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ON CATEGORIES WITH SEMIDIRECT PRODUCTS

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ABSTRACT: We characterize pointed categories having semidirect products in the sense of D. Bourn and G. Janelidze ([3]) providing necessary and sufficient conditions for a pointed category to admit semidirect products and interpreting these conditions in terms of protomodularity and exactness of certain split chains.

KEYWORDS: protomodular, homological, and semi-abelian category, semidirect product.

1. Introduction

The categorical notion of semidirect product in an arbitrary category with split pullbacks was introduced by D. Bourn and G. Janelidze in [3], generalizing the classical notion of semidirect product between a group B and a B-group (X, ξ).

Semi-abelian categories ([5]) admit semidirect products. This is a consequence of Theorem 3.4 in [3]. It is well known that semi-abelianess is not a necessary condition for the existence of semidirect products. Indeed, trivially, any additive category has semidirect products. Furthermore, the same holds for protomodular topological models of semi-abelian varieties as proved in [2], which are in general not exact categories. They are examples of homological categories with binary coproducts that have semidirect products. Other examples are provided in 4.9 below, where we describe a way to construct categories satisfying these conditions that may not be exact.

The existence of a zero object is not necessary for that purpose (see Theorem 3.4 in [3]). Also in [8] B. Metere and A. Montoli prove that it is enough to assume the existence of an initial object and they construct semidirect products for the category of internal groupoids.

In this paper we consider only the pointed case.

Necessary conditions on a category for defining semidirect products are the existence of split pullbacks and split pushouts, in the sense of [3], and

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protomodularity. These are, obviously, far from being sufficient conditions. Example 4.7 shows that even an homological category with binary coproducts may not have semidirect products.

The points in a category $\mathbf{C}$ that are essentially semidirect products, (called the "free split epimorphisms" in [6]), form a full coreflective subcategory of the one of points and have a neat characterization.

The characterization of the full reflective subcategory of the category of internal actions consisting of those actions which arise from points, in a sense we make precise later, are one of the goals of this work.

In our search we identify a particular kind of action that we call strict action. Assuming regularity of the category $\mathbf{C}$ these are exactly the type of actions we are looking for. In particular, we prove that homological categories with binary coproducts have semidirect products if and only if every internal action is a strict action.

2. The setting

Let us recall the basic notions and results.

Definition 2.1. ([3], 3.2) A category $\mathbf{C}$ with split pullbacks has semidirect products if for every morphism $p\colon E \longrightarrow B$, the pullback functor $p^*\colon \mathsf{Pt}_B(\mathbf{C}) \longrightarrow \mathsf{Pt}_E(\mathbf{C})$ is monadic.

A pointed category $\mathbf{C}$ has semidirect products if and only if $i_B{}^* : \mathsf{Pt}_B(\mathbf{C}) \longrightarrow \mathsf{Pt}_0(\mathbf{C}) \cong \mathbf{C}$ is monadic for every $\mathbf{C}$-object B, where $i_B : 0 \longrightarrow B$ is the unique morphism from the zero object 0 to B. This follows from the fact that, for every $p\colon E \longrightarrow B$ in $\mathbf{C}$, the functor $p^*\colon \mathsf{Pt}_B\mathbf{C} \longrightarrow \mathsf{Pt}_E\mathbf{C}$ has a left adjoint and $(i_B)^* \cong (i_E)^* p^*$.

For the definition to make sense, we need to assume the existence of kernels of split epimorphisms, so that we can define the functor $(i_B)^*$ for every object B, as well as the existence of binary coproducts, for $(i_B)^*$ to have a left adjoint.

If (T^B, η^B, μ^B) is the monad defined on $\mathbf{C}$ by the adjunction above, the components of η^B and μ^B are the unique morphisms with $k_0\eta^B{}_X = \iota_X$ and $k_0\mu^B{}_X = [k_0, \iota_B]k'_0$, as displayed in the diagrams

$$\begin{tikzcd} B\flat X \arrow[r, "k_0"] & X+B \\ X \arrow[u, "\eta_X"] \arrow[ur, "\iota_X"'] & \end{tikzcd} \quad , \quad \begin{tikzcd} B\flat(B\flat X) \arrow[r, "k'_0"] \arrow[d, "\mu_X"'] & (B\flat X)+B \arrow[d, "{[k_0,\iota_B]}"] \\ B\flat X \arrow[r, "k_0"] & X+B \end{tikzcd}$$

where k_0 and k_0' denote the kernels of $[0,1]\colon X+B \longrightarrow B$ and of $[0,1]\colon (B\flat X)+B \longrightarrow B$, respectively.

An algebra for this monad is a pair $(X, \xi\colon B\flat X \longrightarrow X)$ with $\xi {\eta^B}_X = 1$ and $\xi {\mu^B}_X = \xi(1\flat\xi)$.

The monadicity of $(i_B)^*$ for every object B in $\mathbf{C}$ is equivalent to the monadicity of the functor $Pt(\mathbf{C}) \longrightarrow \mathbf{C}\times\mathbf{C}$ which assigns to each point (A,α,β,B) the pair $(ker(\alpha), B)$. In more detail, we have an adjunction

$$\mathbf{C}\times\mathbf{C} \underset{\ker}{\overset{+}{\rightleftarrows}} \mathsf{Pt}(\mathbf{C})$$

between pairs and points where "plus" assigns to a pair (X,B) the point

$$X+B \underset{\iota_2}{\overset{[0,1]}{\rightleftarrows}} B$$

and the kernel functor "ker" assigns to each point (A,α,β,B) the pair (X,B) in the split extension

$$X \xrightarrow{\ker(\alpha)} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B\ .$$

Then an action is a triple (X,ξ,B) and a morphism of internal actions is a pair of morphisms $(f_0,f_1)\colon (X,\xi,B) \longrightarrow (X',\xi',B')$ such that $f_0\xi = \xi'(f_1\flat f_0)$, defining the category $\mathsf{Act}(\mathbf{C})$ of internal actions.

The comparison functor $\Phi\colon \mathsf{Pt}(\mathbf{C}) \longrightarrow \mathsf{Act}(\mathbf{C})$ assigns to each point (A,α,β,B) with a specified kernel, say $k\colon X \longrightarrow A$, the triple (X,ξ,B) where ξ is the unique morphism such that $k\xi = [k,\beta]k_0$, as described in [4].

It is well-known that the comparison functor Φ has a left adjoint L if and only if $\mathbf{C}$ has coequalizers of all reflexive pairs of the form $([k_0,\iota_B],\xi+1)$ for every internal action (X,ξ,B).

By the universal property of the coproduct, this is equivalent to the existence of coequalizers of the pairs $k_0, \iota_X\xi\colon B\flat X \longrightarrow X+B$, for every internal action (X,ξ,B).

The semidirect product of B *and the algebra* (X,ξ,B) *is* $L(X,\xi,B)$ ([3]).

So this is the natural setting to work in.

Throughout, for short, we assume that the category $\mathbf{C}$ *is finitely complete, finitely cocomplete and pointed.*

3. A characterization

We are going to describe necessary and sufficient conditions for the monadicity of the kernel functor $(i_B)^*$, that is for the comparison functor

$$\Phi_B \colon \mathsf{Pt}_B(\mathbf{C}) \longrightarrow \mathsf{Act}_B(\mathbf{C})$$

to be an equivalence, for every object B in $\mathbf{C}$.

For fixed B and an object X in $\mathbf{C}$ we consider the canonical split extensions

$$\begin{tikzcd} B\flat X \arrow[r, "k_0"] & X+B \arrow[r, shift left, "{[0,1]}"] & B \arrow[l, shift left, "\iota_B"] \end{tikzcd}$$

as well as morphisms $(\xi, q, 1_B)$ of split chains

$$\begin{tikzcd} B\flat X \arrow[r, "k_0"] \arrow[d, "\xi"'] & X+B \arrow[r, shift left, "{[0,1]}"] \arrow[d, "q"'] & B \arrow[l, shift left, "\iota_B"] \arrow[d, equal] \\ X \arrow[r, "q\iota_X"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd} .$$

We observe that

(i) if $(A, \alpha, \beta) \in Pt_B(\mathbf{C})$ and $q\iota_X = \ker(\alpha)$ then $\Phi_B(A, \alpha, \beta) = (X, \xi)$;

(ii) if $(X, \xi) \in \mathsf{Act}_B(\mathbf{C})$ and $q = \mathsf{Coeq}\,(k_0, \iota_X\xi)$ then $L_B(X, \xi) = (A, \alpha, \beta)$.

Let us also consider the following diagram

$$\begin{tikzcd} B\flat X \arrow[rr, "k_0"] \arrow[dd, "\xi"'] \arrow[dr] & & X+B \arrow[rr, shift left, "{[0,1]}"] \arrow[dr, "q"] \arrow[dd, "{[k,\beta]}"' near end] & & B \arrow[ll, shift left, "\iota_B"] \arrow[dr, equal] & \\ & X' \arrow[rr, "k'"] & & Q \arrow[rr, shift left, "\pi_q"] \arrow[dl, "c"] & & B \arrow[ll, shift left] \\ X \arrow[rr, "k"] \arrow[ur, "u"] & & A \arrow[rr, shift left, "\alpha"] & & B \arrow[ll, shift left, "\beta"] \arrow[ur, equal] & \end{tikzcd} \tag{1}$$

where $(X, \xi) \in \mathsf{Act}_B(\mathbf{C})$, $(A, \alpha, \beta) \in Pt_B(\mathbf{C})$, $q = \mathsf{Coeq}\,(k_0, \iota_X\xi)$, π_q is the unique morphism such that $\pi_q \cdot q = [0, 1]$, $k' = \ker(\pi_q)$ and $k = ker(\alpha)$.

The two undefined morphisms are exactly the components of the unit u and counit c of the adjunction $L_B \dashv \Phi_B$. Indeed,

- for $(X, \xi) \in \mathsf{Act}_B(\mathbf{C})$, the unit $u_{(X,\xi)}$ is the unique morphism such that $k'u = q\iota_X$, and
- for $(A, \alpha, \beta) \in Pt_B(\mathbf{C})$, the counit is the unique morphism $c_{(A,\alpha,\beta)}$ such that $cq = [k, \beta]$.

It is well-known that the counit $c_{(A,\alpha,\beta)}$ is an isomorphism if and only if $[k, \beta]$ is a regular epimorphism, since the later is the counit of the adjunction $- + B \dashv (i_B)^*$. The unit $u = u_{(X,\xi)}$ is an isomorphism if and only if Φ_B preserves coequalizers of pairs $([k_0, \iota_B], \xi + 1)$: in the diagram

$$\begin{tikzcd}
B\flat(B\flat X) \arrow[r, shift left, "\mu_X"] \arrow[r, shift right, "1\flat\xi"'] \arrow[d] & B\flat X \arrow[r, "\xi"] \arrow[d, "k_0"] & X \arrow[r, "u"] \arrow[d, "q\iota_X"] & X' = \Phi L(X,\xi) \arrow[dl, "k'"] \\
(B\flat X) + B \arrow[r, shift left, "{[k_0,\iota_B]}"] \arrow[r, shift right, "\xi+1"'] \arrow[d, shift right] & X + B \arrow[r, "q"] \arrow[d, shift right] & Q \arrow[d, shift right, "\pi"'] & \\
B \arrow[u, shift right] \arrow[r, equal] & B \arrow[u, shift right] \arrow[r, equal] & B \arrow[u, shift right] &
\end{tikzcd} \tag{2}$$

we have that $\xi = \mathsf{Coeq}\,(\mu_X, 1\flat\xi)$, $\mu_X = \Phi_B([k_0, \iota_B])$, $1\flat\xi = \Phi_B(\xi + 1)$ and $\Phi_B(q) = u\xi$.

Protomodularity of a pointed category $\mathbf{C}$ means that $(i_B)^*$ is conservative for every $\mathbf{C}$-object B (because this is equivalent to have p^* conservative for every $p\colon E \longrightarrow B$) and so Φ_B is conservative if and only if $\mathbf{C}$ is protomodular.

In the pointed case, protomodularity is equivalent to the condition that the Split Short Five Lemma holds ([1], Proposition 3.1.2).

Theorem 3.1. *A category* $\mathbf{C}$ *has semidirect products if and only if the following conditions hold:*

(a) $\mathbf{C}$ *satisfies the Split Short Five Lemma;*
(b) *For fixed* B *and internal action* (X, ξ)*, the split chain*

$$\begin{tikzcd} X \arrow[r, "q\iota_X"] & Q \arrow[r, shift left, "\pi"] & B \arrow[l, shift left, "q\iota_B"] \end{tikzcd},$$

where $q = \mathsf{Coeq}\,(k_0, \iota_X\xi)$ *and* $(Q, \pi, q\iota_B) = L(X, \xi)$*, is a split extension (i.e.* $q\iota_X = \ker(\pi)$*).*

Proof: Let us assume that the category $\mathbf{C}$ admits semidirect products, that is that Φ_B is an equivalence for every object B. Then, since $u_{(X,\xi)}$ is an isomorphism, we obtain (b). The fact that $c_{(A,\alpha,\beta)}$ is an isomorphism implies that for every point (A, α, β), with kernel $k\colon X \longrightarrow A$, the induced morphism $[k, \beta]\colon X + B \longrightarrow A$ is a regular epimorphism and so it is a strong epimorphism. Consequently, the Split Short Five Lemma holds in $\mathbf{C}$.

Conversely, from (b) we derive that the unit $u_{(X,\xi)}$ is an isomorphism for every action (X, ξ). Then, in diagram (1) defining u and c, we use the Split

Short Five Lemma to conclude that c is an isomorphism as well. Since L is full and faithful and Φ is conservative we conclude that Φ is an equivalence. ■

This theorem can be easily reformulated as follows:

Theorem 3.2. *The category* $\mathbf{C}$ *has semidirect products if and only if, for every morphism* $(\xi, q, 1_B)$ *of split chains*

$$\begin{tikzcd} B\flat X \arrow[r, "k_0"] \arrow[d, "\xi"'] & X+B \arrow[r, shift left, "{[0,1]}"] \arrow[d, "q"'] & B \arrow[l, shift left, "\iota_B"] \arrow[d, equal] \\ X \arrow[r, "q\iota_X"] & E \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

where (X, ξ) *is an internal action, we have that*

$$q = \mathsf{Coeq}\,(k_0, \iota_X \xi) \Leftrightarrow q\iota_X = \ker(p).$$

The following result is closer to our objectives and motivates the notion of strict action we introduce later.

Theorem 3.3. *Let us assume, in addition, that* $\mathbf{C}$ *is a regular category. Then* $\mathbf{C}$ *has semidirect products if and only if the following two conditions hold:*

(a) $\mathbf{C}$ *satisfies the Split Short Five Lemma;*

(b) *for every* $(X, \xi) \in \mathsf{Act}_B(\mathbf{C})$, *if a commutative square of the form*

$$\begin{tikzcd} B\flat X \arrow[r, "\xi"] \arrow[d, "k_0"'] & X \arrow[d] \\ X+B \arrow[r] & Q \end{tikzcd}$$

is a pushout then it is also a pullback.

Proof: We only need to prove that condition (b) here is equivalent to condition (b) in Theorem 3.1. Let us assume that 3.1(b) holds. If (X, ξ) is an internal action and (q, l) is the pushout of k_0 along ξ then it is easy to prove that $q = \mathsf{Coeq}\,(k_0, k_0\eta_X\xi = \iota_X\xi)$ and $l = q\iota_X$. Then, since $q\iota_X$ is a monomorphism, the square is a pullback.

Conversely, since $(q, q\iota_X)$ is the pushout of (k_0, ξ), the resulting square is also a pullback, by hypothesis. This implies that $q\iota_X$ is a monomorphism and so $u_{(X,\xi)}$ is a monomorphism. Since, in (2), $(u\xi, k_0)$ is the pullback of (q, k') and the category $\mathbf{C}$ is regular then $u\xi$ is a regular epimorphism. Then

$u\xi$ and ξ regular epimorphisms imply that u is also a regular epimorphism. Consequently, $u_{(X,\xi)}$ is an isomorphism and so $q\iota_X$ is the kernel of the induced morphism π_q, as required. ∎

The assumption that the morphism ξ (as in condition (b) of Theorem 3.3) is an internal action cannot be avoided . The following example shows that even in the category of groups, the paradigmatic example in this context, a pushout of a protosplit (i.e. a kernel of a split epimorphism) along a split epimorphism need not be a pullback. Indeed, the fact that pushouts are not always pullbacks in the situation described is a well-known fact. The following is a very simple example of this phenomenon.

Example 3.4. In Gp consider the following square

$$\begin{tikzcd} F[x, yxy^{-1}, y^2xy^{-2}, ...] \arrow[r, "\alpha"] \arrow[d, "k"'] & F[yxy^{-1}, y^2xy^{-2}, ...] \arrow[d, "l"] \\ F[x,y] \arrow[r, "q"] & F[y] \end{tikzcd}$$

where $F[X]$ denotes the free group on a set X, k is an inclusion, α and q send x to the empty word, and l is the zero morphism. Then, k is the kernel of the split epimorphism $F[x,y] \longrightarrow F[y]$, α is split by the inclusion $F[yxy^{-1}, ...] \longrightarrow F[x, yxy^{-1}, ...]$, and the square is a pushout. However it is not a pullback.

4. Internal actions versus strict actions

The functor Φ_B is conservative, if and only if ${i_B}^* : \mathsf{Pt}_B(\mathbf{C}) \longrightarrow \mathsf{Pt}_0(\mathbf{C})$ is conservative, that is if and only if the counit $[k, \beta]$ is an extremal epimorphism, and this means protomodularity of $\mathbf{C}$.

The functor L_B is conservative if and only if $u_{(X,\xi)}$, or equivalently $q\iota_X$, is an extremal monomorphism for every action (X, ξ).

This fact and Theorem 3.3 motivates the following definitions in an arbitrary category.

Definition 4.1. An *exact span* is a diagram

$$\begin{tikzcd} & Y \arrow[dl, "k"'] \arrow[dr, "\alpha"] & \\ A & & X \end{tikzcd}$$

that can be completed into a commutative square which is at the same time a pushout and a pullback.

Definition 4.2. A *generalized action* of (B, p) in (X, α) is an internal structure of the form

$$\begin{tikzcd} Y \arrow[r, shift left, "\alpha"] \arrow[d, "k"'] & X \arrow[l, shift left, "\beta"] \\ A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd} \tag{3}$$

where $\alpha\beta = 1$, $ps = 1$, $k = ker(p)$ and (k, α) is an exact span.

Definition 4.3. A *strict internal action* is a generalized action where the diagram

$$X \xrightarrow{k\beta} A \xleftarrow{s} B$$

is a coproduct diagram.

A structure as in definition 4.2 is a generalized action if and only if the square

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "k"'] & X \arrow[d, "qk\beta"] \\ A \arrow[r, "q"] & Q \end{tikzcd} \tag{4}$$

is a pullback, or equivalently, $qk\beta$ is a monomorphism, where $q = \text{Coeq}(k, k\beta\alpha)$. In particular it is a strict action whenever $A = X + B$ and $q\iota_X$ is a monomorphism.

Proposition 4.4. *Every strict action is an internal action.*

Proof: In this case (4) becomes

$$\begin{tikzcd} B\flat X \arrow[r, "\alpha"] \arrow[d, "k_0"'] & X \arrow[d, "q\iota_X"] \\ X + B \arrow[r, "q"] & Q \end{tikzcd}$$

and, since it is a pullback, the morphism $q\iota_X$ is a monomorphism. On the other hand, this square is always a pushout and so $q = \mathsf{Coeq}\,(k_0, \iota_X\alpha)$. From that we conclude that $\alpha\eta_X = 1$ and $\alpha\mu_X = \alpha(1\flat\alpha)$, as desired. ∎

Proposition 4.5. *In an homological category with binary coproducts there is an equivalence between strict internal actions and points.*

Proof: It follows from 3.3. ■

Corollary 4.6. *An homological category with binary coproducts has semidirect products if and only if every internal action is strict.*

The following is an example due to G. Janelidze of an homological category with binary coproducts and coequalizers, without semidirect products and so where not all internal actions are strict.

Example 4.7. Let **C** be the quasivariety of groups determined by the axiom

$$(xy)^3 = 1 \Rightarrow xy = yx, \tag{5}$$

let $B = \{1, b\}$ be the 2-element cyclic group, $X = \{1, x, x^2\}$ be the 3-element cyclic group, and suppose that B acts on X by $bx = x^2$ (and so $bx^2 = x$). Then:

(i) B+X in **C** will be the same as in the category of groups and so the action above does exist in **C**;
(ii) the semidirect product of B and X in the category of groups will be the symmetric group on three elements;
(iii) the condition (5) will force it to become isomorphic to B, i.e. the semidirect product of B and X in **C** is isomorphic to B;
(iv) therefore the kernel of π (as in condition (b) of Theorem 3.1) will be trivial instead of being X.

Every quasivariety of groups is a homological category and the one defined above is closed under binary coproducts.

This example is a particular instance of a more general fact. We recall that a variety has semidirect products if and only if it is protomodular(Theorem 3.4 in [3]).

Proposition 4.8. *Let $\mathcal{V}$ be a pointed protomodular variety. Then a quasivariety $\mathcal{Q}$ of $\mathcal{V}$, closed under binary coproducts, has semidirect products if and only if it is closed under semidirect products in $\mathcal{V}$.*

Proof: Being a regular epi-reflective subcategory of a pointed protomodular category, $\mathcal{Q}$ is also pointed and protomodular.

Let us consider the diagram

$$\begin{tikzcd}
B\flat X \arrow[r, "\xi"] \arrow[d] & X \arrow[r, equal] \arrow[d, "q\iota_X"'] & X \arrow[d, "q'\iota_X"] \\
X+B \arrow[r, "q"'] \arrow[d, shift left] & Q \arrow[r, "r_Q"'] \arrow[d, shift left, "\pi_\xi"] & R(Q) \arrow[d, shift left, "R(\pi_\xi)"] \\
B \arrow[r, equal] \arrow[u, shift left] & B \arrow[r, equal] \arrow[u, shift left] & B \arrow[u, shift left]
\end{tikzcd}$$

where $(X,\xi) \in \mathsf{Act}_B(\mathcal{Q})$, q is the coequalizer in $\mathcal{V}$ of the pair $(k_0, \iota_X\xi)$ of $\mathcal{Q}$-morphisms and r_Q is the reflection of Q in $\mathcal{Q}$. Then $q' = r_Q q$ is the coequalizer of $(k_0, \iota_X\xi)$ in $\mathcal{Q}$. Consequently, by the Split Short Five Lemma, $q'\iota_X$ is the kernel of the split epimorphism $R(\pi_\xi)$ if and only if r_Q is an isomorphism. ∎

Examples of categories where every action is strict, other than the categories with semidirect products, are

(1) exact ideal-determined Mal'tsev categories (see [7]),
(2) varieties.

In the first case, the Barr-Kock theorem (see e.g. pg. 441 of [1]) can be used to prove that for each internal action the square

$$\begin{tikzcd}
B\flat X \arrow[r, "\xi"] \arrow[d, "k_0"'] & X \arrow[d, "q\iota_X"] \\
X+B \arrow[r, "q"] & Q
\end{tikzcd}$$

is a pullback.

For varieties, the kernel functor $(i_B)^*$ preserves coequalizers of reflexive pairs because underlying functors from varieties to the category of sets preserve them. Indeed, since coequalizers in $\mathsf{Pt}_B(\mathbf{C})$ are constructed at the level of $\mathbf{C}$, it is enough to see that the pullback functor $(i_B)^* : \mathbf{C} \downarrow B \to \mathbf{C}$ preserves coequalizers of reflexive pairs. If $\mathbf{C}$ is a variety and U is the underlying functor, in the commutative diagram

$$\begin{tikzcd}
\mathbf{C}{\downarrow}B \arrow[r, "(i_B)^*"] \arrow[d, "V"'] & \mathbf{C} \arrow[d, "U"] \\
Set{\downarrow}U(B) \arrow[r, "(U(i_B))^*"] & Set
\end{tikzcd}$$

$(U(i_B))^*V$ preserves coequalizers of reflexive pairs because the same holds both for V and for $(U(i_B))^*$, the later preserving all colimits since it is also a left adjoint.Then the conclusion follows because U is conservative.

It is clear that the same holds for monadic categories over a locally cartesian closed category where, in addition, the corresponding forgetful functor preserves coequalizers of reflexive pairs.

In all these examples the categories are exact. The case of protomodular topological models of semi-abelian varieties shows that exactness is not a necessary condition for this purpose. Our last example provides another instance of non-exact categories where all actions are strict.

Let $\mathcal{S}$ be a semi-abelian category and $\mathcal{M} = \mathsf{Mono}(\mathcal{S})$ be the full subcategory of $\mathcal{S}^2 = \mathrm{Mor}(\mathcal{S})$ with objects all triples (A_1, a, A_2) such that $a\colon A_1 \longrightarrow A_2$ is a monomorphism. Then, $\mathcal{M}$ is a regular epi-reflective subcategory of $\mathcal{S}$.

Properties of categories $\mathcal{M} = \mathsf{Mono}(\mathbf{C})$ of monomorphisms in a category $\mathbf{C}$ were investigated by Ana Helena Roque in [9]. In particular she proves there that $\mathsf{Mono}(\mathbf{C})$ is regular whenever $\mathbf{C}$ is regular and that exactness of $\mathbf{C}$ may not be inherited by $\mathsf{Mono}(\mathbf{C})$. This is the case when $\mathbf{C}$ is the category of groups a fact that goes back at least to N. Yoneda [10], whose main example of a non-abelian category is the category of short exact sequences in an abelian category.

A simple example of non-exactness of the category $\mathsf{Mono}(\mathsf{Gp})$ of monomorphisms in the category of groups is provided by any non-trivial group G and the inclusion of the equality relation $(G, 1_G, G)$ on G into the relation $(G \times G, pr_1, pr_2)$ which is a non-effective equivalence relation on the object $(G, 1_G, G)$ of $\mathsf{Mono}(\mathsf{Gp})$.

The category $\mathcal{S}^2$ is semi-abelian and so, in particular, it is pointed and protomodular. Then $\mathsf{Mono}(\mathcal{S})$ is also pointed and protomodular. Consequently, categories of monomorphisms of semi-abelian categories are examples of homological categories that are not exact in general.

If, furthermore, the coproduct of two monomorphisms in $\mathcal{S}$ is a monomorphism, as it is the case in the category of groups and in every abelian category, we can define there internal actions as above.

If (X, ξ) is an internal action in $\mathsf{Act}_B(\mathcal{M})$, considering the diagram in $\mathcal{S}^2$ with $X = (X_1, x, X_2)$, $B = (B_1, b, B_2)$, $(B_1 + X_1, b + x, B_2 + X_2)$ and $(B_1 \flat X_1, b\flat x, B_2 \flat X_2)$ in $\mathsf{Mono}(\mathcal{S}) = \mathcal{M}$, the (regular epi-mono)-factorization

$m\bar{q}$ of $l : Q_1 \longrightarrow Q_2$ in $\mathcal{S}$

$$\begin{tikzcd}
B_1\flat X_1 \arrow[r, shift left, "k_1"] \arrow[r, shift right, "\iota_{X_1}\xi_1"'] \arrow[d, "b\flat x"'] & B_1+X_1 \arrow[r, "q_1"] \arrow[d, "b+x"] & Q_1 \arrow[r, "\bar{q}"] \arrow[d, "l"] & \hat{Q} \arrow[d, "m"] \\
B_2\flat X_2 \arrow[r, shift left, "k_2"] \arrow[r, shift right, "\iota_{X_2}\xi_2"'] & B_2+X_2 \arrow[r, "q_2"'] & Q_2 \arrow[r, "1_{Q_2}"'] & Q_2
\end{tikzcd} .$$

Being an action in $\mathcal{S}^2$, $(q_1\iota_{X_1}, q_2\iota_{X_2})$ is a monomorphism in $\mathcal{S}^2$ and so $q\iota_X = (\bar{q}q_1\iota_{X_1}, q_2\iota_{X_2})$ is a monomorphism in $\mathcal{M}$ because $q_2\iota_{X_2}$ is a monomorphism in $\mathcal{S}$. Consequently, every action is strict.

Thus, by Corollary 4.6, we have there a good theory of semidirect products.

Proposition 4.9. *If $\mathcal{S}$ is a semi-abelian category where the coproduct of two monomorphisms is a monomorphism then* $\mathsf{Mono}(\mathcal{S})$ *has semidirect products.*

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