Abstract: In this paper we characterise descent morphisms in categories of lax algebras via a generalised limit-lifting condition. Furthermore, we apply this result to establish a global van Kampen Theorem in some of these categories.

Keywords: descent morphism, effective descent morphism, lax algebra, van Kampen Theorem.


Introduction

The Theorem of van Kampen is the principal practical tool for calculating the fundamental group $\pi_1(X, x)$ of a topological space $X$ with base point $x$, asserting that $\pi_1(X, x)$ can be obtained as the pushout of the fundamental groups of two sufficiently nice open subspaces. Furthermore, under suitable conditions, this can be equivalently expressed using covering maps, and in \[4\] R. Brown and G. Janelidze prove a very general version of this result which holds not just for topological spaces but in any lextensive category. Another important achievement of \[4\] is the understanding of the rôle of descent theory in this context: the van Kampen Theorem holds for subobjects $X_1 \hookrightarrow X$ and $X_2 \hookrightarrow X$ of $X$ if and only if the induced map $p : X_1 + X_2 \to X$ is of effective descent with respect to a chosen class of morphisms.

The study of descent theory for topological spaces – first treated in \[16\] – turned out to be a difficult subject. Indeed, it essentially concerns intricate conditions on ultrafilter convergence, having as keystone the characterisation of global effective descent morphisms \[24\]. Fortunately, as shown in \[5\], the situation is much simpler in the particular situation of the van Kampen Theorem (with respect to the choice of all continuous maps).

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In fact, \( p : X_1 + X_2 \to X \) is of effective descent if and only if the subspaces \( X_1 \) and \( X_2 \) satisfy a simple topological property.

A topological space is the paradigmatic example of a lax algebra \([9]\), with further examples including ordered sets and metric and approach spaces. Therefore it is natural to ask whether the result of \([5]\) can be extended to this general setting. According to what was said above, this amounts to understanding the notion of descent and effective descent morphism in these categories. A first step in this direction was already done in \([8]\) where effective descent morphisms between (pre)metric spaces are characterised, and where it is shown that open and proper surjections between lax algebras are of effective descent. However, we are not aware of any treatment of descent morphisms on this general level. In this paper we close this gap and show that, under suitable conditions, they are precisely the limit-lifting maps in an appropriate sense, generalising this way the corresponding well-known result for topological spaces. Finally, we show that, under suitable conditions, the intricate condition of \( p : X_1 + X_2 \to X \) being of effective descent is equivalent to \( p \) being a pullback-stable regular epimorphism.

1. Preliminaries

Throughout this paper \( V \) denotes a commutative and unital quantale, with tensor \( \otimes \) and neutral element \( k \). Hence, \( V \) is a complete lattice and \( \otimes : V \times V \to V \) is an associative and commutative binary operation on \( V \) so that

\[
k \otimes u = u = u \otimes k
\]

and which preserves suprema in each variable:

\[
u \otimes \left( \bigvee_{i \in I} u_i \right) = \bigvee_{i \in I} (u \otimes u_i).
\]

We also assume that the complete lattice \( V \) satisfies the frame law, that is,

\[
u \land \left( \bigvee_{i \in I} u_i \right) = \bigvee_{i \in I} (u \land u_i).
\]

Our principal examples are the two-element chain \( 2 = \{\text{false} \leq \text{true}\} \) with tensor given by “and” \& and neutral element \( \text{true} \); and the extended real half-line \([0, \infty]\) ordered by the “greater or equal”-relation \( \geq \), with tensor given by addition (where \( \infty + x = x + \infty = \infty \)) and 0 as neutral element.
The category $\mathbf{V}$-$\text{Rel}$ of $\mathbf{V}$-relations has sets as its objects, and a morphism $r : X \to Y$ in $\mathbf{V}$-$\text{Rel}$ is a function $r : X \times Y \to \mathbf{V}$. Composition of $\mathbf{V}$-relations $r : X \to Y$ and $s : Y \to Z$ is defined as matrix multiplication $(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$, and the identity on $X$ is the $\mathbf{V}$-relation $1_X : X \to X$ which sends all diagonal elements $(x, x)$ to $k$ and all other elements to the bottom element $\bot$ of $\mathbf{V}$. The order of $\mathbf{V}$ induces a complete order relation on $\mathbf{V}$-$\text{Rel}(X, Y)$: for $\mathbf{V}$-relations $r, r' : X \to Y$, one puts $r \leq r' : \iff \forall x \in X \forall y \in Y \ r(x, y) \leq r'(x, y)$.

Composition preserves suprema in each variable since $\otimes$ does so, that is, $\left( \bigvee_{j \in J, i \in I} s_j \cdot r_i \right) = \bigvee_{j \in J} s_j \cdot \bigvee_{i \in I} r_i$.

Therefore $\mathbf{V}$-$\text{Rel}$ is actually a 2-category.

$\mathbf{V}$-$\text{Rel}$ has an order-preserving involution $(\cdot)^\circ$ sending $r : X \to Y$ to its transpose $r^\circ : Y \to X$ defined by $r^\circ(y, x) = r(x, y)$. In fact, this way one defines a contravariant 2-endofunctor on $\mathbf{V}$-$\text{Rel}$.

There is a natural embedding of $\text{Set}$ into $\mathbf{V}$-$\text{Rel}$ leaving objects unchanged and sending a map $f : X \to Y$ to the $\mathbf{V}$-matrix $f(x, y) = \begin{cases} k & \text{if } f(x) = y, \\ \bot & \text{else.} \end{cases}$

In the sequel we will write $f : X \to Y$ rather then $f : X \to Y$ for a $\mathbf{V}$-matrix induced by a $\text{Set}$-map in the sense above. We remark that each $f : X \to Y$ satisfies the inequations $1_X \leq f^\circ \cdot f$ and $f \cdot f^\circ \leq 1_Y$, which just tells us that $f$ is left adjoint to $f^\circ$ in $\mathbf{V}$-$\text{Rel}$.

**Examples 1.1.** Clearly, $2$-$\text{Rel} \cong \text{Rel}$. A morphism $a : X \to Y$ of $[0, \infty]$-$\text{Rel}$ is a generalised distance function $a : X \times Y \to [0, \infty]$. Composition in $[0, \infty]$-$\text{Rel}$ is given by $(b \cdot a)(x, z) = \inf \{a(x, y) + b(y, z) \mid y \in Y\}$, and $1_X : X \to X$ is the discrete distance sending the diagonal to 0 and all other pairs $(x, x')$ to $\infty$. 

Recall that a monad \( \mathbb{T} = (T, e, m) \) on \( \text{Set} \) consists of a functor \( T : \text{Set} \to \text{Set} \) together with natural transformations \( e : 1_{\text{Set}} \to T \) and \( m : TT \to T \) such that
\[
m \cdot Tm = m \cdot m_T \quad \text{and} \quad m \cdot Te = 1_T = m \cdot e_T.
\]
Here the natural transformation \( e \) is the unit of the monad and \( m \) its multiplication. In the sequel we will assume that, besides the quantale \( V \), a monad \( \mathbb{T} = (T, e, m) \) on \( \text{Set} \) is given, and that this monad is suitably extended to \( V\text{-Rel} \). By that we mean that the endofunctor \( T : \text{Set} \to \text{Set} \) is extended to \( V\text{-Rel} \) so that
\[
\begin{align*}
(1) & \quad Ts \cdot Tr \leq T(s \cdot r), \\
(2) & \quad r \leq r' \Rightarrow Tr \leq Tr', \\
(3) & \quad e_Y \cdot r \leq Tr \cdot e_X, \\
(4) & \quad m_Y \cdot TT r \leq Tr \cdot m_X, \\
(5) & \quad (Tr)^\circ = T(r^\circ) \quad \text{(and we write } Tr^\circ),
\end{align*}
\]
for all \( r, r' : X \to Y \) and \( s : Y \to Z \). We remark that (1) becomes an equality whenever \( r = f \) is a function, that is, \( T \) preserves composition of \( V \)-matrices with maps from the right.

For the existence of such an extension it is essential that \( T : \text{Set} \to \text{Set} \) satisfies the Beck-Chevalley Condition (BC) (see [9] for details). We recall that a commutative square in \( \text{Set} \)
\[
\begin{array}{ccc}
& h & \\
\downarrow & & \downarrow \\
k & f & g \\
\downarrow & & \downarrow \\
& &
\end{array}
\]
is said to be a \((BC)\)-square if \( g^\circ \cdot f = h \cdot k^\circ \). A \textit{Set}-endofunctor satisfies (BC) if it preserves (BC)-squares (in particular it is \textit{taut} in the sense of [21]). Moreover, we say that a natural transformation satisfies (BC) if each naturality diagram is a BC-square. \textit{From now on we assume that } \( T \) \text{ satisfies (BC).}

\textbf{Examples 1.2.} The identity monad \( \mathbb{1} = (1, 1, 1) \) on \( \text{Set} \) can be obviously extended to the identity monad on \( V\text{-Rel} \). In the sequel we will only consider this canonical extension of \( \mathbb{1} \).
The \textit{ultrafilter monad} $\mathbb{U} = (U, e, m)$ on $\textbf{Set}$ is induced by the dual adjunction

$$\begin{array}{ccc}
\text{Bool} & \xrightarrow{\eta} & \text{hom}(-,2) \\
\downarrow & & \downarrow \\
\text{hom}(-,2) & \xleftarrow{\epsilon} & \textbf{Set}
\end{array}$$

Explicitly, the ultrafilter functor $U : \textbf{Set} \to \textbf{Set}$ sends a set $X$ to the set $UX$ of all ultrafilters on $X$, and a function $f : X \to Y$ to the function $Uf : UX \to UY$ which assigns to an ultrafilter $\mathfrak{u} \in UX$ the ultrafilter generated by its image $\{f[A] \mid A \in \mathfrak{u}\}$. The natural transformations $e$ and $m$ are given by

$$e_X(x) = \hat{x} = \{A \subseteq X \mid x \in A\} \quad \text{and} \quad m_X(\mathfrak{u}) = \{A \subseteq X \mid A^\# \in \mathfrak{u}\},$$

for all $\mathfrak{u} \in U^2X$ and $x \in X$. Here $A^\#$ denotes the set $\{a \in UX \mid A \in a\}$. We point out that $U$ and $m$ satisfy (BC). It is shown in [2] that the ultrafilter monad $(U, e, m)$ can be naturally extended to $\text{Rel} \cong 2\text{-Rel}$, and in [9] this result is generalised to $V\text{-Rel}$, for a general class of lattices $V$ including $V = [0, \infty]$ (see also [12]). We remark that $m$ becomes a (strict) natural transformation for these extensions and that $U$ extends to a (strict) functor in $\text{Rel}$ and $[0, \infty]\text{-Rel}$. Although $e$ does not satisfy (BC), the naturality diagram

$$\begin{array}{ccc}
X & \xrightarrow{e_X} & UX \\
\downarrow r & & \downarrow Ur \\
Y & \xrightarrow{e_Y} & UY
\end{array}$$

is a (BC)-square, i.e. $Ur^o \cdot e_Y = e_X \cdot r^o$, provided that the relation $r$ has finite fibres in the sense that, for every $y \in Y$, the set

$$\{x \in X \mid \perp < r(x, y)\}$$

is finite.

Another interesting monad in this context is the \textit{free-monoid monad} $\mathbb{L} = (L, e, m)$ on $\textbf{Set}$ where $L : \textbf{Set} \to \textbf{Set}$ sends a set $X$ to the set $LX$ of all finite words $(x_1, \ldots, x_n)$ $(n \in \mathbb{N})$ of elements of $X$, and for a function $f : X \to Y$, $Lf : LX \to LY$ sends $(x_1, \ldots, x_n)$ to $(f(x_1), \ldots, f(x_n))$. Note that $LX$ includes the empty word $(\)$. For every set $X$ there is a map $e_X : X \to LX, x \mapsto (x)$; $e_X$ is the $X$-component of the natural transformation $e : 1 \to L$. An element of $LLX$ is a word of words of $X$; by removing inner brackets we obtain an element of $LX$. This defines the $X$-component $m_X : LLX \to LX$ of $m : LL \to L$. The free-monoid monad is actually
a Cartesian monad on \( \text{Set} \), meaning that the functor \( L : \text{Set} \rightarrow \text{Set} \) sends pullback squares to pullback squares, and every naturality square of \( e \) and of \( m \) is a pullback. In particular, \( L, e \) and \( m \) satisfy (BC). The \( \text{Set} \)-functor \( L \) extends naturally to a functor \( L : \text{Rel} \rightarrow \text{Rel} \) by putting

\[
(x_1, \ldots, x_n) Lr (y_1, \ldots, y_m) \quad \text{if} \quad n = m & (x_1 r y_1) & \ldots & (x_n r y_n),
\]

for every relation \( r : X \rightarrow Y \), \( (x_1, \ldots, x_n) \in LX \) and \( (y_1, \ldots, y_m) \in LY \). With this definition, both \( e \) and \( m \) become natural transformations \( e : 1 \rightarrow L \) and \( m : LL \rightarrow L \) where \( L : \text{Rel} \rightarrow \text{Rel} \).

A \((\mathbb{T}, \mathbb{V})\)-category (called lax algebra in \([7]\)) is a pair \((X, a)\) consisting of a set \( X \) and a \( \mathbb{V} \)-relation \( a : TX \rightarrow X \) such that:

\[
1_X \leq a \cdot e_X \quad \text{and} \quad a \cdot Ta \leq a \cdot m_X,
\]

that is, the map \( a : TX \times X \rightarrow \mathbb{V} \) satisfies the conditions

\[
k \leq a(e_X(x), x) \quad \text{and} \quad Ta(\mathbb{X}, \mathbb{r}) \otimes a(\mathbb{r}, x) \leq a(m_X(\mathbb{X}), x),
\]

for all \( \mathbb{X} \in T^2X \), \( \mathbb{r} \in TX \) and \( x \in X \). For \((\mathbb{T}, \mathbb{V})\)-categories \((X, a)\) and \((Y, b)\), a \((\mathbb{T}, \mathbb{V})\)-functor \( f : (X, a) \rightarrow (Y, b) \) is a map \( f : X \rightarrow Y \) such that \( f \cdot a \leq b \cdot Tf \), that is, for each \( \mathbb{r} \in TX \) and \( x \in X \), \( a(\mathbb{r}, x) \leq b(Tf(\mathbb{r}), f(x)) \). We denote the category of \((\mathbb{T}, \mathbb{V})\)-categories and \((\mathbb{T}, \mathbb{V})\)-functors by \((\mathbb{T}, \mathbb{V})\)-Cat.

A \((\mathbb{T}, \mathbb{V})\)-graph is a pair \((X, a)\) satisfying \( 1_X \leq a \cdot e_X \). Together with \((\mathbb{T}, \mathbb{V})\)-functors, they constitute the category \((\mathbb{T}, \mathbb{V})\)-Gph. Clearly, \((\mathbb{T}, \mathbb{V})\)-Cat is a full subcategory of \((\mathbb{T}, \mathbb{V})\)-Gph, and in the diagram

\[
\begin{array}{ccc}
\text{\((\mathbb{T}, \mathbb{V})\)-Cat} & \rightarrow & \text{\((\mathbb{T}, \mathbb{V})\)-Gph} \\
\downarrow & & \downarrow \\
\text{\texttt{Set}} & & \text{\texttt{Set}}
\end{array}
\]

the forgetful functors into \( \text{Set} \) are topological and the embedding is a bireflection (see \([7]\) for details). In particular, \((\mathbb{T}, \mathbb{V})\)-Cat is complete and cocomplete, and the forgetful functor \((\mathbb{T}, \mathbb{V})\)-Cat \( \rightarrow \text{Set} \) preserves coproducts. This gives immediately that coproducts in \((\mathbb{T}, \mathbb{V})\)-Cat are disjoint. Universality of coproducts follows from the characterisation of coproducts stated below and pullback-stability of open embeddings (see \([20]\) for details).

For \((\mathbb{T}, \mathbb{V})\)-functor \( f : (X, a) \rightarrow (Y, b) \), \( f \cdot a \leq b \cdot Tf \) can be equivalently stated as \( Tf \cdot a^\circ \leq f^\circ \cdot b \); \( f \) is said to be open if the reverse inequality holds, that is \( Tf \cdot a^\circ = f^\circ \cdot b \).
Theorem 1.3 ([20]). For \((\mathbb{T}, \mathbb{V})\)-categories \((X_i, a_i), i \in I\), and \((X, a)\), the following conditions are equivalent:

(i) \((X, a)\) is the coproduct of \((X_i, a_i)_{i \in I}\) in \((\mathbb{T}, \mathbb{V})\)-Cat;
(ii) \(a\)

\(X\) is the coproduct of \((X_i)_{i \in I}\) in \(\text{Set}\);
(b) for each \(i \in I\), the inclusion \(\iota_i : (X_i, a_i) \to (X, a)\) is open.

Corollary 1.4. \((\mathbb{T}, \mathbb{V})\)-Cat is an (infinitely) extensive category.

Examples 1.5. Let us have a look at the \((\mathbb{T}, \mathbb{V})\)-categories for the monad extensions considered in Examples 1.2. For \(\mathbb{T} = \mathbb{1}\) being the identity monad, a \((\mathbb{1}, 2)\)-category consists of a set \(X\) and a binary relation \(\leq\) on \(X\) which is reflexive and transitive, and a \((\mathbb{1}, 2)\)-functor is nothing but a monotone map. That is, \((\mathbb{1}, 2)\)-Cat is equivalent to the category \(\text{Ord}\) of (pre)ordered sets and monotone maps. Furthermore, \((\mathbb{1}, [0, \infty])\)-Cat gives Lawvere’s category \(\text{Met}\) of (pre)metric spaces and non-expansive maps [18].

The main result of [2] describes topological spaces as \((\mathbb{U}, 2)\)-categories. To be more precise, a relation \(r \to x\) between ultrafilters and points of a set \(X\) is the convergence relation of a (unique) topology on \(X\) if and only if

\[ \hat{x} \to x \quad \text{and} \quad (\mathcal{X} \to r \& r \to x) \Rightarrow m_X(\mathcal{X}) \to x, \]

for all \(x \in X\), \(r \in UX\) and \(\mathcal{X} \in UUX\). Since a map between topological spaces is continuous if and only if it preserves ultrafilter convergence, one has \((\mathbb{U}, 2)\)-Cat \(\simeq\) Top. As above, one can trade the quantale 2 for the extended real half-line \([0, \infty]\), and in [7] it is shown that \((\mathbb{U}, [0, \infty])\)-Cat is isomorphic to the category \(\text{App}\) of approach spaces [19].

Finally, a \((\mathbb{L}, 2)\)-category \(X = (X, a)\) is a multi-ordered set, where the structure \(a : LX \rightrightarrows X\) is not just a relation between points but a relation between finite words and points of \(X\) subject to

\[ (x) a x \quad \text{and} \quad ((r_1 a x_1) \& \ldots \& (r_n a x_n) \& (x_1, \ldots, x_n) a x) \Rightarrow \]

\[ (x_1^1, \ldots, x_{m_1}^1, x_1^2, \ldots, x_{m_n}^n) a x, \]

for all \(x \in X\), \((x_1, \ldots, x_n) \in LX\) and \(r_i = (x_1^i, \ldots, x_{m_i}^i) \in LX\) (where \(i = 1, \ldots, n\)). A \((\mathbb{L}, 2)\)-functor \(f : (X, a) \to (Y, b)\) between multi-ordered sets is a function \(f : X \to Y\) preserving the multi-order relation meaning that \((x_1, \ldots, x_n) a x\) implies \((f(x_1), \ldots, f(x_n)) a f(x)\).
2. Descent \((\mathbb{T}, V)\)-functors

Regular epimorphisms in the category of \((\mathbb{T}, V)\)-graphs were characterised in [7] as the final maps, that is the \((\mathbb{T}, V)\)-functors \(f : (X, a) \to (Y, b)\) with \(b = f \cdot a \cdot Tf^\circ\):

\[
\forall \eta \in TY \ \forall y \in Y \ b(\eta, y) = \bigvee_{\substack{Tf(x) = \eta \\ f(x) = y}} a(x, x).
\]

They are exactly the effective descent morphisms, as observed in [17], since \((\mathbb{T}, V)\)-\text{Gph} is locally Cartesian closed (see [10] for details). Hence:

**Proposition 2.1.** Given \((\mathbb{T}, V)\)-graphs \((X, a)\) and \((Y, b)\), for a \((\mathbb{T}, V)\)-functor \(f : (X, a) \to (Y, b)\) the following conditions are equivalent:

(i) \(f\) is a regular epimorphism in \((\mathbb{T}, V)\)-\text{Gph};

(ii) \(f\) is a pullback-stable regular epimorphism in \((\mathbb{T}, V)\)-\text{Gph};

(iii) \(f\) is an effective descent morphism in \((\mathbb{T}, V)\)-\text{Gph};

(iv) \(f\) is a descent morphism in \((\mathbb{T}, V)\)-\text{Gph};

(v) \(f\) is final.

We will now show that \((\mathbb{T}, V)\)-\text{Cat} is finally dense in \((\mathbb{T}, V)\)-\text{Gph}, that is, the sink \((f : X_I \to X)_I\) of all \((\mathbb{T}, V)\)-functors into a fixed \((\mathbb{T}, V)\)-graph whose domains \(X_i\) are \((\mathbb{T}, V)\)-categories is final (for details see e.g. [11]). We will do so with the help of certain “elementary structures”. Consider a set \(X\) together with \(\mathfrak{r} \in TX\) and \(u \in V\). We put \(X^* = X + 1\) where \(1 = \{\ast\}\), and define \(a^* = a_{\mathfrak{r}, u} : TX^* \to X^*\) as follows:

\[
a^*(\eta, y) = \begin{cases} k & \text{if } \eta = e_{X^*}(y), \\ u & \text{if } \eta = \mathfrak{r} \in TX \text{ and } y = \ast, \\ \bot & \text{else} \end{cases}
\]

for every \(\eta \in TX^*\) and \(y \in X^*\). Clearly, \(a^*\) is reflexive; and an additional assumption on the extension of \(\mathbb{T}\) to \(V\)-\text{Rel} guarantees also transitivity of \(a^*\). To see this, let \(\emptyset\) in \(TTX^*\), \(\eta \in TX^*\) and \(y \in X^*\). We write \(i : X \hookrightarrow X^*\) for the inclusion map, and note that \(i\) is an open \((\mathbb{T}, V)\)-functor since in

\[
\begin{CD}
TX @> Ti >> TX^* \\
\downarrow e^*_X \quad @. \quad \downarrow a^* \\
X @> i >> X^*
\end{CD}
\]
one has \((a^*)^\circ \cdot i = Ti \cdot e_X \leq Ti \cdot a^\circ\). From the diagram one obtains also that \(Ti^\circ \cdot Ta^* = Te_X^\circ \cdot TTi^\circ\).

- Assume first \(y \neq \pi\), hence \(y = i(x)\) with \(x \in X\). Therefore we can also assume \(\eta = e_X(i(y))\), since otherwise \(a^*(\eta, y) = \perp\). Then
\[
\perp < Ta^*(\mathcal{Y}, e_X(i(x))) = (e_X^\circ \cdot Ti^\circ \cdot Ta^*)(\mathcal{Y}, x) = (e_X^\circ \cdot Te_X^\circ \cdot TTi^\circ)(\mathcal{Y}, x),
\]
which implies \(\mathcal{Y} = (Te_X^\circ \cdot e_X^\circ)(y)\) and \(Ta^*(\mathcal{Y}, e_X(y)) = k\).

- Assume now \(y = \pi\). If \(\eta = Ti(\mathfrak{r})\) for some \(\mathfrak{r} \in TX\), then we can argue as above:
\[
\perp < Ta^*(\mathcal{Y}, Ti(\mathfrak{r})) = (Ti^\circ \cdot Ta^*)(\mathcal{Y}, \mathfrak{r}) = (Te_X^\circ \cdot TTi^\circ)(\mathcal{Y}, \mathfrak{r})
\]
gives \(\mathcal{Y} = (Te_X \cdot Ti(\mathfrak{r}))\) and \(Ta^*(\mathcal{Y}, Ti(\mathfrak{r})) = k\).

- Finally, assume \(\eta = e_X^\circ(\pi)\). If the naturality square
\[
\begin{array}{ccc}
TX^* & \xrightarrow{e_X^\circ} & TX^* \\
\downarrow a^* & & \downarrow Ta^* \\
X^* & \xrightarrow{e_X^\circ} & TX^*
\end{array}
\]
is a (BC)-square, that is, \((Ta^*)^\circ \cdot e_X^\circ = e_{TX^*} \cdot (a^*)^\circ\) or, equivalently, \(e_X^\circ \cdot Ta^* = a^* \cdot e_{TX^*}\), then
\[
\perp < Ta^*(\mathcal{Y}, e_X^\circ(\pi))
\]
implies \(\mathcal{Y} = e_{TX^*}(\mathfrak{r}')\) for some \(\mathfrak{r}' \in TX^*\) and \(Ta^*(\mathcal{Y}, e_X^\circ(\pi)) = a^*(\mathfrak{r}', \pi)\). Consequently, either \(\mathfrak{r}' = Ti(\mathfrak{r})\) or \(\mathfrak{r}' = e_X(\pi)\), and the assertion follows.

**Proposition 2.2.** Assume that every naturality square \((A)\) is a (BC)-square, for all sets \(X\), \(\mathfrak{r} \in TX\), \(u \in V\) and \(a^* = a_{T,u}\). Then \((\mathbb{T}, V)\)-Cat is finally dense in \((\mathbb{T}, V)\)-Gph. \(\blacksquare\)

**Examples 2.3.** If we start with a monad \(\mathbb{T} = (T, e, m)\) on \(\text{Set}\) where every naturality square of \(e\) satisfies (BC) (for instance, if \(\mathbb{T}\) is Cartesian), then \(e\) is also a natural transformation for the extension of \(\mathbb{T}\) to \(\text{Rel}\), and this remains true for the extension of this monad to \(V\)-\(\text{Rel}\) described in [9, Proposition 4.2 and Lemma 5.2] and [15, Theorem 3.5]. The situation is slightly different for
the ultrafilter monad. Recall from Examples 1.2 that a naturality square

\[
\begin{array}{ccc}
X & \xrightarrow{e_X} & UX \\
\downarrow{r} & & \downarrow{Ur} \\
Y & \xrightarrow{e_Y} & UY \\
\end{array}
\]

for a \( V \)-relation \( r : X \rightarrow Y \) is in general not a (BC)-square; however, it is so provided that \( r \) has finite fibres, i.e. \( \{ x \in X \mid \bot < r(x, y) \} \) is finite for every \( y \in Y \). By definition, every relation \( a_{r,u} : TX^* \rightarrow X^* \) is of this type.

**Theorem 2.4.** Assume that every naturality square of \( e \) with respect to \( V \)-relations with finite fibres is a (BC)-square. Then the following conditions are equivalent, for a morphism \( f : (X, a) \rightarrow (Y, b) \) in \( (\mathcal{T}, V)\)-Cat:

(i) \( f \) is final;
(ii) \( f \) is a pullback-stable regular epimorphism in \( (\mathcal{T}, V)\)-Gph;
(iii) \( f \) is a pullback-stable regular epimorphism in \( (\mathcal{T}, V)\)-Cat;
(iv) \( f \) is a descent morphism in \( (\mathcal{T}, V)\)-Cat.

**Proof:** (i) \( \Rightarrow \) (ii) follows from the proposition above. (ii) \( \Rightarrow \) (iii) follows from the fact that the embedding \( (\mathcal{T}, V)\)-Cat \( \rightarrow \) \( (\mathcal{T}, V)\)-Gph preserves pullbacks, since it is a right adjoint. (iii) \( \Leftrightarrow \) (iv) is valid in every finitely complete category.

(iii) \( \Rightarrow \) (i): Assume now that \( f : X \rightarrow Y \) is a pullback-stable regular epimorphism in \( (\mathcal{T}, V)\)-Cat. Let \( \eta \in TY \) and \( y_0 \in Y \). Consider the \( (\mathcal{T}, V) \)-functor \( g : Y^* \rightarrow Y \) where \( Y^* = (Y^*, b^*) \) is defined as above with \( u = b(\eta, y_0) \) and \( g(y) = y \) for \( y \in Y \) and \( g(\star) = y_0 \). By hypothesis, the pullback \( p_2 : (P, c) \rightarrow (Y^*, b^*) \) of \( f \) along \( g \)

\[
\begin{array}{ccc}
P & \overset{p_2}{\longrightarrow} & Y^* \\
\downarrow{p_1} & & \downarrow{g} \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

is a regular epimorphism in \( (\mathcal{T}, V)\)-Cat. Thanks to the simple structure of \( Y^* \), this is only possible if

\[
u = b^*(\eta, \star) = \bigvee \{ c(p, z) \mid p \in TP, Tp_2(p) = \eta, z \in P, p_2(z) = \star \}
\]

\[
= \bigvee \{ a(r, x) \wedge u \mid r \in TX, Tf(r) = \eta, x \in X, f(x) = y_0 \}
\]

\[
= \left( \bigvee \{ a(r, x) \mid r \in TX, Tf(r) = \eta, x \in X, f(x) = y_0 \} \right) \wedge u,
\]
hence \( b(\eta, y_0) \leq \bigvee \{ a(\mathbf{r}, x) \mid \mathbf{r} \in TX, Tf(\mathbf{r}) = \eta, x \in X, f(x) = y_0 \} \).

**Examples 2.5.** The theorem above generalises the well-known characterisation of pullback-stable regular epimorphisms in the category \( \text{Top} \) of topological spaces and continuous maps (see [13]) as *biquotient maps* in the sense of Michael [22], or *limit-lifting maps* in the sense of Hájek [14]. As consequences of the theorem we have that:

1. As it is well-known, in \( \text{Ord} \) a monotone map \( f : (X, a) \to (Y, b) \) is of descent if, and only if, it is a pullback-stable regular epimorphism, or, equivalently, if it is a regular epimorphism; it is characterised by the condition:
   \[
   \forall y, y' \in Y \ y \leq y' \Rightarrow \exists x \in f^{-1}(y) \exists x' \in f^{-1}(y') : x \leq x'.
   \]

2. In \( \text{Met} \), a non-expansive map \( f : (X, a) \to (Y, b) \) is of descent, or, equivalently, a pullback-stable regular epimorphism, if, and only if,
   \[
   \forall y, y' \in Y \ b(y, y') = \inf \{ a(x, x') ; x \in f^{-1}(y), x' \in f^{-1}(y') \}.
   \]

3. Analogously, a non-expansive map \( f : (X, a) \to (Y, b) \) in \( \text{App} \) is a descent morphism, or, equivalently, a pullback-stable regular epimorphism, if, and only if,
   \[
   \forall \eta \in UY \ \forall y \in Y \ b(\eta, y) = \inf \{ a(\mathbf{r}, x) ; \mathbf{r} \in Uf^{-1}(\eta), x \in f^{-1}(y) \}.
   \]

4. In the category \( \text{MOrd} \) of multi-ordered sets and monotone maps, \( f : (X, a) \to (Y, b) \) is of descent, or equivalently a pullback-stable regular epimorphism, if, and only if,
   \[
   \forall \eta = (y_1, \cdots, y_n) \in LY \ \forall y \in Y \ \eta \ y \Rightarrow
   \exists \mathbf{r} = (x_1, \cdots, x_n) \in f^{-1}(\eta) \ \exists x \in f^{-1}(y) : \mathbf{r} \ a \ x.
   \]

**3. The (global) van Kampen Theorem in \( (\mathbb{T}, V)\text{-Cat} \)**

In this section we will be concerned with the (global) categorical van Kampen Theorem. In [4] it is shown that, for extensive categories, the key property for the van Kampen Theorem to hold is the fact that the morphism \( p : X_1 + X_2 \to X \), from the coproduct of \( X_1 \) and \( X_2 \) into \( X \) induced by the embeddings \( g_1 : X_1 \hookrightarrow X \) and \( g_2 : X_2 \hookrightarrow X \), is of effective descent (see [4] for details).
Since \((\mathbb{T}, \mathbf{V})\)-\text{Cat} is extensive, and complete and cocomplete, in order to state a van Kampen Theorem in this context we will characterise the morphisms \(p : X_1 + X_2 \rightarrow X\) as above which are of effective descent. Here we will restrict ourselves to the case of global effective descent morphisms.

Following Reiterman-Tholen characterisation of effective descent continuous maps in \(\text{Top} \ [24]\), and the subsequent characterisation presented in \[6\], a \((\mathbb{T}, \mathbf{V})\)-functor \(f : (X, a) \rightarrow (Y, b)\) is said to be a \(*\)-quotient if, for every \(y \in T^2 Y, \eta \in TY\) and \(y \in Y\),

\[
Tb(\eta, y) \otimes b(\eta, y) = \bigvee_{x \in (T^2 f)^{-1}(\eta)} Ta(x, x) \otimes a(x, x).
\]

It is shown in \[8\, \text{Theorem 3.3}\] (see also \[11\]) that:

**Proposition 3.1.** A pullback-stable \(*\)-quotient \((\mathbb{T}, \mathbf{V})\)-functor in \((\mathbb{T}, \mathbf{V})\)-\text{Cat} is of effective descent.

It is also shown in \[8\] that open and proper surjections are pullback-stable \(*\)-quotient maps, hence of effective descent, provided that condition \((C1)\) below is fulfilled. We recall that a \((\mathbb{T}, \mathbf{V})\)-functor \(f : (X, a) \rightarrow (Y, b)\) is proper if \(f \cdot a = b \cdot Tf\).

From now on we will consider the following conditions:

\begin{itemize}
  \item[(C0)] every naturality square of \(e\) with respect to \(\mathbf{V}\)-relations with finite fibres is a \((\mathbb{BC})\)-square;
  \item[(C1)] \(T(f \cdot r) = Tf \cdot Tr\), for every map \(f : Y \rightarrow Z\) and \(\mathbf{V}\)-relation \(r : X \rightarrow Y\);
  \item[(C2)] \(m\) satisfies \((\mathbb{BC})\);
  \item[(C3)] naturality diagrams of the natural transformation \(e\) with respect to relations with finite fibres are \((\mathbb{BC})\)-squares;
  \item[(C4)] \(T\) preserves coproducts.
\end{itemize}

We point out that condition \((C1)\) is not a restrictive condition; indeed, it follows easily from a condition widely fulfilled: for any map \(f : Y \rightarrow Z\) and any relation \(r : X \rightarrow Y\), \(T(f \cdot r) = Tf \cdot Tr\). In particular this holds whenever the extension \(T\) is defined by a topological theory in the sense of \[13\]. For an analysis of \((C4)\) see \[3\]. It is straightforward to prove that:
Lemma 3.2. If $T(f \cdot r) = Tf \cdot Tr$ for any map $f$ and any $V$-relation $r$, then $T$ preserves open, proper, and final $(\mathbb{T}, V)$-functors.

Theorem 3.3. Assume that conditions (C0)-(C4) hold. If $(X_1, a_1)$ and $(X_2, a_2)$ are $(\mathbb{T}, V)$-subcategories of $(X, a)$ and $p : X_1 + X_2 \to X$ is the $(\mathbb{T}, V)$-functor induced by their embeddings, then the following conditions are equivalent in $(\mathbb{T}, V)$-Cat:

(i) $p$ is a pullback-stable $*$-quotient map;
(ii) $p$ is of effective descent;
(iii) $p$ is a descent morphism;
(iv) $p$ is final;
(v) for any $x \in TX$ and $x \in X$, either there exists $i \in \{1, 2\}$ such that $x \in Tg_i(TX_i)$ and $x \in g_i(X_i)$, or $a(x, x) = \perp$.

Proof: Proposition 3.1 states that (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) is always true, and (iii) $\Rightarrow$ (iv) follows from Theorem 2.4. In order to show that (iv) $\Leftrightarrow$ (v) and (v) $\Rightarrow$ (i) we first describe the structure $b$ on $X_1 + X_2 = X_1 \times \{1\} \cup X_2 \times \{2\}$: for $\eta \in T(X_1 + X_2)$, and $(x, i) \in X_1 + X_2$,

$$b(\eta, (x, i)) = \begin{cases} a_i(\eta, (x, i)) & \text{if } \eta = T\tau_i(\eta_i) \text{ for } \eta_i \in TX_i \\ \perp & \text{if } \eta \notin T\tau_i(X_i). \end{cases}$$

Hence the embeddings $\tau_i : (X_i, a_i) \to (X_1 + X_2, b)$ are both open and proper.

(iv) $\Leftrightarrow$ (v): Let $x \in TX$ and $X \in X$ with $a(x, x) \neq \perp$. Since $p$ is final

$$a(x, x) = \bigvee_{i : x \in X_i} b(\eta, (x, i))$$

Hence there is $\eta \in T(X_1 + X_2)$ and $i \in \{1, 2\}$ with $Tp(\eta) = x$, $x \in X_i$, and $b(\eta, (x, i)) \neq \perp$. By definition of $b$, $\eta = T\tau_i(\eta_i)$ for some $\eta_i \in TX_i$, and then $TpT\tau_i(\eta_i) = Tg_i(\eta_i) = x$ as claimed. The proof of the reverse implication is straightforward.
Moreover, since a for any \( V \) is:

\[
L \text{ valid for } \exists x \text{ and } x \in X \text{ with }
\]

\[
Ta(\mathfrak{X}, r) = \beta, a(r, x) = \alpha \text{ and } Ta(\mathfrak{X}, r) \otimes a(r, x) = \beta \otimes \alpha \neq \bot.
\]

Finality of \( p \) guarantees that there exists \( i \in \{1, 2\} \) such that \( r \in Tg_i(TX_i) \) and \( x \in g_i(X_i) \). Without loss of generality we assume that \( x \in X_1 \) and that there exists \( r_1 \in TX_1 \) with \( Tg_1(r_1) = r \). Therefore

\[
b(T\tau_1(r_1), (x, 1)) = a_1(r_1, x) = a(r, x) = \alpha.
\]

Moreover, since \( a(m_X(\mathfrak{X}), x) \geq \beta \otimes \alpha \neq \bot \), for some \( j \in \{1, 2\} \) \( x \in X_j \) and there exists \( \eta_j \in TX_j \) with \( Tg_j(\eta_j) = m_X(\mathfrak{X}) \). This implies, by (BC) of \( m \), that \( \mathfrak{X} = T^2g_j(\mathfrak{X}_j) \) with \( \mathfrak{X}_j \in T^2X_j \).

- If \( j = 1 \), then \( Tb(T^2\tau_1(\mathfrak{X}_1), T\tau_1(r_1)) = Ta_1(\mathfrak{X}_1, r_1) = Ta(\mathfrak{X}, r) = \beta \).
- If \( \mathfrak{X} \not\in T^2g_1(T^2X_1) \), we use finality of \( Tp \). Since \( Ta(\mathfrak{X}, r) = \beta \neq \bot \), there exists \( \mathfrak{X} \in T^2(X_1 + X_2) \) and \( \tilde{r} \in T(X_1 + X_2) \) such that \( T^2p(\mathfrak{X}) = \mathfrak{X}, Tp(\tilde{r}) = r \) and \( Tb(\tilde{r}, \tilde{r}) = \beta \). By assumption \( \tilde{r} \in T^2\tau_2(T^2X_2) \) and therefore, since \( T^2\tau_2 \) is proper, \( \tilde{r} \in T^2\tau_2(TX_2) \). This means that there exists \( \mathfrak{X}_2 \in T^2X_2 \) and \( \mathfrak{r}_2 \in TX_2 \) such that

\[
Tb(T^2\tau_2(\mathfrak{X}_2), T\tau_2(\mathfrak{r}_2), (x, 2)) = Ta_2(\mathfrak{X}_2, \mathfrak{r}_2) \otimes a_2(\mathfrak{r}_2, (x, 2))
\]

\[
= Ta(\mathfrak{X}, r) \otimes a(r, x) = \beta \otimes \alpha
\]

and the proof is complete.

Since the identity functor and the ultrafilter functor satisfy (C0)-(C4), this result applies to the categories \( \text{Ord}, \text{Met}, \text{Top}, \text{App} \). Although the free-monoid functor \( L \) does not preserve coproducts, as we show next the result is still valid for \( L \), when we consider the extension of \( L \) to \( \text{Rel} \) defined in [12], that is:

\[
Lr((x_1, \cdots, x_n), (y_1, \cdots, y_m)) = \begin{cases} \otimes_{i=1}^n r(x_i, y_i) & \text{if } m = n \\ \bot & \text{elsewhere,} \end{cases}
\]

for any \( V \)-relation \( r : X \rightarrow Y, (x_1, \cdots, x_n) \in LX \) and \( (y_1, \cdots, y_m) \in LY \).
Theorem 3.4. If \((X_1, a_1)\) and \((X_2, a_2)\) are \((\mathbb{L}, \mathbb{V})\)-subcategories of \((X, a)\) and 
\[ p : X_1 + X_2 \to X \] 
is the \((\mathbb{L}, \mathbb{V})\)-functor induced by their embeddings, then the 
following conditions are equivalent in \((\mathbb{L}, \mathbb{V})\)-Cat:

(i) \(p\) is a pullback-stable \(*\)-quotient map;
(ii) \(p\) is of effective descent;
(iii) \(p\) is a descent morphism;
(iv) \(p\) is final;
(v) for any \(r \in LX\) and \(x \in X\), either there exists \(i \in \{1, 2\}\) such that 
\[ r \in Lg_i(LX_i) \text{ and } x \in g_i(X_i), \] or \(a(r, x) = \perp\).

Proof: We can use the arguments of the proof of the former theorem to show 
all the implications but (iv) \(\Rightarrow\) (i). To prove (iv) \(\Rightarrow\) (i), let 
\(p\) be final, and 
\[ X = (x_1, \ldots, x_n) \in L^2 \] 
with \(x_i = (x_i^1, \ldots, x_i^{m_i})\), \(r = (x_1, \ldots, x_n) \in LX\) and \(x \in X\) such that 
\[ La(\mathcal{X}, r) = \beta, \; a(r, x) = \alpha \text{ and } \beta \otimes \alpha \neq \perp. \]

Finality of \(p\) guarantees that there exists \(j \in \{1, 2\}\) such that 
\[ x_1, \ldots, x_n, x \in X_j. \]

We may assume, without loss of generality, that \(j = 1\). As in the proof above, 
\[ a(m_X(\mathcal{X}), x) \geq \beta \otimes \alpha \neq \perp \] 
assures that \(m_X(\mathcal{X}) \in Lg_1(LX_1)\) or \(m_X(\mathcal{X}) \in Lg_2(LX_2)\).

- If \(m_X(\mathcal{X}) \in Lg_1(LX_1)\), then \(\mathcal{X} \in L^2 g_1(L^2 X_1)\) and the proof is complete.
- If \(\mathcal{X} \in Lg_2(LX_2) \setminus Lg_1(LX_1)\), then necessarily \(x \in g_2(X_2)\) and the proof is complete in case 
\(r = (x_1, \ldots, x_n) \in Lg_2(LX_2)\). If \(r \notin Lg_2(LX_2)\), that is if there exists \(l \in \{1, \ldots, n\}\) with \(x_l \notin X_2\) then we can consider 
\(\mathcal{Y} = (\eta_1, \ldots, \eta_n) \in L^2 X\) defined by 
\[ \eta_i = \begin{cases} 
  r_i & \text{if } i \neq l \\
  (x_l) & \text{if } i = l.
\end{cases} \]

By construction \(Ta(\mathcal{Y}, r) = \otimes_{i \in \{1, \ldots, n\} \setminus \{l\}} \beta_i \neq \perp\) and 
\[ a(m_X(\mathcal{Y}), x) \geq (\otimes_{i \in \{1, \ldots, n\} \setminus \{l\}} \beta_i) \otimes a \neq \perp. \] 
However, under our assumptions, \(m_X(\mathcal{Y}) \notin Lg_1(LX_1) \cup Lg_2(LX_2)\), which contradicts finality of \(p\). \(\blacksquare\)

As a corollary we obtain:
Theorem 3.5 (Global van Kampen Theorem). Let $\mathcal{T} = 1$, or $\mathcal{T} = U$, or $\mathcal{T} = \mathbb{L}$, and let $\mathcal{C} = (\mathcal{T}, \mathcal{V})$-$\text{Cat}$. If the following diagram

```
\begin{array}{ccc}
X & \xrightarrow{g_1} & X_1 \\
\downarrow{g_2} & & \downarrow{f_1}
\end{array}
```

be a pullback, with $g_1$ and $g_2$ embeddings, then the diagram

```
\begin{array}{ccc}
C \downarrow X & \xrightarrow{g_1^*} & (C \downarrow X_1) \times (C \downarrow X_0) \\
\downarrow{g_2^*} & & \downarrow{f_1^*}
\end{array}
```

is a pullback if, and only if, the morphism $p : X_1 + X_2 \to X$, induced by $g_1$ and $g_2$, is a final morphism.

Here $g_1^*$ and $g_2^*$ are the change-of-base functors, and, as in [4], by the latter diagram being a pullback we mean that the functor

$$
\begin{array}{ccc}
C \downarrow X & \xrightarrow{K_{g_1, g_2}} & (C \downarrow X_1) \times (C \downarrow X_0) \\
\end{array}
$$

induced by $(g_1, g_2)$, is an equivalence, where $(C \downarrow X_1) \times (C \downarrow X_0)$ is the category of triples $((A_1, \alpha_1), (A_2, \alpha_2), \varphi)$, with $(A_i, \alpha_i) \in C \downarrow X_i$, $i = 1, 2$, and $\varphi : f_1^*(A_1, \alpha_1) \to f_2^*(A_2, \alpha_2)$ an isomorphism.

Final Remarks 3.6. (1) In the particular case $\mathcal{T} = U$ and $\mathcal{V} = 2$, that is in the category $\text{Top}$, in [5] it is shown that we can add an equivalent condition on Theorem 3.3, namely $p$ being a triquotient map [23]. This notions is purely topological and there is no corresponding notion in categories of lax algebras.

(2) We do not know whether the result of Theorems 3.3 and 3.4 is valid for more general monads. In fact, we do not know any example of monad for which the result does not hold.
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