Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 10–32

UNIVERSAL ENVELOPING ALGEBRAS FOR SIMPLE MALCEV ALGEBRAS

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Abstract: It is our goal to present an universal enveloping algebra for the simple (non-Lie) 7-dimensional Malcev algebra $M(\alpha, \beta, \gamma)$ in order to obtain an universal enveloping algebra for simple Malcev algebras and to study its properties.

Keywords: Simple Malcev algebra; universal enveloping algebra.

Introduction

The universal enveloping algebra of an algebra is a basic tool in the study of its representation theory. Given a algebra A, let us denote by A^- the algebra obtained from A by replacing the product xy by the commutator $[x, y] = xy - yx$, for elements $x, y \in A$. It is known that if A is an associative algebra one obtains a Lie algebra A^- and, conversely, the Poincaré-Birkhoff-Witt Theorem establishes that any Lie algebra is isomorphic to a subalgebra of A^- for some associative algebra A ([1], [2]).

If we start with an alternative algebra A (alternative means that $x(xy) = x^2y$ and $(yx)x = yx^2$, for $x, y \in A$) then A^- is a Malcev algebra. Since any associative algebra is in particular an alternative algebra, then the Malcev algebra A^- is a natural generalization of the Lie algebra A^-. In [4], Pérez-Izquierdo and Shestakov presented an enveloping algebra of Malcev algebras (constructed in a more general way), generalizing the classical notion of universal enveloping algebra for Lie algebras. They proved that for every Malcev algebra M there exist an algebra $U(M)$ and an injective Malcev algebra homomorphism $\iota : M \longrightarrow U(M)^-$ such that the image of M lies into the alternative nucleus of $U(M)$, being $U(M)$ a universal object with respect to such homomorphism. The algebra $U(M)$ has a basis of Poincaré-Birkhoff-Witt Theorem type over M and it inherits good properties, analogous to those of the universal enveloping of Lie algebras, but it is in general not alternative.

Received September 10, 2010.
The first author is supported by the Center of Mathematics of the University of Coimbra/FCT.

In this work we are interested in the universal enveloping algebra of simple Malcev algebras. All simple Malcev algebras are the simple Lie algebras and the 7-dimensional simple (non-Lie) Malcev algebras $M(\alpha, \beta, \gamma)$, with $\alpha\beta\gamma \neq 0$, up to an isomorphism. As the universal enveloping algebra of simple Lie algebras are well known, this work is dedicated to the study of the enveloping algebra of a particular class of simple Malcev algebras: $M(\alpha, \beta, \gamma)$. It is our goal to investigate whether the universal enveloping algebra of an algebra of this particular type is alternative.

1. Classification of simple Malcev algebras

Throughout this paper we consider finite dimensional algebras over a ground field $\mathbb{K}$.

Definition 1.1. Let L be a vector space over $\mathbb{K}$ endowed with a bilinear operation $[,] : L \times L \longrightarrow L$. We say that L is a *Lie algebra* if the following two axioms are satisfied: $\forall x, y, z \in L$

(L1) $[x, y] = -[y, x]$ (anti-symmetry)
(L2) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (Jacobi identity)

Definition 1.2. Let M be a vector space over $\mathbb{K}$ with a bilinear operation $[,] : M \times M \longrightarrow M$. We called M a *Malcev algebra* if the following two axioms are satisfied: $\forall x, y, z \in M$

(M1) $[x, y] = -[y, x]$ (anti-symmetry)
(M2) $[[x, y], [x, z]] = [[[x, y], z], x] + [[[y, z], x], x] + [[[z, x], x], y]$ (Malcev identity)

It is clear from the definitions that any Lie algebra is a Malcev algebra. We say that M is *simple* if it has no ideals except itself and zero, and $[M, M] \neq \{0\}$ (with this last condition we avoid the 1-dimensional algebras). It is known that, if the field $\mathbb{K}$ is of characteristic $\neq 2, 3$, a simple Malcev algebra is either a simple Lie algebra or a 7-dimensional simple (non-Lie) Malcev algebra $M(\alpha, \beta, \gamma)$, with $\alpha\beta\gamma \neq 0$, up to an isomorphism (see Kuzmin [3]).

These last algebras are the only simple Malcev algebras which are not Lie algebras. In particular, if the ground field $\mathbb{K}$ is algebraically closed of characteristic 0, then any simple Malcev algebra is either isomorphic to one simple Lie algebra in the following list: $A_n = \mathfrak{sl}(n+1, \mathbb{K})$, with $n \geq 1$, $B_n = \mathfrak{so}(2n+1, \mathbb{K})$, with $n \geq 2$, $C_n = \mathfrak{sp}(2n, \mathbb{K})$, with $n \geq 3$, $D_n = \mathfrak{so}(2n, \mathbb{K})$, with $n \geq 4$, and the exceptional Lie algebras E_6, E_7, E_8, F_4, G_2, or isomorphic to 7-dimensional simple (non-Lie) Malcev algebra $M(-1, 1, 1)$.

Each algebra $M(\alpha, \beta, \gamma)$ over a ground field $\mathbb{K}$ of characteristic $\neq 2$ is isomorphic to the algebra $C^-/\mathbb{K}$, where C is a suitable Cayley-Dickson algebra over $\mathbb{K}$. Two algebras of this type are isomorphic if and only if the corresponding Cayley-Dickson algebras are isomorphic. The simple (non-Lie) Malcev algebra $M(\alpha, \beta, \gamma)$ has a basis $\{e_1, \ldots, e_7\}$ relative to which the multiplication table is the following (see [3]):

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-\alpha e_2$	e_5	$-\alpha e_4$	$-e_7$	αe_6
e_2	$-e_3$	0	βe_1	e_6	e_7	$-\beta e_4$	$-\beta e_5$
e_3	αe_2	$-\beta e_1$	0	e_7	$-\alpha e_6$	βe_5	$-\alpha\beta e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	γe_1	γe_2	γe_3
e_5	αe_4	$-e_7$	αe_6	$-\gamma e_1$	0	$-\gamma e_3$	$\alpha\gamma e_2$
e_6	e_7	βe_4	$-\beta e_5$	$-\gamma e_2$	γe_3	0	$-\beta\gamma e_1$
e_7	$-\alpha e_6$	βe_5	$\alpha\beta e_4$	$-\gamma e_3$	$-\alpha\gamma e_2$	$\beta\gamma e_1$	0

2. Universal enveloping algebra of a Lie algebra

Since a simple Lie algebra is in particular a simple Malcev algebra, we recall briefly the notion of enveloping algebra of a Lie algebra, its construction and some properties, including the Poincaré-Birkhoff-Witt Theorem ([1], [2]).

Definition 2.1. Let L be a Lie algebra with arbitrary dimension over a field $\mathbb{K}$ of any characteristic. A *universal enveloping algebra* of L is a pair $(\mathfrak{U}, \iota)$, where $\mathfrak{U}$ is an associative algebra with identity element 1 and $\iota : L \longrightarrow \mathfrak{U}^-$ is a Lie homomorphism such that for any associative algebra $\mathfrak{B}$ having an identity element 1 and any Lie homomorphism $\varphi : L \longrightarrow \mathfrak{B}^-$ there exists a unique homomorphism of algebras $\varphi' : \mathfrak{U} \longrightarrow \mathfrak{B}$ such that $\varphi'(1) = 1$ and $\varphi = \varphi' \circ \iota$ or, equivalently, the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[d, "\iota"'] & \mathfrak{B} \\ \mathfrak{U} \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

In the following theorem we collect some properties of the universal enveloping algebra.

Theorem 2.2. *It is verified:*

(1) *The pair $(\mathfrak{U}, \iota)$ is unique (up to isomorphism).*

(2) $\mathfrak{U}$ *is generated by the image* $\iota(L)$ *(as associative algebra).*
(3) *Let* L_1, L_2 *be Lie algebras and* $(\mathfrak{U}_1, \iota_1)$*,* $(\mathfrak{U}_2, \iota_2)$ *its respective universal enveloping algebras and consider the homomorphism* $\alpha : L_1 \longrightarrow L_2$*. Then there exists a unique homomorphism* $\alpha' : \mathfrak{U}_1 \longrightarrow \mathfrak{U}_2$ *such that* $\iota_2 \circ \alpha = \alpha' \circ \iota_1$*.*
(4) *Let* I *be a bilateral ideal in* L *and* $\mathfrak{I}$ *the ideal in* $\mathfrak{U}$ *generated by* $\iota(I)$*. If* $l \in L$ *then* $\jmath : l + I \longrightarrow \iota(l) + \mathfrak{I}$ *is a homomorphism of* L/I *into* $\mathfrak{B}^-$*, where* $\mathfrak{B} = \mathfrak{U}/\mathfrak{I}$*, and* $(\mathfrak{B}, \jmath)$ *is a universal enveloping algebra for* L/I*.*
(5) $\mathfrak{U}$ *has a unique anti-automorphism* π *such that* $\pi \circ \iota = -\iota$*. Moreover* $\pi^2 = 1$*.*
(6) *There is a unique homomorphism* δ *of* $\mathfrak{U}$ *into* $\mathfrak{U} \otimes \mathfrak{U}$ *(the diagonal mapping of* $\mathfrak{U}$*) such that* $\delta(\iota(a)) = \iota(a) \otimes 1 + 1 \otimes \iota(a)$*,* $a \in L$*.*
(7) *If* D *is a derivation in* L *then there exists a unique derivation* D' *in* $\mathfrak{U}$ *such that* $\iota \circ D = D' \circ \iota$*.*

As a consequence of Property (3), instead of studying the representations of Lie algebras, we can study the representations of their enveloping algebras, which becomes simpler since they are associative algebras.

The construction of the universal enveloping algebra $(\mathfrak{U}, \iota)$ of a Lie algebra L appears by considering the tensor algebra of L,

$$T(L) = \bigcup_{i=0}^{\infty} T^i L = \underbrace{T^0 L}_{\mathbb{K}} \cup \underbrace{T^1 L}_{L} \cup \underbrace{T^2 L}_{L\otimes L} \cup \cdots \cup \underbrace{T^m L}_{\substack{L\otimes\cdots\otimes L \\ \text{(m copies)}}} \cup \cdots$$

with the associative product defined on homogeneous generators of $T(L)$ in the following way:

$$(v_1 \otimes \cdots \otimes v_k)(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_m \in T^{m+k} L$$

This makes $T(L)$ an associative algebra with unit element, which is generated by 1 and any basis of V. Let J be the two-sided ideal in $T(L)$ generated by the elements of the form

$$x \otimes y - y \otimes x - [x, y], \forall\, x, y \in L.$$

Define $\mathfrak{U}(L) = T(L)/J$ and $\iota : L \longrightarrow \mathfrak{U}(L)$ the restriction to L of the canonical projection of $T(L) \longrightarrow \mathfrak{U}(L)$. The pair $(\mathfrak{U}(L), \iota)$ is an enveloping algebra of L.

Theorem 2.3 (Poincaré-Birkhoff-Witt)**.** *Let* L *be a Lie algebra (finite or infinite dimensional) and* $\{x_1, x_2, x_3, \ldots\}$ *be an ordered basis of* L*. Then the* *elements*

$$x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k},\ i_1 \leq i_2 \leq \cdots \leq i_k,\ k \in \mathbb{N}$$

with the unit element, form a basis of $\mathfrak{U}(L)$. *In particular, if* $\dim\ L < \infty$ *and* $\{x_1, \ldots, x_n\}$ *a ordered basis of* L *then the elements*

$$x_1^{m_1} \otimes \cdots \otimes x_n^{m_n},\ \textit{with}\ m_i \geq 0\ (i = 1, \ldots, n)$$

form a basis of $\mathfrak{U}(L)$.

3. Universal enveloping algebra of a simple Malcev algebra

In order to obtain the universal enveloping algebra for all simple Malcev algebra and study its properties, it remains to exhibit the universal enveloping algebra of the simple (non-Lie) Malcev algebras $M(\alpha, \beta, \gamma)$. The universal enveloping algebra of a Lie algebra is defined in terms of the basis given by the root decomposition. This algebra is associative and has many interesting properties. It is our goal to see if this structure can be extended to the family of algebras $M(\alpha, \beta, \gamma)$.

The (non-Lie) Malcev algebra $M(\alpha, \beta, \gamma)$ can be decomposed as a direct sum of its root spaces with respect to a Cartan subalgebra, as in Lie algebra case. If $H =< h = e_1 >$, then we have the root space decomposition $M(\alpha, \beta, \gamma) = H \oplus M_{i\sqrt{\alpha}} \oplus M_{-i\sqrt{\alpha}}$, where

$$M_{i\sqrt{\alpha}} = \Big\langle x_1 = e_3 + i\sqrt{\alpha}e_2, x_2 = e_5 + i\sqrt{\alpha}e_4, x_3 = e_7 - i\sqrt{\alpha}e_6 \Big\rangle,$$
$$M_{-i\sqrt{\alpha}} = \Big\langle y_1 = e_3 - i\sqrt{\alpha}e_2, y_2 = e_5 - i\sqrt{\alpha}e_4, y_3 = e_7 + i\sqrt{\alpha}e_6 \Big\rangle.$$

In this new basis the expression of the multiplication in $M(\alpha, \beta, \gamma)$ is given by:

	h	x_1	x_2	x_3	y_1	y_2	y_3
h	0	$i\sqrt{\alpha}x_1$	$i\sqrt{\alpha}x_2$	$i\sqrt{\alpha}x_3$	$-i\sqrt{\alpha}y_1$	$-i\sqrt{\alpha}y_2$	$-i\sqrt{\alpha}y_3$
x_1	$-i\sqrt{\alpha}x_1$	0	$2i\sqrt{\alpha}y_3$	$-2i\sqrt{\alpha}\beta y_2$	$2i\sqrt{\alpha}\beta h$	0	0
x_2	$-i\sqrt{\alpha}x_2$	$-2i\sqrt{\alpha}y_3$	0	$2i\sqrt{\alpha}\gamma y_1$	0	$2i\sqrt{\alpha}\gamma h$	0
x_3	$-i\sqrt{\alpha}x_3$	$2i\sqrt{\alpha}\beta y_2$	$-2i\sqrt{\alpha}\gamma y_1$	0	0	0	$2i\sqrt{\alpha}\beta\gamma h$
y_1	$i\sqrt{\alpha}y_1$	$-2i\sqrt{\alpha}\beta h$	0	0	0	$-2i\sqrt{\alpha}x_3$	$2i\sqrt{\alpha}\beta x_2$
y_2	$i\sqrt{\alpha}y_2$	0	$-2i\sqrt{\alpha}\gamma h$	0	$2i\sqrt{\alpha}x_3$	0	$-2i\sqrt{\alpha}\gamma x_1$
y_3	$i\sqrt{\alpha}y_3$	0	0	$-2i\sqrt{\alpha}\beta\gamma h$	$-2i\sqrt{\alpha}\beta x_2$	$2i\sqrt{\alpha}\gamma x_1$	0

As any Lie algebra appears from an associative algebra by taking commutators, the natural approach to the enveloping algebra of $M(\alpha, \beta, \gamma)$ is imposing alternativity. We recall that an algebra A is *alternative* if $x(xy) = x^2y$ and $(yx)x = yx^2$, for $x, y \in A$.

We want to describe the product in the alternative algebra $U(M(\alpha,\beta,\gamma))$. We consider the set of generators $1, h, x_1, x_2, x_3, y_1, y_2, y_3$ and the multiplication xy satisfying the relation: $xy - yx = [x, y]$, where the commutator $[,]$ is the product in $M(\alpha,\beta,\gamma)$. To make computations, we consider the following relations, that hold in any alternative algebra (see [5]):

$$(x, y, z) = \frac{1}{6}J(x, y, z), \tag{3.1}$$

$$[x, y] \circ (x, y, z) = 0, \qquad \forall\, x, y, z \in U(M(\alpha,\beta,\gamma)), \tag{3.2}$$

where, as usual, $J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y]$ is the Jacobian, $(x, y, z) = (xy)z - x(yz)$ is the associator and $x \circ y = xy + yx$ is the Jordan or symmetric product.

Using the relations above, by means of straightforward calculations, we get the table of the multiplication in $U(M(\alpha,\beta,\gamma))$. For example, equation (3.2) for x_1, x_2, x_3 gives $0 = [x_1, x_2] \circ (x_1, x_2, x_3) = (2i\sqrt{\alpha}y_3) \circ (\frac{2}{3}\alpha\beta\gamma h)$. Since $\alpha\beta\gamma \neq 0$, we have that $y_3 \circ h = 0$. Since we know that $[h, y_3] = -i\sqrt{\alpha}y_3$, we conclude that $hy_3 = -i\frac{\sqrt{\alpha}}{2}y_3 = -y_3h$.

We present the product in $U(M(\alpha,\beta,\gamma))$ in the following table:

	h	x_1	x_2	x_3	y_1	y_2	y_3
h	0	$i\frac{\sqrt{\alpha}}{2}x_1$	$i\frac{\sqrt{\alpha}}{2}x_2$	$i\frac{\sqrt{\alpha}}{2}x_3$	$-i\frac{\sqrt{\alpha}}{2}y_1$	$-i\frac{\sqrt{\alpha}}{2}y_2$	$-i\frac{\sqrt{\alpha}}{2}y_3$
x_1	$-i\frac{\sqrt{\alpha}}{2}x_1$	0	$i\sqrt{\alpha}y_3$	$-i\sqrt{\alpha}\beta y_2$	$i\sqrt{\alpha}\beta h$	0	0
x_2	$-i\frac{\sqrt{\alpha}}{2}x_2$	$-i\sqrt{\alpha}y_3$	0	$i\sqrt{\alpha}\gamma y_1$	0	$i\sqrt{\alpha}\gamma h$	0
x_3	$-i\frac{\sqrt{\alpha}}{2}x_3$	$i\sqrt{\alpha}\beta y_2$	$-i\sqrt{\alpha}\gamma y_1$	0	0	0	$i\sqrt{\alpha}\beta\gamma h$
y_1	$i\frac{\sqrt{\alpha}}{2}y_1$	$-i\sqrt{\alpha}\beta h$	0	0	0	$-i\sqrt{\alpha}x_3$	$i\sqrt{\alpha}\beta x_2$
y_2	$i\frac{\sqrt{\alpha}}{2}y_2$	0	$-i\sqrt{\alpha}\gamma h$	0	$i\sqrt{\alpha}x_3$	0	$-i\sqrt{\alpha}\gamma x_1$
y_3	$i\frac{\sqrt{\alpha}}{2}y_3$	0	0	$-i\sqrt{\alpha}\beta\gamma h$	$-i\sqrt{\alpha}\beta x_2$	$i\sqrt{\alpha}\gamma x_1$	0

Now, we can prove our result:

Proposition 3.1. *The algebra $U(M(\alpha,\beta,\gamma))$ defined by the multiplication table above is alternative if and only if it is trivial.*

Proof: Since $U(M(\alpha,\beta,\gamma))$ is alternative, we have that $h^2x_1 = h(hx_1) = h(i\frac{\sqrt{\alpha}}{2}x_1) = -\frac{\alpha}{4}x_1$. Since $h^2 = 0$ we conclude that $\alpha = 0$. ■

So we can summarize the behavior of universal enveloping algebras of simple Malcev algebras as follows:

Corollary 3.2. *Let M be a simple Malcev algebra of finite dimension over a ground field $\mathbb{K}$ of characteristic $\neq 2, 3$. If M is a simple Lie algebra then there exists a suitable associative algebra A such that $M \subset A^-$. If $M = M(\alpha, \beta, \gamma)$ is a 7-dimensional simple (non-Lie) Malcev algebra then there is no alternative algebra A such that $M \subset A^-$.*

Acknowledgement We thank to Professor Helena Albuquerque, for suggesting us this construction of the universal enveloping algebra for simple Malcev algebras, as well for helpful corrections which allowed the elaboration of the present publication.

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