2(u)u\}\right) \leq K_1 \int_0^\infty |f(x)|^2 dx.$$
Likewise,
$$f'(t) = \int_S H_\alpha f(x) \frac{\partial J_\alpha(tx)(tx)^{\frac{1}{2}}}{\partial t} dx$$
gives
$$|f'(t)|^2 \leq \left(\int_S |H_\alpha f(x)|^2 dx\right)\left(\int_S \left|\frac{\partial J_\alpha(tx)(tx)^{\frac{1}{2}}}{\partial t}\right|^2 dx\right) \leq K_2 \int_0^\infty |f(x)|^2 dx.$$
The rest of the proof completely follows Landau in [5]. ■

Now we will prove the lemmas corresponding to Lemma 1 and Lemma 2 in [5].

Lemma 5. *Let S be bounded and $\{t_k\}$ a set of sampling for $B_\alpha(S)$, whose points are separated by at least $2d > 0$. Let I be any compact set, $I+$ be the set of points whose distance to I is less than d, and $n(I+)$ be the number of points of $\{t_k\}$ contained in $I+$. Then $\lambda_{n(I+)}(I,S) \leq \gamma < 1$, where γ depends on S and $\{t_k\}$ but not in I.*

Proof: The proof goes along the lines of the arguments in [5], but we will need to use the convolution structure associated with the Hankel transform. Let h be a function with support in $[0,d]$ such that we can find constants K_1, K_2 such that its Hankel transform satisfies

$$K_1 s^{\alpha+\frac{1}{2}} \leq (H_\alpha h)(s) \leq K_2 s^{\alpha+\frac{1}{2}},$$

for every $s \in S$. In order to construct such a function, we use the fact (see [18, pag. 482]) that, if A and B are real (not both zero) and $\alpha > -1$, then the function $AJ_\alpha(z) + BzJ'_\alpha(z)$ has all its zeros on the real axis, except that it has two purely imaginary ones when $A/B + \alpha < 0$. Thus, if y_0 is a complex number outside the imaginary and the real axis, then the function

$$\begin{aligned} s^{-\alpha-\frac{1}{2}}\mathcal{R}_\alpha(s,y_0) &= s^{-\alpha-\frac{1}{2}}(sy_0)^{\frac{1}{2}}\frac{J_\alpha(y_0)sJ'_\alpha(s) - J_\alpha(s)y_0J'_\alpha(y_0)}{y_0^2 - s^2} \\ &= y_0^{\frac{1}{2}}\frac{J_\alpha(y_0)s^{-(\alpha-1)}J'_\alpha(s) - s^{-\alpha}J_\alpha(s)y_0J'_\alpha(y_0)}{y_0^2 - s^2} \end{aligned}$$

is bounded away from zero and infinity for every $s \in S$. Since

$$\mathcal{R}(s,x) = H_\alpha(1_{[0,1]}J_\alpha(\cdot x)(\cdot x)^{1/2})(s),$$

then the function h defined via its Hankel transform as

$$(H_\alpha h)(s) = \mathcal{R}_\alpha(ds, y_0)$$

has the desired property. Now define

$$g(x) := f\#h(x) = x^{\alpha+\frac{1}{2}}\left(\frac{\pi}{2}\right)^{\frac{\alpha}{2}}\frac{1}{\Gamma(\alpha+1)}\int_0^\infty f(t)\tau_x h(t)dt$$

Since

$$H_\alpha g(s) = H_\alpha(f\#h)(s) = s^{-\left(\alpha+\frac{1}{2}\right)}(2\pi)^{\frac{\alpha}{2}}H_\alpha f(s)H_\alpha h(s)$$

and $f \in B_\alpha(S)$ then clearly also $g \in B_\alpha(S)$. It follows that

$$\|g\|_2^2 \lesssim \sum_k |g(t_k)|^2,$$

where the notation $\|g\|_2^2 \lesssim \sum_k |g(t_k)|^2$ stands for: there exists a constant K such that $\|g\|_2^2 \leq K \sum_k |g(t_k)|^2$, for all g. Since $K_1 s^{\alpha+\frac{1}{2}} \leq H_\alpha h(s)$, then

$$\|f\|_2^2 \lesssim \|g\|_2^2,$$

By Proposition 1 of Section 2, $\operatorname{supp} \tau_x h \subset [\max\{0, x-d\}, x+d]$ and one can write

$$\begin{aligned}
|g(x)|^2 &\leq \left(x^{\alpha+\frac{1}{2}} \left(\frac{\pi}{2}\right)^{\frac{\alpha}{2}} \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(t)\tau_x h(t)dt \right)^2 \\
&\leq x^{2\alpha+1} \left(\left(\frac{\pi}{2}\right)^{\frac{\alpha}{2}} \frac{1}{\Gamma(\alpha+1)} \right)^2 \left(\int_{x-d}^{x+d} f(t)\tau_x h(t)dt \right)^2 \\
&\lesssim x^{2\alpha+1} \left(\int_{x-d}^{x+d} f(t)\tau_x h(t)dt \right)^2 \\
&\lesssim \left(\int_{x-d}^{x+d} \tau_x h(t)^2 x^{2\alpha+1} dt \right) \int_{x-d}^{x+d} |f(t)|^2 dt.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{x-d}^{x+d} \tau_x h(t)^2 x^{2\alpha+1} dt &= x^{2\alpha+1} \|\tau_x h\|_2^2 \\
&= x^{2\alpha+1} \|j_\alpha(x\cdot) H_\alpha h(\cdot)\|_2^2 \\
&= \int_S x^{2\alpha+1} j_\alpha^2(xs) H_\alpha h(s)^2 ds \\
&= \int_S \left((xs)^{\alpha+\frac{1}{2}} j_\alpha(xs) \right)^2 \left(\frac{H_\alpha h(s)}{s^{\alpha+\frac{1}{2}}} \right)^2 ds \\
&= C'_\alpha \int_S xs J_\alpha(xs)^2 \left(\frac{H_\alpha h(s)}{s^{\alpha+\frac{1}{2}}} \right)^2 ds \\
&\lesssim \mathrm{m}(S) \int_S xs J_\alpha(xs)^2 ds \\
&\lesssim \int_0^{r_0} xs J_\alpha(xs)^2 ds, \text{ for some } r_0. \\
&\leq C,
\end{aligned}$$

for some constant $C > 0$. We have thus shown that

$$|g(x)|^2 \lesssim \int_{x-d}^{x+d} |f(t)|^2 dt.$$

Now we impose the $n(I+)$ orthogonality conditions:

$$\int_0^\infty f(t)\tau_{t_k} h(t) dt = 0$$

for every $t_k \in I+$. This gives $g(t_k) = 0$, for every $t_k \in I+$.Finally, using the separation of $\{t_k\}$ and the definition of $I+$,

$$\|f\|_2^2 \leq \|g\|_2^2 \leq K \sum_{t_k \notin I+} |g(t_k)|^2 \leq K' \sum_{t_k \notin I+} \int_{t_k-d}^{t_k+d} |f(t)|^2 dt \leq K' \int_{\mathbb{R}^+ \setminus I} |f(t)|^2 dt$$

$$\frac{1}{\|f\|_2^2} \int_I |f(t)|^2 dt = 1 - \frac{1}{\|f\|_2^2} \int_{\mathbb{R}^+ \setminus I} |f(t)|^2 dt \leq 1 - \frac{1}{K'} < 1.$$

Since K' is independent of I, the Lemma is proved. ■

Lemma 6. *Let S be bounded and $\{t_k\}$ a set of interpolation for $B_\alpha(S)$, whose points are separated by at least $d > 0$. Let I be any compact set, $I-$ be the set of points whose distance to the complement of I exceeds $\frac{d}{2}$, and $n(I-)$ be the number of points of $\{t_k\}$ contained in $I-$. Then $\lambda_{n(I-)-1}(I, S) \geq \delta > 0$, where δ depends on S and $\{t_k\}$ but not in I.*

Proof: We have shown in Proposition 4 that the interpolation can be done in a stable way, thus

$$\|g\|_2^2 \leq K \sum_k |g(t_k)|^2.$$

Now, for each t_l let $\phi_l \in B(S)$ be the interpolating function that is 1 at t_l and 0 in the rest of the t_k, all these functions are linearly independent. Let h be the same as in the proof of Lemma 1 and define $\psi_l \in B_\alpha(S)$ by

$$(H_\alpha \psi_l)(s) = (2\pi)^{\frac{2}{\alpha}} (H_\alpha \phi_l)(s) \left(\frac{s^{\alpha+\frac{1}{2}}}{(H_\alpha h)(s)} \right),$$

for every $s \in S$ (recall that $s^{-\alpha-\frac{1}{2}} H_\alpha h$ is bounded away from zero in S). By Hankel transform,

$$\phi_k = \psi_k \# h.$$

Given $f \in span\{\psi_k\}_{k \in I-}$, let $g = f\#h$. Then g is a linear combination of ϕ_k with $t_k \in I-$, thus $g(t_k) = 0$ for $t_k \notin I-$. We get, using the proof of Lemma 5,

$$\|f\|_2^2 \leq K\|g\|_2^2 \leq K' \sum_{t_k \in I-} |g(t_k)|^2 \leq K'' \sum_{t_k \in I-} \int_{t_k-d}^{t_k+d} |f(t)|^2 dt \leq K'' \int_I |f(t)|^2 dt$$

so

$$\lambda_{k-1}(I, S) \geq \inf_{f \in span\{\psi_k\}_{k \in I-}} \frac{\int_I |f(t)|^2 dt}{\int_0^\infty |f(t)|^2 dt} \geq \frac{1}{K''}.$$

Once again K'' does not depend on I and we are done. ■

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Luís Daniel Abreu

CMUC, Department of Mathematics, University of Coimbra, School of Science and Technology (FCTUC) 3001-454 Coimbra

E-mail address: daniel@mat.uc.pt

URL: http://www.mat.uc.pt/~daniel/

Afonso Bandeira

Department of Mathematics, University of Coimbra, School of Science and Technology (FCTUC) 3001-454 Coimbra

and

PACM, Princeton University, Princeton, NJ 08544, USA

E-mail address: ajsb@princeton.edu

URL: http://www.mat.uc.pt/~mat0603/