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# A NOTE ON THE CATEGORICAL VAN KAMPEN THEOREM

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Abstract: In this note we show that for the canonical map $X_1 + X_2 \to X$, from the topological sum of two subspaces $X_1, X_2$ of $X$ into $X$, to be of effective descent it is sufficient to be hereditarily a quotient map. Further, making use of the Brown-Janelidze categorical van Kampen Theorem, this translates into a van Kampen-type result.

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## 1. Introduction

In [3] Brown and Janelidze identified the key property for obtaining the classical van Kampen Theorem as a result on Descent Theory. We recall that the van Kampen Theorem states that, under suitable conditions, the fundamental group $\Pi_1(X, x)$ of a space $X$ at a point $x \in X$ can be obtained as the pushout of the fundamental groups $\Pi_1(X_1, x)$, $\Pi_1(X_2, x)$ of two open subspaces $X_1$ and $X_2$ of $X$ covering $X$.

**General van Kampen Theorem.** *Let $\mathsf{C}$ be an extensive category with finite limits, and let the following diagram*

$$\begin{tikzcd}
& X & \\
X_1 \arrow[ur, "g_1"] & & X_2 \arrow[ul, "g_2"'] \\
& X_0 \arrow[ul, "f_1"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

*be a pullback, with $g_1$ and $g_2$ monomorphisms. Let $\mathsf{F}$ be a pullback-stable class of morphisms of $\mathsf{C}$ containing the isomorphisms. Then*

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*the diagram*

$$\begin{tikzcd}[column sep=small] & \mathsf{F}\downarrow X \arrow[dl, "g_1^*"'] \arrow[dr, "g_2^*"] & \\ \mathsf{F}\downarrow X_1 \arrow[dr, "f_1^*"'] & & \mathsf{F}\downarrow X_2 \arrow[dl, "f_2^*"] \\ & \mathsf{F}\downarrow X_0 & \end{tikzcd}$$

*is a pullback if, and only if, the morphism $p : X_1 + X_2 \to X$, induced by $g_1$ and $g_2$, is an effective $\mathsf{F}$-descent morphism.*

Here $g_1^*$ and $g_2^*$ are the change-of-base functors, and, as in [3], by the latter diagram being a pullback we mean that the functor

$$\mathsf{F}\downarrow X \xrightarrow{K_{g_1,g_2}} (\mathsf{F}\downarrow X_1)\times_{\mathsf{F}\downarrow X_0}(\mathsf{F}\downarrow X_2),$$

induced by $(g_1, g_2)$, is an equivalence, where $(\mathsf{F}\downarrow X_1)\times_{\mathsf{F}\downarrow X_0}(\mathsf{F}\downarrow X_2)$ is the category of triples $((A_1,\alpha_1),(A_2,\alpha_2),\varphi)$, with $(A_i,\alpha_i)\in\mathsf{F}\downarrow X_i$, $i=1,2$, and $\varphi : f_1^*(A_1,\alpha_1)\to f_2^*(A_2,\alpha_2)$ an isomorphism. Moreover, we recall from [4] that a finite-complete category $\mathsf{C}$ with coproducts is said to be *extensive* if finite coproducts are disjoint and universal; equivalently, if for any pair $X, Y$ of objects of $\mathsf{C}$ the functor

$$(\mathsf{C}\downarrow X)\times(\mathsf{C}\downarrow Y) \xrightarrow{\;+\;} (\mathsf{C}\downarrow X+Y)$$

is an equivalence. A morphism $f : Y \to X$ is *effective for* $\mathsf{F}$*-descent* if the change-of-base functor $f^* : \mathsf{F}\downarrow X \to \mathsf{F}\downarrow Y$ is monadic.

In this paper we will focus on the study of the global situation, that is when the class $\mathsf{F}$ in the Theorem is the class of all morphisms of $\mathsf{C}$. Although this approach diverges from the classical Theorem, our goal here is to show that for the morphism $p$ of the Theorem to be of effective descent some easy criteria are available. Indeed, the study of effective descent morphisms in topological settings, more precisely in the category of topological spaces and continuous maps, started in [12], has revealed that morphisms effective for descent are in general very difficult to be identified. Although global effective descent continuous maps were characterized by Reiterman and Tholen in [16], their characterization is based on intricate conditions on convergence, which gave rise to subsequent contributions towards a better understanding of these maps: see [11, 6, 8, 7].

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## 2. The global van Kampen Theorem in topological spaces

Throughout we will be working in the category **Top** of topological spaces and continuous maps. We start by recalling some special properties of continuous maps. Given a topological space $X$, we denote by $\Omega X$ the set of open subsets of $X$.

**Definitions.** A continuous map $f : Y \to X$ is said to be:

(1) a *universal quotient map* if the pullback of $f$ along any continuous map is a quotient map;
(2) a *hereditary quotient map* if the pullback of $f$ along any embedding is a quotient map.

Universal quotient maps were characterized by Day and Kelly [9], and also by Michael [13], who called them *biquotient maps*. Further, Michael introduced in [14] a smaller class of quotient maps, the triquotient maps, which includes both open and proper surjections (that is, closed surjections with compact fibres: see [1]). This notions relies on the existence of a map $(\ )^\sharp : \Omega Y \to \Omega X$ fulfilling a list of properties which are shared by the direct-image map $f(\ ) : \Omega Y \to \Omega X$ in the case $f$ is a surjective open map, and by $f^\sharp : \Omega Y \to \Omega X$, defined by $f^\sharp(U) = X \setminus f(Y \setminus U)$ in the case $f$ is a proper surjection.

**Definition.** A continuous map $f : Y \to X$ is a *triquotient map* if there exists a map $(\ )^\sharp : \Omega Y \to \Omega X$ such that:

(T1) $(\forall U \in \Omega Y)\ U^\sharp \subseteq f(U)$,
(T2) $Y^\sharp = X$,
(T3) $(\forall U, V \in \Omega Y)\ U \subseteq V \Rightarrow U^\sharp \subseteq V^\sharp$,
(T4) $(\forall U \in \Omega Y)\ (\forall x \in U^\sharp)\ (\forall \Sigma \subseteq \Omega Y \text{ directed})\ f^{-1}(x) \cap U \subseteq \bigcup \Sigma \Rightarrow \exists S \in \Sigma : x \in S^\sharp$.

Every triquotient map is of effective descent — as shown by Plewe [15]—, while every effective descent morphism is a universal quotient map. It was proved in [12] that universal quotient maps are exactly the *descent maps*, that is those continuous maps $f$ such that the functor $f^*$ is premonadic.

That these three notions do not coincide can be checked already at the level of finite spaces, as shown in [11] (see also [5]). As observed in [11] for the case of finite spaces, if the fibres of $f$ are finite, in (T4) we can replace the directed set of open subsets $\Sigma$ of $Y$ by a single open subset $S$ of $Y$.

**Theorem.** *If $X_1$ and $X_2$ are subspaces of $X$, and $p : X_1 + X_2 \to X$ is the continuous map induced by their subspace embeddings, then the following conditions are equivalent:*

(i) *$p$ is a triquotient map;*
(ii) *$p$ is an effective descent map;*
(iii) *$p$ is a descent map;*
(iv) *$p$ is a universal quotient map;*
(v) *$p$ is an hereditary quotient map;*
(vi) *$X_1 \supseteq \overline{X \setminus X_2}$ and $X_2 \supseteq \overline{X \setminus X_1}$.*

*Proof*: For any continuous map $p$, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v).
(v) $\Rightarrow$ (vi): Following [10, Exercise 2.4.F], $p : X_1 + X_2 \to X$ is an hereditary quotient map if, and only if, for any $x \in X$ and any open subset $U$ of $X_1 + X_2$ containing the fibre $p^{-1}(x)$ of $x$, $x \in \mathrm{int}(p(U))$ (where we denote by $\mathrm{int}(A)$ the interior of $A \subseteq X$ in $X$). Hence, if $p$ is hereditarily a quotient map and $x \in X \setminus \mathrm{int}(X_2)$, then $p^{-1}(x) \not\subseteq X_2$, that is $x \in X_1$ and (vi) follows.

(vi) $\Rightarrow$ (i): As a disjoint union, let $X_1 + X_2 = X_1 \times \{1\} \cup X_2 \times \{2\}$, and consider the inclusion maps $\iota_i : X_i \to X_1 + X_2$, $i = 1, 2$. For each subset $S$ of $X_1 + X_2$, let $S_i := \iota_i^{-1}(S)$, $i = 1, 2$. We recall that $S$ is open in $X_1 + X_2$ if, and only if, $S_i$ is open in $X_i$, $i = 1, 2$. Define, for each open subset $U$ of $X_1 + X_2$,

$$U^\sharp := \mathrm{int}_X(U_1) \cup \mathrm{int}_X(U_2) \cup \{x \in X_1 \cap X_2\ ;\ p^{-1}(x) \subseteq U\}.$$

Then clearly (T1), (T2) and (T3) hold. To check (T4), let $x \in U^\sharp$ and $S \in \Omega(X_1 + X_2)$ with $p^{-1}(x) \cap U \subseteq S$. If $x \in \mathrm{int}_X(U_1)$, then $x \in \mathrm{int}_X(X_1)$ and $(x, 1) \in U$, hence $(x, 1) \in S$. Since $x \in S_1$ and $S_1$ is open in $X_1$, $x \in \mathrm{int}_X(S_1) \subseteq S^\sharp$. For $x \in \mathrm{int}_X(U_2)$ we argue analogously. If $x \in X_1 \cap X_2$ and $p^{-1}(x) \subseteq U$, then $p^{-1}(x) \subseteq S$. By definition of $S^\sharp$, $x \in S^\sharp$.

It remains to be shown that $U^\sharp$ is an open subset of $X$. Let $x \in U_1 \cap U_2$ with $p^{-1}(x) \subseteq U$. Since $U_1$ and $U_2$ are open in $X_1$ and $X_2$ respectively, there exist $V, W \in \Omega X$ such that $V \cap X_1 = U_1$ and $W \cap X_2 = U_2$. Clearly $x \in V \cap W$, and our proof will be completed once we check that $V \cap W \subseteq U^\sharp$: if $y \in V \cap W$ and $y \notin X_2$, then $y \in U_1$ and $y \in \mathrm{int}_X(X_1)$, which implies that $y \in \mathrm{int}_X(U_1)$; if $y \notin X_1$, we conclude similarly that $y \in \mathrm{int}_X(U_2)$. If $y \in X_1 \cap X_2$, then $y \in U_1$ and $y \in U_2$, that is $p^{-1}(y) \subseteq U$ and the result follows. ∎

**Corollary.** *Given a pullback diagram in* **Top**

$$\begin{tikzcd} & X & \\ X_1 \arrow[ur, "g_1"] & & X_2 \arrow[ul, "g_2"'] \\ & X_0 \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

*where $g_1, g_2$ are embeddings, the diagram*

$$\begin{tikzcd} & \mathbf{Top} \downarrow X \arrow[dl, "g_1^*"'] \arrow[dr, "g_2^*"] & \\ \mathbf{Top} \downarrow X_1 \arrow[dr, "f_1^*"'] & & \mathbf{Top} \downarrow X_2 \arrow[dl, "f_2^*"] \\ & \mathbf{Top} \downarrow X_0 & \end{tikzcd}$$

*is a pullback diagram if, and only if, $X_1 \supseteq \overline{X \setminus X_2}$ and $X_2 \supseteq \overline{X \setminus X_1}$.* ∎

We point out that the condition obtained for $p : X_1 + X_2 \to X$ to be of effective descent is weaker than Brown's condition for the classical Van Kampen Theorem (see [2, 9.1.2]), where $X$ has to be covered by $\mathrm{int}(X_1)$ and $\mathrm{int}(X_2)$. This is the case, for instance, when $X = [0, 2]$, $X_1 = [0, 1]$ and $X_2 = [1, 2]$. However, our result does not generalise the classical theorem since ours focus on effective global descent morphisms while the classical one uses effective descent morphisms with respect to a class of covering maps (see [2, 3]).

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