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MULTIPLICITY-FREE SKEW SCHUR FUNCTIONS WITH INTERVAL SUPPORT

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Abstract: It is known that the Schur expansion of a skew Schur function runs over the interval of partitions, equipped with dominance order, defined by the least and the most dominant Littlewood-Richardson filling of the skew-shape. We characterize skew Schur functions (and therefore the product of two Schur functions) which are multiplicity–free and the resulting Schur expansion runs over the whole interval of partitions, *i.e.*, skew Schur functions having Littlewood–Richardson coefficients always equal to 1 over the full interval.

Keywords: Skew Schur functions, Littlewood–Richardson coefficients, multiplicity–free, dominance order, interval.
AMS Subject Classification (2000): 05A17, 05E05, 05E10, 20C30.

1. Introduction

A function which can be written as a non–negative linear combination of Schur functions is said to be Schur–positive. Skew Schur functions and, in particular, the product of two Schur functions are examples of Schur–positive functions.

In [21] it is completely classified the products of Schur functions that are multiplicity-free, i.e., products for which every coefficient in the resulting Schur function expansion is either 0 or 1, which is equivalent, when the considered Schur functions allow only finitely many variables, to a classification of all multiplicity-free tensor products of irreducible representations of $\mathrm{GL}(n)$ or $\mathrm{SL}(n)$, or in other word, it is completely determined when the outer products of characters of the symmetric groups do not have multiplicity. Afterwards, in [4] it is achieved the analogous classification for Schur P-functions, which solved a similar problem for (projective) outer products of spin characters of double covers of the symmetric groups, and finally in [19] it is solved the multiplicity–free problem for the expansion of Schur P-functions in terms of the Schur basis, which in turn yields criteria for when an irreducible spin character of the twisted symmetric groups in the product of a

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basic spin character with an irreducible character of the symmetric groups is 0 or 1. Moreover, recently in [17] it is studied the problem of the Schur–positivity for ribbon Schur function differences, while the characterization of multiplicity–free skew Schur functions was solved in [10, 23]. Furthermore, using a different combinatorial model, namely the hive model, in [7] results similar to that ones investigated in [10, 21, 23] are obtained.

We remark that multiplicity–free representations have many applications, typically based on the fact that their centralizer algebras are commutative, or that their irreducible decompositions are canonical, see the survey article [12], as well as, e.g. [22], and the references therein, and they are also closely related to Gelfand pairs, see e.g. [6, 24] and the references therein.

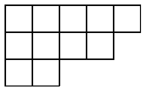
If A is a skew shape, the support $\mathrm{supp}(A)$ of the skew Schur function s_A is the set of the conjugate of those partitions ν for which there exists an Littlewood-Richardson (LR for short)–filling of A with content ν; equivalently, the Schur function s_ν appears with nonzero coefficient in the linear expansion of s_A in terms of Schur functions. The dominance order "$\preceq$" on partitions has been used in the study of Schur functions to prove that the monomial $x^\mu = x_1^{\mu_1} x_2^{\mu_2} \dots$ occurs in s_λ if and only if $\mu \preceq \lambda$, see [13, 14], and to deduce necessary conditions on the support of a skew Schur function s_A, namely, that the LR–filling contents of the skew shape A vary respectively between the ones defined by the least and the most dominant LR–fillings of A, see [1, 16, 26]. The set of the conjugates of all LR filling contents of A is an interval equipped with the dominance order which we call Schur interval of A. The lower bound of that interval is the partition of the column lengths of A sorted by decreasing order, denoted $\mathbf{w}$, and the upper bound is defined similarly considering the conjugate partition of the row lengths of A, denoted $\mathbf{n}$.

In this paper we characterize all the skew Schur functions (and therefore product of two Schur functions) which are both multiplicity–free and such that the resulting Schur function expansion runs over the whole Schur interval, i.e. skew Schur functions having Littlewood–Richardson coefficients always equal to 1 over the full interval. Equivalently, given the triple of partitions (μ, ν, λ), with $\mu \subseteq \lambda$, we answer the questions on under which conditions we have, $c^\lambda_{\mu\,\nu} = 1$ if and only if $\mathbf{w} \preceq \nu' \preceq \mathbf{n}$; or $c^\lambda_{\mu\,\nu} = 1$ if and only if $\mu \cup \nu \preceq \lambda \preceq \mu + \nu$. Furthermore, necessary conditions on the skew shape are given so that all Littlewood–Richardson coefficients are positive, that is, they are all equal or greater than 1 in the Schur interval. These conditions allow us to get rid of a lot of skew shapes which do not achieve the full interval. Our study is based on a procedure described in [1] which produces all the LR fillings from the least to the most dominant one.

The paper is organised in five sections. In the next, which in turn is divided in four subsections, we give necessary definitions regarding partitions, skew shapes and operations on them; the lattice of integer partitions with dominance order; Littlewood-Richardson tableaux using the notion of complete sequence of strings introduced in [1]; Schur, skew Schur functions, and, following the presentation given in [7], the classification of multiplicity-free skew Schur functions due to Thomas and Yong [23], and Gutschwager [10], and therefore the multiplicity-free Schur function products due to Stembridge [21]. In section three the notion of Schur interval and support of a skew diagram and therefore of skew Schur function, considering the conjugate of the content of an LR tableau, are introduced. Algorithm 1 in [1], the main tool of this work, is introduced. Some other related results in [1, 25, 11] are also recalled. In section four, divided in three subsections, general skew shapes whose support does not achieve the Schur interval are deduced, and the main result is provided. In Subsection 4.1, we stress Lemma 4.5 and Corollary 4.6 which will be extensively used in the remaining of the paper. In particular, we conclude that the support of a disconnected skew shape is equal to the Schur interval only if its components are ribbon shapes. (Although this not not sufficient.) In Subsection 4.2 a sequence of results are derived to be used, in the next subsection, to prove the main result, Theorem 4.16 which makes the list of all multiplicity-free skew shapes whose support is the Schur interval. A graphical list of all those shapes is provided at the end of this Subsection 4.3. We remark that the strategy in the proof of Theorem 4.16 follows closely the one used in Lemma 7.1 [7] as, roughly speaking, we have to *shrink* the multiplicity-free skew shapes in order to fit the full Schur interval. The last section is dedicated to the classification of the multiplicity-free Schur function products whose support is an interval. This is a consequence of Theorem 4.16.

2. Preliminaries

2.1. Partitions and diagrams. Let $\mathbb{N}$ denote the set of non negative integers. A weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)})$, whose sum is n is said to be a *partition* of n, denoted $\lambda \vdash n$. We say that n is the *size* of λ, denoted $|\lambda|$, and we call the λ_i the *parts* of λ and $\ell(\lambda)$ the *length* of λ. It is convenient to set $\lambda_k = 0$ for $k > \ell(\lambda)$. We also let 0 denote the partition with length 0. The set of all partitions of $n \in \mathbb{N}$ is denoted by P_n. If $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+j-1} = a$, we denote the sublist $\lambda_i, \ldots, \lambda_{i+j-1}$ by a^j, for $j \geq 0$. We identify a partition $\lambda \vdash n$ with its *Young diagram*, in "English notation", which we also denote by λ: containing λ_i left justified boxes in the ith row, for $1 \leq i \leq \ell(\lambda)$, and use the matrix-type coordinates to refer to the boxes. For example, if $\lambda = (5, 4, 2)$, which we often abbreviate to 542, the Young diagram is



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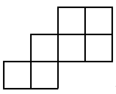
A partition with at most one part size is called a *rectangle*, and a partition with exactly two different part sizes is called a *fat hook*. A fat hook is said to be a *near rectangle* if it becomes a rectangle when one suppresses one row or one column, and just a *hook* if it becomes a one row rectangle when we suppress a column. If μ is another partition, we write $\lambda \supseteq \mu$ whenever μ is contained in λ as Young diagrams, or, equivalently, $\mu_i \leq \lambda_i$, for all $i \geq 1$. In this case, we define the skew diagram λ/μ which is obtained from λ by removing μ_i boxes from the ith row of λ, for $i = 1, \ldots, \ell(\mu)$. In particular, $\lambda/0 = \lambda$. The size of λ/μ is $|\lambda| - |\mu|$, denoted $|\lambda/\mu|$. A *connected skew-diagram* is a skew-diagram such that it can not be formed by two skew-diagrams so that the bottom left cell of one skew-diagram is immediately above and right of the top right cell of the other one, otherwise, it is said to be *disconnected*. A *ribbon shape* is a connected skew-diagram with no blocks of 2×2 squares. A skew-diagram forms a *vertical (respect. horizontal) strip* if it has no two boxes in the same row (respect. column). In particular, they are disconnected skew-diagrams.

Example 2.1. The skew diagram for $\lambda/\mu = (4, 4, 2)/(2, 1)$ is



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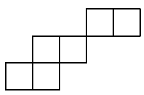
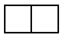
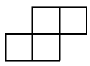
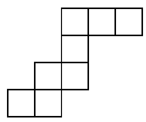
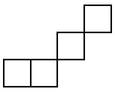
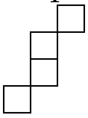
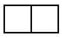
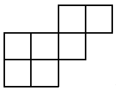
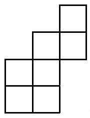
which is connected but it is not a ribbon shape. Instead,  is a disconnected skew-diagram with two components  and ; and 

is a ribbon shape. The following are respectively horizontal and vertical strips

 ,  .

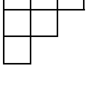
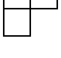
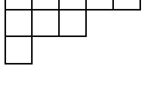
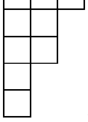
The π-*rotation* of a skew diagram λ/μ, denoted $(\lambda/\mu)^\pi$, is obtained rotating λ/μ through π radians. Denote by λ' the partition obtained by transposing the diagram of λ, called *conjugate partition* of λ, and set $(\lambda/\mu)' := \lambda'/\mu'$. If $\lambda \subseteq m^n$, then define its m^n-*complement* as $\lambda^* = (m^n)/\lambda$, where $\lambda^*_k = m - \lambda_{n-k+1}$ for $k = 1, 2, \ldots, n$. In particular, $(\lambda^*)^\pi$ is a partition.

Example 2.2. If $\lambda/\mu = (4,4,2)/(2,1)$, then the 4^3-complement of λ, the π-rotation, and the transposition are respectively

$\lambda^* =$ , $(\lambda/\mu)^\pi =$ , and $(\lambda/\mu)' =$ .

A partition $\lambda \subseteq m^n$ naturally defines a lattice path from the southwest to the northeast corner points of the rectangle. Following Thomas and Yong [23] we let the m^n-*shortness* of λ to be the length of the shortest straight line segment of the path of length $m + n$ from the southwest to northeast corner of m^n that separates λ from the π-rotation of λ^*. For instance, if $\lambda = (4,4,2)$ then the path from the southwest to the northeast corner of the 4^3 rectangle that borders λ is $(2,1,2,2)$, and therefore the 4^3-shortness of λ is 1.

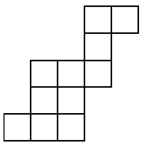
The *sum* $\lambda + \mu$ of two partitions λ and μ, is the partition whose parts are equal to $\lambda_i + \mu_i$, with $i = 1, \ldots, \max\{\ell(\lambda), \ell(\mu)\}$. Using conjugation, we define the *union* $\lambda \cup \mu := (\lambda' + \mu')'$. Equivalently, $\lambda \cup \mu$ is obtained by taking all parts of λ jointly with those of μ and rearranging all these parts in descending order.

Example 2.3. Let $\lambda =$  and $\mu =$ . Then, $\lambda + \mu =$  and $\lambda \cup \mu =$  .

Fix a positive integer n, and let λ and μ be two partitions with length $\leq n$. The *product* $\lambda^\pi \bullet_n \mu$ of two partitions λ and μ is defined as

$$(\lambda_1 + \mu_1, \ldots, \lambda_1 + \mu_n)/(\lambda^*)^\pi,$$

where $\lambda^* = \lambda_1^n/\lambda$. (When it is clear from the context we shall avoid in the notation $\bullet_n$ the subindex n.) Graphically, place λ^π in the southeast corner of the rectangle $\lambda_1 \times n$, and place μ in the northwest corner of the rectangle $\mu_1 \times n$. Then, form the rectangle $(\lambda_1 + \mu_1) \times n$ by gluing together the rectangles $\lambda_1 \times n$ and $\mu_1 \times n$ in this order. The outcome diagram is a connected skew-diagram when $n < \ell(\lambda) + \ell(\mu)$ and a disconnected one otherwise. As illustrated below with $\lambda = (3, 2^2, 0^2)$ and $\mu = (2, 1^2, 0^2)$, we obtain $\lambda^\pi \bullet_5 \mu = (5, 4^3, 3^2)/(3^2, 1^2)$,

$\lambda^\pi \bullet \mu =$  .

2.2. Dominance order on partitions.

Definition 2.1. The *dominance order* on partitions $\lambda, \mu \vdash n$ is defined by setting $\lambda \preceq \mu$ if

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i,$$

for $i = 1, \ldots, \min\{\ell(\lambda), \ell(\mu)\}$.

$(P_n, \preceq)$ is a lattice with maximum element (n) and minimum element (1^n), and is self dual under the map which sends each partition to its conjugate, *i.e.* $(\lambda')' = \lambda$. Graphically, $\lambda \preceq \mu$ if and only if the diagram of λ is obtained by "lowering" at least one box in the diagram of μ. Clearly $\lambda \preceq \mu$ if and only if $\mu' \preceq \lambda'$. Moreover, μ *covers* λ, written as $\lambda \lhd \mu$, if and only if μ is obtained from λ by lifting exactly one box in the diagram of λ to the next available position such that the transfer must be from some λ_{i+1} to λ_i , or from λ'_{i-1} to λ'_i. The interval $[\lambda, \mu]$ denotes the set of all partitions ν such that $\lambda \preceq \nu \preceq \mu$. The chain $\lambda = \lambda^0 \preceq \lambda^1 \preceq \cdots \preceq \lambda^k = \mu$, $k \geq 0$, is said to be *saturated* if λ^i covers λ^{i-1}, for $i = 1, \ldots, k$ [5, 9].

2.3. Tableaux and Littlewood-Richardson tableaux. A *semi-standard Young tableau* (SSYT) T of shape λ/μ is a filling of the boxes in the diagram λ/μ with integers such that: (i) the entries of each row weakly increase when read from left to right, and (ii) the entries of each column strictly increase when read from top to bottom. The *reading word* w of a SSYT T is the word obtained by reading the entries of T from right to left and top to bottom [20]. If, for all positive integers i and j, the first j letters of w includes at least as many i's as $(i+1)'$s, then we say that w is a *lattice*. If α_i is the number of i's appearing in T, and therefore in w, then the sequence $(\alpha_1, \alpha_2, \ldots)$ is called the *content* of T, and of w. Clearly, the content of a lattice word is a partition. A SSYT T whose word is a lattice is said to be a Littlewood-Richardson tableau (LR tableau for short). Given the partition $m = (m_1, \ldots, m_s)'$ the set of all lattice words with content m is equal to the set of all shuffles of the s words $12 \ldots m_1$, $12 \ldots m_2, \ldots, 12 \ldots m_s$.

Example 2.4. The following is a SSYT of shape $\lambda/\mu = (5,4,2)/(3,1)$, content $m = (4,2,1) = (3,2,1,1)'$ and reading word $w = 1121132$:

			1	1
	1	1	2	
2	3			

.

The reading word $w = 1121132$ is a lattice word, and it is a shuffle of the four words 123, 12, 1 and 1. Therefore this SSYT is an LR tableau.

Taking into account the shuffle property of a lattice word, we give another characterization of an LR tableau that we shall rather use in this work. This is based on §3 of [1], especially Definitions 5 and 6 and Theorem 5, and we refer to it for proofs and further details.

Definition 2.2. Given a semi-standard tableau T, $S_k = (y_1, y_2, \ldots, y_k)$ is a *k–string* (or just string, for short, when there is no ambiguity) of T if $y_1 < \cdots < y_k$ and the rightmost box in row y_j is labeled with j; the corresponding strip, denoted by $st\,(S_k)$, is the union of all rightmost boxes in rows y_j, for all $j = 1, \ldots, k$.

We say that $S_k = (y_1, y_2, \ldots, y_k) \leq S_t = (z_1, z_2, \ldots, z_t)$ if $k \geq t$ and $y_j \leq z_j$ for all $j = 1, \ldots, t$.

We define in a recursive way $(S_{m_1}, S_{m_2}, \ldots S_{m_s})$ a *complete sequence of strings* of the tableau T having content $(m_1, m_2, \ldots, m_s)'$ (note that we are writing the content in terms of its conjugate) if S_{m_1} is a string of T and $(S_{m_2}, \ldots S_{m_s})$ is a complete sequence of strings of the tableau $T \setminus st\,(S_{m_1})$ (when this set is not empty) having content $(m_2, \ldots, m_s)'$.

In other word, $(S_{m_1}, S_{m_2}, \ldots S_{m_s})$ is a complete sequence of strings of T if S_{m_j} is a string for $T \setminus \{\bigcup_{k=1}^{j-1} st\,(S_{m_k})\}$ for all $j = 1, \ldots, s$.

Example 2.5.

$$T = \begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ & 2 & 3 & \\ 1 & 4 & & \\ 3 & & & \end{array}$$

is an LR tableau with content $(4,2,2,1) = (4,3,1,1)'$ and it admits $S_4 = (1,2,3,4) \leq S_3 = (1,3,5) \leq S_1 = (2) \leq S_1 = (4)$ as complete sequence of strings.

In fact, we have

$$\begin{aligned} T &= \begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ & 2 & 3 & \\ 1 & 4 & & \\ 3 & & & \end{array} \\ T \setminus st\,(S_4) &= \begin{array}{ccc} & & 1 \\ & 1 & \\ & 2 & \\ 1 & & \\ 3 & & \end{array} \\ (T \setminus st\,(S_4)) \setminus st\,(S_3) &= \begin{array}{cc} & 1 \\ 1 & \end{array} \\ ((T \setminus st\,(S_4)) \setminus st\,(S_3)) \setminus st\,(S_1) &= \begin{array}{c} 1 \end{array}. \end{aligned}$$

The following result holds, which is nothing but Theorem 5 in [1].

Proposition 2.1. *A semi-standard tableau with content $m = (m_1, m_2, \ldots, m_s)'$ is an LR tableau if and only if it has a complete sequence of strings $S_{m_1}, S_{m_2}, \ldots, S_{m_s}$; and, in particular, there is always one satisfying $S_{m_1} \leq S_{m_2} \leq \cdots \leq S_{m_s}$.*

■

2.4. Schur functions, skew Schur functions and free-multiplicity. Let Λ denote the ring of symmetric functions in the variables $x = (x_1, x_2, \ldots)$ over $\mathbb{Q}$, say. The Schur functions s_λ form an orthonormal basis for Λ, with respect to the Hall inner product, and may be defined in terms of SSYT by

$$s_\lambda = \sum_T x^T \in \Lambda, \tag{2.1}$$

where the sum is over all SSYT of shape λ and x^T denotes the monomial

$$x_1^{\#1's\ in\ T} x_2^{\#2's\ in\ T} \cdots .$$

Replacing λ by λ/μ in (2.1) gives the definition of the skew Schur function $s_{\lambda/\mu} \in \Lambda$, where now the sum is over all SSYT of shape λ/μ. For instance, the SSYT shown in the Example 2.4 above contributes with the monomial $x_1^4 x_2^2 x_3^1$ to $s_{542/31}$.

The product of two Schur functions s_μ and s_ν can be written as a positive linear combination of Schur functions by the *Littlewood-Richardson rule* which states

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda,$$

where the *Littlewood-Richardson coefficient* $c^\lambda_{\mu\nu}$ is the number of LR tableaux with shape λ/μ and content ν [15]. The Littlewood-Richardson coefficients can also be used to expand skew Schur functions $s_{\lambda/\mu}$ in terms of Schur functions:

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu.$$

If $c^\lambda_{\mu\nu}$ is 0 or 1 for all λ (resp. all ν), then we say that the product of Schur functions $s_\mu s_\nu$ (resp. the skew Schur function $s_{\lambda/\mu}$) is *multiplicity-free*.

The Littlewood-Richardson coefficients satisfy a number of symmetry properties [20], including:

$$c^\lambda_{\mu\nu} = c^\lambda_{\nu\mu} \quad \text{and} \quad c^\lambda_{\mu\nu} = c^{\lambda'}_{\mu'\nu'}. \tag{2.2}$$

Moreover, we have

$$s_\lambda = s_{\lambda^\pi} \quad \text{and} \quad s_{\lambda/\mu} = s_{(\lambda/\mu)^\pi}. \tag{2.3}$$

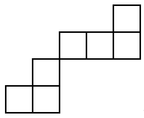
Another useful fact about skew Schur functions is that

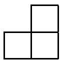
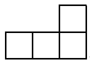
$$s_{\lambda/\mu} = s_{\tilde{\lambda}/\tilde{\mu}},$$

where $\tilde{\lambda}/\tilde{\mu}$ is the skew Young diagram obtained from λ/μ by deleting any empty rows and any empty columns. A skew Schur function without empty rows or empty columns is said to be *basic* [7]. Therefore, the previous identity allows each skew Schur function to be expressed as a basic skew Schur function.

If λ/μ is not connected, and consists of two components A and B, and may themselves be either Young diagrams or skew Young diagrams, then ([20, 7])

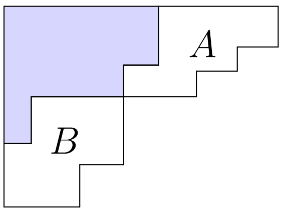
$$s_{\lambda/\mu} = s_A s_B = s_B s_A.$$

Example 2.6. If $\lambda/\mu = 5522/421 =$ , we have the disconnected components

$A =$  and $B =$ .

Therefore, $s_{5522/421} = s_{22/1} s_{33/2}$.

Any product $s_A s_B$ of skew Schur functions s_A and s_B is again a skew Schur function, as the figure below makes evident,



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In particular, a product of Schur functions $s_\mu s_\nu$ may be seen as a skew Schur function with $A = \mu$ and $B = \nu$ in the previous picture.

For the following characterization of the basic multiplicity-free skew Schur function, jointly due to Gutschwager and to Thomas and Yong, we follow [7].

Theorem 2.2 (Gutschwager [10], Thomas and Yong [23])**.** *The basic skew Schur function $s_{\lambda/\mu}$ is multiplicity-free if and only if one or more of the following is true:*

R0 μ or λ^* is the zero partition 0;
R1 μ or λ^* is a rectangle of m^n-shortness 1;
R2 μ is a rectangle of m^n-shortness 2 and λ^* is a fat hook (or vice versa);
R3 μ is a rectangle and λ^* is a fat hook of m^n-shortness 1 (or vice versa);
R4 μ and λ^* are rectangles;

where λ^* is the m^n-complement of λ with $m = \lambda_1$ and $n = \lambda'_1$.

In particular, for partitions μ and ν, the product $s_\mu s_\nu$ of Schur functions is a skew Schur function, and we get the following characterization of the multiplicity-free product of skew-Schur functions, due to Stembridge, as a corollary of the above theorem.

Corollary 2.3 (Stembridge [21]). *The Schur function product $s_\mu s_\nu$ is multiplicity-free if and only if one or more of the following is true:*

P0 μ or ν is the zero partition 0;
P1 μ or ν is a one-line rectangle;
P2 μ is a two-line rectangle and ν is a fat hook (or vice versa);
P3 μ is a rectangle and ν is a near rectangle (or vice versa);
P4 μ and ν are rectangles.

3. The Schur interval

Given partitions $\mu \subseteq \lambda$, let A denote the skew-diagram λ/μ. We associate to the skew-diagram A two partitions: $rows(A)$ obtained by sorting the row lengths of A into weakly decreasing order, and similarly $cols(A)$ by sorting column lengths [1, 16]. It is known that $cols(A) \preceq rows(A)'$ [14, 13, 26, 1, 16]. For abbreviation, we write $\mathbf{w} := cols(A)$ and $\mathbf{n} := rows(A)'$. (When there is danger of confusion we write respectively $\mathbf{w}(A)$ and $\mathbf{n}(A)$.) If A consists of two disconnected partitions ϕ and θ then $\mathbf{w} = \phi \cup \theta$ and $\mathbf{n} = \phi + \theta$.

Definition 3.1. The interval $[\mathbf{w}, \mathbf{n}] = \{\nu \in \mathbf{P}_{|\mathbf{A}|} : \mathbf{w} \preceq \nu \preceq \mathbf{n}\}$ is called the *Schur interval* of A.

The Schur interval of A and A^π is the same, and due to the equivalence $\mathbf{w} \preceq \nu \preceq \mathbf{n}$ if and only if $\mathbf{n}' \preceq \nu' \preceq \mathbf{w}'$, the Schur interval of A' is $[\mathbf{n}', \mathbf{w}']$.

Suppose $\mathbf{n} = (n_1, \ldots, n_s)$. We may decompose $A = \lambda/\mu$ into a sequence of n_i-vertical strips, for all i, as follows:

1. First consider the n_1-vertical strip V_1 formed by the rightmost box of each row in A, and let $A \setminus V_1$ be the skew diagram obtained by removing that strip, described in partitions with λ^1/μ.
2. Repeat the previous step with the diagram $A \setminus V_1$.

Let $V = (V_1, \ldots, V_s)$ denote the sequence of vertical strips obtained by the previous process, called the *V-sequence of A*. To construct the strip V_i we subtract the rightmost box of each row in λ^{i-1}/μ, and to describe in partitions the new skew shape, we put $A \setminus (V_1 \cup V_2 \cup V_i) = \lambda^i/\mu$, for $i = 1, \ldots s-1$, with $\lambda^0 := \lambda$. This means that each entry of $\mathbf{n}$ is obtained by successively pushing up some boxes in each entry of $\mathbf{w}$: n_1 is the number of non empty rows of A; n_2 is the number of rows of A of length at least 2, …, and n_s is the number of rows of length s. Hence, n_s is the number of rows of the strip

$V_s = A \setminus (V_1 \cup \cdots \cup V_{s-1})$, consisting of the leftmost boxes in each row of A with longest length $s = \ell(\mathbf{n})$. Each vertical strip V_i intersects the rows of A with longest length s, and, therefore, $\ell(\mathbf{w}) \geq \ell(\mathbf{n})$.

We shall denote the minimum and the maximum of $\mathrm{supp}(\lambda^i/\mu)$ respectively by $\mathbf{w}^i$ and $\mathbf{n}^i$, for $i = 1, \dots, s-1$. If $\mathbf{w} = (w_1, \dots, w_r)$, then $r = \ell(\mathbf{w})$ is the number of non empty columns of A, and w_1 is the length of the longest column of A.

Definition 3.2. Given the skew-diagram A, we define the support *supp*(A) of A or of s_A to be the set of those partitions ν' for which s_ν appears with nonzero coefficient when we expand s_A in terms of Schur functions. Equivalently,

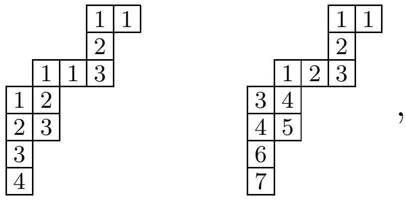
$$supp(A) = \{\nu' : c^{\lambda}_{\mu\nu} > 0\}.$$

Notice that we have defined the support of A in terms of the conjugate of the contents of the LR fillings of A. From [14, 13, 26, 1, 16] we know that $\nu \in supp(A)$ only if $cols(A) \preceq \nu \preceq rows(A)'$ and therefore $\mathrm{supp}(A) \subseteq [\mathbf{w}, \mathbf{n}]$. Thanks to the rotation symmetry (2.3), the support of a skew diagram A equals the support of $(A)^\pi$. Also, by the conjugation symmetry of the Littlewood-Richardson coefficients (2.2) and the equivalence $\lambda \preceq \mu \Leftrightarrow \mu' \preceq \lambda'$, we know that $\nu \in \mathrm{supp}(A)$ if and only if the $\nu' \in \mathrm{supp}(A')$.

Moreover it is known that $\mathbf{w}$ and $\mathbf{n}$ are in $\mathrm{supp}(A)$ and the coefficients $c^{\lambda}_{\mu,\mathbf{w}'}$ and $c^{\lambda}_{\mu,\mathbf{n}'}$ are both equal to 1 (see [1, 16]). The only LR tableau with shape A and content $\mathbf{w}'$ is obtained by filling the boxes of each column, from top to bottom, with the integers $1, 2, \dots$. To describe the only LR tableau with shape A and content $\mathbf{n}'$, let $V = (V_1, \dots, V_s)$ be the V-sequence of A. The LR tableau with shape A and content $\mathbf{n}'$ is obtained by filling each vertical strip V_i with the integers $1, \dots, n_i$, $i = 1, \dots, s$.

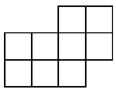
Although $\mathbf{w}'$ and $\mathbf{n}'$ are respectively the most and the least dominant LR filling contents of A, since $\mathbf{w}$ and $\mathbf{n}$ are the minimum and maximum of the *supp*(A), we will refer to the corresponding LR fillings of A as the minimum and maximum ones. The reason for this terminology comes from the fact that the lattice words with those contents $\mathbf{w}'$ and $\mathbf{n}'$ are respectively shuffles of the words $12\dots w_i$ and $12\dots n_i$, $i \geq 1$, and the partitions defined by their lengths satisfy $\mathbf{w} = (w_1, \dots w_r) \preceq \mathbf{n} = (n_1, \dots, n_s)$.

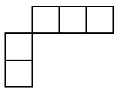
Example 3.1. The LR fillings of $A = 5442211/331$ with the least and most dominant conjugate contents respectively $\mathbf{w} = cols(A)$ and $\mathbf{n} = rows(A)'$, are



where $\mathbf{w} = (4,3,3,1,1) \preceq \mathbf{n} = (7,4,1)$. The lattice word of content $\mathbf{w}'$ is a shuffle of the words 1234, 123, 123, 1, 1 with lengths given by $\mathbf{w}$; and the lattice word of content $\mathbf{n}'$ is a shuffle of the words 1234567, 1234, 1 with lengths given by $\mathbf{n}$.

The example below shows that in general $\text{supp}(A) \subsetneq [\mathbf{w}, \mathbf{n}]$.

Example 3.2. (1) The support of $A =$  is only the set $\{\mathbf{w} = 3222, \mathbf{n} = 3321\}$ and therefore $s_A = s_{441} + s_{432}$. In this case, $supp(A) = \{\mathbf{w} = 3222, \mathbf{n} = 3321\} = [\mathbf{w}, \mathbf{n}]$.

(2) The Schur interval of $A =$  is the chain $\mathbf{w} = 2111 \prec 221 \prec \mathbf{n} = 311$ while $\text{supp}(A) = \{21^3, 31^2\} \subsetneq [\mathbf{w}, \mathbf{n}]$.

The next algorithm provides a procedure to construct systematically all partitions in $\text{supp}(\lambda, \mu) \cap [\mathbf{w}, \mathbf{n}]$. Along the process, all LR tableaux of shape λ/μ are also exhibited. We remark that the algorithm is essentially a rephrasing of Lemma 3 and 4 and Algorithm 4 in §4 of [1], so we refer there for further details and proofs.

Algorithm 1.

Procedure 1.
Input of the procedure: tableau T of shape λ/μ and content $m = (m_1, m_2, \dots, m_s)'$ (therefore it admits a complete sequence of strings $(S_{m_1}, S_{m_2}, \dots S_{m_s})$ since Proposition 2.1).

If for all $j = 1, \dots, s$ $st\left(S_{m_j}\right)$ achieves (i.e. intersects) all rows of the tableau $T \setminus \{\bigcup_{k=1}^{j-1} st\,(S_{m_k})\}$ then *Output of the procedure:* T (i.e. the procedure does nothing)

else

Begin

$t := \min\{j = 1, \ldots, s$ s. t. $st\left(S_{m_j}\right)$ does not achieve all rows of the tableau $T \setminus \{\bigcup_{k=1}^{j-1} st\,(S_{m_k})\}\}$.

$t_1 := \min\{j$ s.t. the row j in the tableau $T \setminus \{\bigcup_{k=1}^{t-1} st\,(S_{m_k})\}$ is not achieved by $st\,(S_{m_t})\}$.

$X :=$ set of boxes made of the rightmost (with respect to the tableau $T \setminus \{\bigcup_{k=1}^{t-1} st\,(S_{m_k})\}$) box in rows $(t_1 \bigcup \{j > t_1$ s.t. $j \in S_{m_t}\})$
(i.e. $X :=$ rightmost (with respect to the tableau $T \setminus \{\bigcup_{k=1}^{t-1} st\,(S_{m_k})\}$) box in
row $t_1 \bigcup (st\,(S_{m_t})$ below row $t_1))$.

Output of the procedure: $T_1 :=$ tableau obtained by T increasing by one
the filling of the set X.

End

Input of the algorithm: $\mu \subseteq \lambda$.

$\mathbf{n} := (\mathbf{rows}(\lambda/\mu))'$.

$\mathbf{w} := \mathbf{cols}(\lambda/\mu)$.

$n :=$ number of columns of λ/μ.

for $i = 0, 1, \ldots, n-1$ do $(\lambda/\mu)^{n-i} :=$ the skew diagram defined by the $n-i, n-i+1, \ldots, n$ columns of λ/μ.

$T^{\{0\}} :=$ the LR tableau with shape λ/μ and content $\mathbf{w}'$.

$T^{[n]} :=$ the LR tableau of shape $(\lambda/\mu)^n$.

$i := 0$.

Repeat

Begin

To each LR tableau $T \in T^{[n-i]}$, adjoin to the leftmost column of T the $(n-i-1)$-th column of $T^{\{0\}}$ such that the LR tableau obtained is of shape $(\lambda/\mu)^{n-i-1}$.

Apply the Procedure to construct all LR tableaux of shape $(\lambda/\mu)^{n-i-1}$ containing $T \in T^{[n-i]}$, and denote this set by $T^{[n-i-1]}$.

Add the remaining columns of $T^{\{0\}}$ to each LR tableau $T^{[n-i-1]}$, obtaining a set, denoted by $T^{\{i+1\}}$, of LR tableaux of shape λ/μ.

Output of the algorithm: set $T^{\{i+1\}}$.

$i := i + 1$.

End

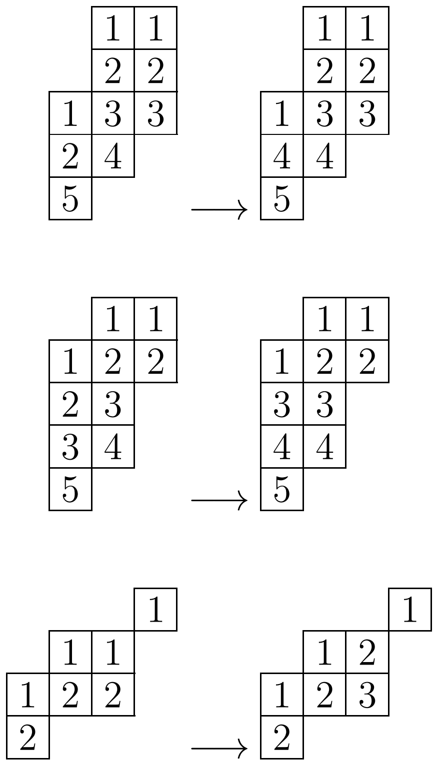
until $i = n$.

This algorithm produces a sequence of sets of LR tableaux of shape λ/μ

$$T^{\{0\}} \subseteq T^{\{1\}} \subseteq T^{\{2\}} \subseteq \cdots \subseteq T^{\{n\}}, \tag{3.1}$$

such that if G is in $T^{\{i\}}$ with conjugate content γ, and B is in $T^{\{i-1\}}$ with conjugate content β then $\beta \preceq \gamma$, for all $i = 0, \ldots, n$.

Example 3.3. To make things clear we present here some instances of application of the Procedure 1.



Note that in the first two instances the conjugate content of the output covers in the dominance order the conjugate content of the input, whereas in the third one it does not.

As an easy consequence of the algorithm above, we exhibit a chain in *supp*(A), with respect to the dominance order, that goes from $\mathbf{w} = (w_1, \ldots, w_r)$ to $\mathbf{n} = (n_1, \ldots, n_s)$. Start with the minimum LR filling of A, that is, the only filling of A with content $\mathbf{w}'$. If one fills the vertical strip V_1 with $12 \ldots n_1$

then λ^1/μ has the minimum LR filling, and one gets an LR filling of A with conjugate content

$$\sigma^1 := (n_1) \cup \mathbf{w}^1,$$

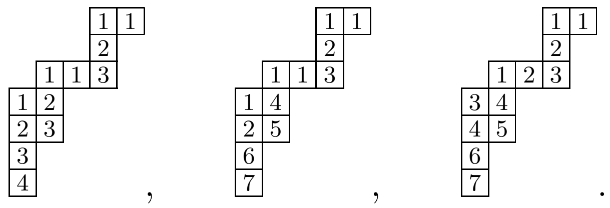
where $\mathbf{w}^1$ is the minimum of the $supp(\lambda^1/\mu)$. From Algorithm 1 one knows that $\mathbf{w} \preceq \sigma^1 \preceq \mathbf{n}$. (It worths to notice that the partition σ^1 is obtained by subtraction from the entries $2, \dots, r$ of $\mathbf{w}$, and adding those nonnegative quantities to the first entry. Graphically, those operations correspond to lift the rightmost box in some of the rows of A, to the first row, and thus one has $\mathbf{w} \preceq \sigma^1$.) One may now repeat the above argument with the skew diagram λ^1/μ and we are led a to a chain of partitions

$$\mathbf{w} \preceq \sigma^1 \preceq \sigma^2 \preceq \cdots \preceq \sigma^{s-2} \preceq \sigma^{s-1} = \mathbf{n} \tag{3.2}$$

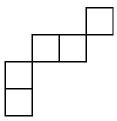
where $\sigma^i := (n_1, \cdots, n_i) \cup \mathbf{w}^i \in \text{supp}(\lambda/\mu)$ and $\mathbf{w}^i$ the minimum of $\text{supp}(\lambda^i/\mu)$, for $1 \leq i \leq s-1$. One has $\mathbf{w}^{i-1} \preceq (n_i) \cup \mathbf{w}^i$, for $1 \leq i \leq s-1$, with $\mathbf{w}^0 := \mathbf{w}$.

The next example illustrates this construction.

Example 3.4. Consider the skew diagram A in Example 3.1 and its sequence $V = (V_1, V_2, V_3)$ of vertical strips. Start with the minimum LR filling of A where $\mathbf{w} = (4, 3, 3, 1, 1)$, and apply Algorithm 1 to produce LR tableaux such that: the vertical strip V_1 is filled with the word 1234567; and the vertical strips V_1 and V_2 are filled with 1234567 and 1234 (hence V_3 is filled with 1) respectively

$T^{\{0\}} =$ 

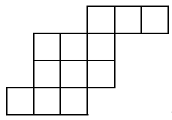
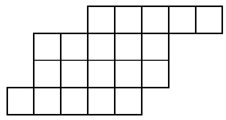
The conjugate contents are respectively: $\mathbf{w}$; $\sigma^1 = (72111) = (7) \cup (2111)$, with $\mathbf{w}^1 = 2111$ the conjugate content of the minimum LR filling of $A \setminus V_1 =$



; and $\mathbf{n} = (741) = (74) \cup (1)$, with $\mathbf{w}^2 = (1)$ the conjugate content of the minimum LR filling of $A \setminus (V_1 \cup V_2) = V_3 = \square$. One has $\mathbf{w} \prec \sigma^1 \prec \sigma^2 = \mathbf{n}$.

Definition 3.3. The skew diagrams A and B are said to be equal up to a block of maximal depth if for some $x \in \mathbb{N}$, and $n \geq \ell(A), \ell(B)$, $A = u^\pi \bullet_n v$ and $B = [u^\pi + (x^n)] \bullet_n v$, with u and v partitions. Similarly they are said to be equal up to a block of maximal width if $A = (u^\pi \bullet_n v)'$ and $B = ([u + (x^n)]^\pi \bullet_n v)'$.

Example 3.5. Let $u = (3, 2, 2, 0)$, $v = (3, 1, 1, 0)$ and $n = 4$. Then

$A = u^\pi \bullet v =$  , $B = [u + (2^4)]^\pi \bullet v =$  ,

are equal up to a block of maximal depth (2^4).

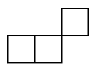
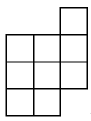
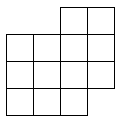
Lemma 3.1. *Let A, B be skew diagrams, and let u, v be partitions with $\ell(v), \ell(v) \leq n$. Let $x \in \mathbb{N}$.*

(a) *If $A = u^\pi \bullet_n v$ and $B = [u^\pi + (x^n)] \bullet_n v$, then $[\mathbf{w}(B), \mathbf{n}(B)] = [n^x \cup \mathbf{w}(A), n^x \cup \mathbf{n}(A)]$. If $A = (u^\pi \bullet_n v)'$ and $B = ([u^\pi + (x^n)] \bullet_n v)'$, then $[\mathbf{w}(B), \mathbf{n}(B)] = [x^n + \mathbf{w}(A), x^n + \mathbf{n}(A)]$.*

(b) ([1], Lemma 5) *If $A = u^\pi \bullet_n v$ and $B = u^\pi + (x^n) \bullet_n v$, then $c \in \operatorname{supp}(B)$ if and only if $c = b \cup (n^x)$ with $b \in \operatorname{supp}(A)$. If $A = (u^\pi \bullet_n v)'$ and $B = (u^\pi + (x^n) \bullet_n v)'$, then $c \in \operatorname{supp}(B)$ if and only if $c = b + (x^n)$ with $b \in \operatorname{supp}(A)$.*

Note that this follows from the conjugation symmetry and Algorithm 1. In the application of this algorithm, the rectangle (x^n) of length n is always filled with x words $12 \cdots n$. Therefore, if c is an element of the support of B then it has the form $b \cup (n^x)$ for some partition b in the support of A.

Example 3.6. The following skew diagrams are equal up to a block of maximal depth and maximal width

$A =$  , $B =$  , $C =$  ,

$\mathbf{w}(A) = 111 \lhd \mathbf{n}(A) = 21$; $\mathbf{w}(B) = 333 = 2^3 + \mathbf{w}(A) \lhd \mathbf{n}(B) = 2^3 + \mathbf{n}(A) = 432$; $\mathbf{w}(C) = 4333 = 4^1 \cup \mathbf{w}(B) \lhd \mathbf{n}(C) = 4 \cup \mathbf{n}(B) = 4432$.

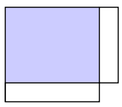
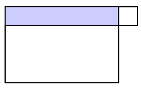
Next the skew shapes whose support has only one element, that is, $\mathbf{w} = \mathbf{n}$, are characterized, and, therefore, the skew Schur functions which are Schur functions. Note that, in (3.2), one has $\sigma^i = \mathbf{w}$, $1 \leq i \leq s - 1$, if and only if A or A^π is a partition.

Proposition 3.2. *Let A be a skew diagram and let u, v and ν be partitions. Then,*

(a) ([1], Theorem 3, 16) $\mathbf{w} = \mathbf{n} = \nu$ *if and only if* $A = \nu$ *or* $A = \nu^\pi$. *In this case,* $\operatorname{supp}(A) = \{\mathbf{w} = \mathbf{n}\} = [\mathbf{w}, \mathbf{n}]$.

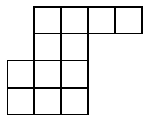
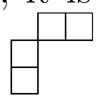
(b) (van Willigenburg [25]) $s_A = s_\nu$ *if and only if* $A = \nu$ *or* $A = \nu^\pi$.

In the next proposition we characterize the skew diagrams A whose support has only two elements, that is, $\operatorname{supp}(A) = \{\mathbf{w}, \mathbf{n}\}$, and, in particular, those whose Schur interval has only two elements, $[\mathbf{w}, \mathbf{n}] = \{\mathbf{w}, \mathbf{n}\}$. This was shown in [1] by means of Algorithm 1. In [11] such a description is also given. Consider the skew diagrams: $F1 = ((a+1)^x, a)/(a^x)$, and $\tilde{F}1 = (a+1, a^x)/(a)$, a, $x \geq 1$:

$F1$  $\widetilde{F}1$  . (3.3)

Proposition 3.3. ([1], Theorem 16) *Let A be a skew diagram with* $\mathbf{w} \precneqq \mathbf{n}$. *Then,* $\operatorname{supp}(A) = \{\mathbf{w}, \mathbf{n}\}$ *if and only if, up to a π-rotation/ or conjugation and up to a block of maximal width or maximal depth, A either is an $F1$ or an $\widetilde{F}1$ configuration. In particular, if A is a disconnected two column (row) diagram, with one connected component a single box, then one has* $\mathbf{w} \lhd \mathbf{n}$ *and* $\operatorname{supp}(A) = [\mathbf{w}, \mathbf{n}] = \{\mathbf{w}, \mathbf{n}\}$.

Note that, when applying Algorithm 1 to $F1$, we find that the only string in the minimum LR filling of $F1$ that we can stretch is the string of length x, which can only be stretched in one way, which gives rise to the maximum LR filling. Therefore, the support of $F1$ is formed only by $\mathbf{w} = (x, 1^a) \preceq \mathbf{n} = (x+1, 1^{a-1})$. When a, $x \geq 2$, the partition $\mathbf{n}$ does not cover $\mathbf{w}$, for instance $\xi = (x, 2, 1^{a-2}) \in [\mathbf{w}, \mathbf{n}]$. The proof is similar for $\tilde{F}_1$, considering its π-rotation.

Example 3.7. Proposition 3.3 can be used together with Lemma 3.1 to show that the support of the skew diagram $\lambda/\mu = (5, 3^3)/(1^2) =$  is not the entire Schur interval. By Lemma 3.1, it is enough to consider the support of the simple skew diagram $\alpha/\beta =$  obtained from λ/μ by removing the block (2^4). Since the resulting diagram is the π-rotation of an

$F1$ configuration with $a, x \geq 2$, the support of λ/μ is strictly contained in its Schur interval.

4. Full interval linear expansion of free-multiplicity skew Schur functions

In the rest of the paper, the general philosophy of the application of Algorithm 1 to a skew diagram A consists in the prolongation of its strings, starting with the minimum LR filling of A, in any possible way. We remark that in [1] it is characterized when in Algorithm 1 the partition corresponding to the content of the output covers, in the dominance order, the partition corresponding to the content of the input. Thus, when applying a step of the algorithm we can check whether the partition, corresponding to the new content, covers the preceding one. If not, then we have a suspicious interval that may contain a partition not in the support of A.

4.1. Bad configurations. We start this section with the analysis of some particular configurations of boxes such that their appearance in a skew diagram A implies that $\text{supp}(A) \subsetneq [\mathbf{w}, \mathbf{n}]$.

Lemma 4.1. *If A is a skew-diagram and the support of $A \setminus V_1$ is not the entire Schur interval, then neither is the support of A.*

Proof: Let $[\mathbf{w}^1, \mathbf{n}^1]$ be the Schur interval of $A \setminus V_1$ and $\mathbf{w}^1 \preceq \xi \preceq \mathbf{n}^1$ such that $\xi \notin \text{supp}(A \setminus V_1)$. Then $\mathbf{w} \preceq (n_1) \cup \xi \preceq \mathbf{n}$ and $(n_1) \cup \xi \notin \text{supp}(A)$ since the only way to put the string $n_1 \cdots 21$ in A is to fill the strip V_1 with it and what remains is $A \setminus V_1$. ■

We observe that if $\text{supp}(A \setminus V_1)$ attains the Schur interval this does not mean that the same happens to A, as one can see in the next example.

Lemma 4.2. *Let A be a skew diagram with two or more connected components. If there is a component containing a two by two block of boxes, then the support of A is not the entire Schur interval.*

Proof: Let $\mathbf{n} = (n_1, \ldots, n_s)$. Recall that $\mathbf{n}^{i-1} = (n_i, \ldots, n_s)$ is the maximum of $\text{supp}(A \setminus \bigcup_{k=1}^{i-1} V_k)$, $i = 2, \cdots s$, and that n_i is the number of rows of $A \setminus \bigcup_{k=1}^{i-1} V_k$, for all i. Since there is a 2 by 2 block in one of the connected components of A, there must exists a column in $A \setminus V_1$ whose length is at least 2. Let $\mathbf{w}^1 = (\overline{w}_1, \ldots, \overline{w}_\ell, 1^q)$, with $\overline{w}_\ell \geq 2$ for some $\ell \geq 1$ and $q \geq 0$.

Clearly $(n_2, \dots, n_s) \not\succcurlyeq (\overline{w}_1, \dots, \overline{w}_\ell - 1, 1^{q+1})$, and from (3.2) the partition

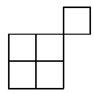
$$\sigma^1 = (n_1, \overline{w}_1, \dots, \overline{w}_\ell, 1^q) \in [\mathbf{w}, \mathbf{n}].$$

Note that $\ell(\mathbf{w}) \geq \ell(\mathbf{w}^1) + o$ where o is the number of components of A. Since A has at least two components, one has $\ell(\mathbf{w}) \geq \ell(\sigma) + 2$. Then, the partition

$$\xi := (n_1, \overline{w}_1, \dots, \overline{w}_\ell - 1, 1^{q+1})$$

clearly satisfy $\mathbf{w} \preceq \xi \preceq \sigma$. Moreover, since $(\overline{w}_1, \dots, \overline{w}_\ell - 1, 1^{q+1}) \precneqq \mathbf{w}^1$ it follows that $(\overline{w}_1, \dots, \overline{w}_\ell - 1, 1^{q+1}) \notin \mathrm{supp}(A \setminus V_1)$, and therefore we conclude that $\xi \notin \mathrm{supp}(A)$. ■

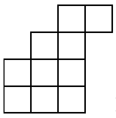
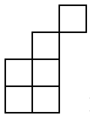
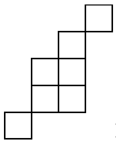
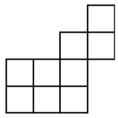
Corollary 4.3. *If A is a skew diagram with two or more components and the support of A is all the Schur interval $[\mathbf{w}, \mathbf{n}]$, then the components of A are ribbon shapes.*

Example 4.1. The support of the skew diagram $A =$  is not the entire Schur interval, since it has two connected components, and one of them has a 2 by 2 block. We may follow the proof of the previous lemma to get a partition in the Schur interval that does not belong to the support of A. Note that $\mathbf{w} = (2, 2, 1)$, $\mathbf{n} = (3, 2)$, $\mathbf{w}^1 = (2) = \mathbf{n}^1$, $\mathbf{w} \preceq \sigma^1 = (3, 2) = \mathbf{n}$, and $[\mathbf{w}, \mathbf{n}] = \{\mathbf{w} = 221, \xi = 311, \mathbf{n} = 32\}$ with $\xi \notin \mathrm{supp}(A) = \{\mathbf{w}, \mathbf{n}\}$. Note also that $A \setminus V_1 = V_2$ and V_2 is a column of A.

Corollary 4.4. *Let A be a skew diagram such that $\ell(\mathbf{w}) > \ell(\mathbf{n}) = s$ (equivalently, it has no block of maximal width), and the strip V_s is a column of A of length greater than, or equal to 2. Then, the support of A is not $[\mathbf{w}, \mathbf{n}]$.*

Proof: Since $\ell(\mathbf{w}) > \ell(\mathbf{n}) = s$, A is not a partition, and since V_s is a column, A has precisely $|V_s| \geq 2$ rows of length $s \geq 2$ such that they form a rectangle, therefore, containing a 2×2 block of boxes. If A is disconnected it is done. Otherwise, as all vertical strips V_i $1 \leq i \leq s$, transverse that rectangle, we may delete the vertical strips $V_1, \dots, V_k$, for some $1 \leq k < s - 1$, until getting a disconnected skew diagram. At this point, we are in conditions of Lemma 4.2, and, from Lemma 4.1, we are done. ■

Example 4.2. For instance, it follows from Corollary 4.4 that the supports of the skew diagrams

$A =$  ; $B =$  ; $C =$  ; $D =$ 

are strictly contained in the Schur interval $[\mathbf{w}, \mathbf{n}]$. In the first, for instance, one has $\mathbf{w} = (4, 3, 2, 1)$ and $\mathbf{n} = (4, 4, 2)$, where V_3 is a column of A of length two, and $\ell(\mathbf{w}) = \ell(\mathbf{n}) + 1$. The Schur interval is $[\mathbf{w}, \mathbf{n}] = \{\mathbf{w} = (4, 3, 2, 1); (4, 4, 1, 1); (4, 3, 3); \mathbf{n} = (4, 4, 2)\}$ and the partition $\xi = (4, 4, 1, 1) \notin \text{supp}(A) = \{\mathbf{w} = (4321), (433); \mathbf{n} = (442)\}$. Therefore $s_A = s_{\mathbf{w}} + s_{433} + s_{\mathbf{n}}$. In the last, one has $\xi = 4311 \in [\mathbf{w} = 32^3, \mathbf{n} = 432] \setminus \text{supp}(D)$.

Lemma 4.5. *Let A be a connected skew diagram such that*

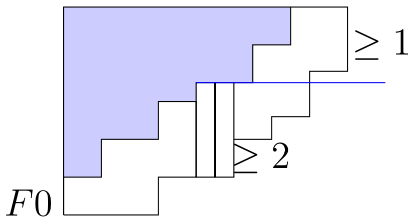
$$\mathbf{w} = (w_1, \dots, w_r) \preceq \sigma^1 = (n_1) \cup \mathbf{w}^1 = (n_1, w'_2, \dots, w'_\ell, w_{\ell+1}, \dots, w_r) \preceq$$
$$\preceq \mathbf{n} = (n_1, \dots, n_s)$$

for some $3 \leq \ell \leq r$ such that $w'_k \leq w_k$ for $k = 1, \dots, \ell$ and $0 < w'_\ell < w_\ell$. Moreover, assume the existence of two integers $2 \leq i < j \leq \ell$ such that $w'_i \geq w'_j + 2$ and $w_j > w'_j$. Then the support of λ/μ is not the entire Schur interval.

Proof: Consider the partition ξ obtained from σ by replacing the entries w'_i and w'_j by $w'_i - 1$ and $w'_j + 1$, respectively. It is clear that $\xi \preceq \sigma \preceq \mathbf{n}$. Note also that while ξ is obtained from σ by lowering one box, from one row of length w'_i to one of length w'_j, $\mathbf{w}$ is obtained from σ by lowering $n_1 - w_1 = w_2 + \cdots + w_l - (w'_2 + \cdots + w'_l \geq 1)$ boxes from the first row to some rows in which is included that one of length w'_j, since $w_j > w'_j$. Thus, $\mathbf{w}$ can be obtained from ξ by lowering $k = n_1 - w_1$ boxes, in particular, $w_i - w'_i + 1$ boxes to row i and $w_j - w'_j + 1$ to row i. Thus $\mathbf{w} \preceq \xi \preceq \sigma$.

Moreover, $\xi^1 \prec \mathbf{w}^1$, where ξ^1 denotes the partition obtained from ξ removing the first entry. Thus, $\xi^1 \notin \text{supp}(A \setminus V_1)$. From Lemma 4.1, we conclude that $\xi \notin \text{supp}(A)$. ■

As a consequence of the last lemma we describe below a large group of skew diagrams whose support is strictly contained in the Schur interval. Let $F0$ be a skew diagram having two columns W_i and W_{i+1} with the same length, and starting and ending on the same rows, say x and y, and such that to its right all columns end at least two rows above row y, with at least one of these columns starting at least one row above row x, as illustrated by:



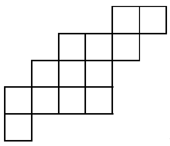
Denote by $F0', F0^\pi$ and $F0^{\pi'}$ the skew diagrams which are, respectively, the conjugate, the π-rotation, and the conjugate of the π-rotation of an $F0$ skew diagram.

Corollary 4.6. *The support of the skew diagrams $F0$, $F0'$, $F0^\pi$, or $F0^{\pi'}$ is strictly contained in the Schur interval.*

Proof: Thanks to the conjugate symmetry and to the π-rotation symmetry, it is enough to consider the $F0$ configuration. Denote by $a+b$ the common length of the two columns W_i, W_{i+1} of $F0$, where $b \geq 0$ is the number of boxes that column W_{i+1} shares with the column to its right, and $a \geq 2$ is the number of boxes of W_{i+1} with no right neighbour.

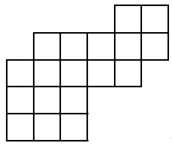
Consider $F0 \setminus V_1$ and $\mathbf{w}^1 = (w_2', \dots, w_\ell', w_{\ell+1}, \dots, w_q)$ with $w_f' = a+b$ and $w_g' = b$ satisfying $w_f' \geq w_g' + 2$, for some integers $2 \leq f < g \leq \ell \leq q$. By Lemma 4.5, it follows that supp$(F0)$ is strictly contained in the Schur interval. ■

Example 4.3. To illustrate the previous corollary, consider the skew diagram

$\lambda/\mu =$  ,

with $\mathbf{w} = (3,3,2,2,2,1)$ and $\mathbf{n} = (5,3,2,1^3)$. Clearly, λ/μ is a $F0$ configuration since the third and fourth columns have the same length and starts on the same row, and to its right all columns end 2 rows above the ending row of these columns and there are columns that starts one row above the starting row of these columns. Thus, by the previous corollary, the support of λ/μ is not the entire Schur interval. Moreover, following the proof of Lemma 4.5, we construct the partition $\tilde{\xi} = (5, 2^3, 1^2)$, which belongs to the interval $[\mathbf{w}, \mathbf{n}]$ but is not an element of supp(λ/μ).

Consider now the skew-diagram

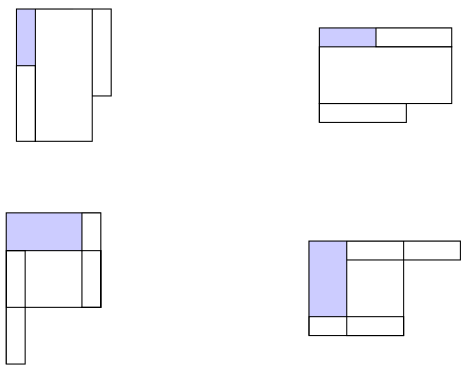
$A =$  ,

One has $\mathbf{w} = 443322 \preceq \xi = 533322 \preceq \sigma = 543222$ and $\xi \notin \mathrm{supp}(A)$.

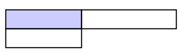
4.2. Recognise full configurations. Next we examine some particular configurations needed in order to characterize the multiplicity-free skew Schur functions that achieve the full interval. We start with the two row (column) skew diagram. (The one row (column) case is a partition and, therefore, the support is $[\mathbf{w}, \mathbf{n}] = \{\mathbf{w} = \mathbf{n}\}$.)

Proposition 4.7. ([1], Theorem 15) *If λ/μ has exactly two rows, then* $\mathrm{supp}(\lambda/\mu) = [\mathbf{w}, \mathbf{n}]$ *and it is a saturated chain. More generally, if λ/μ is a skew diagram with one of the configurations* (4.1)*, then the support of λ/μ is the entire interval* $[\mathbf{w}, \mathbf{n}]$*, and it is a saturated chain.*

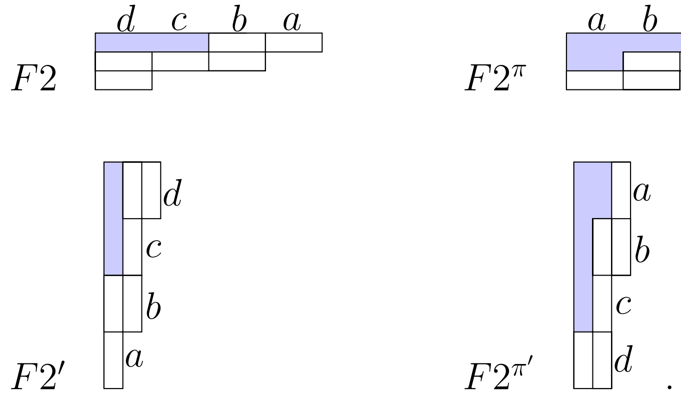
Lemma 3.1 shows that the support of a skew diagram λ/μ having exactly two columns, obtained by inserting a block (x^n) of maximal length between the two columns of λ/μ, or/and adding the partition (n^y) to both λ and μ, is again equal to its entire Schur interval. This construction, together with its conjugates, yields four cases, whose schematic representations are shown below. These diagrams have been arranged so that the ones in the right column are the conjugates of the ones in the left column.

 (4.1)

A two column or a two row disconnected skew diagram is called an A1 configuration,

 . (4.2)

Consider now skew diagrams λ/μ with three rows (columns) where $\mu = (d+c)$ is a one row (column) rectangle and $\lambda^* = (a+b+c, a)$ is a fat hook, or vice versa, for some integers $a, d \geq 1$ and $b, c \geq 0$. There are four cases, as illustrated by:



(4.3)

The diagrams have been arranged so that those on the right hand are the π-rotations of the diagrams in the left hand, and the diagrams in the second row are the conjugates of the ones in the first row.

If these diagrams (4.3) *satisfy the additional conditions* $a \leq c+1$ *and* $d \leq b+1$*, then they are called* $A2, A2^\pi, A2'$ *and* $A2^{\pi'}$ *configurations, as illustrated above, replacing the letter* F *by* A*.*

Proposition 4.8. *Let* λ/μ *be one of the skew diagrams* (4.3)*. Then, the support of* λ/μ *coincides with its Schur interval if and only if it is a* $A2$ *configuration.*

Proof: By assumption $a, d \geq 1$ and $b, c \geq 0$. When $a = 0$ or $d = 0$ we are in the case of two columns (rows) were already studied in Proposition 4.7. Thanks to the rotation and conjugation symmetry, we only consider case $F2'$. We will start by showing that when $a > c+1$ or $d > b+1$, the support of λ/μ is not the entire Schur interval. For the first case, $a > c+1$, just note that with $k := a-(c+1)$, the partition

$$\xi = (d+c+b+k, d+b+c+1)$$

belongs to the Schur interval of λ/μ, since the first entry of $\mathbf{w}$ and $\mathbf{n}$ is $\min\{d+c+b, b+a\}$ and $d+c+b+a$, respectively. Moreover, $\xi \notin \operatorname{supp}(\lambda/\mu)$, since when placing in λ/μ the string of length $d+c+b+k$ we must place the integer $d+b+c+1$ in the first column of the diagram, leaving no room to place the second string.

For the case $d > b+1$ note that the entries of $\mathbf{w}$ are $b+c+d, a+b$ and d, and that $\sigma = (a+b+c+d, d, b)$ with $d \geq b+2$. By Lemma 4.5 it follows that the support of λ/μ is not the entire Schur interval.

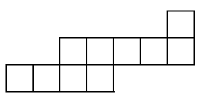
Thus, when $a > c+1$ or $d > b+1$ the support of λ/μ does not coincide with its Schur interval. For the rest of the proof assume $a \leq c+1$ and $d \leq b+1$. Then,

$$\mathbf{w} = (d+c+b, b+a, d) \preceq \mathbf{n} = (d+c+b+a, d+b).$$

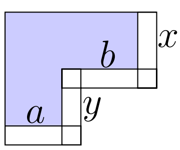
Let $\xi = (\xi_1, \xi_2, \xi_3) \in [\mathbf{w}, \mathbf{n}]$. We will show that $\xi \in \operatorname{supp}(\lambda/\mu)$. Since $\mathbf{w} \preceq \xi \preceq \mathbf{n}$, we must have

$$d+c+b \leq \xi_1 \leq d+c+b+a \quad \text{and} \quad d+c+2b+a \leq \xi_1+\xi_2 \leq 2d+c+2b+a.$$

Then $\xi_1 = d+c+b+k$ and $a+b-k \leq \xi_2 \leq d+a+b-k$, for some $k \in \{0, \ldots, a\}$. From the Algorithm 1, this means that, for each $k \in \{0, \ldots, a\}$, after placing (in the unique possible way) the string of length ξ_1 in the diagram λ/μ, we must insert the strings of length ξ_2, ξ_3 in the skew diagram $\tilde{\lambda}/\mu$, obtained by removing the boxes of the string of length ξ_1. The Schur interval of $\tilde{\lambda}/\mu$ is $[\tilde{\mathbf{w}} = (b+a-k, d), \tilde{\mathbf{n}} = (d+b+a-k)]$. Since $\tilde{\lambda}/\mu$ has two columns, from Proposition 4.7, $[\tilde{\mathbf{w}} = (b+a-k, d), \tilde{\mathbf{n}} = (d+b+a-k)] = \operatorname{supp}(\tilde{\lambda}/\mu)$. On the other hand, $\tilde{\mathbf{w}} \preceq (\xi_2, \xi_3) \preceq \tilde{\mathbf{n}}$, we have $(\xi_2, \xi_3) \in \operatorname{supp}(\tilde{\lambda}/\mu)$. Finally, note that when $k \geq 0$, from the inequality $a-k < c+1$, it follows that $\xi_2 \leq d+b+a-k < d+c+b+1$. Therefore we also have $\xi \in \operatorname{supp}(\lambda/\mu)$. ■

Example 4.4. The skew diagram $\lambda/\mu =$  is an $A2^\pi$ configuration with $d=1$ and $c=b=a=2$. Therefore, its Schur interval coincides with its support.

The next configuration that we consider is the ribbon skew diagram λ/μ, with $\mu = ((a+b+1)^x, a^y)$ and $\lambda^* = ((b+1)^{y+1})$, for some integers $a, b, x, y \geq 1$, as illustrated by:

$F3$ 

. (4.4)

A $F3$ configuration (4.4) with $a = x = 1$, or $a = 1$ and $x \leq y+1$, or $a \leq b+1$ and $x = 1$, is called an $A3$ configuration.

Proposition 4.9. *Let λ/μ be a skew diagram with a configuration (4.4). Then its support equals its Schur interval if and only if it is an $A3$ configuration. Moreover, when $a = x = 1$, the support of $A3$ is $[\mathbf{w}, \mathbf{n}] = \{\mathbf{w}, \xi_1 = (y+3, 1^{b+2}), \xi_2 = (y+2, 2, 2, 1^{b-1}), \xi_3 = (y+2, 3, 1^b), \mathbf{n}\}$, and the skew Schur function $s_{\lambda/\mu} = s_{\mathbf{w}'} + s_{\xi_1'} + s_{\xi_2'} + s_{\xi_3'} + s_{\mathbf{n}'}$ has exactly five components all with multiplicity 1.*

Proof: We start by noticing that when both integers a and x are strictly greater than 1, then the minimum and maximum of the $\mathrm{supp}(\lambda/\mu)$ are given by

$$\mathbf{w} = (w_1, w_2, 1^{a+b}) \preceq \mathbf{n} = (x+y+2, 2^{\min\{b+1,a\}}, 1^{a+b+1-2\min\{b+1,a\}}),$$

with $w_1 = \max\{y+2, x+1\}$, $w_2 = \min\{y+2, x+1\}$ and $\min\{b+1, a\} \geq 2$. Therefore, we can consider the partition

$$\xi := (w_1, w_2, 3, 1^{a+b-3}).$$

It is straightforward to check that $\mathbf{w} \preceq \xi \preceq \mathbf{n}$ and that ξ is not in the support of λ/μ.

In the case $a = x = 1$, in which the minimum and maximum of the support are given by

$$\mathbf{w} = (y+2, 2, 1^{b+1}) \preceq \mathbf{n} = (y+3, 2, 1^b),$$

the Schur interval is $[\mathbf{w}, \mathbf{n}] = \{\mathbf{w}, (y+3, 1^{b+2}), (y+2, 2, 2, 1^{b-1}), (y+2, 3, 1^b), \mathbf{n}\}$, and we can check directly that this interval is equal to $\mathrm{supp}(\lambda/\mu)$.

So we are left with the case $a = 1$ and $x > 1$, since the remaining case is obtained by the conjugation symmetry. The minimum and maximum of the support are

$$\mathbf{w} = (w_1, w_2, 1^{b+1}) \preceq \mathbf{n} = (x+y+2, 2, 1^b),$$

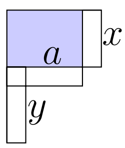
where w_1 and w_2 are defined as in the initial case. Thus, if ξ is a partition in the Schur interval, it must satisfy $\xi = (\xi_1, \xi_2, \xi_3, 1^b)$, for some integers ξ_i. Moreover, we must have

$$\begin{array}{rcccl} \max\{y+2, x+1\} & \leq & \xi_1 & \leq & x+y+2, \\ x+y+3 & \leq & \xi_1+\xi_2 & \leq & x+y+4, \\ & & \text{and} & & \\ x+y+4 & \leq & \xi_1+\xi_2+\xi_3 & \leq & x+y+5. \end{array}$$

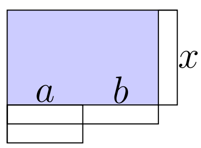
There are two possibilities for the sum $\xi_1+\xi_2$. As $\xi_3 \geq 1$, when this sum equals $x+y+3$, it follows that ξ_3 must be either 1 or 2, and when $\xi_1+\xi_2 = x+y+4$ then $\xi_3 = 1$. Consider $\xi_1+\xi_2 = x+y+3$. We have $\tilde{\mathbf{w}}_1 := (w_1, w_2) \preceq (\xi_1, \xi_2) \preceq \tilde{\mathbf{n}}_1 := (x+y+2, 1)$, with $\tilde{\mathbf{w}}_1$ and $\tilde{\mathbf{n}}_1$, respectively, the minimum and maximum LR fillings of the two column skew diagram $\tilde{A}$ obtained from λ/μ by removing all except the second and last columns. Since, by Proposition 4.7, we have $\operatorname{supp}(\tilde{A}) = [\tilde{\mathbf{w}}_1, \tilde{\mathbf{n}}_1]$, we conclude that we may place strings of length ξ_1 and ξ_2 using only the second and last columns of λ/μ. As the remaining entries of ξ are equal to 1, and, at most one, equal to 2, it follows that $\xi \in \operatorname{supp}(\lambda/\mu)$.

Assume now that $\xi_1+\xi_2 = x+y+4$. In this case, $(\xi_1, \xi_2) \in [\tilde{\mathbf{w}}_2, \tilde{\mathbf{n}}_2]$, where $\tilde{\mathbf{w}}_2 = (w_1, w_2, 1)$ and $\tilde{\mathbf{n}}_2 = (x+y+2, 2)$ are the minimum and maximum LR fillings of the skew diagram B obtained from λ/μ removing all columns except the first two and the last one. Note that B is an $F2^{\pi'}$ (4.3), and, by Proposition 4.8, $\operatorname{supp}(B) = [\tilde{\mathbf{w}}_2, \tilde{\mathbf{n}}_2]$ if and only if $x \leq y+1$. Thus, when $x \leq y+1$ we find that $(\xi_1, \xi_2) \in \operatorname{supp}(B)$ and similarly, as before, it follows that $\xi \in \operatorname{supp}(\lambda/\mu)$. If, on the other hand, we have $x > y+1$, then, by Proposition 4.8, we can consider a partition $(\sigma_1, \sigma_2) \in [\tilde{\mathbf{w}}_2, \tilde{\mathbf{n}}_2]$ which is not in the set $\operatorname{supp}(B)$. It follows that $\sigma := (\sigma_1, \sigma_2, 1^b) \in [\mathbf{w}, \mathbf{n}]$ but $\sigma \notin \operatorname{supp}(\lambda/\mu)$. ■

In the next lemmas we analyse some families of skew diagrams needed in the sequel. We start with skew diagrams λ/μ of types $F4$ and $\widetilde{F}4$ defined respectively by the partitions $\lambda = ((a+2)^x, a+1, 1^y)$ and $\mu = ((a+1)^x)$ for some a, x, $y \geq 1$ such that not both x and y are equal to 1, and by the partitions $\lambda = ((a+b+1)^x, a+b, a)$ and $\mu = ((a+b)^x)$, for some integers $b \geq 1$ and $a, x > 1$, as illustrated by:



$F4$



$\widetilde{F}4$. (4.5)

Note that if we let $x = y = 1$ in an $F4$ configuration, or $x = 1$ in an $\widetilde{F}4$ configuration, then we get an $F2$ configuration.

An $F4$ configuration with $a = 1$ and $x \leq y+1$, or $a \geq 2$ and $x = 1$, is called an $A4$ configuration.

Proposition 4.10. (i) *If λ/μ is a skew diagram with configuration $F4$, then its support is equal to the Schur interval if and only if it is an $A4$ configuration. Moreover, when $a \geq 2$ and $x = 1$, the support of $A4$ is $[\mathbf{w}, \mathbf{n}] = \{\mathbf{w}, \xi = (y+1, 2, 1^{a-1}), \mathbf{n}\}$ and the skew Schur function $s_{\lambda/\mu} = s_{\mathbf{w}'} + s'_{\xi} + s'_{\mathbf{n}}$ has exactly three components all with multiplicity* 1.

(ii) *The support of $\widetilde{F}4$ is strictly contained in the Schur interval.*

Proof: We start with an $F4$ configuration. If $a \geq 2$ then the minimum and maximum of $\mathrm{supp}(\lambda/\mu)$ are

$$\mathbf{w} = (w_1, w_2, 1^a) \preceq \mathbf{n} = (x+y+1, 1^a),$$

with $w_1 = \max\{x, y+1\}$, $w_2 = \min\{x, y+1\}$ and $\ell(\mathbf{w}) = \ell(\mathbf{n}) + 1$. When $x \geq 2$ the partition $\xi := (w_1, w_2, 2, 1^{a-2})$ satisfy $\ell(\xi) = \ell(\mathbf{n})$, and thus $\mathbf{w} \preceq \xi \preceq \mathbf{n}$, but is not in the support of λ/μ, since the strings of length w_1 and w_2 must fill the first and last columns, leaving no space for the string of length 2. On the other hand, if $x = 1$, then the Schur interval of λ/μ is

$$[\mathbf{w}, \mathbf{n}] = \{\mathbf{w}, (y+1, 2, 1^{a-1}), \mathbf{n}\} = \mathrm{supp}(\lambda/\mu).$$

Assume now that $a = 1$. If $x > y+1$ then $\mathbf{w} = (x, y+1, 1) \preceq \mathbf{n} = (x+y+1, 1)$, and $\xi := (x, y+2) \in [\mathbf{w}, \mathbf{n}]$ but clearly $\xi \notin \mathrm{supp}(\lambda/\mu)$. If otherwise, we have $x \leq y+1$, then the minimum and the maximum of λ/μ are given by

$$\mathbf{w} = (y+1, x, 1) \preceq \mathbf{n} = (x+y+1, 1).$$

A partition $\xi = (\xi_1, \xi_2, \xi_3) \in [\mathbf{w}, \mathbf{n}]$ must satisfy $y+1 \leq \xi_1 \leq x+y+1$ and $x+y+1 \leq \xi_1 + \xi_2 \leq x+y+2$ with $\xi_3 \in \{0, 1\}$. Let $\xi_1 = y+1+k$, for some $k \in \{0, \ldots, y+1\}$. Then, we get $x - k \leq \xi_2 \leq x+1-k$, and since $x \leq y+1$ it follows that the partition ξ belongs to the support of λ/μ.

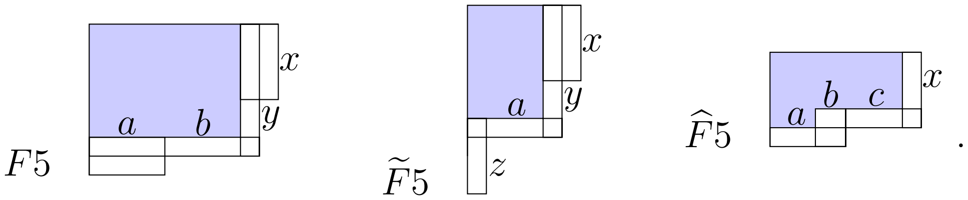
Finally, suppose next that λ/μ is an $\widetilde{F}4$ configuration. Then the minimum and maximum of $\mathrm{supp}(\lambda/\mu)$ are

$$\mathbf{w} = (x, 2^a, 1^b) \preceq \mathbf{n} = (x+2, 2^{a-1}, 1^b),$$

and, since $a \geq 2$, $\xi := (x+2, 2^{a-2}, 1^{b+2})$ is a partition and satisfy $\mathbf{w} \preceq \xi \preceq \mathbf{n}$, but $\xi \notin \mathrm{supp}(\lambda/\mu)$. ∎

The next skew diagrams λ/μ are respectively $F5$, $\widetilde{F}5$ and $\widehat{F}5$, defined by the partitions: $\mu = ((a+b)^{x+y})$ and $\lambda^* = (b+2, 1^{y+1})$, with $a, x \geq 2$ and $b, y \geq 0$; $\mu = ((a+1)^{x+y})$ and $\lambda^* = ((a+2)^z, 1^{y+1})$, with $a \geq 2, y \geq 0$ and

$x, z \geq 1$; and $\mu = ((a+b+c)^x, a)$ and $\lambda^* = (c+1)$, with $x \geq 2$ and either $a, b, c \geq 1$ or $b = 0$ and $a, c \geq 1$ with $a + c \geq 3$, as illustrated by:



(4.6)

Lemma 4.11. *The support of* $F5$, $\widetilde{F}5$ *or* $\widehat{F}5$ *is strictly contained in the Schur interval.*

Proof: The minimum and maximum of the support of $F5$ are respectively

$$\mathbf{w} = (x+y+1, x, 2^a, 1^b) \preceq \mathbf{n} = (x+y+2, x+2, 2^{a-2}, 1^{b+1}),$$

with $\ell(\mathbf{w}) = \ell(\mathbf{n}) + 1$. Since a, $x > 1$, we may consider the partition

$$\xi := (x+y+1, x+1, 3, 2^{a-2}, 1^b)$$

which is an element of the Schur interval but it is not in the support of $F5$.

In the case of $\widetilde{F}5$, we have

$$\mathbf{w} = (w_1, w_2, w_3, 1^a) \preceq \mathbf{n} = (x+y+z+1, x+1, 1^a),$$

with w_1, w_2, w_3 the lengths of the first and the two last columns of $\widetilde{F}5$ by decreasing order. Take

$$\xi := (x+y+z+1, x, 2, 1^{a-1}) \in [\mathbf{w}, \mathbf{n}].$$

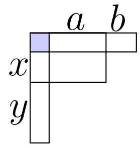
Since after placing the strings of length $x+y+z+1$ and x, in the unique possible positions in $\widetilde{F}5$, we are left with only a single row, it follows that $\xi \notin \operatorname{supp}(\widetilde{F}5)$.

Finally, we are in the case of $\widehat{F}5$. If $b \geq 1$, the minimum and the maximum of the support are given by

$$\mathbf{w} = (x+1, 2^b, 1^{a+c}) \preceq \mathbf{n} = (x+2, 2^b, 2^{\min\{a-1,c\}}, 1^{a-1+c-2\min\{a-1,c\}}).$$

It is easy to check that the partition $\xi := (x+1, 3, 2^{b-1}1^{a+c-1})$ is an element of the Schur interval but is not in the support of $\widehat{F}5$. Similarly, when $b = 0$ the partition $\xi = (x+1, 3, 1^{a+c-3})$ belongs to the Schur interval but not to the support of $\widehat{F}5$. ■

The next family of skew diagrams λ/μ is designated by $F6$ and it is defined by partitions $\lambda = (a+b+1, (a+1)^x, 1^y)$ and $\mu = (1)$, for some integers $a, x > 0$ and $b, y \geq 1$,

$F6$ 

. (4.7)

When $b = y = 1$ an $F6$ configuration is called an $A6$ configuration.

Proposition 4.12. *The support of a skew diagram λ/μ with an $F6$ configuration equals the Schur interval if and only if it is an $A6$ configuration. Moreover, in this case, the support is $[\mathbf{w}, \mathbf{n}] = \{\mathbf{w} = ((x+1)^{a+1}, 1), \xi = (x+2, (x+1)^{a-1}, x, 1), \mathbf{n} = (x+2, (x+1)^a) : a,\ x \geq 1\}$, and the skew Schur function $s_{\lambda/\mu} = s_{\mathbf{w}'} + s_{\xi'} + s_{\mathbf{n}'}$ has three components all with multiplicity 1.*

Proof: We consider the case $b \geq 2$. Notice that the case $y \geq 2$ is the conjugate of the former. In our case, the minimum and the maximum of the support are respectively

$$\mathbf{w} = (x+y, (x+1)^a, 1^b) \preceq \mathbf{n} = (x+y+1, (x+1)^a, 1^{b-1}),$$

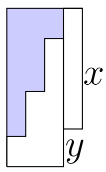
and we can consider the partition $\xi := (x+y, (x+1)^a, 2, 1^{b-2}) \in [\mathbf{w}, \mathbf{n}]$, which clearly does not belong to the support of λ/μ.

For the remaining case $y = b = 1$, notice that the Schur interval is given by

$$[\mathbf{w}, \mathbf{n}] = \{\mathbf{w} = ((x+1)^{a+1}, 1), \xi = (x+2, (x+1)^{a-1}, x, 1), \mathbf{n} = (x+2, (x+1)^a)\}.$$

Since $\xi \in \mathrm{supp}(\lambda/\mu)$ the result follows. ■

In the next lemmas we analyse the support of a skew diagram λ/μ, with $\ell + 1 \geq 4$ columns, having the form

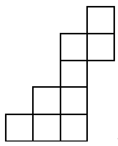
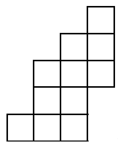
$F7$ 

, (4.8)

where the first ℓ columns end in the same row and have pairwise distinct lengths, x is the length of the last column and $\lambda^* = (y)$. Moreover by Lemma 3.1, we assume without loss of generality that the last but one column starts at least one row below the topmost box of the last column, and that the first column has length $\leq y$. Denote by $(w_1, \ldots, w_\ell)$ the partition formed by the first ℓ columns of the diagram, and let $k \geq 0$ be the number of rows that the last two columns share.

A skew diagram $F7$ such that $\ell = 3$, $x = w_2$, $w_3 = 1$ and $k = w_2 - 1$ is called an $A7$ configuration.

We distinguish two cases: either (i) $w_2 \leq y$, or (ii) $w_2 > y$. In the first case, the last column shares rows only with column ℓ, and, in the second case, the last column shares rows with at least columns ℓ and $\ell - 1$. Examples of $A7$ configurations of types (i) and (ii) are shown below:

(i)  , (ii)  .

In the next two lemmas we show that the support of an $A7$ configuration is the entire Schur interval.

Lemma 4.13. *Let λ/μ be an $F7$ configuration* (4.8) *such that $w_2 \leq y$. Then, the support of λ/μ equals the Schur interval if and only if $\ell = 3$, $w_3 = 1$, $x = w_2$ and $k = w_2 - 1$.*

Proof: The condition $w_2 \leq y$ means that the last column of λ/μ shares rows with at most the last but one column. Then, the minimum and maximum of the support of λ/μ are given respectively by

$$\mathbf{w} = (w_1, \ldots, w_\ell) \cup \{x\} \preceq \mathbf{n} = (w_1 + x - k, w_2 + k, w_3, \ldots, w_\ell),$$

with $\ell(\mathbf{w}) = \ell(\mathbf{n}) + 1$.

Note that when $w_\ell \geq 2$, it follows, from Corollary 4.4, that $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$, and when $\ell \geq 4$ and $w_\ell = 1$, the partition $\xi := (w_1 + x - k, w_2 + k, w_3, \ldots, w_{\ell-1} - 1, 1, 1)$ shows that $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$. Moreover, if $k = 0$ the skew diagram is disconnected, and, by Lemma 4.2, the support of λ/μ is not the entire Schur interval.

So, assuming $\ell = 3$, $k > 0$ and $w_3 = 1$, we have

$$\mathbf{w} = (w_1, w_2, 1) \cup \{x\} \preceq \mathbf{n} = (w_1 + x - k, w_2 + k, 1).$$

If $x < w_2$ then $1 \leq k < x < w_2$, and, in particular, we get $w_2 \geq k + 2$. Since w_2 and k are the lengths of the second and third columns of λ^1/μ, it follows, from Lemma 4.5, that $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$. The same situation happens if $x > w_2$, since in this case the partition $\xi := (w_1, w_2+1) \cup (x)$ satisfies $\mathbf{w} \preceq \xi \preceq \mathbf{n}$, but it does not belong to the support of λ/μ.

We are, therefore, left with the case $\ell = 3, k > 0, x = w_2$ and $w_3 = 1$. Note that $1 \leq k \leq w_2 - 1$. When $k < w_2 - 1$, the second and third columns of λ^1/μ have lengths w_2 and k, respectively, and by Lemma 4.5, we find that the support of λ/μ is not the entire Schur interval. So we must also consider $k = w_2 - 1$. In this case, the minimum and the maximum of the support are given by

$$\mathbf{w} = (w_1, w_2, w_2, 1) \preceq \mathbf{n} = (w_1+1, w_2+w_2-1, 1).$$

Let $\xi := (\xi_1, \xi_2, \xi_3, \xi_4) \in [\mathbf{w}, \mathbf{n}]$, and note that $w_1 \leq \xi_1 \leq w_1 + 1$.

If $\xi_1 = w_1$, then from the inequalities $\mathbf{w} \preceq \xi \preceq \mathbf{n}$ it follows that $\alpha \preceq (\xi_2, \xi_3, \xi_4) \preceq \beta$, where α and β are the minimum and the maximum of the support of the skew diagram A obtained from λ/μ by removing the third column. Since A is an $A2^{\pi'}$ configuration, it follows that $(\xi_2, \xi_3, \xi_4) \in \mathrm{supp}(A)$, and therefore $\xi \in \mathrm{supp}(\lambda/\mu)$.

For the remaining case $\xi_1 = w_1 + 1$ the situation is analogous, since in this case we have $\mathbf{w}^1 \preceq (\xi_2, \xi_3, \xi_4) \preceq \mathbf{n}^1$, where $\mathbf{w}^1$ and $\mathbf{n}^1$ give the minimum and the maximum LR filling of $A \setminus V_1$. Since this diagram is also an $A2^{\pi'}$ configuration, we find that $\xi \in \mathrm{supp}(\lambda/\mu)$. ■

Lemma 4.14. *Let λ/μ be an $F7$ configuration* (4.8) *such that $w_2 > y$. Then, the support of λ/μ equals the Schur interval if and only if $\ell = 3$, $w_3 = 1$, $x = w_2$ and $k = w_2 - 1$.*

Proof: We start the proof by showing that if the conditions $\ell = 3$, $w_3 = 1$, $x = w_2$ and $k = w_2 - 1$ are not satisfied then the support of λ/μ is not the entire Schur interval. Consider the minimal and maximal fillings of the diagram

$$\mathbf{w} = (w_1, w_2, \ldots, w_\ell) \cup (x) \preceq \mathbf{n} = (n_1, n_2, \ldots, n_s),$$

where $n_1 = x + y$, $n_2 = w_1$ and $\ell(\mathbf{w}) > \ell(\mathbf{n})$.

Note that when $k = 0$, the diagram λ/μ is disconnected, with one of the connected components having a 2 by 2 block. In this case, by Lemma 4.2, $\mathrm{supp}(\lambda/\mu)$ is not the entire Schur interval. So, assume $k > 0$. If $x > w_2$ then the partition $\xi := (w_1, \ldots, w_{\ell-2}, w_{\ell-1}+1, w_\ell - 1) \cup (x)$ shows that $\mathrm{supp}(\lambda/\mu)$ is strictly contained in the Schur interval $[\mathbf{w}, \mathbf{n}]$. If, on the other hand, $x < w_2$ then $1 \leq k < x < w_2$, and this implies $w_2 \geq k + 2$. Since k and w_2 are the lengths of two columns of λ^1/μ, it follows, from Lemma 4.5, that also in this case the support of λ/μ is not the entire Schur interval.

So, for the rest of the proof we assume $x = w_2$, and therefore

$$\mathbf{w} = (w_1, w_2, w_2, w_3, \dots, w_\ell) \preceq \mathbf{n} = (n_1, n_2, \dots, n_s),$$

with $n_1 = w_2 + y$ and $n_2 = w_1$. Note that this implies $y \geq 2$.

If $\ell \geq 4$ then the partition $\xi := (w_1, w_2, w_2, \dots, w_{\ell-2}, w_{\ell-1} + 1, w_\ell - 1)$ clearly shows that $\mathrm{supp}(\lambda/\mu)$ is not the entire Schur interval. Thus, consider $\ell = 3$ and note that since $1 \leq k \leq w_2 - 1$, it follows, from Lemma 4.5, that $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$ except if $k = w_2 - 1$.

So, assume now that $\ell = 3, x = w_2$ and $k = w_2 - 1$. Then, $n_1 = w_2 + y = w_1 + 1$, and the minimal and maximal fillings are now

$$\mathbf{w} = (w_1, w_2, w_2, w_3) \preceq \mathbf{n} = (w_1 + 1, w_1, w_3 + h),$$

where $h > 0$ is the number of rows that the second and the last columns share. It follows that if $w_3 \geq 2$ the partition $\xi := (w_1, w_2 + 1, w_2 + 1, w_3 - 2)$ satisfies $\mathbf{w} \preceq \xi \preceq \mathbf{n}$ but it is not in the support of λ/μ.

To finish the proof consider $\ell = 3, w_3 = 1, x = w_2$ and $k = w_2 - 1$, and let $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ be a partition in the Schur interval $[\mathbf{w}, \mathbf{n}]$. Using the same argument used in the proof of the previous Lemma 4.13, it is easy to show that ξ belongs to the support of λ/μ, and it follows that in this case the support of λ/μ is the entire Schur interval. ■

From lemmas 4.13 and 4.14 we deduce the following result.

Corollary 4.15. *If the skew diagram λ/μ is an $F7$ configuration, then its support is the Schur interval if and only if λ/μ is an $A7$ configuration.*

4.3. Main Theorem. We are now ready to identify the basic multiplicity-free skew Schur functions whose support is the entire interval $[\mathbf{w}, \mathbf{n}]$. Our strategy and terminology follows closely the one used in the proof of Lemma 7.1 in [7].

Theorem 4.16. *The basic skew Schur function $s_{\lambda/\mu}$ is multiplicity-free and its support is the entire Schur interval $[\mathbf{w}, \mathbf{n}]$ if and only if, up to a block of maximal width or maximal depth, and up to a π-rotation and/or conjugation, one or more of the following is true:*

(i) μ or λ^* is the zero partition 0.

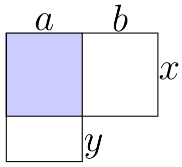
(ii) λ/μ is a two column or a two row diagram ($A1$ configuration).

(*iii*) λ/μ *is an* $A2$ *configuration.*
(*iv*) λ/μ *is an* $A3$ *configuration.*
(*iv*) λ/μ *is an* $A4$ *configuration.*
(*iv*) λ/μ *is an* $A6$ *configuration.*
(*iv*) λ/μ *is an* $A7$ *configuration.*

Proof: If λ/μ satisfy one or more of the conditions listed in the theorem, then the corresponding Schur function is multiplicity-free. They are particular instances of the configurations $\mathbf{R0} - \mathbf{R4}$ described in Theorem 2.2 as follows : $A1$ is in $R4$; $A2$ is in $R1$, or $R3$; $A3$ and $A4$ are in $R3$; $A6$ is in $R1$ or $R3$; and $A7$ is in $R1$. The strategy for the reciprocal is to use Corollary 4.6 to analyse the support of the basic multiplicity-free skew Schur function $s_{\lambda/\mu}$ listed under cases $\mathbf{R0} - \mathbf{R4}$ in Theorem 2.2. We consider the five cases in turn.

R0. If μ or λ^* is the zero partition 0 then either λ/μ or $(\lambda/\mu)^\pi$ is a partition. This means that the minimum and maximum LR fillings of λ/μ coincide and therefore, $\mathrm{supp}(\lambda/\mu) = [\mathbf{w}, \mathbf{n}] = \{\mathbf{n} = \mathbf{w}\}$.

R4. In this case both μ and λ^* are rectangles, and thus λ is a fat hook. Thanks to Lemma 3.1, we may assume that $\lambda = ((a+b)^x, b^y)$ and $\mu = (a^x, 0^y)$ with $a, b, x, y \geq 1$, as illustrated below:

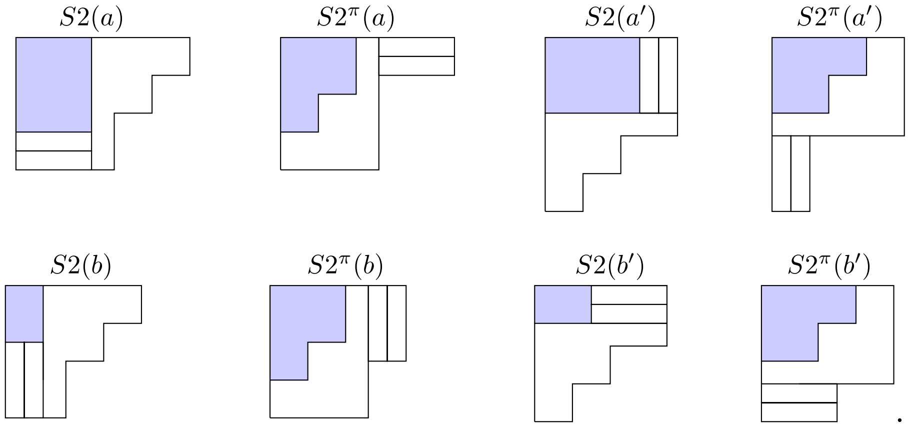
$\lambda/\mu =$ 

.

Since λ/μ has two disconnected components, by Lemma 4.2, it follows that if any of the components has a 2 by 2 block then its support is not the entire Schur interval. Thus, we are left with four cases to analyse. When $a = b = 1$ or when $x = y = 1$ we get an a two column or a two row diagram. In both cases, by Proposition 4.7, the Schur interval coincides with the support of λ/μ. In any other case we get either an $F1$ or an $F1^\pi$ configuration, and, by Proposition 3.3, we find that the support of λ/μ is strictly contained in its Schur interval.

Therefore, by Lemma 3.1, we find that if both μ and λ^* are rectangles, then $\mathrm{supp}(\lambda/\mu) = [\mathbf{w}, \mathbf{n}] = \{\mathbf{n}, \mathbf{w}\}$ if and only if λ/μ satisfy conditions (*ii*) of the theorem.

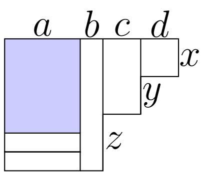
R2. The two main subcases, μ a rectangle of m^n-shortness 2 and λ^* a fat hook, and vice versa, are denoted by **S2** and $\mathbf{S2}^\pi$, respectively. Each has four subcases, as illustrated by:



.

These skew Young diagrams are arranged so that those of type $S2^\pi$ are the π-rotations of those of type $S2$, and the right-hand block of four is the conjugate of the left-hand block of four. Thanks to the rotation symmetry and the conjugate symmetry, one has only to consider two cases, which we select to be **S2**(**a**) and **S2**(**b**).

S2(**a**). In this case $\lambda = ((a+b+c+d)^x, (a+b+c)^y, (a+b)^z)$ and $\mu = (a^{x+y+z-2})$ with $x+y+z \geq 4$, for some integers such that $a \geq 2$, $c, d, x, y, z \geq 1$ and $b \geq 0$, as illustrated in the following figure:



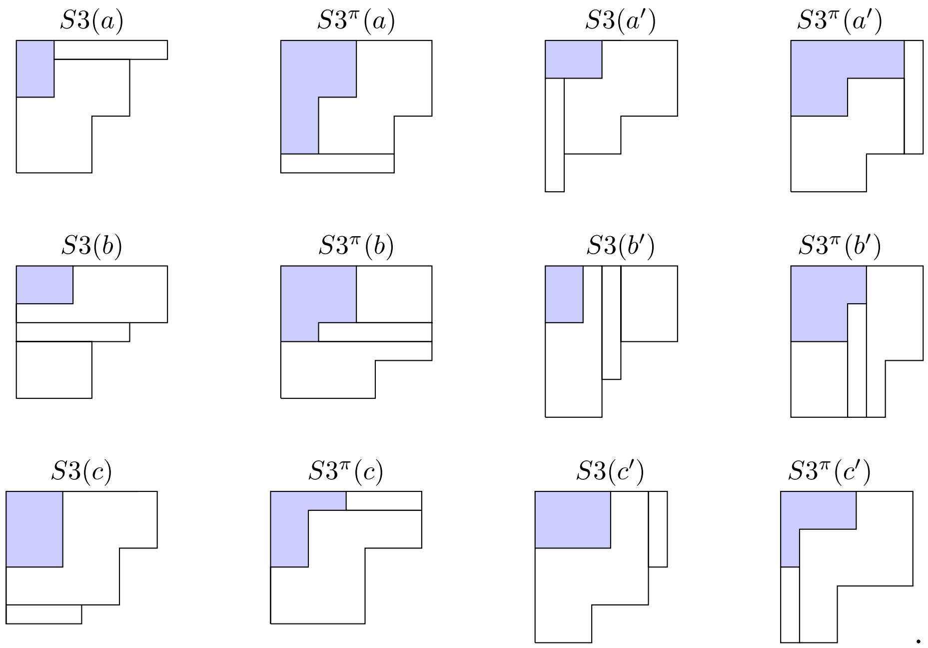
.

By Lemma 3.1, we may assume that $b = 0$. We start by noticing that when $z = 2$ the skew diagram λ/μ is disconnected with a 2 by 2 block, in which case we have $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$, by Lemma 4.2. Assume now that $z = 1$, and note that by the hypothesis on λ/μ, the length of the diagram is greater than, or equal to 4. Therefore, an $F0^\pi$ or an $F0'$ configuration appears when $c \geq 2$ or when $x, d \geq 2$, respectively. Again in this cases the support of λ/μ is strictly contained in the Schur interval by Corollary 4.6. Now, when $c = x = 1$, we get an $F5$ configuration, and when $c = d = 1$, we get the conjugate of an $F5$ configuration. By Lemma 4.11, $\mathrm{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$.

Therefore, in all subcases, the support of the skew diagram **S2**(**a**) is strictly contained in its Schur interval.

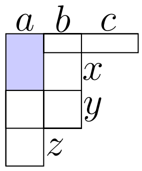
The analysis of case **S2**(**b**) is completely analogous to the previous one.

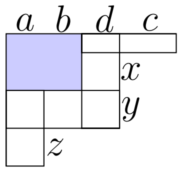
R3. The two main subcases, μ a rectangle and λ^* a fat hook of m^n-shortness 1, and vice versa, are denoted by **S3** and $\mathbf{S3}^\pi$, respectively. Each has six subcases, as illustrated by:



As before, these skew Young diagrams are arranged so that those of type $\mathbf{S3}^\pi$ are the π-rotations of those of type **S3**. This time the right-hand block of six is the conjugation of the left-hand block of six. Thanks to the rotation symmetry and the conjugation symmetry, we have only to consider three cases. We choose to be **S3**(**a**), **S3**(**b′**) and **S3**(**c′**).

S3(**a**). There are two subcases which, by Lemma 3.1, may be reduced to (i) $\mu = (a^{x+1})$ and $\lambda^* = ((b+c)^z, c^{x+y})$ for some integers $a, b, c, z \geq 1$ and $x, y \geq 0$ such that $x + y \geq 1$; and (ii) $\mu = ((a + b)^{x+1})$ and $\lambda^* = ((b + c + d)^z, c^{y+x})$ where $a, b, c, z, y \geq 1$ and $d, x \geq 0$, as illustrated below:

(i) 

(ii) 

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In the subcase **S3**(**a**)(i) we start by identifying an $F0, F0^{\pi}$ or an $F0^{\pi'}$ configuration in the diagram whenever $a, z \geq 2$ or when $x \geq 1$ and $b \geq 2$, or even when $a, y \geq 2$. In these cases, the support of the skew diagram is not the entire Schur interval by Corollary 4.6. There remains thus six cases to consider.

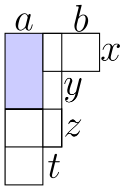
When $a = 1$ and $x = 0$ we get an $F6$, and by Lemma 4.7 $\mathrm{supp}(\lambda/\mu) = [\mathbf{w}, \mathbf{n}]$ if and only if λ/μ is an $A6$ configuration. When $a = b = 1$ and $x > 0$, we get the conjugate of the π-rotation of an $\widehat{F}5$ configuration if $c \geq 2$, and an $F2'$ configuration if $c = 1$. By Proposition 4.8 and Lemma 4.11, we find that in these cases, $\mathrm{supp}(\lambda/\mu) = [\mathbf{w}, \mathbf{n}]$ if and only if λ/μ is an $A2$ configuration.

Assume now that $a \geq 2$ and $z = 1$. Then, if $y = x = 0$ we get a two row skew diagram, the configuration (ii) of the statement under proof, and thus its support equals the Schur interval. If $y = 0$ and $b = 1$ we get the conjugate of an $F4$ configuration, and, by Lemma 4.10, its support is equal to the Schur interval if and only if λ/μ is an $A4$ configuration. Finally, if $y = 1$ and $x = 0$ we get an $F2$ configuration, and when $y = 1$ and $b = 1$ we get the transpose of an $\widetilde{F}5$ configuration. By Proposition 4.8 and Lemma 4.11, we find that the support is equal to the Schur interval if and only if λ/μ is an $A2$ configuration.

Consider now the subcase **S3**(**a**)(ii). When $a, z \geq 2$, we have an $F0$ configuration, and when $x \geq 1$ and $d \geq 2$, we have an $F0^{\pi}$ configuration. Note also that if $y \geq 2$, we have an $F0'^{\pi}$ configuration. So we are left with six cases to analyse, all having $y = 1$.

If $d = 0$ and $z = 1$, we get an $F2$ configuration (recall that one restricts to basic skew Schur functions); if $d = 0$ and $a = 1$, we get either an $F2$ or an $\widehat{F}5$ configuration; if $d = 1$ and $z = 1$, we also get the conjugate of an $\widetilde{F}5$ configuration; if $d = 1$ and $a = 1$, we also get the π-rotation of an $F3$ configuration; if $x = 0$ and $z = 1$, we also get an $F2$ configuration; and finally, if $x = 0$ and $a = 1$, we also get the π-rotation of an $\widehat{F}5$ configuration. Using propositions 4.8, 4.9 and Lemma 4.11, it follows that the support of λ/μ is the full Schur interval if and only if λ/μ is an $A2$ or an $A3$ configuration.

S3(b′). Using Lemma 3.1, we may assume that $\lambda = ((a+b+1)^x, (a+1)^{y+z}, a^t)$ and $\mu = (a^{x+y})$, for some integers $a, b, x, t \geq 1$, and $y, z \geq 0$ such that $y + z \geq 1$, as illustrated by

$\lambda/\mu =$ 

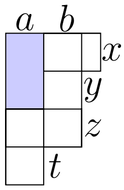
.

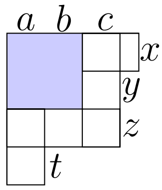
When $a, t \geq 2$ or $a, z \geq 2$ we get an $F0$ or an $F0^{\pi'}$ configuration, in which cases we know from Corollary 4.6 that its support is not the entire Schur interval. An $F0'$ configuration also appears whenever $b, x \geq 2$, and again the support of λ/μ is strictly contained in its Schur interval. So we are left with six cases to analyse.

If $a = b = 1$, we get an $F2'$ configuration, and if $a = x = 1$, we get either an $F2'$ configuration (when $b = 1$), or the conjugate of the π-rotation of an $\widehat{F}5$ configuration (otherwise). By Proposition 4.8 and Lemma 4.11, it follows that in this cases, the support of λ/μ is equal to the entire Schur interval if and only if λ/μ is an $A2$ configuration.

Assume now $a \geq 2$ and $t = z = 1$. If also $b = 1$, we get respectively an $F5$, and if $x = 1$, we get the conjugate of an $\widetilde{F}5$ configuration. Again by Lemma 4.11, it follows that in these cases the support of λ/μ is strictly contained in the Schur interval. For the remaining two cases, assume that $a \geq 2$, $t = 1$ and $z = 0$. When $b = 1$, we get the conjugate of an $\widetilde{F}4$ configuration, and when $x = 1$, we get the conjugate of an $F4$ configuration. In these cases, by Lemma 4.10, the support of λ/μ is the full Schur interval if and only if λ/μ is the conjugate of an $A4$ configuration.

S3(c′). Thanks to Lemma 3.1 there are only two subcases to study: (i) $\mu = (a^{x+y})$ and $\lambda^* = ((b+1)^t, 1^{z+y})$ for some integers $a, b, x, t \geq 1$ and $y, z \geq 0$ such that $y + z \geq 1$; and (ii) $\mu = ((a+b)^{x+y})$ and $\lambda^* = ((b+c+1)^t, 1^{z+y})$ with $a, b, x, t, z \geq 1$ and $y, c \geq 0$, as illustrated below:

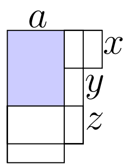
(i) 

(ii) 

.

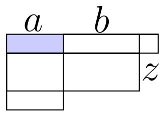
In subcase $\mathbf{S3}(\mathbf{c}')(i)$, we have $F0$ and $F0^{\pi}$ configurations whenever $a, t \geq 2$ or $b, x + y \geq 2$, respectively. In these cases, by Corollary 4.6 we have $\text{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$. It remains to consider four cases. When $a = b = 1$, we get an $F2'$ configuration, and when $a = x+y = 1$, we get an $F6$ configuration. Then, Proposition 4.8 and Lemma 4.7 show that, in each case, the support of λ/μ is equal to the Schur interval if and only if λ/μ is an $A2'$ or an $A6$ configuration.

Assume now $a \geq 2$ and $t = 1$. If $b = 1$ we get the diagram



.

Now, an $F0'$ appears if $z \geq 2$, and, when $z = 1$, we get the conjugate of an $\widetilde{F}5$ configuration. By Theorem 4.6 and Lemma 4.11, it follows that in these cases the Schur interval contains strictly the support of λ/μ. On the other hand, if $x + y = 1$, then we must have $x = 1$ and $y = 0$, and thus λ/μ has the form



.

As before, an $F0'$ appears if $z \geq 2$, and, when $z = 1$, we get an $F2$ configuration, in which case, respectively, by Corollary 4.6 and Proposition 4.8, we find that the Schur interval equals the support of λ/μ if and only it is an $A2$ configuration.

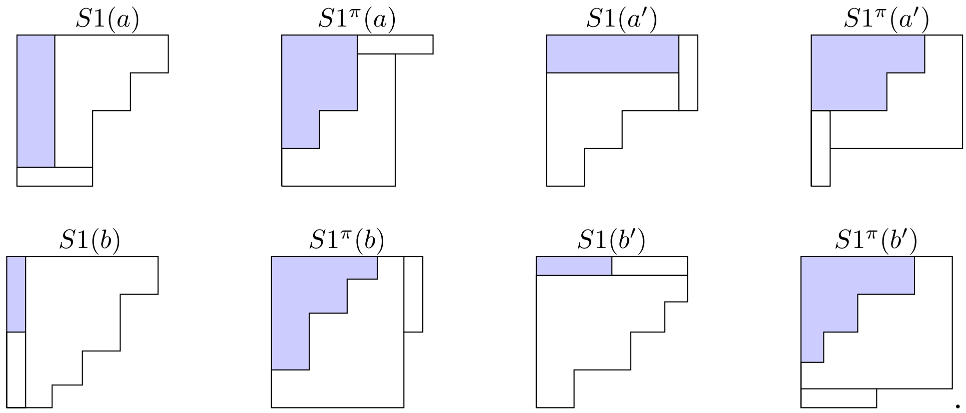
Consider now the subcase $\mathbf{S3}(\mathbf{c}')(ii)$. If $a, t \geq 2$ or $z \geq 2$ or $c, x + y \geq 2$ we get respectively $F0, F0^{\pi'}$ or $F0^{\pi}$ configurations on λ/μ. In all these cases, using Corollary 4.6, we find that $\text{supp}(\lambda/\mu) \subsetneq [\mathbf{w}, \mathbf{n}]$. So we may assume $z = 1$.

If we have $c = 0$, we get an $F4$ or an $\widetilde{F}4$ configuration, respectively, when $a = 1$ and $t = 1$, and, by Lemma 4.10, the support of λ/μ is equal to its entire Schur interval if and only if the λ/μ is an $A4$ configuration.

If $c = 1$ then we get either an $F5$, or an $\widetilde{F}5$ or an $F3$ configuration, respectively, when $a, x \geq 2$ and $t = 1$, or when $a = 1$ and $t, x \geq 2$, or when $a = t = x = 1$. In these cases, by Proposition 4.9 and Lemma 4.11, the support of the skew diagram is equal to the Schur interval if and only if λ/μ is an $A3$ configuration.

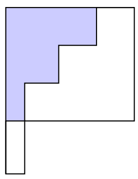
Finally, assume $c \geq 2$ and $x + y = 1$. When $t = 1$, we get an $F3$ configuration, and, when $a = 1$ and $t \geq 2$, we get the π-rotation of an $\widehat{F}5$. By Proposition 4.8 and Lemma 4.11, we find that, in these cases, $\mathrm{supp}(\lambda/\mu) = [\mathbf{w}, \mathbf{n}]$ if and only if λ/μ is an $A3$ configuration.

R1. There are two main subcases. We denote them by $S1$ and $S1^\pi$ in which μ and λ^*, respectively, are rectangles of m^n-shortness 1. Each has four subcases, as illustrated by:



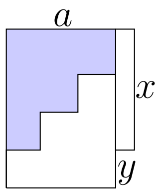
These skew Young diagrams are arranged so that those of type $S1^\pi$ are the π-rotations of their left-hand neighbour of type $S1$. Moreover, the right-hand block of four are the conjugates of the left-hand block. Thanks to the rotation symmetry and the conjugation symmetry, it is therefore only necessary to consider two cases. We select to be $\mathbf{S1}^\pi(\mathbf{a}')$ and $\mathbf{S1}^\pi(\mathbf{b})$.

$\mathbf{S1}^\pi(\mathbf{a}')$. Using Lemma 3.1, we may assume that

$\lambda/\mu =$  .

Moreover, we may assume that the diagram has at least 4 columns, three amongst the last but one have different sizes, otherwise we are in case **R3** or **R4**. In this case, λ/μ is disconnected and the largest component as a 2 by 2 block. By Lemma 4.2, it follows that the support of λ/μ is not the full Schur interval.

S1$^\pi$(**b**). We may use again Lemma 3.1 to reduce our study to the skew diagrams of the form

$\lambda/\mu =$ 

,

where $a+1$ is the number of columns of the diagram, which we assume that amongst the first a there is at least three distinct lengths (otherwise we are in previous cases), x is the length of the last column and $\lambda^* = (y)$. Moreover, we assume without loss of generality that the last but one column starts at least one row below the topmost box of the last column, and that the first column has length $\leq y$. Note that if there are at least two columns, among the first a ones, having the same length and $y \geq 2$, then λ/μ is an $F0$ configuration. We are left with two cases: either (i) $y = 1$, or (ii) the first a columns have pairwise distinct lengths. In the first case, the minimal and maximal fillings of λ/μ are

$$\mathbf{w} = (x, n_1^{s_1}, \ldots, n_r^{s_r}, 1^{s_{r+1}}) \preceq \mathbf{n} = (x+1, n_1^{s_1}, \ldots, n_r^{s_r}, 1^{s_{r+1}-1}),$$

with $x \geq n_1 > \cdots > n_r > 1$ and $r \geq 2$. Therefore, the partition

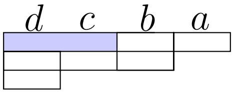
$$\xi := (x, n_1^{s_1}, \ldots, n_r + 1, n_r^{s_r - 1}, 1^{s_{r+1}-1})$$

shows that the support of λ/μ is not the entire admissible interval.

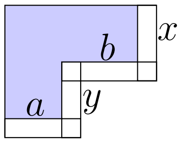
Finally, assume that amongst the first a columns of λ/μ there are no two with the same length, and that $y \geq 2$. By Corollary 4.15 it follows that the support of λ/μ is equal to the Schur interval if and only if λ/μ is an $A7$ configuration. ■

We give the graphical representation of the skew shapes whose support achieve the entire Schur interval, up to a block of maximal width or maximal depth, and up to a π-rotation and/or conjugation:

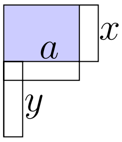
The Full Interval Multiplicity-Free Skew shapes up to a block of maximal width or maximal depth, and up to a π-rotation and/or conjugation

$A2$: 

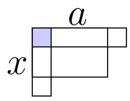
$1 \leq a \leq c+1,$
$1 \leq d \leq b+1,$
$b, c \geq 1.$

$A3$: 

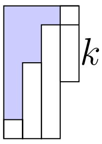
$a = x = 1$
$b, y \geq 1;$ or
$a = 1, 1 \leq x \leq y+1$
$b, y \geq 1.$

$A4$: 

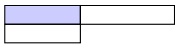
$a = 1,\ 1 \leq x \leq y+1,$
$y \geq 1;$ or
$a \geq 2,\ x = 1,$
$y \geq 1.$

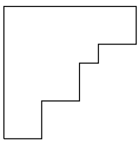
$A6$: 

$a,\ x \geq 1.$

$A7$: 

$\mathbf{w} = (w_1, k+1, k+1, 1)$
$w_1 \geq k+2,$
$k \geq 2.$

$A1$:  .

partition :  .

Example 4.5. Let $\lambda/\mu = ((b+3)^2, 2^{y+1})/(b+2, 1^{y+1})$ be an $A3$ configuration with $\mathbf{w} = (y+2, 2, 1^{b+1}) \preceq \mathbf{n} = (y+3, 2, 1^b)$. Then

$$s_{\lambda/\mu} = s_{(y+2,2,1^{b+1})'} + s_{(y+2,2,2,1^{b-1})'} + s_{(y+2,3,1^b)'} + s_{(y+3,1^{b+2})'} + s_{(y+3,2,1^b)'}.$$

Example 4.6. Let $\lambda/\mu = (4^4, 3^2)/(3, 2, 1^3)$ be an $A7$ configuration with $\mathbf{w} = (5, 4, 4, 1) \preceq \mathbf{n} = (6, 5, 3)$. Then

$$s_{\lambda/\mu} = s_{(5,4,4,1)'} + s_{(5,5,3,1)'} + s_{(6,4,3,1)'} + s_{(5,5,4)'} + s_{(6,5,2,1)'} + s_{(6,4,4)'} + s_{(6,5,3)'}.$$

5. The multiplicity-free Schur function products with full interval support

As a consequence of the characterization of the multiplicity-free skew Schur functions that achieve the full interval, given in Theorem 4.16, and of Corollary 2.3, we get the characterization of the multiplicity-free Schur function products that achieve the full interval. They are precisely given by the configurations above: partition, $A1$ and $A4$.

Corollary 5.1. *The Schur function product $s_\mu s_\nu$ is multiplicity-free and its support is the entire Schur interval if and only if one or more of the following is true:*

(a) *μ or ν is the zero partition.*
(b) *μ and ν are both rows or both columns.*
(c) *$\mu = (1^x)$ is a one-column rectangle and $\nu = (a, 1^y)$ is a hook such that either $a = 2$ and $1 \le x \le y+1$, or $a \ge 3$ and $x = 1$ (or vice versa).*
(c') *$\mu = (x)$ is a one-row rectangle and $\nu = (z, 1^a)$ is a hook such that either $a = 1$ and $1 \le x \le z$, or $a \ge 2$ and $x = 1$ (or vice versa).*

The case (c') is the conjugate of (c).

We list now explicitly the partitions (μ, ν, λ) for which $c^\lambda_{\mu\,\nu} = 1$ for all $\lambda \in [\mu \cup \nu,\, \mu + \nu]$. Recall that the Pieri rule expresses the product of a Schur function and a single row (column) Schur function in terms of Schur functions [18, 8, 20]. These are precisely the cases where the Hasse diagram of the interval $[\mu \cup \nu,\, \mu + \nu]$ is given by the Pieri rule.

Corollary 5.2. *Let $(\mu,\, \nu, \lambda)$ be a triple of partitions.*

(a) *If μ or ν is the zero partition, $c^\lambda_{\mu,0} = 1$ if and only if $\mu = \lambda$*
(b) *If $\mu = (1^x)$ and $\nu = (1^y)$ (or vice versa), with $x \ge y \ge 1$, $c^\lambda_{\mu,\nu} = 1$ if and only if $\lambda \in [(1^{x+y});\, (2^y,\, 1^{x-y})]$.*
(b') *If $\mu = (x)$ and $\nu = (y)$ (or vice versa), with $x \ge y \ge 1$, $c^\lambda_{\mu,\nu} = 1$ if and only if $\lambda \in [(x,\, y);\, (x+y)]$.*
(c) *If $\mu = (1^x)$ and $\nu = (2, 1^y)$ is such that $1 \le x \le y+1$ (or vice versa), $c^\lambda_{\mu,\nu} = 1$ if and only if $\lambda \in [(2,\, 1^{x+y});\, (3,\, 2^{x-1},\, 1^{y-x+1})]$.*
(c') *If $\mu = (x)$ and $\nu = (z, 1)$ is such that $1 \le x \le z$, (or vice versa), $c^\lambda_{\mu,\nu} = 1$ if and only if $\lambda \in [(z,\, x,\, 1);\, (z+x+1, 1)]$.*
(d) *If $\mu = (1)$ and $\nu = (a, 1^y)$ is such that $a \ge 3$, $y \ge 1$ (or vice versa), $c^\lambda_{\mu,\nu} = 1$ if and only if $\lambda \in [(a,\, 1^{y+1});\, (a+1,\, 1^y)]$.*
(d') *If $\mu = (1)$ and $\nu = (z, 1^a)$ is such that $a \ge 2$, $z \ge 1$ (or vice versa), $c^\lambda_{\mu,\nu} = 1$ if and only if $\lambda \in [(z,\, 1^{a+1});\, (z+1,\, 1^a)]$.*

Remark 5.1. We may also list explicitly the multiplicity-free Schur function products whose support is an interval, that is, those whose number of components (summands) is the cardinal of the Schur interval.

(a) $s_0 s_\nu = s_\nu$ has 1 component;

(*b*) The conjugate Schur interval $[(1^{x+y}); (2^y, 1^{x-y})]$ is a saturated chain.

$$s_{(1^x)}s_{(1^y)} = s_{(2^y,1^{x-y})} + s_{(2^{y-1},1^{x-y+2})} + \cdots + s_{(2^1,1^{x+y-2})} + s_{(1^{y+x})}, \quad x \geq y \geq 1,$$

has $y+1$ components. In particular, when $y=1$, $s_{(1^x)}s_{(1)} = s_{(2,1^{x-1})} + s_{(1^{1+x})}$ has 2 components.

(*b′*) It is the conjugate of (*b*).

(*c*) There are two cases for the conjugate Schur interval $[(2, 1^{x+y}), (3, 2^{x-1}, 1^{y-x+1})]$, with $1 \leq x \leq y+1$. When $y = x-1$,

$$s_{1^x}s_{2\,1^{x-1}} = s_{(x\,x\,1)'} + s_{(x+1\,x-1\,1)'} + s_{(x+1\,x)'} + s_{(x+2\,x-2\,1)'} + s_{(x+2\,x-1)'} + s_{(x+3\,x-3\,1)'} + \cdots + s_{(2x-1\,1\,1)'} + s_{(2x-1\,2)'} + s_{(2x\,1)'}, \ x \geq 1$$

and when $y > x-1$,

$$s_{(2,1^{x-1+k})}s_{(1^x)} = s_{(x+k,\,x,\,1)'} + s_{(x+k,\,x+1)'} + s_{(x+1+k,\,x-1,\,1)'} + s_{(x+1+k,\,x)'} + s_{(x+2+k,\,x-2,\,1)'} + s_{(x+2+k,\,x-1)'} + s_{(x+3+k,\,x-3,\,1)'} + \cdots + s_{(2x-1+k,\,1,\,1)'} + s_{(2x-1+k,\,2)'} + s_{(2x+k,\,1)'}, \quad x \geq 1, \ k > 0.$$

(*c′*) It is the conjugate of (*c*).

(*d*) The conjugate Schur interval $[(a, 1^{y+1}); (a+1, 1^y)]$, $a \geq 3$, $y \geq 1$, has 3 elements.

$$s_{(a,1^y)}s_1 = s_{(a,1^{y+1})} + s_{(a,2,1^{y-1})} + s_{(a+1,1^y)}, \quad a \geq 3, \ y \geq 1.$$

(*d′*) It is the conjugate of (*d*).

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