Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 10–54

(DISCRETE) ALMANSI TYPE DECOMPOSITIONS: AN UMBRAL CALCULUS FRAMEWORK BASED ON $\mathfrak{osp}(1|2)$ SYMMETRIES

NELSON FAUSTINO AND GUANGBIN REN

Abstract: We introduce the umbral calculus formalism for hypercomplex variables starting from the fact that the algebra of multivariate polynomials $\mathbb{R}[\underline{x}]$ shall be described in terms of the generators of the Weyl-Heisenberg algebra. The extension of $\mathbb{R}[\underline{x}]$ to the algebra of Clifford-valued polynomials $\mathcal{P}$ give rise to an algebra of Clifford-valued operators whose canonical generators are isomorphic to the orthosymplectic Lie algebra $\mathfrak{osp}(1|2)$.

This extension provides an effective framework in continuity and discreteness that allows us to establish an alternative version of Almansi decomposition comprising continuous and discrete versions of classical Almansi theorem in Clifford analysis (c.f. [38, 33]) that corresponds to a meaningful generalization of Fischer decomposition for the subspaces $\ker(D')^k$.

We will discuss afterwards how the symmetries of $\mathfrak{sl}_2(\mathbb{R})$ (even part of $\mathfrak{osp}(1|2)$) are ubiquitous on the recent approach of Render (c.f. [37]), showing that they can be interpreted as the method of separation of variables for the Hamiltonian operator in quantum mechanics.

Keywords: Almansi theorem, Clifford analysis, hypercomplex variables, orthosymplectic Lie algebras, umbral calculus.
AMS Subject Classification (2000): 30G35;35C10;39A12;70H05.

1. Introduction

1.1. The Scope of Problems. In the last two decades considerable attention has been given to the study of polynomial sequences for hypercomplex variables in different contexts. For example, in the approach proposed by Faustino & Kähler (c.f. [20]), rising and lowering factorials that yield e.g. the classical Bernoulli and Euler polynomials (c.f. [44]) are the discrete analogues of homogeneous polynomials that appear in Fischer's decomposition involving difference Dirac operators. The hypercomplex generalization of this polynomials was studied recently in [35] by Malonek & Tomaz in connection with Pascal matrices.

Roughly speaking, the construction of hypercomplex Bernoulli polynomials shall be obtained via Appell sets [8, 34]. These set of polynomials studied

Received December 29, 2010.

at an early stage by ABUL-EZ & CONSTALES in [1] in terms of basic sets of hypercomplex polynomials shall be interpreted as a Cauchy-Kovaleskaya extension of the rising factorials considered in [20]. For a fully explanation of Cauchy-Kovaleskaya extension we refer to [14] (Subsection II.5); a meaningful characterization of Cauchy-Kovaleskaya's extension in interplay with Segal-Bargmann spaces can be found in [10] (Subsection 2.2).

Along with the construction of basic polynomial sequences in hypercomplex variables, other directions have been followed to construct Appell sets, namely by taking the Fueter-Sce extension of complex monomials z^k (c.f. [26]), using a Fourier series expansion of square integrable monogenic functions on the unit ball (c.f. [5]) or alternatively using a Gelfand-Tsetlin basis approach (c.f. [6]) that essentially combines Fischer decomposition with the Cauchy-Kovalevskaya extension.

With respect to the discrete setting, it was further developed in the Ph.D thesis of FAUSTINO (c.f. [22]) that families of discrete polynomials can be constructed as blending between *continuum* and discrete Clifford analysis giving an affirmative answer to the paper of MALONEK & FALCÃO (c.f. [34]).

According to this proposal, discrete Clifford operators underlying the orthogonal group $O(n)$ were introduced by means of representations of the Lie superalgebra $\mathfrak{osp}(1|2)$. Moreover, the refinement of discrete harmonic analysis follows from the representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ as the even part of $\mathfrak{osp}(1|2)$ while the blending between *continuum* and discrete Clifford analysis was obtained via a Sheffer map (c.f.[40, 39]) that essentially maps the homogeneous polynomials onto basic polynomial sequence of binomial type.

This approach combines the radial algebra based approach proposed by SOMMEN in [43] with the umbral calculus approach postponed in [16] by DI BUCCHIANICO,LOEB & ROTA. The main novelty of this new approach rests mostly from the fact that continuous and discrete Clifford analysis are described as realizations of the well-known Wigner quantum systems (c.f. [45]) on which the Sheffer map shall be interpreted as a gauge transformation that keep invariant the symmetries of both systems (see e.g. [23] for a sketch of this approach).

There were still consider alternative constructions of discrete Clifford analysis using different type of symmetries. One of them proposed in the preprint of FAUSTINO & KÄHLER (c.f.[21]), Clifford analysis on symmetric lattices appears as a mimetic description of Hermitian Clifford analysis on which the unitary group $U(n)$ appear as the natural candidate to the induced representations for the algebra of Clifford-valued operators (c.f. [7]). The major obstacle arising this construction follows from the fact that the multiplication operators $x_jT_h^{-j}$ and $x_jT_h^{+j}$ do not commute and hence there is no chance to get a radial algebra structure (c.f. [43]). For a complete survey besides this drawback we refer to [22] (Section 3).

Quite recently, in the recent approach of De Ridder, De Schepper, Kähler & Sommen (c.f.[13]) the Weyl-Heisenberg symmetries encoded in the forward/backward finite difference operators $\partial_h^{\pm j}$ and multiplication operators $x_jT_h^{\mp j}$ were replaced by 'skew'-Weyl symmetries with the purpose to get, in analogy with the Hermitian setting, linear independence between the vector multiplication operators $X^+ = \sum_{j=1}^n \mathbf{e}_j^+ X_j^+$ and $X^- = \sum_{j=1}^n \mathbf{e}_j^- X_j^-$. As a result, the authors show that the Euler polynomials are the resulting discrete polynomials that yield a Fischer decomposition for $D_h^+ + D_h^- = \sum_{j=1}^n \mathbf{e}_j^+ \partial_h^{+j} + \mathbf{e}_j^- \partial_h^{-j}$. Besides this approach there is an open question regarding the symmetries of such system.

Let us turn now our attention for the Almansi decomposition state of art in Clifford and harmonic analysis. The theorem formulated below:

Almansi's Theorem (cf. [2, 3]) *If f is polyharmonic of degree k in a starlike domain with center 0, then there exist uniquely defined functions $f_0, \cdots, f_{k-1}$, each harmonic in Ω such that*

$$f(x) = f_0(\underline{x}) + |x|^2 f_2(\underline{x}) + \cdots + |x|^{2(k-1)} f_{k-1}(\underline{x}).$$

corresponds to the Almansi decomposition for poly-harmonic functions.

One can find important applications and generalizations of this result for several complex variables in the monograph of Aronszajn, Creese & Lipkin, [3], e.g. concerning functions holomorphic in the neighborhood of the origin in $\mathbb{C}^n$. Generalizations of Almansi's Theorem can be found in [38, 33, 12, 36, 37].

For the harmonic case, the importance of this result was recently explored by Render in [37], showing that for functions belonging to the real Bargmann space, there is an intriguing connection between the existence of a Fischer inner pair (c.f. [37]) the problem of uniqueness of polyharmonic functions posed by Hayman in [27] (c.f. [37], Section 9) as well as a characterization for the entire for the Dirichlet problem (c.f. [37], Section 10).

In the Clifford setting, the Almansi theorem shall be understood as a meaningful generalization of Fischer decomposition for hypercomplex variables without requiring a Fischer inner product *a-priori* (c.f [14], pp. 204-207). This result plays a central role in the study of polymonogenic functions likewise in the study of polyharmonic functions as refinements of polymonogenic functions. This was consider in the begining of $90's$ by RYAN [38] to study invariance of iterated Dirac operator in relation to Möbius transformations on manifolds. On the last decade MALONEK & REN established a general framework which describe the decomposition of iterated kernels for different function classes [33, 36]. Besides the approach of COHEN, COLONNA, GOWRISANKARAN & SINGMAN regarding polyharmonic functions on trees and the approaches on Fischer decomposition for difference Dirac operators proposed by FAUSTINO & KÄHLER [20] and DE RIDDER, DE SCHEPPER, KÄHLER & SOMMEN [13], as far as we know, there is no established framework on Almansi-type theorems as a general method for obtaining special representations for discrete hypercomplex functions.

1.2. Motivation of this approach. The umbral calculus formalism proposed by ROMAN & ROTA (c.f. [39, 40]) have received on the last fifteen years the attention of mathematicians and physicists. Besides the papers of DI BUCCHIANICO & LOEB (c.f.[15]) and DI BUCCHIANICO, LOEB, & ROTA (c.f. [16]) devoted to classical aspects of umbral calculus, further applications were developed after the papers of SMIRNOV & TURBINER (c.f. [42]) and DIMAKIS, MÜLLER-HOISSEN & STRIKER in the mid of the $90's$ (c.f. [19]) with special emphasize to systematic discretization of Hamiltonian operators preserving Weyl-Heisenberg symmetries (c.f. [32, 31]), to the construction of Appell sets (c.f. [44]) and complete orthogonal systems of polynomials (c.f. [17]) based on the theory of Sheffer sets likewise in the solution of the Boson-Normal ordering problem in quantum mechanics by combinatorial identities based on binomial sums with the construction of coherent states (c.f. [4]).

When we take the tensor product between the algebra of multivariate polynomials $\mathbb{R}[\underline{x}]$ with the Clifford algebra of signature $(0,n)$ in $\mathbb{R}^n$, the resulting algebra of Clifford-valued polynomials is described in Lie symmetries underlying the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ and the Lie superalgebra $\mathfrak{osp}(1|2)$ (see [18, 23, 11] and the references given there) while the Fischer decomposition of the algebra of homogeneous Clifford-valued polynomials in terms of spherical harmonics and spherical monogenics follows from the Howe dual pair technique (see [6] and references given there) applied to $\mathfrak{sl}_2(\mathbb{R}) \times O(n)$ and $\mathfrak{osp}(1|2) \times O(n)$, respectively. The details of such technique can be found on the papers of Howe (c.f. [29]) and Cheng & Zhang (c.f.[9]).

Although on the last years the Lie (super)algebra framework was successfully applied in Clifford analysis, these kind of algebras are also ubiquitous e.g. in the old works of Wigner (c.f. [45]) and Turbiner (c.f. [41]):

In [45] it was shown that the so-called Wigner quantum systems that describe a motion of a particle on the ambient space $\mathbb{R}^n$ may be characterized in terms of symmetries of $\mathfrak{osp}(1|2n)$; on [41] the eigenfunctions for the Hamiltonian operators were computed explicitly by taking into account the $\mathfrak{sl}_2(\mathbb{R})$ symmetries of such system while the eigenvalues were described as an infinite number of (unplaited) sheets lying on a Riemann surface. Quite recently, in [46] Zhang also apply this framework to the study of quantum analogues for the Kepler problem in the superspace setting.

1.3. Organization of the paper. In this paper we will derive an umbral counterpart for the well known Almansi type decomposition for hypercomplex variables by employing combinatorial and algebraic/geometric techniques regarding umbral calculus (c.f. [19, 16]), radial algebras (c.f. [43]) and the Howe dual pair technique confining nonharmonic analysis and quantum physics (c.f. [28, 30]).

We will start to introduce the umbral calculus framework in the algebra of Clifford-valued polynomials $\mathcal{P} := \mathbb{R}[\underline{x}] \otimes C\ell_{0,n}$ as well as the symmetries preserved under the action of the Sheffer map, showing that there is a mimetic transcription of classical Clifford analysis to discrete Clifford analysis that generalizes complex analysis to higher dimensions (c.f. [14, 25]). Roughly speaking, in umbral calculus the algebra of polynomials $\mathbb{R}[\underline{x}]$ can be recognized as being isomorphic to the algebra generated by position and momentum operators x'_j and O_{x_j}, respectively, satisfying the Weyl-Heisenberg relations

$$[O_{x_j}, O_{x_k}] = 0 = [x'_j, x'_k], \quad [O_{x_j}, x'_k] = \delta_{jk}\mathbf{id}. \tag{1}$$

Here and elsewhere $[\mathbf{a}, \mathbf{b}] := \mathbf{ab} - \mathbf{ba}$ denotes the commuting bracket between $\mathbf{a}$ and $\mathbf{b}$.

Moreover, if we take $\mathbf{e}_1, \ldots, \mathbf{e}_n$ as the Clifford algebra generators satisfying the anti-commuting relations $\{\mathbf{e}_j, \mathbf{e}_k\} := \mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = -2\delta_{jk}$, umbral Clifford analysis (c.f. [23]) deals with the study of the algebra of differential operators

$$\mathrm{Alg}\left\{x'_j, O_{x_j}, \mathbf{e}_j \ : \ j=1,\ldots,n\right\},$$

For a complete survey besides this approach we refer to [22] (Section 3).

Moreover, taking into account the action of the orthosymplectic Lie superalgebra of type $\mathfrak{osp}(1|2)$ on the subspaces $(x')^s \ker D'$ we will derive some recursive relations (Lemma 3.1 and Proposition 3.1) and inversion formulae (Lemmata 3.3 and 3.4) which allow us to decompose the subspace $\ker(D')^k$ (the so-called umbral polymonogenic functions of degree k) as a direct sum of subspaces of the type $P_s(x')\ker D'$, for $s=0,1,\ldots,k-1$, where $P_s(x')$ stands a polynomial type operator of degree s satisfying the mapping property $P_s(x') : \ker(D')^s \rightarrow \ker D'$.

This in turn gives an alternative interpretation for the results obtained by Ryan (c.f. [38]), Malonek & Ren ([33, 36]) and Faustino & Kähler [20] in terms of the symmetries of the Lie superalgebra $\mathfrak{osp}(1|2)$ and the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$.

Finally, in Subsection 3.2 we will give an interpretation for the recent approach of Render (c.f. [37]) showing that for a special choice of potential operator $V_\hbar(x')$ the Almansi decomposition encoded in the quantized Fischer inner pair $\left((2V_\hbar(x'))^k, (\Delta')^k\right)$ is nothing else than a $\mathfrak{sl}_2(\mathbb{R})$ based diagonalization of the Hamiltonian $\mathcal{H}' = -\frac{1}{2}\Delta' + V_\hbar(x')$.

2. Umbral Clifford Analysis

2.1. Umbral calculus revisited. In this section we will review some basic notions regarding umbral calculus. The proof of further results that we will omit can be found in [40, 39, 15] or alternatively in [22], Chapter 1.

In the following, we will set by $\mathbb{R}[\underline{x}]$ the ring of polynomials over $\underline{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, by $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$ the multi-index over $\mathbb{N}_0^n$ and by $\underline{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ the monomial over $\underline{x}$ The partial derivative with respect to x_j will be denoted by $\partial_{x_j} := \frac{\partial}{\partial x_j}$ while the gradient $\partial_{\underline{x}}$ corresponds to the $n-$tuple $\partial_{\underline{x}} := (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$.

Here and elsewhere, we will also adopt the following notations:

$$\partial_{\underline{x}}^\alpha := \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\ldots\partial_{x_n}^{\alpha_n}, \ \ \alpha! = \alpha_1!\alpha_2!\ldots\alpha_n!, \ \ \left(\begin{array}{c}\beta\\ \alpha\end{array}\right) = \frac{\beta!}{\alpha!(\beta-\alpha)!}, \ \ |\alpha| = \sum_{j=1}^n \alpha_j.$$

By means of the differentiation formulae $\partial_{\underline{x}}^\alpha \underline{x}^\beta = 0$ for $|\alpha| > |\beta|$ and $\partial_{\underline{x}}^\alpha \underline{x}^\beta = \frac{\beta!}{\alpha!}\ \underline{x}^{\beta-\alpha}$ for $|\alpha| \leq |\beta|$, it turns out the representation of the binomial formula in terms of the gradient operator $\partial_{\underline{x}}$:

$$(\underline{x}+\underline{y})^{\beta}=\sum_{|\alpha|=0}^{|\beta|}\begin{pmatrix}\beta\\ \alpha\end{pmatrix}\underline{x}^{\alpha}\underline{y}^{\beta-\alpha}=\sum_{|\alpha|=0}^{\infty}\frac{[\partial_{\underline{x}}^{\alpha}\underline{x}^{\beta}]_{\underline{x}=\underline{y}}}{\alpha!}\underline{x}^{\alpha} \tag{2}$$

Linearity arguments shows that the extension of the above formula to the ring of polynomials $\mathbb{R}[\underline{x}]$ corresponds to $f(\underline{x}+\underline{y})=\exp(\underline{y}\cdot\partial_{\underline{x}})f(\underline{x})$, where $\exp(\underline{y}\cdot\partial_{\underline{x}})=\sum_{|\alpha|=0}^{\infty}\frac{\underline{y}^{\alpha}}{\alpha!}\partial_{\underline{x}}^{\alpha}$ denotes the formal power series representation of the shift operator $T_{\underline{y}}f(\underline{x})=f(\underline{x}+\underline{y})$.

An operator $Q\in\mathrm{End}(\mathbb{R}[\underline{x}])$ is shift-invariant if and only if it commutes with $T_{\underline{y}}=\exp(\underline{y}\cdot\partial_{\underline{x}})$ for all $P\in\mathbb{R}[\underline{x}]$ and $\underline{y}\in\mathbb{R}^{n}$:

$$[Q,T_{\underline{y}}]P(\underline{x}):=Q(T_{\underline{y}}P(\underline{x}))-T_{\underline{y}}(Q\ P(\underline{x}))=0.$$

Under the shift-invariance condition for Q, the first expansion theorem (c.f. [15]) states that any linear operator $Q:\mathbb{R}[\underline{x}]\to\mathbb{R}[\underline{x}]$ is shift-invariant if and only if Q is given by the following formal power series:

$$Q=\sum_{|\alpha|=0}^{\infty}\frac{a_{\alpha}}{\alpha!}\partial_{\underline{x}}^{\alpha},\ \ \text{with } a_{\alpha}=[Q\underline{x}^{\alpha}]_{\underline{x}=\underline{0}}\,.$$

Set $O_{\underline{x}}=(O_{x_1},O_{x_2},\ldots,O_{x_n})$ as a multivariate operator. We say that $O_{\underline{x}}$ is shift-invariant operator if and only if $O_{x_1},O_{x_2},\ldots,O_{x_n}$ are shift-invariant too. Moreover, $O_{\underline{x}}$ is a multivariate delta operator if and only if there is a non-vanishing constant c such that $O_{x_j}(x_k)=c\delta_{jk}$, holds for all $j,k=1,2,\ldots,n$. It can be shown that if $O_{\underline{x}}$ is a multivariate delta operator, each O_{x_j} lowers the degree of a polynomial $P(\underline{x})\in\mathbb{R}[\underline{x}]$. In particular $O_{x_j}(c)=0$ for each non-vanishing constant c (c.f. [22], Lemmata 1.1.8 and 1.1.9) and hence, any multivariate delta operator $O_{\underline{x}}$ uniquely determines a polynomial sequence of binomial type $\{V_{\alpha}(\underline{x})\ :\ \alpha\in\mathbb{N}_0^n\}$, (c.f.[22] Theorems 1.1.12 and 1.1.13):

$$V_{\beta}(\underline{x}+\underline{y})=\sum_{|\beta|=0}^{|\alpha|}\begin{pmatrix}\beta\\ \alpha\end{pmatrix}V_{\alpha}(\underline{x})V_{\beta-\alpha}(\underline{y}) \tag{3}$$

such that $V_{\underline{0}}(\underline{x})=1$, $V_{\alpha}(\underline{0})=\delta_{\alpha,0}$ and $O_{x_j}V_{\alpha}(\underline{x})=\alpha_jV_{\alpha-\mathbf{v}_j}(\underline{x})$, where $\mathbf{v}_j$ stands the $j-$element of the canonical basis of $\mathbb{R}^n$.

The Pincherle derivative of O_{x_j} with respect to x_j is defined formally as the commutator between O_{x_j} and x_j:

$$O'_{x_j}f(\underline{x}):=[O_{x_j},x_j]f(\underline{x})=O_{x_j}(x_jf(\underline{x}))-x_j(O_{x_j}f(\underline{x})),$$

This canonical operator plays an important role in the construction of basic polynomial sequences (c.f. [19, 17, 4]). The subsequent results allows us to determine in which conditions the operator $(O'_{x_j})^{-1}$ exists.

We will start with the following lemma:

Lemma 2.1. *The Pincherle derivative of a shift-invariant operator Q is shift-invariant.*

Regardless the last lemma one looks to shift-invariant operators Q as formal power series $Q(\underline{x}) = \sum_\alpha \frac{a_\alpha}{\alpha!}\underline{x}^\alpha$ obtained *viz* the replacement of $\underline{x}$ by the gradient operator $\partial_{\underline{x}}$, i.e. $\iota[Q(\underline{x})] = Q(\partial_{\underline{x}})$ where $\iota : \widehat{\mathbb{R}[\underline{x}]} \to \mathrm{End}(\mathbb{R}[\underline{x}])$ is defined as a mapping between the algebra of formal power series $\widehat{\mathbb{R}[\underline{x}]}$ and the algebra of linear operators acting on $\mathbb{R}[\underline{x}]$ $\mathrm{End}(\mathbb{R}[\underline{x}])$. According to the isomorphism theorem (see [39], page 7, Theorem 2.1.1.), ι is one-to-one and onto. This in turn leads to the following proposition:

Proposition 2.1. *An shift-invariant operator Q has its inverse if and only if $Q1 \neq 0$.*

From Lemma 2.1 notice that O'_{x_j} is shift-invariant whenever O_{x_j} is shift-invariant. Since from the definition $O'_{x_j}(\mathbf{1}) = O_{x_j}(x_j)$ and $O_{x_j}(x_j)$ is a non-vanishing constant, Proposition 2.1 asserts that $(O'_{x_j})^{-1}$ exists locally as a formal series expansion involving multi-index derivatives $\partial^\alpha_{\underline{x}}$.

The former description in terms of Pincherle derivatives allows us to determine basic polynomial sequences $V_\alpha(\underline{x})$ as polynomial sequences obtained from the action of $(\underline{x}')^\alpha := \prod_{k=1}^n (x'_k)^{\alpha_k}$ with $x'_k := x_k(O'_{x_k})^{-1}$ on the constant polynomial $\Phi = \mathbf{1}$ (see also [39], page 51, Corollary 3.8.2):

$$V_\alpha(\underline{x}) = (\underline{x}')^\alpha \mathbf{1}. \tag{4}$$

The properties of basic polynomial sequences are naturally characterized within the extension of the mapping property $\Psi_{\underline{x}} : \underline{x}^\alpha \mapsto V_\alpha(\underline{x})$ to $\mathbb{R}[\underline{x}]$. According to [39], this mapping is the well-know Sheffer map that link two basic polynomial sequences of binomial type. It is clear from the construction that $\Psi_{\underline{x}}^{-1}$ exists and it is given by the linear extension of $\Psi_{\underline{x}}^{-1} : V_\alpha(\underline{x}) \mapsto \underline{x}^\alpha$ to $\mathbb{R}[\underline{x}]$ leading to the following properties on $\mathbb{R}[\underline{x}]$:

$$O_{x_j} = \Psi_{\underline{x}}\partial_{x_j}\Psi_{\underline{x}}^{-1}, \quad \text{and} \quad x'_j = \Psi_{\underline{x}} x_j \Psi_{\underline{x}}^{-1}.$$

From the border view of quantum mechanics, the $2n+1$ operators $x'_1, \ldots, x'_n$, $O_{x_1}, \ldots, O_{x_n}$ and **id** generate the Bose algebra isomorphic to $\mathbb{R}[\underline{x}]$ (c.f. [16]).

Indeed for $\Phi = \mathbf{1}$ (the so-called vacuum vector) $O_{x_j}(\Phi) = 0$ holds for each $j = 1, \ldots, n$ while the raising and lowering operators, $x'_j : V_\alpha \mapsto V_{\alpha+\mathbf{v}_j}$ and $O_{x_j} : V_\alpha \mapsto \alpha_j \, V_{\alpha-\mathbf{v}_j}$ respectively, satisfy the Weyl-Heisenberg relations given by (1).

Due to this correspondence, we would like to stress that the quantum mechanical description of umbral calculus give us many degrees of freedom for constructing raising operators $x'_j : V_\alpha(\underline{x}) \mapsto V_{\alpha+\mathbf{v}_j}(\underline{x})$ in such way that the commuting relations (1) fulfil. In particular, in [19, 17] it was postponed the importance to consider the following symmetrized versions of the raising operators $x_j(O'_{x_j})^{-1}$:

$$x'_j = \frac{1}{2}(x_j(O'_{x_j})^{-1} + (O'_{x_j})^{-1}x_j) \tag{5}$$

as a special type of canonical discretization.

2.2. Basic operators. In what follows we will use the notation introduced in Section 2.1. Additionally we will define $C\ell_{0,n}$ as the algebra determined by the set of vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ satisfying the graded anti-commuting relations

$$\{\mathbf{e}_j, \mathbf{e}_k\} = -2\delta_{jk}, \tag{6}$$

where $\{\mathbf{a}, \mathbf{b}\} = \mathbf{ab} + \mathbf{ba}$ denotes the anti-commuting bracket between $\mathbf{a}$ and $\mathbf{b}$.

The above algebra is commonly known in literature as the Clifford algebra of signature $(0, n)$ (c.f. [14], Chapters 0 & I; [25], Chapter 1) which corresponds particular example of an algebra of radial type since the anti-commutator $\{\mathbf{e}_j, \mathbf{e}_k\}$ (scalar-valued quantity) commutes with all the basic vectors $\mathbf{e}_j$:

$$\{ \ [\mathbf{e}_j, \mathbf{e}_k] \, , \mathbf{e}_l \ \} = 0, \ \text{ for all } j, k, l = 1, \ldots, n. \tag{7}$$

For further details concerning the construction of $C\ell_{0,n}$ as an algebra of radial type we refer to [43].

Additionally, we will denote by $\mathcal{P} = \mathbb{R}[\underline{x}] \otimes C\ell_{0,n}$ the algebra of all Clifford-valued polynomials and by $\mathrm{End}(\mathcal{P})$ the algebra of all linear operators acting on $\mathcal{P}$. The Weyl-Heisenberg character of the operators x'_j and O_{x_j} combined with the radial character of the generators underlying the Clifford algebra $C\ell_{0,n}$ allows us to define umbral Clifford analysis as the study of the algebra of differential operators

$$\mathrm{Alg}\left\{x'_j, O_{x_j}, \mathbf{e}_j \ : \ j = 1, \ldots, n\right\} \tag{8}$$

Furthermore, the umbral counterparts Dirac operator, vector variable and Euler operator, D', x' and E' respectively, given by

$$D' = \sum_{j=1}^{n} \mathbf{e}_j O_{x_j}, \tag{9}$$

$$x' = \sum_{j=1}^{n} \mathbf{e}_j x'_j, \tag{10}$$

$$E' = \sum_{j=1}^{n} x'_j O_{x_j}, \tag{11}$$

correspond to linear combinations of the elements of the algebra (8).

In this context, the operators (9)-(11) shall be understood as basic left endomorphisms acting on the algebra End($\mathcal{P}$):

$$x' : F(\underline{x}) \mapsto \sum_{j=1}^{n} \mathbf{e}_j x'_j(F(\underline{x})),$$

$$x' : F(\underline{x}) \mapsto \sum_{j=1}^{n} \mathbf{e}_j O_{x_j}(F(\underline{x})),$$

$$E' : F(\underline{x}) \mapsto \sum_{j=1}^{n} x'_j O_{x_j}(F(\underline{x})).$$

Along this paper we will use several times the notation $\Delta' := \sum_{j=1}^{n} O_{x_j}^2$ when we refer to the umbral counterpart of the Laplace operator $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$.

The next lemma naturally follows from straightforward computations obtained by direct combination of the relations (1) and (6):

Lemma 2.2 (c.f. [22], Lemma 3.4.3, pp. 68)**.** *The operators $x', D', E' \in$ End($\mathcal{P}$) satisfy the following anti-commutation relations*

$$\{x', x'\} = -2\sum_{j=1}^{n}(x'_j)^2, \;\; \{D', D'\} = -2\Delta', \;\; \{x', D'\} = -2E' - n\mathbf{id}.$$

When acting on the polynomial $\Phi = \mathbf{1}$, from the first relation of Lemma 2.2, the quantity $-(x')^2(\mathbf{1})$ is scalar-valued while from the second relation of Lemma 2.2 $\Delta' := -(D')^2$ is a second order operator satisfying the vanishing condition $\Delta'(\mathbf{1}) = 0$. On the other hand, the third relation of Lemma 2.2, the action of the umbral Euler operator on the algebra $\mathrm{End}(\mathcal{P})$ we can recast E' in terms of the graded anti-commuting relation between x' and D':

$$E' = \sum_{j=1}^{n} x'_j O_{x_j} = -\frac{1}{2}\left(\{x', D'\} + n\mathbf{id}\right). \tag{12}$$

The acting both sides of the above identity on the polynomial $\Phi = \mathbf{1}$, we recast the dimension of the ambient space $\mathbb{R}^n$ as $n = -D'(x'\mathbf{1})$. So, the polynomial $\Phi = \mathbf{1}$ shall be interpreted as the corresponding ground level eigenstate while the dimension of the ambient space $\mathbb{R}^n$ appears as twice of the ground level energy associated to the harmonic oscillator containing n degrees of freedom.

It is also clear from Lemma 2.2 that the anti-commutators $\{x', x'\}$, $\{D', D'\}$ and $\{x', D'\}$ are scalar operators. Thus, from (9),(10) and (12) $(x')^2$, $-(D')^2$ and E' shall be view as generalizations for the norm squared of a vector variable in the Euclidean space, the Laplacian operator and Euler operator, respectively. From the border point of quantum mechanics, the operators $-\frac{1}{2}(x')^2$ and $-\frac{1}{2}\Delta'$ describe a spherical potential and kinetic energy, respectively.

Here we would like also to stress that the umbral Euler operator E' (see identity (11)) comprises at the same time the concept of directional derivative introduced by Howe (c.f. [28]) for quantum groups with the concept of non-shift-invariant mixed/number operator given Di Bucchianico, Loeb & Rota (c.f. [16]).

We will end this subsection by showing and discussing some examples regarding the construction of the operators (9)-(11).

Example 2.1. *If we take $O_{x_j} = \partial_{x_j}$, D' and x' coincide with the standard Dirac and coordinate variable operators, respectively:*

$$D = \sum_{j=1}^{n} \mathbf{e}_j \partial_{x_j}, \quad x = \textstyle\sum_{j=1}^{n} \mathbf{e}_j x_j.$$

while $E = \sum_{j=1}^{n} x_j \partial_{x_j}$ corresponds to the classical Euler operator.

Furthermore, the continuum Hamiltonian $\frac{1}{2}\left(-\Delta + |\underline{x}|^2\right)$ can we rewritten as

$$\frac{1}{2}\left(-\Delta + |\underline{x}|^2\right) = \frac{1}{2}(D^2 - x^2),$$

where the quantity $\frac{1}{2}|\underline{x}|^2 = -\frac{1}{2}x^2$ *corresponds to a spherical symmetric potential operator.*

Example 2.2. *Next we will consider* D' *as a difference Dirac operator given in terms of the forward differences* $\partial_h^{+j} f(\underline{x}) = \frac{f(\underline{x}+h\mathbf{v}_j)-f(\underline{x})}{h}$*:*

$$D' = \sum_{j=1}^{n} \mathbf{e}_j \partial_h^{+j} = \sum_{j=1}^{n} \mathbf{e}_j \frac{T_{h\mathbf{v}_j} - \mathbf{id}}{h}.$$

The square of D' *corresponds to* $(D')^2 = -\frac{1}{h^2}\sum_{j=1}^{n}(T_{2h\mathbf{v}_j} - 2T_{h\mathbf{v}_j} + \mathbf{id})$. *On the other hand, the formal series expansion for* ∂_h^{+j} *is given by* $\partial_h^{+j} = \frac{1}{h}\left(\exp(h\partial_{x_j}) - \mathbf{id}\right)$, *and* $[\partial_h^{+j}, x_j] = T_{h\mathbf{v}_j} = \exp(h\partial_{x_j})$ *is the Pincherle derivative for* ∂_h^{+j}.

Thus the operator x' *corresponds to*

$$x' = \sum_{j=1}^{n} \mathbf{e}_j \ x_j T_{-h\mathbf{v}_j} = \sum_{j=1}^{n} \mathbf{e}_j \ x_j \exp(-h\partial_{x_j}).$$

Alternatively, using relation (5), the operator x' *can also be taken as*

$$x' = \frac{1}{2}\sum_{j=1}^{n} \mathbf{e}_j \ \left(x_j T_{-h\mathbf{v}_j} + T_{-h\mathbf{v}_j} x_j\right) = \frac{1}{2}\sum_{j=1}^{n} \mathbf{e}_j \ \left(x_j \exp(-h\partial_{x_j}) + \exp(-h\partial_{x_j}) x_j\right).$$

Then, take into account the definition (11) and the relation (12), we can consider two different constructions for E'*:*

- $E' = \sum_{j=1}^{n} x_j T_{-h\mathbf{v}_j} \partial_h^{+j} = \sum_{j=1}^{n} x_j \partial_h^{-j}$;
- $E' = \frac{1}{2}\sum_{j=1}^{n} \left(x_j T_{-h\mathbf{v}_j} + T_{-h\mathbf{v}_j} x_j\right) \partial_h^{+j} = \frac{1}{2}\sum_{j=1}^{n} x_j \partial_h^{-j} + \frac{1}{2}\sum_{j=1}^{n} T_{-h\mathbf{v}_j} x_j \partial_h^{+j}$.

Hereby $\partial_h^{-j} = \frac{1}{h}(\mathbf{id} - T_{-\mathbf{v}_j})$ *corresponds to the backward finite difference operator.*

Example 2.3. *Now we will replace the forward finite differences* ∂_h^{+j} *used in the definition of* D' *in Example 2.2 by a central difference operator on the equidistant grid* $h\mathbb{Z}^n$*:*

$$O_{x_j} f(\underline{x}) = \frac{f(\underline{x} + h\mathbf{v}_j) - f(\underline{x} - h\mathbf{v}_j)}{2h}.$$

The formal series expansion for these operators correspond to

$$O_{x_j} = \frac{1}{2h}\left(\exp\left(h\partial_{x_j}\right) - \exp\left(-h\partial_{x_j}\right)\right) = \frac{1}{2h}\sinh\left(h\partial_{x_j}\right)$$

and moreover the formal series expansion for D' is given by

$$D' = \frac{1}{2h}\sum_{j=1}^{n} \mathbf{e}_j \ \sinh\left(h\partial_{x_j}\right).$$

The square of $(D')^2$ splits the star laplacian on a equidistant grid with meshwidth $2h$:

$$-(D')^2 = \sum_{j=1}^{n} \frac{\exp(2h\partial_{x_j}) - 2\mathbf{id} + \exp(-2h\partial_{x_j})}{4h^2} = \sum_{j=1}^{n} \frac{T_{2h\mathbf{v}_j} - 2\mathbf{id} + T_{-2h\mathbf{v}_j}}{4h^2}.$$

Therefore, the construction of the operators x' and E' shall be take into account the following formal series expansion for O'_{x_j} :

$$O'_{x_j} f(\underline{x}) = \frac{f\left(\underline{x} + h\mathbf{v}_j\right) + f\left(\underline{x} - h\mathbf{v}_j\right)}{2} = \cosh\left(h\partial_{x_j}\right) f(\underline{x}).$$

Using the relation $\cosh\left(h\partial_{x_j}\right) = \frac{1}{h}\exp\left(-h\partial_{x_j}\right)\left(\mathbf{id} - \exp\left(2h\partial_{x_j}\right)\right)$ combined with the standard Von Neumann series expansion of $\left(\mathbf{id} - \exp\left(2h\partial_{x_j}\right)\right)^{-1}$ in the $C^\infty-$topology, we get the following asymptotic expansion for $(O'_{x_j})^{-1}$:

$$(O'_{x_j})^{-1} = -h\left(\mathbf{id} - \exp\left(2h\partial_{x_j}\right)\right)^{-1}\exp\left(h\partial_{x_j}\right) = -h\sum_{k=0}^{\infty}\exp\left((2k+1)h\partial_{x_j}\right),$$

or equivalently $(O'_{x_j})^{-1} = -h\sum_{k=0}^{\infty} T_{(2k+1)h\mathbf{v}_j}$. The above inverse only exists whenever $\left\|T_{2h\mathbf{v}_j}\right\| = \left\|\exp\left(2h\partial_{x_j}\right)\right\| < 1$.

Alternatively, we can express $(O'_{x_j})^{-1}$ by taking into account the following formal integral representation in terms of the Laplace transform $(\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t)\ dt$ (c.f. [16]):

$$(O'_{x_j})^{-1} = -h\int_0^\infty e^{-st}\exp(h(2t+1)\partial_{x_j})\ dt = -h\int_0^\infty e^{-st} T_{h(2t+1)\mathbf{v}_j}\ dt.$$

The umbral Dirac operator introduced in Example 2.2 corresponds to the forward difference Dirac operator introduced by Faustino & Kähler in [20]. Here we would like to notice that in Example 2.2, the square $(D')^2$ does not split the star laplacian Δ_h defined below:

$$\Delta_h f(\underline{x}) = \sum_{j=1}^{n} \frac{f(\underline{x} + h\mathbf{v}_j) + f(\underline{x} - h\mathbf{v}_j) - 2f(\underline{x})}{h^2},$$

which means that discrete harmonic analysis can not be refined in terms of discrete Dirac operators involving only forward differences (c.f. [20]).

As we see in Example 2.3, the computation of the inverse for $O'_{x_j} = \cosh\left(h\partial_{x_j}\right)$ is cumbersome and involves infinite sums or integral representations. However, in the case when periodic boundary conditions of the type $\underline{x} + hN\mathbf{v}_j = \underline{x}$ for certain $N \in \mathbb{N}$ are imposed on the equidistant lattice $h\mathbb{Z}^n$, in [19] (see Section 5) the authors compute explicitly $(O'_{x_j})^{-1}$ as a finite sum involving powers of $T_{h\mathbf{v}_j} = \exp\left(h\partial_{x_j}\right)$.

On the other hand, contrary to Example 2.2, the operators x' and D' obtained as mimetic transcription of classical Clifford analysis to the discrete setting *viz* intertwining properties on the algebra of Clifford-valued polynomials $\mathcal{P} = \mathbb{R}[\underline{x}] \otimes C\ell_{0,n}$:

$$\Psi_{\underline{x}} D = D'\Psi_{\underline{x}}, \quad \Psi_{\underline{x}} x = x'\Psi_{\underline{x}}, \quad \Psi_{\underline{x}} E = E'\Psi_{\underline{x}} \tag{13}$$

concern with the nearest neighbor points together with all the points contained in each direction $h\mathbf{v}_j$. Hereby, $\Psi_{\underline{x}}$ is the Sheffer map introduced in Section 2.1.

Here we would also like to stress that $\Delta_{2h} = -(D')^2$ is defined on the equidistant lattice $(2h)\mathbb{Z}^n$. So, the periodicity as well as the coarsening of lattice is the price that we must pay in order to get discrete Clifford analysis as a refinement of discrete harmonic analysis underlying the orthogonal group $O(n)$.

2.3. Orthosymplectic Lie Algebra Representation. The main objective of this subsection is to gather a fully description of the Clifford operators defined on Subsection 2 as a representation of the orthosymplectic Lie algebra $\mathfrak{osp}(1|2)$. We will start to recall some basic definitions underlying the Lie algebra setting. A comprehensive survey of this topic can be found in [28, 24].

The orthosymplectic Lie algebra of type $\mathfrak{osp}(1|2)$ is defined as a direct sum of linear spaces

$$\text{span}\left\{\mathbf{p}^-, \mathbf{p}^+, \mathbf{q}\right\} \oplus \text{span}\left\{\mathbf{r}^-, \mathbf{r}^+\right\}$$

equipped with the standard graded commutator $[\cdot,\cdot]$ such that $\mathbf{p}^-, \mathbf{p}^+, \mathbf{q}, \mathbf{r}^-$ and $\mathbf{r}^+$ satisfy the following standard commutation relations (see e.g. [24]):

$$\begin{aligned}
\left[\mathbf{q}, \mathbf{p}^{\pm}\right] &= \pm\mathbf{p}^{\pm}, \quad [\mathbf{p}^+, \mathbf{p}^-] = 2\mathbf{q}, \\
\left[\mathbf{q}, \mathbf{r}^{\pm}\right] &= \pm\frac{1}{2}\mathbf{p}^{\pm}, \quad [\mathbf{r}^+, \mathbf{r}^-] = \tfrac{1}{2}\mathbf{q}, \\
\left[\mathbf{p}^{\pm}, \mathbf{r}^{\mp}\right] &= -\mathbf{r}^{\pm}, \quad [\mathbf{r}^{\pm}, \mathbf{r}^{\pm}] = \pm\tfrac{1}{2}\mathbf{p}^{\pm}.
\end{aligned}$$

Here we would like to point out that on above construction, the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ appears as a refinement of $\mathfrak{osp}(1|2)$ in the sense that the canonical generators $\mathbf{p}^-, \mathbf{p}^+, 2\mathbf{q}$ itself generate $\mathfrak{sl}_2(\mathbb{R})$. In particular $\mathfrak{sl}_2(\mathbb{R})$ corresponds to the even part of $\mathfrak{osp}(1|2)$.

The subsequent lemma leads up to an isomorphism to a canonical realization of the orthosymplectic Lie algebra of type $\mathfrak{osp}(1|2)$ in terms of the operators x', D' and $E' + \frac{n}{2}\mathbf{id}$. We leave the proof of the following lemma to Appendix A.

Lemma 2.3 (See Appendix A). *The operators x',D' and $E' + \frac{n}{2}\mathbf{id}$ generate a finite-dimensional Lie algebra in End($\mathcal{P}$). The remaining commutation relations are*

$$\begin{array}{lll}
\left[x', (x')^2\right] = 0, & [x', -\Delta'] = -2D', & [E' + \dfrac{n}{2}\mathbf{id}, x'] = x' \\
\left[D', (x')^2\right] = 2x', & [D', -\Delta'] = 0, & [E' + \dfrac{n}{2}\mathbf{id}, D'] = -D' \\
\left[(x')^2, -\Delta'\right] = 4\left(E' + \dfrac{n}{2}\mathbf{id}\right), & \left[E' + \frac{n}{2}\mathbf{id}, -(x')^2\right] = -2(x')^2, & \left[E' + \dfrac{n}{2}\mathbf{id}, -\Delta'\right] = 2\Delta'
\end{array}$$

Furthermore, the standard commutation relations for $\mathfrak{osp}(1|2)$ are obtained *viz* the following normalization

$$\mathbf{p}^- = -\tfrac{1}{2}\Delta', \ \ \mathbf{p}^+ = -\tfrac{1}{2}(x')^2, \ \ \mathbf{q} = \tfrac{1}{2}\left(E' + \tfrac{n}{2}\mathbf{id}\right) \ \ \mathbf{r}^+ = \tfrac{1}{2\sqrt{2}}ix', \ \ \mathbf{r}^- = \tfrac{1}{2\sqrt{2}}iD'$$

and hence, $\mathbf{p}^+ = \frac{1}{2}(x')^2$, $\mathbf{p}^- = \frac{1}{2}\Delta$ and $2\mathbf{q} = E' + \frac{n}{2}\mathbf{id}$ correspond to the canonical generators of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$.

In brief, the above description establishes a parallel with the *continuum* versions of Clifford analysis (c.f [14]) and harmonic analysis (c.f. [30]) as representations of $\mathfrak{osp}(1|2)$ and $\mathfrak{sl}_2(\mathbb{R})$, respectively. This also establishes the with the celebrated Wigner Quantum systems introduced by Wigner in [45] in the sense that $\mathfrak{osp}(1|2)$ encode the $n-$dimensional discrete harmonic oscillator.

3. Almansi-type theorems in (discrete) Clifford analysis

3.1. Main Result. In this section, we will derive an Almansi-type theorem based on replacements of the operators $(x')^k$ by polynomial type operators $P_k(x')$ such that $(D')^k P_k(x') : \ker D' \to \ker D'$ is an isomorphism. For a sake a simplicity, we leave for Appendix B the proofs of the technical results regarding the proof of our main result.

We will start to pointing out the following definitions:

Definition 3.1. *Let Ω be a domain in $\mathbb{R}^n$ and $k \in \mathbb{N}$. A function $f : \Omega \longrightarrow C\ell_{0,n}$ is Umbral polymonogenic of degree k if $(D')^k f(\underline{x}) = 0$ for all $\underline{x} \in \Omega$. If $k = 1$, it is called umbral monogenic function.*

Definition 3.2. *A domain $\Omega \subset \mathbf{R}^n$ is a starlike domain with center 0 if with $\underline{x} \in \Omega$ also $t\underline{x} \in \Omega$ holds for any $0 \leq t \leq 1$.*

For any $k \in \mathbb{N}$, we denote by $Q'_k : \ker D' \to \ker D'$ the right inverse of $(D')^k P_k(x')$, i.e.

$$(D')^k P_k(x')(Q'_k f) = f, \quad \text{for all } f \in \ker D'.$$

Thus, the Almansi theorem can be formulated as follows:

Theorem 3.1. *Let Ω be a starlike domain in $\mathbb{R}^n$ with center 0. If f is a umbral polymonogenic function in Ω of degree k, then there exist unique functions $f_0, f_1, \dots, f_{k-2}, f_{k-1}$, each umbral monogenic in Ω, such that*

$$f(\underline{x}) = P_0(x')f_0(\underline{x}) + P_1(x')f_1(\underline{x}) + \cdots + P_{k-1}(x')f_{k-1}(\underline{x}). \tag{14}$$

Moreover the umbral monogenic functions $f_0, f_1, \dots, f_{k-2}, f_{k-1}$ are given by the following formulas:

$$\begin{array}{rcr} f_{k-1}(\underline{x}) & = & Q'_{k-1}(D')^{k-1}f(\underline{x}) \\ f_{k-2}(\underline{x}) & = & Q'_{k-2}(D')^{k-2}(\mathbf{id} - P_{k-1}(x')Q'_{k-1}(D')^{k-1})f(\underline{x}) \\ & \vdots & \\ f_1(\underline{x}) & = & Q_1 D'(\mathbf{id} - P_2(x')Q_2(D')^2)\cdots(\mathbf{id} - P_{k-1}(x')Q_{k-1}(D')^{k-1})f(\underline{x}) \\ f_0(\underline{x}) & = & (\mathbf{id} - P_1(x')Q'_1 D')(\mathbf{id} - P_2(x')Q'_2(D')^2)\cdots(\mathbf{id} - P_{k-1}(x')Q_{k-1}(D')^{k-1})f(\underline{x}). \end{array} \tag{15}$$

Conversely, the sum in (14), with $f_0, f_1, \dots, f_{k-2}, f_{k-1}$ umbral monogenic in Ω, defines a umbral polymonogenic function in Ω.

Before proving Theorem 3.1, we need some preliminary results.

Lemma 3.1 (See Appendix B). *Let Ω be a starlike domain in $\mathbb{R}^n$ with center 0. For any Clifford-valued function $f(\underline{x})$ in Ω, the following relations hold for each $s \in \mathbb{N}$:*

$$D'((x')^s f(\underline{x})) = -2(x')^{s-1}T'_s f(\underline{x}) + (-1)^s (x')^s D' f(\underline{x}), \tag{16}$$

where

$$T'_s = \begin{cases} k\,\mathbf{id}, & \text{if } s = 2k \\ E' + (\frac{n}{2} + k)\mathbf{id}, & \text{if } s = 2k+1 \end{cases}.$$

From the above lemma, the next proposition naturally follows:

Proposition 3.1 (See Appendix B). *The iterated umbral Dirac operator $(D')^k$ has the mapping property*

$$(D')^k : (x')^s \ker D' \to (x')^{s-k} \ker D'$$

for any $s \geq k$. Hereby, for each $f(\underline{x}) \in \ker D'$,

$$(D')^k \left((x')^s f(\underline{x})\right) = (-2)^k (x')^{s-k} T'_{s-k+1} \ldots T'_{s-1} T'_s f(\underline{x}), \tag{17}$$

where the operators U'_j are defined in Lemma 3.1.

Let Ω be a starlike domain with center 0. For any $s > 0$, we define the operator $I_s : C^1(\Omega, C\ell_{0,n}) \longrightarrow C^1(\Omega, C\ell_{0,n})$ by

$$I_s f(\underline{x}) = \int_0^1 f(t\underline{x}) t^{s-1} dt. \tag{18}$$

In addition, we set $E_s = s\mathbf{id} + E$. For $s = 0$ we write E instead of E_0.

Lemma 3.2 (c.f.[33]). *Let $\underline{x} \in \mathbb{R}^n$ and Ω be a domain with $\Omega \supset [0, \underline{x}]$. If $s > 0$ and $f \in C^1(\Omega, C\ell_{0,n})$, then*

$$f(\underline{x}) = I_s E_s f(\underline{x}) = E_s I_s f(\underline{x}). \tag{19}$$

Sloppily speaking, the family of maps $I_s : C^1(\Omega, C\ell_{0,n}) \to C^1(\Omega, C\ell_{0,n})$ can be viewed as certain sort of right inverse for the operator $Dx = \sum_{j,k=1}^n \mathbf{e}_j \mathbf{e}_k \partial_{x_j} x_k$ in $\ker D$. Indeed, if $f \in \ker D$ (i.e. f is monogenic), from Lemma 2.2

$$D(xf(\underline{x})) = x(Df(\underline{x})) + D(xf(\underline{x})) = -2Ef(\underline{x}) - nf(\underline{x}) = -2E_{n/2} f(\underline{x})$$

holds whenever $O_{\underline{x}} = \partial_{\underline{x}} := (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$.

Finally, from Lemma 3.2, $f(\underline{x}) = -2E_{n/2}\left(-\frac{1}{2} I_{n/2} f(\underline{x})\right) = -\frac{1}{2} D(x I_{n/2} f(\underline{x}))$, showing that the operator $-\frac{1}{2} I_{n/2}$ is a right inverse for Dx on the range $\ker D$.

On the other hand, when restricted to the space $\mathcal{P} = \oplus_{k=0}^{\infty}\mathcal{P}_k$, where each $f_k \in \mathcal{P}_k$ is a Clifford-valued homogeneous polynomial of degree k (i.e. $f_k(t\underline{x}) = t^k f_k(\underline{x})$), the family of mappings I_s satisfy the equation

$$I_s f_k(\underline{x}) = \int_0^1 f_k(\underline{x}) t^{k+s-1} dt = \frac{1}{k+s} f_k(\underline{x}), \text{ for all } f_k \in \mathcal{P}_k$$

This corresponds to the simplest case of a local coordinate system that contains along any point $\underline{x} \in \mathbb{R}^n$ the curve $t^s\underline{x} = (t^s x_1, t^s x_2, \ldots, t^s x_n)$ tangent to the family of vector-fields $E_s = s\mathbf{id} + E$ satisfying the eigenvalue equation $E_s f_k(\underline{x}) = (k+s) f_k(\underline{x})$. Hence, Lemma 3.2 remains true for $E' + s\mathbf{id} = \Psi_{\underline{x}}^{-1} E_s \Psi_{\underline{x}}$ in $\mathcal{P}$, since it holds componentwise.

Lemma 3.3 (See Appendix B). *There exists $I'_s : \mathcal{P} \to \mathcal{P}$ such that*

$$(E' + s\mathbf{id}) I'_s = \mathbf{id} = I'_s (E' + s\mathbf{id}).$$

The next lemma will be also important on the sequel

Lemma 3.4 (See Appendix B). *If $f \in \mathcal{P}$, then*

$$D' I'_s f(\underline{x}) = I'_{s+1} D' f(\underline{x}). \tag{20}$$

For any $k \in \mathbb{N}_0$, denote by $Q'_k = \left(-\frac{1}{2}\right)^k (U'_k)^{-1} (U'_{k-1})^{-1} \ldots (U'_1)^{-1}$, where

$$(U'_s)^{-1} = \begin{cases} \frac{1}{k}\ \mathbf{id}, & \text{if } s = 2k \\ \\ I'_{\frac{n}{2}+k}, & \text{if } s = 2k+1 \end{cases}. \tag{21}$$

As direct consequence of (20), we find that $I'_s f(\underline{x})$ and is umbral monogenic whenever $f(\underline{x})$ is umbral monogenic in Ω. From the definition of Q'_k, we thus obtain

$$Q'_k(\ker D') = \ker D'. \tag{22}$$

Then the following lemma holds

Lemma 3.5 (See Appendix B). *For any umbral monogenic function f in Ω,*

$$(D')^k \left[(x')^k Q'_k f(\underline{x})\right] = f(\underline{x}), \quad \underline{x} \in \Omega.$$

Now we come to the proof of our main theorem for $P_k(x') = (x')^k$:
Proof: [Proof of Theorem 3.1] It is sufficient to show that

$$\ker(D')^k = \ker(D')^{k-1} + P_{k-1}(x') \ker D', \quad k \in \mathbb{N},$$

where $P_{k-1}(x')=(x')^{k-1}$. Notice that Lemma 3.3 states that

$$(D')^k P_k(x')Q'_k = \mathbf{id}. \tag{23}$$

We divide the proof into two parts:

(i) $\ker(D')^k \supset \ker(D')^{k-1} + P_{k-1}(x')\ker D'$. Since $\ker(D')^{k-1} \subset \ker(D')^k$, we need only to show $P_{k-1}(x')\ker D' \subset \ker(D')^k$. For any $g \in \ker D'$, by (23) and (22) we have

$$(D')^k(P_{k-1}(x')g) = D'((D')^{k-1}P_{k-1}(x')Q'_{k-1})(Q'_{k-1})^{-1}g = D'(Q'_{k-1})^{-1}g = 0.$$

(ii) $\ker(D')^k \subset \ker(D')^{k-1} + P_{k-1}(x')\ker D'$.

For any $f \in \ker(D')^k$, we have the decomposition

$$f = (\mathbf{id} - P_{k-1}(x')Q'_{k-1}(D')^{k-1})f + P_{k-1}(x')(Q'_{k-1}(D')^{k-1}f).$$

We will show that the first summand above is in $\ker(D')^{k-1}$ and the item in the braces of the second summand is in $\ker D'$. This can be verified directly. First,

$$\begin{aligned} (D')^{k-1}(\mathbf{id} - P_{k-1}(x')Q_{k-1}(D')^{k-1})f = \\ = ((D')^{k-1} - ((D')^{k-1}P_{k-1}(x')Q'_{k-1})(D')^{k-1})f \\ = ((D')^{k-1} - (D')^{k-1})f = 0. \end{aligned}$$

Next, since $(D')^{k-1}f \in \ker D'$ and $Q'_{k-1}\ker D' \subset \ker D'$, we have $Q'_{k-1}(D')^{k-1}f \in \ker D'$, as desired.

This proves that $\ker(D')^k = \ker(D')^{k-1} + P_{k-1}(x')\ker D'$. By induction, we can easily deduce that $\ker(D')^k = \ker D' + x'\ker D' + (x')^2\ker D' + \ldots + (x')^{k-1}\ker D'$.

Next we prove that for any $f \in \ker(D')^k$ the decomposition

$$f = g + P_{k-1}(x')f_k, \quad g \in \ker(D')^{k-1}, f_k \in \ker D'$$

is unique. In fact, for such a decomposition, applying $(D')^{k-1}$ on both sides we obtain

$$\begin{aligned} (D')^{k-1}f &= (D')^{k-1}g + (D')^{k-1}P_{k-1}(x')f_k \\ &= (D')^{k-1}P_{k-1}(x')Q'_{k-1}(Q'_{k-1})^{-1}f_1 \\ &= (Q'_{k-1})^{-1}f_k. \end{aligned}$$

Therefore $f_k = Q'_{k-1}(D')^{k-1}f$, so that

$$g = f - P_{k-1}(x')f_k = (\mathbf{id} - P_{k-1}(x')Q'_{k-1}(D')^{k-1})f.$$

Thus equations (14) and (15) follows by induction.

To prove the converse, we see from equation (17) of Lemma 3.1 that $(D')^{k+1}(x')^k \ker D' = \{0\}$ holds for any $k \in \mathbb{N}$.

Replacing k by j, we have

$$(D')^k(x')^j \ker D' = \{0\}$$

for any $k > j$. ■

The proof of the above theorem can be interpreted as the following infinite triangle on which the subspaces $\ker(D')^k$ are despicted into columns. Each element of the triangle given by Proposition 2.3 corresponds to the action of $\mathfrak{osp}(1|2) \times O(n)$ on rows and columns:

$$\begin{array}{ccccccccc}
\{0\} & \ker D' & & \ker(D')^2 & & \ker(D')^3 & & \ker(D')^4 & \dots \\
 & & & & & & & & \\
\{0\} & \ker D' & \xrightarrow{x'} & x' \ker D' & \xrightarrow{x'} & (x')^2 \ker D' & \xrightarrow{x'} & (x')^3 \ker D' & \dots \\
 & \downarrow D' & & \downarrow D' & & \downarrow D' & & \downarrow D' & \\
 & \{0\} & & \ker D' & \xrightarrow{x'} & x' \ker D' & \xrightarrow{x'} & (x')^2 \ker D' & \dots \\
 & & & \downarrow D' & & \downarrow D' & & \downarrow D' & \\
 & & & \{0\} & & \ker D' & \xrightarrow{x'} & x' \ker D' & \dots \\
 & & & & & \downarrow D' & & \downarrow D' & \\
 & & & & & \{0\} & & \ker D' & \dots \\
 & & & & & & & \downarrow D' & \\
 & & & & & & & \{0\} & \dots \\
 & & & & & & & & \\
 & & & & & & & & \dots
\end{array}$$

In those actions, the operator x' shifts all the spaces in the same row to the left while the operator D' shifts all the spaces in the same column down. In particular, the $(k+1)-$line of the above diagram corresponds to the action of $(D')^k$ on the subspaces $(x')^s \ker D'$ represented in $(s+1)-$column.

The next important step is the passage from the homogeneous operator of degree k, $(x')^k$, to a general polynomial type operator $P_k(x')$ with the mapping property $P_k(x') : \ker(D')^k \to \ker D'$. The corollary below gives us a possible generalization for the construction of $P_k(x')$:

Corollary 3.1. *If $P_k(x') = A'_k\ (x')^k + R_k(x')$ where A'_k is a Hilbert-Schmidt operator that satisfy the graded commuting relation $[A'_k, D'] = a_k D'$ for some*

$a_k \in \mathbb{R}$ *and*

$$(D')^k \left(R_k(x')f(\underline{x})\right) = 0, \quad \text{for all } f \in \ker D',$$

then Theorem 3.1 fulfils whenever the eigenvalues of the operator A'_k *are greater than* ka_k*.*

Proof: Starting from the definition of $P_k(x')$ and using induction on $k \in \mathbb{N}_0$, the assumptions for A'_k and $R_k(x')$ lead to

$$(D')^k(P_k(x')f(\underline{x})) = (D')^k(A'_k(x')^k f(\underline{x})) = (-ka_k\mathbf{id} + A'_k)(D')^k\left((x')^k f(\underline{x})\right).$$

whenever f belongs to $\ker D'$ (i.e. f is umbral monogenic).

From direct application of Lemma 17, the later equation becomes then

$$(D')^k(P_k(x')f(\underline{x})) = (-2)^k(-ka_k\mathbf{id} + A'_k)T'_1 \ldots T'_{k-1}T'_k f(\underline{x})),$$

where the operators U'_j are defined in Lemma 3.1.

Replacement of $f(\underline{x})$ by $S'_k f(\underline{x}) = \left(-\frac{1}{2}\right)^k (T'_k)^{-1}(T'_{k-1})^{-1}\ldots(T'_0)^{-1}f(\underline{x})$, on the above equation results in

$$(D')^k(P_k(x')S'_k f(\underline{x})) = (-ka_k\mathbf{id} + A'_k)f(\underline{x}).$$

Hereby $(U'_s)^{-1}$ are defined viz equation (21).

Now it remains to show that $-ka_k\mathbf{id} + A'_k$ is invertible ensuring that $Q'_k = S'_k(-ka_k\mathbf{id} + A'_k)^{-1}$ is a right inverse for $(D')^k P_k(x') : \ker D' \to \ker D'$.

If A'_k is a multiple of the identity operator $\mathbf{id}$, $a_k = 0$ and hence A'_k is invertible and the proof of corollary holds.

Otherwise, since A'_k is a Hilbert-Schmidt operator we conclude that A'_k has discrete spectra. Then, analogously to the proof of Lemma 3.3 (see Appendix B), A'_k is given by the series expansion

$$A'_k f(\underline{x}) = \sum_{s=0}^{\infty} \lambda_{k,s} f_s(\underline{x}),$$

where $\lambda_{k,s} \in \mathbb{R}$ are the eigenvalues of A'_k.

Thus $-ka_k\mathbf{id} + A'_k$ is invertible whenever $-ka_k + \lambda_{k,s}$ is positive, that is $\lambda_{k,s} > ka_k$.

Finally, using the same order of ideas of the proof of Theorem 3.1, induction arguments lead to the following infinite triangle

$$
\begin{array}{ccccccccc}
\{0\} & \ker D' & & \ker(D')^2 & & \ker(D')^3 & & \ker(D')^4 & \dots \\
 & & & & & & & & \\
\{0\} & P_0(x')\ker D' & \stackrel{x'}{\longrightarrow} & P_1(x')\ker D' & \stackrel{x'}{\longrightarrow} & P_2(x')\ker D' & \stackrel{x'}{\longrightarrow} & P_3(x')\ker D' & \dots \\
 & \downarrow D' & & \downarrow D' & & \downarrow D' & & \downarrow D' & \\
 & \{0\} & & P_0(x')\ker D' & \stackrel{x'}{\longrightarrow} & P_1(x')\ker D' & \stackrel{x'}{\longrightarrow} & P_2(x')\ker D' & \dots \\
 & & & \downarrow D' & & \downarrow D' & & \downarrow D' & \\
 & & & \{0\} & & P_0(x')\ker D' & \stackrel{x'}{\longrightarrow} & P_1(x')\ker D' & \dots \\
 & & & & & \downarrow D' & & \downarrow D' & \\
 & & & & & \{0\} & & P_0(x')\ker D' & \dots \\
 & & & & & & & \downarrow D' & \\
 & & & & & & & \{0\} & \dots \\
 & & & & & & & & \\
 & & & & & & & & \dots
\end{array}
$$

This gives the following direct decomposition of $\ker(D')^k$:

$$
\begin{aligned}
\ker(D')^k &= \ker(D')^{k-1} \oplus P_{k-1}(x')\ker D' \\
&= \ker(D')^{k-2} \oplus P_{k-2}(x')\ker D' \oplus P_{k-1}(x')\ker D' \\
&= \dots \\
&= P_0(x')\ker D' \oplus P_1(x')\ker D' \oplus \dots \oplus P_{k-1}(x')\ker D'.
\end{aligned}
$$

concluding in this way the proof of Corollary 3.1. ■

We will end this section by establishing a parallel between our approach and the approaches of Ryan (c.f. [38]), Malonek & Ren (c.f. [33, 36]) and Faustino & Kähler (c.f. [20]).

Recall that Fischer decomposition ([14], Theorem 1.10.1) states the spaces of homogeneous polynomials $\mathcal{P}_k$ are splitted in spherical monogenics pieces with degree lower than k:

$$
\mathcal{P}_k = \sum_{s=0}^{k} \bigoplus x^s \left(\mathcal{P}_{k-s} \cap \ker D\right).
$$

Moreover, from the mapping property given by Lemma 3.5 each $P_k \in \mathcal{P}_k$ belongs to $\ker D^{k+1}$ and hence from the intertwining property given by relations (13) the Clifford-valued polynomial $\Psi_{\underline{x}} P_k(\underline{x}) = P_k(\underline{x}')\mathbf{1}$ belongs to $\ker(D')^{k+1}$ and hence the Almansi decomposition

$$
\ker(D')^{k+1} = \ker D' \oplus x' \ker D' \oplus (x')^2 \ker D' \oplus \dots \oplus (x')^k \ker D',
$$

comprise the approaches of Ryan (c.f. [38]), Malonek & Ren (c.f. [33]) (i.e. for $\Psi_{\underline{x}} = \mathbf{id}$) as well as the Fischer decomposition in terms of forward Dirac operators if we consider the operators introduced in Example 2.2. The flexibility of this approach allows us also to get the Fischer decomposition for several classes of difference operators like the difference operators considered in Example 2.3.

Remark 3.1. *The replacement of $(x')^k$ by the polynomial type operators $P_k(x')$ given by Corollary 3.1 gives a parallel in continuum as the decomposition in terms iterated kernels obtained by* Ren & Malonek *(c.f.* [33]*) on which the operators $P_k(x')$ shall be interpreted as quantization of a Clifford-valued polynomial of degree k.*

3.2. Parallelism with the Quantum Harmonic Oscillator. We will finish this section by turning out our attention to the quantum harmonic oscillator given by the following Hamiltonian with written in terms of the potential operator $V_\hbar(x') = -\frac{1}{2}(x')^2 - \frac{\hbar}{2}x' + \frac{\hbar^2}{8}\left(\Gamma' - \frac{n}{2}\mathbf{id}\right)$:

$$\mathcal{H}'_\hbar = -\frac{1}{2}\Delta' + V_\hbar(x'), \ \text{ with } \hbar \in \mathbb{R}.$$

Hereby $\Gamma' = -x'D' - E'$ corresponds to the umbral counterpart of the spherical Dirac operator (c.f. [14]).

In order to analyze the $\mathfrak{sl}_2(\mathbb{R})$ symmetries of $\mathcal{H}'$, we further introduce the auxiliar operator

$$\mathcal{J}'_\hbar = \frac{\hbar}{4}D + \frac{1}{2}\left(E' + \frac{n}{2}\mathbf{id}\right)$$

on the algebra of Clifford-valued polynomials $\mathcal{P}$. The subsequent proposition gives a description of the Lie algebra symmetries underlying $\mathcal{H}'$, showing that $\mathbf{p}^+ = V_\hbar(x')$, $\mathbf{p}^- = -\frac{\Delta'}{2}$ and $\mathbf{q} = \frac{\hbar}{4}D + E + \frac{n}{2}\mathbf{id}$ are the canonical generators of $\mathfrak{sl}_2(\mathbb{R})$.

We start with the following lemma:

Lemma 3.6. *When acting on $\mathcal{P}$, the operator Γ' commute with the operators E' and Δ', i.e.*

$$[E', \Gamma'] = 0, \ \ [\Delta', \Gamma'] = 0.$$

Proof: For the proof of $[E', \Gamma'] = 0$ it remains to show that $[E', x'D'] = 0$ and since from definition $[E', \Gamma'] = [E', -x'D' - E'] = -[E', x'D']$.

From Lemma 2.3 we get $[E', x'] = x'$ and $[E', D'] = -D'$. This leads to

$$E'(x'D') = (x' + x'E')D' = x'D' + (-x'D' + x'D'E') = (x'D')E',$$

or equivalently $[E', x'D'] = 0$, as desired.

In order to show that $[\Gamma', \Delta'] = 0$, we recall the relations $[\Delta', x'] = 2D'$, $[\Delta', E'] = 2\Delta'$ $[\Delta', D'] = 0$ follow from Lemma 2.3. This shows that

$$\Delta'(x'D') = (2D' + x'\Delta')D' = -2\Delta' + (x'D')\Delta',$$

and hence,

$$\Delta'(x'D' + E') = -2\Delta' + (x'D')\Delta' + 2\Delta' + E'\Delta' = (x'D' + E')\Delta'.$$

Finally, taking into account the definition of Γ' the above equation is equivalent to $[\Delta', \Gamma'] = 0$, as desired. ■

Lemma 3.7. *When acting on $\mathcal{P}$, the elements $\frac{\Delta'}{2}$, $V_\hbar(x')$ and $\mathcal{J}_\hbar$ are the canonical generators of the Lie algebra. The remaining commutation relations are*

$$\left[\frac{\Delta'}{2}, V_\hbar(x')\right] = \mathcal{J}'_\hbar, \quad [\mathcal{J}'_\hbar, V_\hbar(x')] = V_\hbar(x'), \quad \left[\mathcal{J}'_\hbar, \frac{\Delta'}{2}\right] = \frac{\Delta'}{2}.$$

Proof: Recall that from Lemma 2.3, $\mathbf{p}^- = -\frac{\Delta'}{2}$ $\mathbf{p}^+ = \frac{(x')^2}{2}$ and $\mathbf{q} = \mathcal{J}'_\hbar$ are the canonical generators of $\mathfrak{sl}_2(\mathbb{R})$:

$$\left[\mathbf{p}^-, \mathbf{p}^+\right] = \mathbf{q}, \quad [\mathbf{q}, \mathbf{p}^-] = -\mathbf{p}^-, \quad \left[\mathbf{q}, \mathbf{p}^+\right] = \mathbf{q}.$$

and moreover $[x', \mathbf{p}^-] = -D'$, $[D', \mathbf{p}^+] = x'$, $[\mathbf{q}, x'] = \frac{1}{2}x'$ and $[D', \mathbf{p}^-] = 0 = [x', \mathbf{p}^+]$. Taking into account that

$$V_\hbar(x') = -\mathbf{p}^+ - \frac{\hbar}{2}x' + \frac{\hbar^2}{8}\left(\Gamma' - \frac{n}{2}\mathbf{id}\right) \text{ and } \mathcal{J}'_\hbar = \mathbf{q} + \frac{\hbar}{4}D',$$

combination of the above relations with Lemma 3.6 results in the following identities in terms graded commuting relations:

$$\begin{array}{rcl} [-\mathbf{p}^-, V_\hbar(x')] & = & [\mathbf{p}^-, \mathbf{p}^+] + \left[\mathbf{p}^-, -\frac{\hbar}{2}x' + \frac{\hbar^2}{8}\left(\Gamma' - \frac{n}{2}\mathbf{id}\right)\right] \\ & = & \mathbf{q} + \frac{\hbar}{2}D'; \\ & & \\ [\mathcal{J}'_\hbar, V_\hbar(x')] & = & -[\mathbf{q}, \mathbf{p}^+] - \frac{\hbar}{2}[\mathbf{q}, x'] - \frac{\hbar}{4}[D', \mathbf{p}^+] - (\frac{\hbar}{4})^2[D', x'] \\ & = & -\mathbf{p}^+ - \frac{\hbar}{2}x' + \frac{\hbar^2}{8}\left(\Gamma' - \frac{n}{2}\mathbf{id}\right) \\ & = & V_\hbar(x'); \\ & & \\ [\mathcal{J}'_\hbar, -\mathbf{p}^-] & = & [\mathbf{q}, \mathbf{p}^-] - \left[\frac{\hbar}{4}D', \mathbf{p}^-\right] \\ & = & -\mathbf{p}^-. \end{array}$$

This proves Lemma 3.7. ■

Proposition 3.2. *The operators $\mathcal{J}'_\hbar, \mathcal{H}'_\hbar \in End(\mathcal{P})$ are interrelated by the following intertwining property:*

$$\mathcal{H}'_\hbar \exp\left(V_\hbar(x')\right) \exp\left(-\frac{\Delta'}{2}\right) = -\exp\left(V_\hbar(x')\right) \exp\left(-\frac{\Delta'}{2}\right) \mathcal{J}'_\hbar.$$

Proof: From Lemma 3.7, the elements $\frac{\Delta'}{2}$, $V_\hbar(x')$ and $\mathcal{J}_\hbar$ correspond to the canonical generators of $\mathfrak{sl}_2(\mathbb{R})$

From the above relations, it follows from induction over k that

$$\left[\frac{\Delta'}{2}, V_\hbar(x')^k\right] = k\mathcal{J}'_\hbar \left(V_\hbar(x')\right)^{k-1}, \quad \left[\mathcal{J}'_\hbar, V_\hbar(x')^k\right] = kV_\hbar(x')^{k-1}\left(V_\hbar(x')\right)^{k-1},$$

leading to

$$\left[\frac{\Delta'}{2}, \exp\left(V_\hbar(x')\right)\right] = \mathcal{J}'_\hbar \exp\left(V_\hbar(x')\right), \quad [\mathcal{J}'_\hbar, \exp\left(V_\hbar(x')\right)] = V_\hbar(x') \exp\left(V_\hbar(x')\right).$$

Combining the above two relations we get

$$\left[\frac{\Delta'}{2} + \mathcal{J}'_\hbar, \exp\left(V_\hbar(x')\right)\right] = \left(\mathcal{J}'_\hbar + V_\hbar(x')\right) \exp\left(V_\hbar(x')\right).$$

which is equivalent to $\left(\frac{\Delta'}{2} - V_\hbar(x')\right) \exp\left(V_\hbar(x')\right) = \exp\left(V_\hbar(x')\right)\left(\frac{\Delta'}{2} + \mathcal{J}'_\hbar\right)$.

Not it remains to show that $\left(\frac{\Delta'}{2} + \mathcal{J}'_\hbar\right)\exp\left(-\frac{\Delta'}{2}\right) = \exp\left(-\frac{\Delta'}{2}\right)\mathcal{J}'_\hbar$.

This statement is then immediate from the fact that $\left[\mathcal{J}'_\hbar, \exp\left(-\frac{\Delta'}{2}\right)\right] = \left(-\frac{\Delta'}{2}\right)\exp\left(-\frac{\Delta'}{2}\right)$.

Therefore

$$\begin{aligned}\mathcal{H}_\hbar \exp\left(V_\hbar(x')\right)\left(-\frac{\Delta'}{2}\right) &= -\exp\left(V_\hbar(x')\right)\left(\frac{\Delta'}{2}+\mathcal{J}'_\hbar\right)\left(-\frac{\Delta'}{2}\right)\\ &= -\exp\left(V_\hbar(x')\right)\left(-\frac{\Delta'}{2}\right)\mathcal{J}'_\hbar,\end{aligned}$$

as desired. ■

We will finish this section by establishing a parallel with the recent approach of Render (c.f. [37]):

According to the definition of Fischer inner pair ([37], page 315) Corollary 3.1 shows that when A'_k is a multiple of the identity operator **id**, the pair $(P_{2k}(x'), (\Delta')^k)$ corresponds to a quantization of the Fischer inner pair that completely determines a (discrete) Almansi decomposition for polyharmonic functions. Indeed, if f is umbral poly-harmonic of degree k on Ω (i.e. $(\Delta')^k f(\underline{x}) = 0$ holds on Ω). If we take $P_{2k}(x') = V_\hbar(x')^k$ it is straightforward from Corollary 3.1 that the following decompositions holds:

$$f(\underline{x}) = P_0(x')f_0(\underline{x}) + P_2(x')f_1(\underline{x}) + \cdots + P_{2k-2}(x')f_{k-1}(\underline{x})$$

where $f_0, f_1, \ldots, f_{k-2}, f_{k-1}$ are umbral harmonic functions on Ω (i.e. $\Delta' f_j(\underline{x}) = 0$ holds on Ω for each $j = 0, \ldots, k-1$).

Thus, it is also possible to obtain explicit formulae analogue to (14) for umbral harmonic functions $f_0, f_1, \ldots, f_{k-1}$ by taking into account the quantized Fischer inner pair $(P_{2k}(x'), (\Delta')^k)$.

Remark 3.2. *Based on Lemma 3.7, it is clear from the above construction that the functions $f_0, f_1, \ldots, f_{k-1}$ obtained viz Corollary 3.1 are solutions of the coupled system of equations:*

$$\Delta' f_k = 0, \quad \mathcal{J}'_\hbar f_k = \left(\tfrac{1}{2}k + \tfrac{n}{4}\right)f$$

In addition, from Proposition 3.2 the composite action of $-\exp\left(V_\hbar(x')\right)\left(-\frac{\Delta'}{2}\right)$ on each f_k span the eigenfunctions of $\mathcal{H}'_\hbar$.

On the other hand, it is clear that the above coupled system of equations approximate the the umbral counterpart of spherical harmonics in the limit $\hbar \leftarrow 0$. However, from the following graded commuting property:

$$\left[\frac{1}{2}\left(E' + \frac{n}{2}\mathbf{id}\right), \exp\left(\frac{\hbar}{2}D\right)\right] = -\frac{\hbar}{2}D\exp\left(\frac{\hbar}{2}D\right)$$

follows from straightforward computations combined Lemma 2.3 with induction over the natural numbers. This yields the following intertwining property when restricted to the algebra $\mathcal{P}$*:*

$$\frac{1}{2}\left(E' + \frac{n}{2}\mathbf{id}\right)\exp\left(-\frac{\hbar}{2}D\right) = \exp\left(-\frac{\hbar}{2}D\right)\mathcal{J}'_{\hbar}.$$

showing that $\exp\left(-\frac{\hbar}{2}D\right)$ *maps the umbral harmonic polynomials of degree* k *onto umbral counterparts of spherical harmonics.*

In this case the action $\exp\left(-\frac{\hbar}{2}D\right)$ *on* $\mathcal{P}$ *plays a role similar to the inversion of the Wick operator in Segal-Bargmann spaces underlying nilpotent Lie groups like* $SL_2(\mathbb{R})$ *(c.f.* [10]*).*

4. Concluding Remarks and Open Problems

In this paper we introduce an algebraic framework that can be seen as a comprised model for Clifford analysis underlying the orthogonal group $O(n)$. This makes it possible to construct the associated equations and solutions in the discrete setting starting from the equations and their solutions in the continuous setting. The intertwining properties given by relations (13) at the level of the algebra $\mathrm{End}(\mathcal{P})$ gives us a meaningful interpretation of classical and discrete Clifford analysis as two quantal systems on which the Sheffer operator $\Psi_{\underline{x}}$ acts as a gauge transformation preserving the canonical relations between both systems.

This approach can be viewed as a merge between radial algebra approach proposed by Sommen [43] to define Clifford analysis with the quantum mechanical approach for umbral calculus described by Dimakis, Hoissen & Striker (c.f. [19]) and Levi, Tempesta & Winternitz (c.f. [31]). Based on the recent paper of Tempesta (c.f. [44]) we believe that this approach shall also be useful to construct polynomials in hypercomplex variables possessing the Appell set property. In this direction, the recent approaches of Malonek & Tomaz (c.f. [35]) De Ridder,De Schepper, Kähler & Sommen (c.f.[13]) and Bock, Gürlebeck, Lávička & V. Souček (c.f. [6]) are beyond to the Sheffer set property.

Here we would like to stress that contrary to the approaches of Malonek & Tomaz and Bock, Gürlebeck, Lávička & Souček on it is almost clear that the considered operators are generators of $\mathfrak{sl}_2(\mathbb{R})$ (or alternatively $\mathfrak{sl}_2(\mathbb{C})$) and $\mathrm{osp}(1|2)$ while the Appell sets are invariant under the action of the orthogonal group $O(n)$, in the approach of De Ridder,De Schepper, Kähler & Sommen it was not yet realized which kind of symmetries are encoded and for which group the Appell sets (or more generally, the Sheffer sets) are invariant.

Based on the recent paper of Brackx, De Schepper,Eelbode (c.f. [6]) and the preprint of Faustino & Kähler (c.f. [21]), we conjecture the following:

'*All Hermitian operators represented in terms of* $\mathfrak{sl}_2(\mathbb{R})$ *and* $\mathfrak{osp}(1|2)$ *generators in continuum cannot be represented by* $\mathfrak{sl}_2(\mathbb{R})$ *and* $\mathfrak{osp}(1|2)$ *generators in discrete but instead by quantum deformations of it'.*

For a nice motivation on this direction we refer to [29], Section 2 and also [22], Subsection 3.3, on which such gap was undertaken.

In the proof of Almansi decomposition (Theorem 3.1 and Corollary 3.1), the iterated (umbral) Dirac operators $(D')^k$ play a central role. In comparison with [14, 20, 22, 13] we prove a similar result using the decomposition of the subspaces $\ker(D')^k$ viz resolutions of $\mathfrak{osp}(1|2) \times O(n)$ instead of considering *a-priori* a Fischer inner product.

With this framework, Theorem 3.1 shows that the decomposition of $\ker(D')^k$ in terms of $\mathfrak{osp}(1|2) \times O(n)$ pieces yield the subspaces $(x')^s \ker D'$ for $s = 0, 1, \ldots, k-1$. Moreover, the replacement of $(x')^k$ by a polynomial type operator $P_k(x')$ in Corollary 3.1 gives an alternative interpretation for decomposing kernel approach proposed by Malonek & Ren (c.f. [36]) as well as refines the Fischer inner pair technique used by Render in [37] to prove the Almansi decomposition in terms umbral polyharmonic functions.

As it was observed along this paper the resulting approach based on representation of the Lie algebra $\mathfrak{osp}(1|2)$ as a refinement $\mathfrak{sl}_2(\mathbb{R})$ has a core of applications in quantum mechanics that can further be consider to study special functions in Clifford analyis that belong to Bargmann-Fock spaces (see [11] and references therein). From the border view of physics, we have shown in Subsection 3.2 that the Almansi decomposition approach obtained by Render shall be described using a diagonalization in terms of $\mathfrak{sl}_2(\mathbb{R})$. Indeed, Proposition 3.2 and Remark 3.2 explains the parallel between the Almansi decomposition of the subspaces $\ker(\Delta')^k$ and the separation of variables of quantum harmonic oscillators (c.f. [41, 42, 46]).

One may further bring this technique in the future to construct new families of Appell/Sheffer sets for hypercomplex variables as well as to study

Schrödinger equations on grids. At this stage, new families of discrete Clifford-valued polynomials like e.g. hypercomplex generalizations of Kravchuk polynomials (c.f. [32]) should appear.

Acknowledgments

N. Faustino was supported by *Fundação para a Ciência e a Tecnologia* under the fellowship SFRH/BPD/63521/2009 and the project

PTDC/MAT/114394/2009; G. Ren was supported by the *Center of Research and Development in Mathematics and Applications* of University of Aveiro, and by the NNSF of China (No. 10771201), NCET-05-0539.

The major part of this work was developed in the *Center of Research and Development in Mathematics and Applications* of University of Aveiro (Portugal). The authors would like also to thank to the members of the *Complex and Hypercomplex analysis* research group for the kind hospitality and for the active-research environment during their stay.

Appendix A. Umbral Clifford Analysis

A.1. Proof of Lemma 2.3. Proof: Notice that the relations $[D', \Delta'\ ] = 0 = \left[x', -(x')^2\right]$ are then fulfilled since $(x')^2$ and $-\Delta'$ commute with all elements of $\mathrm{End}(\mathcal{P})$ (first and second relations of Lemma 2.2).

The proof of $\left[E' + \frac{n}{2}\mathbf{id}, x'\right] = x'$ and $\left[E' + \frac{n}{2}\mathbf{id}, D'\right] = -D'$ follow straightforward from the Weyl-Heisenberg character of operators x'_j and O_{x_j}. Straightforward application of the above relations naturally leads to

$$\begin{aligned}\left[E' + \frac{n}{2}\mathbf{id}, (x')^2\right] &= \left(x' + x'\left(E' + \frac{n}{2}\mathbf{id}\right)\right)x' - x'\left(-x' + \left(E' + \frac{n}{2}\mathbf{id}\right)x'\right) \\ &= 2(x')^2,\end{aligned}$$

$$\begin{aligned}\left[E' + \frac{n}{2}\mathbf{id}, -\Delta'\right] &= \left(-D' + D'\left(E' + \frac{n}{2}\mathbf{id}\right)\right)D' - D'\left(D' + E' + \frac{n}{2}D'\right) \\ &= 2\Delta'.\end{aligned}$$

Furthermore the relations $\left[E' + \frac{n}{2}\mathbf{id}, x'\right] = x'$, $\left[E' + \frac{n}{2}\mathbf{id}, D'\right] = -D'$ together with the third anti-commuting relation of Lemma 2.2 lead to

$$\begin{aligned} D'(x')^2 - (x')^2 D' &= \left(-2\left(E' + \frac{n}{2}\mathbf{id}\right) - x'D'\right)x' - x'\left(-2\left(E' + \frac{n}{2}\mathbf{id}\right) - D'x'\right) \\ &= -2\left[E' + \frac{n}{2}\mathbf{id}, x'\right] \\ &= -2x' \end{aligned}$$

$$\begin{aligned} -x'\Delta' + \Delta' x' &= \left(-2\left(E' + \frac{n}{2}\mathbf{id}\right) - D'x'\right)D' - D'\left(-2\left(E' + \frac{n}{2}\mathbf{id}\right) - x'D'\right) \\ &= -2\left[E' + \frac{n}{2}\mathbf{id}, D'\right] \\ &= 2D'. \end{aligned}$$

Finally, the combination of the relations $[E' + \frac{n}{2}\mathbf{id}, x'] = x'$ and $[E' + \frac{n}{2}\mathbf{id}, D'] = -D'$ with the third anti-commuting relation of Lemma 2.2 leads to

$$\begin{aligned} -\Delta'(x')^2 &= D'\left(-2x' + (x')^2 D'\right) \\ &= -2\{x', D'\} - (x')^2\Delta' \\ &= 4\left(E' + \frac{n}{2}\mathbf{id}\right) - (x')^2\Delta'. \end{aligned}$$

■

Appendix B. Almansi-type theorems in (discrete) Clifford analysis

B.1. Proof of Lemma 3.1. **Proof:** We use induction to prove (16). Since $\{x', D'\} = x'D' + D'x' = -2\left(E' + \frac{n}{2}\mathbf{id}\right)$ and $Dg(\underline{x}) = 0$, we have

$$D'(x'g(\underline{x})) = -2\left(E' + \frac{n}{2}\mathbf{id}\right)g(\underline{x}). \tag{24}$$

Next we show that, for any $\underline{x} \in \Omega$ and $k \in \mathbb{N}$,

$$\begin{aligned} D'((x')^{2k}g(\underline{x})) &= -2k(x')^{2k-1}g(x); \\ D'((x')^{2k-1}g(\underline{x})) &= -2(x')^{2(k-1)}\left(E' + (\tfrac{n}{2} + k - 1)\mathbf{id}\right)g(\underline{x}). \end{aligned} \tag{25}$$

This can be checked by induction. Assuming that (25) holds for k. we shall now prove it also holds for $k+1$. We now apply the operator $x'D' + D'x' =$

$-2\left(E'+\frac{n}{2}\mathbf{id}\right)$ to the function $(x')^{2k}g(x)$:

$$x'D'((x')^{2k}g(\underline{x}))+D'x'((x')^{2k}g(\underline{x}))=-2\left(E'+\frac{n}{2}\mathbf{id}\right)((x')^{2k}g(\underline{x})). \tag{26}$$

By the hypothesis of induction, the first term in the left is equal to $-2k(x')^{2k}g(\underline{x})$, while the left side equals $-2x'^{2k}\left(E'+(\frac{n}{2}+2k)\mathbf{id}\right)g(\underline{x})$ due to the fact that

$$\left(E'+\left(\frac{n}{2}+s\right)\mathbf{id}\right)x'-x'\left(E'+\left(\frac{n}{2}+s\right)\mathbf{id}\right)=\left[E'+\frac{n}{2}\mathbf{id},x'\right]=x',$$

holds for all $s>0$.

As a result,

$$\begin{array}{rcl} D'((x')^{2k+1}g(\underline{x})) & = & -x'D'(x'^{2k}g(\underline{x}))-2\left(E'+\frac{n}{2}\mathbf{id}\right)((x')^{2k}g(\underline{x})) \\ & = & 2k(x)'^{2k}g(\underline{x})-2(x')^{2k}\left(E'+\left(\frac{n}{2}+2k\right)\mathbf{id}\right)g(\underline{x}) \\ & = & -2(x')^{2k}\left(E'+\left(\frac{n}{2}+k\right)\mathbf{id}\right)g(\underline{x}). \end{array} \tag{27}$$

This proves the second equality of (25). The first equality of (25) can be proved similarly. This proves the identities (25). ■

B.2. Proof of Proposition 3.1. Proof: In order to prove the above mapping property, we will derive (17) using induction over $k\in\mathbb{N}$. First notice that for $k=1$, relation (17) automatically fulfils according to Lemma 3.1.

Next we assume that (17) holds for any $k\in\mathbb{N}$. Hence, the action of D' on both sides of (17) combined with Lemma 3.1 results in

$$\begin{aligned} &(D')^k\left((x')^sf(\underline{x})\right)= \\ &=(-2)^{k+1}(x')^{s-k-1}T'_{s-k}T'_{s-k+1}\dots T'_{s-1}T'_sf(\underline{x})+(-1)^{s-k}(x')^{s-k}D'g_{s-k}(\underline{x}), \end{aligned}$$

with $g_{s-k}(\underline{x})=(-2)^kU'_{s-k}U'_{s-k+1}\dots U'_{s-1}U'_sf(\underline{x})$.

Now it remains to show that $D'g_{s-k}(\underline{x})=0$. If j is even, $U'_j=\frac{j}{2}\mathbf{id}$ and hence $[U'_j,D']=0$. Otherwise, from the second relations of 2.3 $\left[E'+\frac{n}{2},D'\right]=D'$ combined with the definition of U'_j for j odd results in $[U'_j,D']=D'$.

Thus, we have $[U'_j,D']=\frac{1-(-1)^j}{2}D'$ for each $j\in\mathbb{N}$ and moreover for each $g\in\ker D'$ the action of D' on $U'_jg(\underline{x})$ is equal to

$$D'(U'_jg(\underline{x}))=-\frac{1-(-1)^j}{2}D'g(\underline{x})+U'_j(D'g(\underline{x}))=0,$$

that is, $D'U'_j(\ker D')\subset\ker D'$ for each $j\in\mathbb{N}$.

Finally, recursive application of the above relation leads to

$$D'U'_{s-k}U'_{s-k+1}\dots U'_{s-1}U'_s\left(\ker D'\right)\subset \ker D'.$$

and this results in $D'g_{s-k}(\underline{x})=0$, as desired. ■

B.3. Proof of Lemma 3.3. Proof: Take $f_k(\underline{x})\in\mathcal{P}$ such that $E'f_k(\underline{x})=kf_k(\underline{x})$ holds for each $k\in\mathbb{N}_0$. Hence for $s>0$, the operator $E'+s\mathbf{id}$ has only positive eigenvalues of the form $\lambda=k+s$ which shows that the inverse of $E'+s\mathbf{id}$ tactically exists.

For any $f(\underline{x})=\sum_{k=0}^{\infty}f_k(\underline{x})\in\mathcal{P}$ and $s>0$, define $I'_s:\mathcal{P}\to\mathcal{P}$ as the operator given by the series expansion

$$I'_sf(\underline{x})=\sum_{k=0}^{\infty}\frac{1}{k+s}f_k(\underline{x}).$$

Then we have

$$(E'+s\mathbf{id})(I'_sf(\underline{x}))=\sum_{k=0}^{\infty}\frac{1}{k+s}\left((E'+s\mathbf{id})f_k(\underline{x})\right)=\sum_{k=0}^{\infty}f_k(\underline{x})=f(\underline{x})$$

Recalling the definition of $\Psi_{\underline{x}}$ in Section 2.1, the Clifford-valued polynomial $\left(\Psi_{\underline{x}}^{-1}f_k\right)(\underline{x})$ is homogeneous of degree k and hence

$$I'_sf_k(\underline{x})=\Psi_{\underline{x}}\left(I_s\Psi_{\underline{x}}^{-1}f_k(\underline{x})\right)=\Psi_{\underline{x}}\left(\frac{1}{k+s}\Psi_{\underline{x}}^{-1}f_k(\underline{x})\right)=\frac{1}{k+s}f_k(\underline{x})$$

leads to

$$I'_s\left((E'+s\mathbf{id})f(\underline{x})\right)=\sum_{k=0}^{\infty}I'_s((k+s)f_k(\underline{x}))=\sum_{k=0}^{\infty}f_k(\underline{x})=f(\underline{x}).$$

This shows that I'_s is an inverse for the operator $E'+s\mathbf{id}$, as desired. ■

B.4. Proof of Lemma 3.4. Proof: Starting from Lemma 2.3, we have $-D'=[E'+\frac{n}{2}\mathbf{id},D']$, or equivalently,

$$-D'=E'D'-D'E'=\left(E'+s\mathbf{id}\right)D'-D'\left(E'+s\mathbf{id}\right)$$

by adding and subtracting $(s-\frac{n}{2})D'$ on both sides of the first equation. This is equivalent to $D'E'_s=E'_{s+1}D'$, where $s\mapsto E'_s=E'+s\mathbf{id}$.

Using the fact that $E'_s=({I'}_s)^{-1}$, we end up with

$$D'I'_s=I'_{s+1}E'_{s+1}D'I'_s=I'_{s+1}D'E'_sI'_s=I'_{s+1}D'.$$

■

B.5. Proof of Lemma 3.5. Proof: Denote $g(\underline{x}) = Q'_k f(\underline{x})$. From (22) and the definition of Q'_k, g is umbral monogenic in Ω and hence from equation (17) of Proposition 3.1, we know that

$$(D')^k((x')^k g(\underline{x})) = (-2)^k U'_1 \dots U'_{k-1} U'_k g(\underline{x}).$$

Thus $(D')^k((x')^k Q'_k f(\underline{x})) = f(\underline{x})$ follows directly from the above induced formulas. ■

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Nelson Faustino
Centre for Mathematics, University of Coimbra, Largo D. Dinis, Apartado 3008, P-3001 – 454 Coimbra, Portugal

E-mail address: nelson@mat.uc.pt

Guangbin Ren
University of Science and Technology of China, Department of Mathematics, Hefei, Anhui 230026, P. R. China

E-mail address: rengb@ustc.edu.cn