Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 11–01

DESCENT MORPHISMS AND A VAN KAMPEN THEOREM IN CATEGORIES OF LAX ALGEBRAS

MARIA MANUEL CLEMENTINO AND DIRK HOFMANN

Dedicated to Dikran Dikranjan on the occasion of his sixtieth birthday

Abstract: In this paper we characterise descent morphisms in categories of lax algebras via a generalised limit-lifting condition. Furthermore, we apply this result to establish a global van Kampen Theorem in some of these categories.

Keywords: descent morphism, effective descent morphism, lax algebra, van Kampen Theorem.
AMS Subject Classification (2010): 18B30, 18A30, 18C20, 54B30, 54C10.

Introduction

The Theorem of van Kampen is the principal practical tool for calculating the fundamental group $\pi_1(X, x)$ of a topological space X with base point x, asserting that $\pi_1(X, x)$ can be obtained as the pushout of the fundamental groups of two sufficiently nice open subspaces. Furthermore, under suitable conditions, this can be equivalently expressed using covering maps, and in [4] R. Brown and G. Janelidze prove a very general version of this result which holds not just for topological spaces but in any lextensive category. Another important achievement of [4] is the understanding of the rôle of descent theory in this context: the van Kampen Theorem holds for subobjects $X_1 \hookrightarrow X$ and $X_2 \hookrightarrow X$ of X if and only if the induced map $p : X_1 + X_2 \to X$ is of effective descent with respect to a chosen class of morphisms.

The study of descent theory for topological spaces – first treated in [16] – turned out to be a difficult subject. Indeed, it essentially concerns intricate conditions on ultrafilter convergence, having as keystone the characterisation of global effective descent morphisms [24]. Fortunately, as shown in [5], the situation is much simpler in the particular situation of the van Kampen Theorem (with respect to the choice of all continuous maps).

Received January 14, 2011.
Partial financial assistance by Centro de Matemática da Universidade de Coimbra/FCT, by Centro de Investigação e Desenvolvimento em Matemática e Aplicações da Universidade de Aveiro/FCT and by the project MONDRIAN (under the contract PTDC/EIA-CCO/108302/2008) is gratefully acknowledged.

In fact, $p : X_1 + X_2 \to X$ is of effective descent if and only if the subspaces X_1 and X_2 satisfy a simple topological property.

A topological space is the paradigmatic example of a lax algebra [9], with further examples including ordered sets and metric and approach spaces. Therefore it is natural to ask whether the result of [5] can be extended to this general setting. According to what was said above, this amounts to understanding the notion of descent and effective descent morphism in these categories. A first step in this direction was already done in [8] where effective descent morphisms between (pre)metric spaces are characterised, and where it is shown that open and proper surjections between lax algebras are of effective descent. However, we are not aware of any treatment of descent morphisms on this general level. In this paper we close this gap and show that, under suitable conditions, they are precisely the limit-lifting maps in an appropriate sense, generalising this way the corresponding well-known result for topological spaces. Finally, we show that, under suitable conditions, the intricate condition of $p : X_1 + X_2 \to X$ being of effective descent is equivalent to p being a pullback-stable regular epimorphism.

1. Preliminaries

Throughout this paper V denotes a *commutative and unital quantale*, with tensor $\otimes$ and neutral element k. Hence, V is a complete lattice and $\otimes : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ is an associative and commutative binary operation on V so that

$$k \otimes u = u = u \otimes k$$

and which preserves suprema in each variable:

$$u \otimes \left(\bigvee_{i \in I} u_i \right) = \bigvee_{i \in I} (u \otimes u_i).$$

We also assume that the complete lattice V satisfies the *frame law*, that is,

$$u \wedge \left(\bigvee_{i \in I} u_i \right) = \bigvee_{i \in I} (u \wedge u_i).$$

Our principal examples are the two-element chain $\mathsf{2} = \{\mathsf{false} \leq \mathsf{true}\}$ with tensor given by "and" & and neutral element true; and the extended real half-line $[0, \infty]$ ordered by the "greater or equal"-relation $\geqslant$, with tensor given by addition (where $\infty + x = x + \infty = \infty$) and 0 as neutral element.

The category $\mathsf{V}\text{-}\mathsf{Rel}$ of V-*relations* has sets as its objects, and a morphism $r : X \longrightarrow\!\!\!\!\!\!\!+\;\, Y$ in $\mathsf{V}\text{-}\mathsf{Rel}$ is a function $r : X \times Y \to \mathsf{V}$. Composition of V-relations $r : X \longrightarrow\!\!\!\!\!\!\!+\;\, Y$ and $s : Y \longrightarrow\!\!\!\!\!\!\!+\;\, Z$ is defined as matrix multiplication

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z),$$

and the identity on X is the V-relation $1_X : X \longrightarrow\!\!\!\!\!\!\!+\;\, X$ which sends all diagonal elements (x, x) to k and all other elements to the bottom element $\perp$ of V. The order of V induces a complete order relation on $\mathsf{V}\text{-}\mathsf{Rel}(X, Y)$: for V-relations $r, r' : X \longrightarrow\!\!\!\!\!\!\!+\;\, Y$, one puts

$$r \le r' \;:\iff\; \forall x \in X\ \forall y \in Y\ \ r(x, y) \le r'(x, y).$$

Composition preserves suprema in each variable since $\otimes$ does so, that is,

$$\bigvee_{j \in J, i \in I} s_j \cdot r_i = \bigvee_{j \in J} s_j \cdot \bigvee_{i \in I} r_i.$$

Therefore $\mathsf{V}\text{-}\mathsf{Rel}$ is actually a *2-category*.

$\mathsf{V}\text{-}\mathsf{Rel}$ has an order-preserving *involution* $(-)^\circ$ sending $r : X \longrightarrow\!\!\!\!\!\!\!+\;\, Y$ to its transpose $r^\circ : Y \longrightarrow\!\!\!\!\!\!\!+\;\, X$ defined by $r^\circ(y, x) = r(x, y)$. In fact, this way one defines a contravariant 2-endofunctor on $\mathsf{V}\text{-}\mathsf{Rel}$.

There is a natural embedding of Set into $\mathsf{V}\text{-}\mathsf{Rel}$ leaving objects unchanged and sending a map $f : X \to Y$ to the V-matrix

$$f(x, y) = \begin{cases} k & \text{if } f(x) = y, \\ \perp & \text{else.} \end{cases}$$

In the sequel we will write $f : X \to Y$ rather then $f : X \longrightarrow\!\!\!\!\!\!\!+\;\, Y$ for a V-matrix induced by a Set-map in the sense above. We remark that each $f : X \to Y$ satisfies the inequations $1_X \le f^\circ \cdot f$ and $f \cdot f^\circ \le 1_Y$, which just tells us that f is left adjoint to f° in $\mathsf{V}\text{-}\mathsf{Rel}$.

Examples 1.1. Clearly, $\mathsf{2}\text{-}\mathsf{Rel} \cong \mathsf{Rel}$. A morphism $a : X \longrightarrow\!\!\!\!\!\!\!+\;\, Y$ of $[0, \infty]\text{-}\mathsf{Rel}$ is a generalised distance function $a : X \times Y \to [0, \infty]$. Composition in $[0, \infty]\text{-}\mathsf{Rel}$ is given by

$$(b \cdot a)(x, z) = \inf\{a(x, y) + b(y, z) \mid y \in Y\},$$

and $1_X : X \longrightarrow\!\!\!\!\!\!\!+\;\, X$ is the discrete distance sending the diagonal to 0 and all other pairs (x, x') to ∞.

Recall that a *monad* $\mathbb{T} = (T, e, m)$ on Set consists of a functor $T : \mathsf{Set} \to \mathsf{Set}$ together with natural transformations $e : 1_{\mathsf{Set}} \to T$ and $m : TT \to T$ such that

$$m \cdot Tm = m \cdot m_T \qquad \text{and} \qquad m \cdot Te = 1_T = m \cdot e_T.$$

Here the natural transformation e is the *unit* of the monad and m its *multiplication*. In the sequel we will assume that, besides the quantale V, a monad $\mathbb{T} = (T, e, m)$ on Set is given, and that this monad is suitably extended to V-Rel. By that we mean that the endofunctor $T : \mathsf{Set} \to \mathsf{Set}$ is extended to V-Rel so that

(1) $Ts \cdot Tr \leq T(s \cdot r)$,
(2) $r \leq r' \;\Rightarrow\; Tr \leq Tr'$,
(3) $e_Y \cdot r \leq Tr \cdot e_X$,
(4) $m_Y \cdot TTr \leq Tr \cdot m_X$,
(5) $(Tr)^\circ = T(r^\circ)$ (and we write Tr°),

for all $r, r' : X \looparrowright Y$ and $s : Y \looparrowright Z$. We remark that (1) becomes an equality whenever $r = f$ is a function, that is, T preserves composition of V-matrices with maps from the right.

For the existence of such an extension it is essential that $T : \mathsf{Set} \to \mathsf{Set}$ satisfies the *Beck-Chevalley Condition (BC)* (see [9] for details). We recall that a commutative square in Set

$$\begin{tikzcd} \bullet \arrow[r, "h"] \arrow[d, "k"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "f"] & \bullet \end{tikzcd}$$

is said to be a *(BC)-square* if $g^\circ \cdot f = h \cdot k^\circ$. *A* Set*-endofunctor satisfies (BC)* if it preserves (BC)-squares (in particular it is *taut* in the sense of [21]). Moreover, we say that *a natural transformation satisfies (BC)* if each naturality diagram is a BC-square. *From now on we assume that T satisfies (BC).*

Examples 1.2. The *identity monad* $\mathbb{1} = (1, 1, 1)$ on Set can be obviously extended to the identity monad on V-Rel. In the sequel we will only consider this canonical extension of $\mathbb{1}$.

The *ultrafilter monad* $\mathbb{U} = (U, e, m)$ on Set is induced by the dual adjunction

$$\mathsf{Bool} \overset{\eta}{\Rightarrow} \underset{\hom(-,2)}{\overset{\hom(-,2)}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathsf{Set}.$$

Explicitly, the ultrafilter functor $U : \mathsf{Set} \to \mathsf{Set}$ sends a set X to the set UX of all ultrafilters on X, and a function $f : X \to Y$ to the function $Uf : UX \to UY$ which assigns to an ultrafilter $\mathfrak{x} \in UX$ the ultrafilter generated by its image $\{f[A] \mid A \in \mathfrak{x}\}$. The natural transformations e and m are given by

$$e_X(x) = \dot{x} = \{A \subseteq X \mid x \in A\} \quad \text{and} \quad m_X(\mathfrak{X}) = \{A \subseteq X \mid A^\# \in \mathfrak{X}\},$$

for all $\mathfrak{X} \in U^2X$ and $x \in X$. Here $A^\#$ denotes the set $\{\mathfrak{a} \in UX \mid A \in \mathfrak{a}\}$. We point out that U and m satisfy (BC). It is shown in [2] that the ultrafilter monad (U, e, m) can be naturally extended to $\mathsf{Rel} \cong \mathsf{2\text{-}Rel}$, and in [9] this result is generalised to $\mathsf{V\text{-}Rel}$, for a general class of lattices V including $\mathsf{V} = [0, \infty]$ (see also [12]). We remark that m becomes a (strict) natural transformation for these extensions and that U extends to a (strict) functor in Rel and $[0,\infty]\text{-}\mathsf{Rel}$. Although e does not satisfy (BC), the naturality diagram

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[d, "r"', "|" marking] & UX \arrow[d, "Ur", "|" marking] \\
Y \arrow[r, "e_Y"] & UY
\end{tikzcd}$$

is a (BC)-square, i.e. $Ur^\circ \cdot e_Y = e_X \cdot r^\circ$, provided that the relation r has *finite fibres* in the sense that, for every $y \in Y$, the set

$$\{x \in X \mid \bot < r(x, y)\}$$

is finite.

Another interesting monad in this context is the *free-monoid monad* $\mathbb{L} = (L, e, m)$ on Set where $L : \mathsf{Set} \to \mathsf{Set}$ sends a set X to the set LX of all finite words $(x_1, \ldots, x_n)$ $(n \in \mathbb{N})$ of elements of X, and for a function $f : X \to Y$, $Lf : LX \to LY$ sends $(x_1, \ldots, x_n)$ to $(f(x_1), \ldots, f(x_n))$. Note that LX includes the empty word $()$. For every set X there is a map $e_X : X \to LX, x \mapsto (x)$; e_X is the X-component of the natural transformation $e : 1 \to L$. An element of LLX is a word of words of X; by removing inner brackets we obtain an element of LX. This defines the X-component $m_X : LLX \to LX$ of $m : LL \to L$. The free-monoid monad is actually a Cartesian monad on Set, meaning that the functor $L : \mathsf{Set} \to \mathsf{Set}$ sends pullback squares to pullback squares, and every naturality square of e and of m is a pullback. In particular, L, e and m satisfy (BC). The Set-functor L extends naturally to a functor $L : \mathsf{Rel} \to \mathsf{Rel}$ by putting

$$(x_1, \dots, x_n)\, Lr\, (y_1, \dots, y_m) \quad \text{if} \quad n = m \,\&\, (x_1\, r\, y_1) \,\&\, \dots \,\&\, (x_n\, r\, y_n),$$

for every relation $r : X {\longrightarrow\!\!\!\!\!\!\mapsto}\, Y$, $(x_1, \dots, x_n) \in LX$ and $(y_1, \dots, y_m) \in LY$. With this definition, both e and m become natural transformations $e : 1 \to L$ and $m : LL \to L$ where $L : \mathsf{Rel} \to \mathsf{Rel}$.

A $(\mathbb{T}, \mathsf{V})$-*category* (called *lax algebra* in [7]) is a pair (X, a) consisting of a set X and a V-relation $a : TX {\longrightarrow\!\!\!\!\!\!\mapsto}\, X$ such that:

$$1_X \le a \cdot e_X \qquad \text{and} \qquad a \cdot Ta \le a \cdot m_X,$$

that is, the map $a : TX \times X \to \mathsf{V}$ satisfies the conditions

$$k \le a(e_X(x), x) \qquad \text{and} \qquad Ta(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \le a(m_X(\mathfrak{X}), x),$$

for all $\mathfrak{X} \in T^2X$, $\mathfrak{x} \in TX$ and $x \in X$. For $(\mathbb{T}, \mathsf{V})$-categories (X, a) and (Y, b), a $(\mathbb{T}, \mathsf{V})$-*functor* $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that $f \cdot a \le b \cdot Tf$, that is, for each $\mathfrak{x} \in TX$ and $x \in X$, $a(\mathfrak{x}, x) \le b(Tf(\mathfrak{x}), f(x))$. We denote the category of $(\mathbb{T}, \mathsf{V})$-categories and $(\mathbb{T}, \mathsf{V})$-functors by $(\mathbb{T}, \mathsf{V})$-Cat.

A $(\mathbb{T}, \mathsf{V})$-*graph* is a pair (X, a) satisfying $1_X \le a \cdot e_X$. Together with $(\mathbb{T}, \mathsf{V})$-functors, they constitute the category $(\mathbb{T}, \mathsf{V})$-Gph. Clearly, $(\mathbb{T}, \mathsf{V})$-Cat is a full subcategory of $(\mathbb{T}, \mathsf{V})$-Gph, and in the diagram

$$\begin{tikzcd} (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat} \arrow[rr] \arrow[dr] & & (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Gph} \arrow[dl] \\ & \mathsf{Set} & \end{tikzcd}$$

the forgetful functors into Set are topological and the embedding is a bireflection (see [7] for details). In particular, $(\mathbb{T}, \mathsf{V})$-Cat is complete and cocomplete, and the forgetful functor $(\mathbb{T}, \mathsf{V})$-$\mathsf{Cat} \to \mathsf{Set}$ preserves coproducts. This gives immediately that coproducts in $(\mathbb{T}, \mathsf{V})$-Cat are disjoint. Universality of coproducts follows from the characterisation of coproducts stated below and pullback-stability of open embeddings (see [20] for details).

For $(\mathbb{T}, \mathsf{V})$-functor $f : (X, a) \to (Y, b)$, $f \cdot a \le b \cdot Tf$ can be equivalently stated as $Tf \cdot a^\circ \le f^\circ \cdot b$; f is said to be *open* if the reverse inequality holds, that is $Tf \cdot a^\circ = f^\circ \cdot b$.

Theorem 1.3 ([20]). *For $(\mathbb{T},\mathsf{V})$-categories (X_i,a_i), $i\in I$, and (X,a), the following conditions are equivalent:*

(i) (X,a) *is the coproduct of* $(X_i,a_i)_{i\in I}$ *in* $(\mathbb{T},\mathsf{V})$-Cat*;*
(ii) (a) X *is the coproduct of* $(X_i)_{i\in I}$ *in* Set*;*
(b) *for each* $i\in I$*, the inclusion* $\iota_i:(X_i,a_i)\to(X,a)$ *is open.* ■

Corollary 1.4. $(\mathbb{T},\mathsf{V})$-Cat *is an (infinitely) extensive category.* ■

Examples 1.5. Let us have a look at the $(\mathbb{T},\mathsf{V})$-categories for the monad extensions considered in Examples 1.2. For $\mathbb{T}=\mathbb{1}$ being the identity monad, a $(\mathbb{1},2)$-category consists of a set X and a binary relation $\leq$ on X which is reflexive and transitive, and a $(\mathbb{1},2)$-functor is nothing but a monotone map. That is, $(\mathbb{1},2)$-Cat is equivalent to the category Ord of (pre)ordered sets and monotone maps. Furthermore, $(\mathbb{1},[0,\infty])$-Cat gives Lawvere's category Met of (pre)metric spaces and non-expansive maps [18].

The main result of [2] describes topological spaces as $(\mathbb{U},2)$-categories. To be more precise, a relation $\mathfrak{x}\to x$ between ultrafilters and points of a set X is the convergence relation of a (unique) topology on X if and only if

$$\dot{x}\to x \qquad\text{and}\qquad (\mathfrak{X}\to\mathfrak{x}\;\&\;\mathfrak{x}\to x)\;\Rightarrow\; m_X(\mathfrak{X})\to x,$$

for all $x\in X$, $\mathfrak{x}\in UX$ and $\mathfrak{X}\in UUX$. Since a map between topological spaces is continuous if and only if it preserves ultrafilter convergence, one has $(\mathbb{U},2)$-$\mathsf{Cat}\simeq\mathsf{Top}$. As above, one can trade the quantale 2 for the extended real half-line $[0,\infty]$, and in [7] it is shown that $(\mathbb{U},[0,\infty])$-Cat is isomorphic to the category App of approach spaces [19].

Finally, a $(\mathbb{L},2)$-category $X=(X,a)$ is a *multi-ordered set*, where the structure $a:LX{\longrightarrow\kern-0.8em\mapsto}X$ is not just a relation between points but a relation between finite words and points of X subject to

$$(x)\,a\,x \;\text{ and }\; ((\mathfrak{x}_1\,a\,x_1)\,\&\,\ldots\,\&\,(\mathfrak{x}_n\,a\,x_n)\,\&\,(x_1,\ldots,x_n)\,a\,x)\;\Rightarrow\; (x^1_1,\ldots,x^1_{m_1},x^2_1,\ldots,x^n_{m_n})\,a\,x,$$

for all $x\in X$, $(x_1,\ldots,x_n)\in LX$ and $\mathfrak{x}_i=(x^i_1,\ldots,x^i_{m_i})\in LX$ (where $i=1,\ldots,n$). A $(\mathbb{L},2)$-functor $f:(X,a)\to(Y,b)$ between multi-ordered sets is a function $f:X\to Y$ preserving the multi-order relation meaning that $(x_1,\ldots,x_n)\,a\,x$ implies $(f(x_1),\ldots,f(x_n))\,a\,f(x)$.

2. Descent $(\mathbb{T},\mathsf{V})$-functors

Regular epimorphisms in the category of $(\mathbb{T},\mathsf{V})$-graphs were characterised in [7] as the *final maps*, that is the $(\mathbb{T},\mathsf{V})$-functors $f:(X,a)\to(Y,b)$ with $b=f\cdot a\cdot Tf^\circ$:

$$\forall\mathfrak{y}\in TY\ \forall y\in Y\ \ b(\mathfrak{y},y)=\bigvee_{\substack{Tf(\mathfrak{x})=\mathfrak{y}\\ f(x)=y}} a(\mathfrak{x},x).$$

They are exactly the effective descent morphisms, as observed in [17], since $(\mathbb{T},\mathsf{V})$-Gph is locally Cartesian closed (see [10] for details). Hence:

Proposition 2.1. *Given $(\mathbb{T},\mathsf{V})$-graphs (X,a) and (Y,b), for a $(\mathbb{T},\mathsf{V})$-functor $f:(X,a)\to(Y,b)$ the following conditions are equivalent:*

(i) *f is a regular epimorphism in $(\mathbb{T},\mathsf{V})$-Gph;*
(ii) *f is a pullback-stable regular epimorphism in $(\mathbb{T},\mathsf{V})$-Gph;*
(iii) *f is an effective descent morphism in $(\mathbb{T},\mathsf{V})$-Gph;*
(iv) *f is a descent morphism in $(\mathbb{T},\mathsf{V})$-Gph;*
(v) *f is final.* ∎

We will now show that $(\mathbb{T},\mathsf{V})$-Cat is *finally dense* in $(\mathbb{T},\mathsf{V})$-Gph, that is, the sink $(f:X_I\to X)_I$ of all $(\mathbb{T},\mathsf{V})$-functors into a fixed $(\mathbb{T},\mathsf{V})$-graph whose domains X_i are $(\mathbb{T},\mathsf{V})$-categories is final (for details see e.g. [1]). We will do so with the help of certain "elementary structures". Consider a set X together with $\mathfrak{x}\in TX$ and $u\in\mathsf{V}$. We put $X^\star=X+1$ where $1=\{\star\}$, and define $a^\star=a_{\mathfrak{x},u}:TX^\star\,{\to\!\!\!\!\!\mapsto}\,X^\star$ as follows:

$$a^\star(\mathfrak{y},y)=\begin{cases}k & \text{if } \mathfrak{y}=e_{X^\star}(y),\\ u & \text{if } \mathfrak{y}=\mathfrak{x}\in TX \text{ and } y=\star,\\ \bot & \text{else}\end{cases}$$

for every $\mathfrak{y}\in TX^\star$ and $y\in X^\star$. Clearly, $a^\star$ is reflexive; and an additional assumption on the extension of $\mathbb{T}$ to V-Rel guarantees also transitivity of $a^\star$. To see this, let $\mathfrak{Y}$ in $TTX^\star$, $\mathfrak{y}\in TX^\star$ and $y\in X^\star$. We write $i:X\hookrightarrow X^\star$ for the inclusion map, and note that i is an open $(\mathbb{T},\mathsf{V})$-functor since in

$$\begin{tikzcd} TX \arrow[r, "Ti"] \arrow[d, "e_X^\circ"', "\mid" marking] & TX^\star \arrow[d, "a^\star", "\mid" marking] \\ X \arrow[r, "i"] & X^\star \end{tikzcd}$$

one has $(a^\star)^\circ \cdot i = Ti \cdot e_X \leq Ti \cdot a^\circ$. From the diagram one obtains also that $Ti^\circ \cdot Ta^\star = Te_X^\circ \cdot TTi^\circ$.

– Assume first $y \neq \star$, hence $y = i(x)$ with $x \in X$. Therefore we can also assume $\mathfrak{y} = e_{X^\star}(y)$, since otherwise $a^\star(\mathfrak{y}, y) = \bot$. Then

$$\bot < Ta^\star(\mathfrak{Y}, e_{X^\star}(i(x))) = (e_X^\circ \cdot Ti^\circ \cdot Ta^\star)(\mathfrak{Y}, x) = (e_X^\circ \cdot Te_X^\circ \cdot TTi^\circ)(\mathfrak{Y}, x),$$

which implies $\mathfrak{Y} = (Te_{X^\star} \cdot e_{X^\star})(y)$ and $Ta^\star(\mathfrak{Y}, e_{X^\star}(y)) = k$.

– Assume now $y = \star$. If $\mathfrak{y} = Ti(\mathfrak{x})$ for some $\mathfrak{x} \in TX$, then we can argue as above:

$$\bot < Ta^\star(\mathfrak{Y}, Ti(\mathfrak{x})) = (Ti^\circ \cdot Ta^\star)(\mathfrak{Y}, \mathfrak{x}) = (Te_X^\circ \cdot TTi^\circ)(\mathfrak{Y}, \mathfrak{x})$$

gives $\mathfrak{Y} = (Te_{X^\star} \cdot Ti)(\mathfrak{x})$ and $Ta^\star(\mathfrak{Y}, Ti(\mathfrak{x})) = k$.

– Finally, assume $\mathfrak{y} = e_{X^\star}(\star)$. If the naturality square

$$\begin{tikzcd} TX^\star \arrow[r, "e_{TX^\star}"] \arrow[d, "a^\star"', "\shortmid" marking] & TTX^\star \arrow[d, "Ta^\star", "\shortmid" marking] \\ X^\star \arrow[r, "e_{X^\star}"] & TX^\star \end{tikzcd} \tag{A}$$

is a (BC)-square, that is, $(Ta^\star)^\circ \cdot e_{X^\star} = e_{TX^\star} \cdot (a^\star)^\circ$ or, equivalently, $e_{X^\star}^\circ \cdot Ta^\star = a^\star \cdot e_{TX^\star}^\circ$, then

$$\bot < Ta^\star(\mathfrak{Y}, e_{X^\star}(\star))$$

implies $\mathfrak{Y} = e_{TX^\star}(\mathfrak{y}')$ for some $\mathfrak{y}' \in TX^\star$ and $Ta^\star(\mathfrak{Y}, e_{X^\star}(\star)) = a^\star(\mathfrak{y}', \star)$. Consequently, either $\mathfrak{y}' = Ti(\mathfrak{x})$ or $\mathfrak{y}' = e_{X^\star}(\star)$, and the assertion follows.

Proposition 2.2. *Assume that every naturality square* (A) *is a (BC)-square, for all sets X, $\mathfrak{x} \in TX$, $u \in \mathsf{V}$ and $a^\star = a_{\mathfrak{x},u}$. Then* $(\mathbb{T}, \mathsf{V})$-Cat *is finally dense in* $(\mathbb{T}, \mathsf{V})$-Gph. ∎

Examples 2.3. If we start with a monad $\mathbb{T} = (T, e, m)$ on Set where every naturality square of e satisfies (BC) (for instance, if $\mathbb{T}$ is Cartesian), then e is also a natural transformation for the extension of $\mathbb{T}$ to Rel, and this remains true for the extension of this monad to V-Rel described in [9, Proposition 4.2 and Lemma 5.2] and [15, Theorem 3.5]. The situation is slightly different for the ultrafilter monad. Recall from Examples 1.2 that a naturality square

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "r"', "\shortmid" marking] & UX \arrow[d, "Ur", "\shortmid"' marking] \\ Y \arrow[r, "e_Y"] & UY \end{tikzcd}$$

for a V-relation $r : X \to\!\!\!\!\!\!\shortmid\;\, Y$ is in general not a (BC)-square; however, it is so provided that r *has finite fibres*, i.e. $\{x \in X \mid \bot < r(x,y)\}$ is finite for every $y \in Y$. By definition, every relation $a_{\mathfrak{x},u} : TX^\star \to\!\!\!\!\!\!\shortmid\;\, X^\star$ is of this type.

Theorem 2.4. *Assume that every naturality square of e with respect to* V*-relations with finite fibres is a (BC)-square. Then the following conditions are equivalent, for a morphism $f : (X,a) \to (Y,b)$ in $(\mathbb{T},\mathsf{V})$-*Cat*:*

(i) *f is final;*
(ii) *f is a pullback-stable regular epimorphism in $(\mathbb{T},\mathsf{V})$-*Gph*;*
(iii) *f is a pullback-stable regular epimorphism in $(\mathbb{T},\mathsf{V})$-*Cat*;*
(iv) *f is a descent morphism in $(\mathbb{T},\mathsf{V})$-*Cat*.*

Proof: (i) $\Rightarrow$ (ii) follows from the proposition above. (ii) $\Rightarrow$ (iii) follows from the fact that the embedding $(\mathbb{T},\mathsf{V})$-Cat $\to$ $(\mathbb{T},\mathsf{V})$-Gph preserves pullbacks, since it is a right adjoint. (iii) $\Leftrightarrow$ (iv) is valid in every finitely complete category.

(iii) $\Rightarrow$ (i): Assume now that $f : X \to Y$ is a pullback-stable regular epimorphism in $(\mathbb{T},\mathsf{V})$-Cat. Let $\mathfrak{y} \in TY$ and $y_0 \in Y$. Consider the $(\mathbb{T},\mathsf{V})$-functor $g : Y^\star \to Y$ where $Y^\star = (Y^\star, b^\star)$ is defined as above with $u = b(\mathfrak{y}, y_0)$ and $g(y) = y$ for $y \in Y$ and $g(\star) = y_0$. By hypothesis, the pullback $p_2 : (P,c) \to (Y^\star, b^\star)$ of f along g

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y^\star \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a regular epimorphism in $(\mathbb{T},\mathsf{V})$-Cat. Thanks to the simple structure of $Y^\star$, this is only possible if

$$\begin{aligned} u = b^\star(\mathfrak{y},\star) &= \bigvee\{c(\mathfrak{p},z) \mid \mathfrak{p} \in TP,\, Tp_2(\mathfrak{p}) = \mathfrak{y},\, z \in P,\, p_2(z) = \star\} \\ &= \bigvee\{a(\mathfrak{x},x) \wedge u \mid \mathfrak{x} \in TX,\, Tf(\mathfrak{x}) = \mathfrak{y},\, x \in X,\, f(x) = y_0\} \\ &= \Big(\bigvee\{a(\mathfrak{x},x) \mid \mathfrak{x} \in TX,\, Tf(\mathfrak{x}) = \mathfrak{y},\, x \in X,\, f(x) = y_0\}\Big) \wedge u, \end{aligned}$$

hence $b(\mathfrak{y}, y_0) \le \bigvee\{a(\mathfrak{x}, x) \mid \mathfrak{x} \in TX,\ Tf(\mathfrak{x}) = \mathfrak{y},\ x \in X,\ f(x) = y_0\}$. ■

Examples 2.5. The theorem above generalises the well-known characterisation of pullback-stable regular epimorphisms in the category Top of topological spaces and continuous maps (see [13]) as *biquotient maps* in the sense of Michael [22], or *limit-lifting maps* in the sense of Hájek [14]. As consequences of the theorem we have that:

(1) As it is well-known, in Ord a monotone map $f : (X, a) \to (Y, b)$ is of descent if, and only if, it is a pullback-stable regular epimorphisms, or, equivalently, if it is a regular epimorphism; it is characterised by the condition:

$$\forall y, y' \in Y\ y \le y' \ \Rightarrow\ \exists x \in f^{-1}(y)\ \exists x' \in f^{-1}(y')\ :\ x \le x'.$$

(2) In Met, a non-expansive map $f : (X, a) \to (Y, b)$ is of descent, or, equivalently, a pullback-stable regular epimorphism, if, and only if,

$$\forall y, y' \in Y\ b(y, y') = \inf\{a(x, x')\,;\, x \in f^{-1}(y),\, x' \in f^{-1}(y')\}.$$

(3) Analogously, a non-expansive map $f : (X, a) \to (Y, b)$ in App is a descent morphism, or, equivalently, a pullback-stable regular epimorphism, if, and only if,

$$\forall \mathfrak{y} \in UY\ \forall y \in Y\ b(\mathfrak{y}, y) = \inf\{a(\mathfrak{x}, x)\,;\, \mathfrak{x} \in Uf^{-1}(\mathfrak{y}),\, x \in f^{-1}(y)\}.$$

(4) In the category MOrd of multi-ordered sets and monotone maps, $f : (X, a) \to (Y, b)$ is of descent, or equivalently a pullback-stable regular epimorphism, if, and only if,

$$\forall \mathfrak{y} = (y_1, \cdots, y_n) \in LY\ \forall y \in Y\ \mathfrak{y}\, b\, y\ \Rightarrow$$
$$\exists \mathfrak{x} = (x_1, \cdots, x_n) \in f^{-1}(\mathfrak{y})\ \exists x \in f^{-1}(y)\ :\ \mathfrak{x}\, a\, x.$$

3. The (global) van Kampen Theorem in $(\mathbb{T}, \mathsf{V})$-Cat

In this section we will be concerned with the (global) categorical van Kampen Theorem. In [4] it is shown that, for extensive categories, the key property for the van Kampen Theorem to hold is the fact that the morphism $p : X_1 + X_2 \to X$, from the coproduct of X_1 and X_2 into X induced by the embeddings $g_1 : X_1 \hookrightarrow X$ and $g_2 : X_2 \hookrightarrow X$, is of effective descent (see [4] for details).

Since $(\mathbb{T},\mathsf{V})$-Cat is extensive, and complete and cocomplete, in order to state a van Kampen Theorem in this context we will characterise the morphisms $p : X_1 + X_2 \to X$ as above which are of effective descent. Here we will restrict ourselves to the case of global effective descent morphisms.

Following Reiterman-Tholen characterisation of effective descent continuous maps in Top [24], and the subsequent characterisation presented in [6], a $(\mathbb{T},\mathsf{V})$-functor $f : (X,a) \to (Y,b)$ is said to be a $*$-*quotient* if, for every $\mathfrak{Y} \in T^2Y$, $\mathfrak{y} \in TY$ and $y \in Y$,

$$Tb(\mathfrak{Y},\mathfrak{y}) \otimes b(\mathfrak{y},y) = \bigvee_{\substack{\mathfrak{X}\in(T^2f)^{-1}(\mathfrak{Y})\\ \mathfrak{x}\in(Tf)^{-1}(\mathfrak{y})\\ x\in f^{-1}(y)}} Ta(\mathfrak{X},\mathfrak{x}) \otimes a(\mathfrak{x},x).$$

It is shown in [8, Theorem 3.3] (see also [11]) that:

Proposition 3.1. *A pullback-stable $*$-quotient $(\mathbb{T},\mathsf{V})$-functor in $(\mathbb{T},\mathsf{V})$-Cat is of effective descent.* ∎

It is also shown in [8] that open and proper surjections are pullback-stable $*$-quotient maps, hence of effective descent, provided that condition (C1) below is fulfilled. We recall that a $(\mathbb{T},\mathsf{V})$-functor $f : (X,a) \to (Y,b)$ is *proper* if $f \cdot a = b \cdot Tf$.

From now on we will consider the following conditions:

- (C0) every naturality square of e with respect to V-relations with finite fibres is a (BC)-square;
- (C1) $T(f\cdot r) = Tf\cdot Tr$, for every map $f : Y \to Z$ and V-relation $r : X \to\!\!\!\!\!\!\mapsto Y$;
- (C2) m satisfies (BC);
- (C3) naturality diagrams of the natural transformation e with respect to relations with finite fibres are (BC)-squares;
- (C4) T preserves coproducts.

We point out that condition (C1) is not a restrictive condition; indeed, it follows easily from a condition widely fulfilled: for any map $f : Y \to Z$ and any relation $r : X \to\!\!\!\!\!\!\mapsto Y$, $T(f \cdot r) = Tf \cdot Tr$. In particular this holds whenever the extension T is defined by a topological theory in the sense of [15]. For an analysis of (C4) see [3]. It is straightforward to prove that:

Lemma 3.2. *If* $T(f \cdot r) = Tf \cdot Tr$ *for any map* f *and any* V*-relation* r*, then* T *preserves open, proper, and final* $(\mathbb{T}, \mathsf{V})$*-functors.* ■

Theorem 3.3. *Assume that conditions (C0)-(C4) hold. If* (X_1, a_1) *and* (X_2, a_2) *are* $(\mathbb{T}, \mathsf{V})$*-subcategories of* (X, a) *and* $p : X_1 + X_2 \to X$ *is the* $(\mathbb{T}, \mathsf{V})$*-functor induced by their embeddings, then the following conditions are equivalent in* $(\mathbb{T}, \mathsf{V})$-Cat*:*

(i) p *is a pullback-stable* $*$*-quotient map;*
(ii) p *is of effective descent;*
(iii) p *is a descent morphism;*
(iv) p *is final;*
(v) *for any* $\mathfrak{x} \in TX$ *and* $x \in X$*, either there exists* $i \in \{1, 2\}$ *such that* $\mathfrak{x} \in Tg_i(TX_i)$ *and* $x \in g_i(X_i)$*, or* $a(\mathfrak{x}, x) = \bot$*.*

Proof: Proposition 3.1 states that (i) ⇒ (ii), (ii) ⇒ (iii) is always true, and (iii) ⇒ (iv) follows from Theorem 2.4. In order to show that (iv) ⇔ (v) and (v) ⇒ (i) we first describe the structure b on $X_1 + X_2 = X_1 \times \{1\} \cup X_2 \times \{2\}$: for $\mathfrak{y} \in T(X_1 + X_2)$, and $(x, i) \in X_1 + X_2$,

$$b(\mathfrak{y}, (x, i)) = \begin{cases} a_i(\mathfrak{y}_i, (x, i)) & \text{if } \mathfrak{y} = T\tau_i(\mathfrak{y}_i) \text{ for } \mathfrak{y}_i \in TX_i \\ \bot & \text{if } \mathfrak{y} \notin T\tau_i(X_i). \end{cases}$$

Hence the embeddings $\tau_i : (X_i, a_i) \to (X_1 + X_2, b)$ are both open and proper.

(iv) ⇔ (v): Let $\mathfrak{x} \in TX$ and $X \in X$ with $a(\mathfrak{x}, x) \neq \bot$. Since p is final

$$a(\mathfrak{x}, x) = \bigvee_{\substack{Tp(\mathfrak{y}) = \mathfrak{x} \\ i : x \in X_i}} b(\mathfrak{y}, (x, i))$$

Hence there is $\mathfrak{y} \in T(X_1 + X_2)$ and $i \in \{1, 2\}$ with $Tp(\mathfrak{y}) = \mathfrak{x}$, $x \in X_i$, and $b(\mathfrak{y}, (x, i)) \neq \bot$. By definition of b, $\mathfrak{y} = T\tau_i(\mathfrak{x}_i)$ for some $\mathfrak{x}_i \in TX_i$, and then $TpT\tau_i(\mathfrak{x}_i) = Tg_i(\mathfrak{x}_i) = \mathfrak{x}$ as claimed. The proof of the reverse implication is straightforward.

(iv) $\Rightarrow$ (i): Since final maps are pullback-stable, we only have to show that p is a $*$-quotient map provided that it is final. Let $\mathfrak{X} \in T^2X$, $\mathfrak{x} \in TX$ and $x \in X$ with

$$Ta(\mathfrak{X}, \mathfrak{x}) = \beta, a(\mathfrak{x}, x) = \alpha \text{ and } Ta(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) = \beta \otimes \alpha \neq \bot.$$

Finality of p guarantees that there exists $i \in \{1, 2\}$ such that $\mathfrak{x} \in Tg_i(TX_i)$ and $x \in g_i(X_i)$. Without loss of generality we assume that $x \in X_1$ and that there exists $\mathfrak{x}_1 \in TX_1$ with $Tg_1(\mathfrak{x}_1) = \mathfrak{x}$. Therefore

$$b(T\tau_1(\mathfrak{x}_1), (x, 1)) = a_1(\mathfrak{x}_1, x) = a(\mathfrak{x}, x) = \alpha.$$

Moreover, since $a(m_X(\mathfrak{X}), x) \geq \beta \otimes \alpha \neq \bot$, for some $j \in \{1, 2\}$ $x \in X_j$ and there exists $\mathfrak{y}_j \in TX_j$ with $Tg_j(\mathfrak{y}_j) = m_X(\mathfrak{X})$. This implies, by (BC) of m, that $\mathfrak{X} = T^2g_j(\mathfrak{X}_j)$ with $\mathfrak{X}_j \in T^2X_j$.

– If $j = 1$, then $Tb(T^2\tau_1(\mathfrak{X}_1), T\tau_1(\mathfrak{x}_1)) = Ta_1(\mathfrak{X}_1, \mathfrak{x}_1) = Ta(\mathfrak{X}, \mathfrak{x}) = \beta$.
– If $\mathfrak{X} \notin T^2g_1(T^2X_1)$, we use finality of Tp. Since $Ta(\mathfrak{X}, \mathfrak{x}) = \beta \neq \bot$, there exists $\tilde{\mathfrak{X}} \in T^2(X_1 + X_2)$ and $\tilde{\mathfrak{x}} \in T(X_1 + X_2)$ such that $T^2p(\tilde{\mathfrak{X}}) = \mathfrak{X}$, $Tp(\tilde{\mathfrak{x}}) = \mathfrak{x}$ and $Tb(\tilde{\mathfrak{X}}, \tilde{\mathfrak{x}}) = \beta$. By assumption $\tilde{\mathfrak{X}} \in T^2\tau_2(T^2X_2)$ and therefore, since $T^2\tau_2$ is proper, $\tilde{\mathfrak{x}} \in T\tau_2(TX_2)$. This means that there exists $\mathfrak{X}_2 \in T^2X_2$ and $\mathfrak{x}_2 \in TX_2$ such that

$$\begin{aligned} Tb(T^2\tau_2(\mathfrak{X}_2), T\tau_2(\mathfrak{x}_2)) \otimes b(T\tau_2(\mathfrak{x}_2), (x, 2)) &= Ta_2(\mathfrak{X}_2, \mathfrak{x}_2) \otimes a_2(\mathfrak{x}_2, (x, 2)) \\ &= Ta(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) = \beta \otimes \alpha \end{aligned}$$

and the proof is complete. ■

Since the identity functor and the ultrafilter functor satisfy (C0)-(C4), this result applies to the categories Ord, Met, Top, App. Although the free-monoid functor L does not preserve coproducts, as we show next the result is still valid for L, when we consider the extension of L to Rel defined in [12], that is:

$$Lr((x_1, \cdots, x_n), (y_1, \cdots, y_m)) = \begin{cases} \otimes_{i=1}^n r(x_i, y_i) & \text{if } m = n \\ \bot & \text{elsewhere,} \end{cases}$$

for any V-relation $r : X \to\!\!\!\!\!\!\mapsto Y$, $(x_1, \cdots, x_n) \in LX$ and $(y_1, \cdots, y_m) \in LY$.

Theorem 3.4. *If* (X_1, a_1) *and* (X_2, a_2) *are* $(\mathbb{L}, \mathsf{V})$*-subcategories of* (X, a) *and* $p : X_1 + X_2 \to X$ *is the* $(\mathbb{L}, \mathsf{V})$*-functor induced by their embeddings, then the following conditions are equivalent in* $(\mathbb{L}, \mathsf{V})$-Cat*:*

(i) *p is a pullback-stable $*$-quotient map;*
(ii) *p is of effective descent;*
(iii) *p is a descent morphism;*
(iv) *p is final;*
(v) *for any $\mathfrak{x} \in LX$ and $x \in X$, either there exists $i \in \{1, 2\}$ such that $\mathfrak{x} \in Lg_i(LX_i)$ and $x \in g_i(X_i)$, or $a(\mathfrak{x}, x) = \perp$.*

Proof: We can use the arguments of the proof of the former theorem to show all the implications but (iv) $\Rightarrow$ (i). To prove (iv) $\Rightarrow$ (i), let p be final, and $\mathfrak{X} = (\mathfrak{x}_1, \cdots, \mathfrak{x}_n) \in L^2X$, with $\mathfrak{x}_i = (x^i_1, \cdots, x^i_{m_i})$, $\mathfrak{x} = (x_1, \cdots, x_n) \in LX$ and $x \in X$ such that

$$La(\mathfrak{X}, \mathfrak{x}) = \beta,\ a(\mathfrak{x}, x) = \alpha \text{ and } \beta \otimes \alpha \neq \perp.$$

Finality of p guarantees that there exists $j \in \{1, 2\}$ such that

$$x_1, \cdots, x_n, x \in X_j.$$

We may assume, without loss of generality, that $j = 1$. As in the proof above, $a(m_X(\mathfrak{X}), x) \geq \beta \otimes \alpha \neq \perp$ assures that $m_X(\mathfrak{X}) \in Lg_1(LX_1)$ or $m_X(\mathfrak{X}) \in Lg_2(LX_2)$.

- If $m_X(\mathfrak{X}) \in Lg_1(LX_1)$, then $\mathfrak{X} \in L^2g_1(L^2X_1)$ and the proof is complete.
- If $\mathfrak{X} \in Lg_2(LX_2)\backslash Lg_1(LX_1)$, then necessarily $x \in g_2(X_2)$ and the proof is complete in case $\mathfrak{x} = (x_1, \cdots, x_n) \in Lg_2(LX_2)$. If $\mathfrak{x} \notin Lg_2(LX_2)$, that is if there exists $l \in \{1, \cdots, n\}$ with $x_l \notin X_2$ then we can consider $\mathfrak{Y} = (\mathfrak{y}_1, \cdots, \mathfrak{y}_n) \in L^2X$ defined by

$$\mathfrak{y}_i = \begin{cases} \mathfrak{x}_i & \text{if } i \neq l \\ (x_l) & \text{if } i = l. \end{cases}$$

  By construction $Ta(\mathfrak{Y}, \mathfrak{x}) = \otimes_{i \in \{1, \cdots, n\} \backslash \{l\}} \beta_i \neq \perp$ and $a(m_X(\mathfrak{Y}), x) \geq (\otimes_{i \in \{1, \cdots, n\} \backslash \{l\}} \beta_i) \otimes a \neq \perp$. However, under our assumptions, $m_X(\mathfrak{Y}) \notin Lg_1(LX_1) \cup Lg_2(LX_2)$, which contradicts finality of p. ∎

As a corollary we obtain:

Theorem 3.5 (Global van Kampen Theorem)**.** *Let* $\mathbb{T} = \mathbb{1}$*, or* $\mathbb{T} = \mathbb{U}$*, or* $\mathbb{T} = \mathbb{L}$*, and let* $\mathbb{C} = (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$*. If the following diagram*

$$\begin{tikzcd}
 & X & \\
X_1 \arrow[ur, "g_1"] & & X_2 \arrow[ul, "g_2"'] \\
 & X_0 \arrow[ul, "f_1"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

be a pullback, with g_1 *and* g_2 *embeddings, then the diagram*

$$\begin{tikzcd}
 & \mathbb{C} \downarrow X \arrow[dl, "g_1^*"'] \arrow[dr, "g_2^*"] & \\
\mathbb{C} \downarrow X_1 \arrow[dr, "f_1^*"'] & & \mathbb{C} \downarrow X_2 \arrow[dl, "f_2^*"] \\
 & \mathbb{C} \downarrow X_0 &
\end{tikzcd}$$

is a pullback if, and only if, the morphism $p : X_1 + X_2 \to X$*, induced by* g_1 *and* g_2*, is a final morphism.*

Here g_1^* and g_2^* are the change-of-base functors, and, as in [4], by the latter diagram being a pullback we mean that the functor

$$\mathbb{C} \downarrow X \xrightarrow{K_{g_1,g_2}} (\mathbb{C} \downarrow X_1) \times_{\mathbb{C}\downarrow X_0} (\mathbb{C} \downarrow X_2),$$

induced by (g_1, g_2), is an equivalence, where $(\mathbb{C} \downarrow X_1) \times_{\mathbb{C}\downarrow X_0} (\mathbb{C} \downarrow X_2)$ is the category of triples $((A_1, \alpha_1), (A_2, \alpha_2), \varphi)$, with $(A_i, \alpha_i) \in \mathbb{C} \downarrow X_i$, $i = 1, 2$, and $\varphi : f_1^*(A_1, \alpha_1) \to f_2^*(A_2, \alpha_2)$ an isomorphism.

Final Remarks 3.6.
(1) In the particular case $\mathbb{T} = \mathbb{U}$ and $\mathsf{V} = 2$, that is in the category Top, in [5] it is shown that we can add an equivalent condition on Theorem 3.3, namely p being a *triquotient map* [23]. This notions is purely topological and there is no corresponding notion in categories of lax algebras
(2) We do not know whether the result of Theorems 3.3 and 3.4 is valid for more general monads. In fact, we do not know any example of monad for which the result does not hold.

References

[1] Jiří Adámek, Horst Herrlich, and George E. Strecker. *Abstract and concrete categories*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1990. The joy of cats, A Wiley-Interscience Publication.

[2] Michael Barr. Relational algebras. In *Reports of the Midwest Category Seminar, IV*, pages 39–55. Lecture Notes in Mathematics, Vol. 137. Springer, Berlin, 1970.

[3] Reinhard Börger. Coproducts and ultrafilters. *J. Pure Appl. Algebra*, 46(1):35–47, 1987.

[4] Ronald Brown and George Janelidze. Van Kampen theorems for categories of covering morphisms in lextensive categories. *J. Pure Appl. Algebra*, 119(3):255–263, 1997.

[5] Maria Manuel Clementino. A note on the categorical van Kampen Theorem. *Topology Appl.*, to appear (`doi:10.1016/j.topol.2011.01.011`).

[6] Maria Manuel Clementino and Dirk Hofmann. Triquotient maps via ultrafilter convergence. *Proc. Amer. Math. Soc.*, 130(11):3423–3431, 2002.

[7] Maria Manuel Clementino and Dirk Hofmann. Topological features of lax algebras. *Appl. Categ. Structures*, 11(3):267–286, 2003.

[8] Maria Manuel Clementino and Dirk Hofmann. Effective descent morphisms in categories of lax algebras. *Appl. Categ. Structures*, 12(5-6):413–425, 2004.

[9] Maria Manuel Clementino and Dirk Hofmann. On extensions of lax monads. *Theory Appl. Categ.*, 13:No. 3, 41–60, 2004.

[10] Maria Manuel Clementino, Dirk Hofmann, and Walter Tholen. Exponentiability in categories of lax algebras. *Theory Appl. Categ.*, 11:No. 15, 337–352, 2003.

[11] Maria Manuel Clementino and and George Janelidze. A note on effective descent morphisms of topological spaces and relational algebras. Technical Report, University of Coimbra, 10-55, 2010.

[12] Maria Manuel Clementino and Walter Tholen. Metric, topology and multicategory—a common approach. *J. Pure Appl. Algebra*, 179(1-2):13–47, 2003.

[13] B. J. Day and G. M. Kelly. On topologically quotient maps preserved by pullbacks or products. *Proc. Cambridge Philos. Soc.*, 67:553–558, 1970.

[14] Otomar Hájek. Notes on quotient maps. *Comment. Math. Univ. Carolinae*, 7:319–323, 1966.

[15] Dirk Hofmann. Topological theories and closed objects. *Adv. Math.*, 215(2):789–824, 2007.

[16] George Janelidze and Walter Tholen. How algebraic is the change-of-base functor? In *Category theory (Como, 1990)*, volume 1488 of *Lecture Notes in Math.*, pages 174–186. Springer, Berlin, 1991.

[17] George Janelidze and Walter Tholen. Facets of descent. I. *Appl. Categ. Structures*, 2(3):245–281, 1994.

[18] F. William Lawvere. Metric spaces, generalized logic, and closed categories. *Rend. Sem. Mat. Fis. Milano*, 43:135–166 (1974), 1973. Also in: Repr. Theory Appl. Categ. No. 1 (2002), 1–37.

[19] Robert Lowen. *Approach spaces.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1997. The missing link in the topology-uniformity-metric triad, Oxford Science Publications.

[20] Mojgan Mahmoudi, Christoph Schubert, and Walter Tholen. Universality of coproducts in categories of lax algebras. *Appl. Categ. Struct.*, 14(3):243–249, 2006.

[21] Ernest G. Manes. Taut monads and $T0$-spaces. *Theoret. Comput. Sci.*, 275(1-2):79–109, 2002.

[22] Ernest Michael. Bi-quotient maps and Cartesian products of quotient maps. *Ann. Inst. Fourier (Grenoble)*, 18(fasc. 2):287–302 vii (1969), 1968.

[23] Ernest Michael. Complete spaces and tri-quotient maps. *Illinois J. Math.*, 21(3):716–733, 1977.

[24] Jan Reiterman and Walter Tholen. Effective descent maps of topological spaces. *Topology Appl.*, 57(1):53–69, 1994.

Maria Manuel Clementino
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: mmc@mat.uc.pt

Dirk Hofmann
Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail address: dirk@ua.pt