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THE ARL OF MODIFIED SHEWHART CONTROL CHARTS FOR CONDITIONALLY HETEROSKEDASTIC MODELS

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ABSTRACT: In this paper we consider the modified Shewhart control chart for ARCH processes and introduce it for threshold ARCH (TARCH) ones. For both charts, we determine bounds for the distribution of the in-control run length (RL) and, consequently, for its average (ARL), both depending only on the distribution of the generating white noise, the model parameters and the critical value. For the ARCH model, we compare our bounds with others available in literature and show how they improve the existing ones. We present a simulation study to assess the quality of the bounds calculated for the ARL.

KEYWORDS: Shewhart control chart, average run length, time series, stationarity, ARCH model, TARCH model.
AMS SUBJECT CLASSIFICATION (2000): 62L10, 62M10, 60G10.

1. Introduction

Control charts are an important tool of Statistical Process Control (SPC) extensively used in industry. They allow monitoring whether an observed process diverts from a supposed target process by issuing out-of-control alerts. These schemes were introduced by Shewhart in the 1920's (Shewhart 1931) and his original design is still widely used, despite multiple alternatives that have appeared in literature, like EWMA and CUSUM charts. Several studies show that Shewhart type charts are still the best to detect large shifts.

In the last decades, the applicability of control schemes has been extended from independent processes to time series, namely, with the appearance of charts which incorporate the time series structure into its design, named modified charts. Vasilopoulos and Stamboulis (1978) were the precursors, suggesting the modified Shewhart chart for AR processes. The first modified chart for conditionally heteroskedastic models was presented by Severin and Schmid (1999), who considered the generalized ARCH model. The conditionally heteroskedastic models are particulary well-suited for modeling financial

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time series, so combining them with control charts opens the door to the use of SPC techniques also to practitioners of the financial field.

Given the variety of control charts available to monitor a time series, raises the question of evaluating the best design to detect a deviation from target as soon as possible. The average run length (ARL) is widely used as a performance measure for control charts and, when dealing with time series, is defined as the average number of instants that must go by before one indicates an out-of-control condition.

Formally, let $Y = (Y_t, t \in \mathbb{Z})$ be the target process, (weakly) stationary, such that $E(Y_t) = \mu_0$ and $V(Y_t) = \sigma_Y^2$. The observed process $X = (X_t, t \in \mathbb{N})$ is in-control if $X_t = Y_t$, for $t \geq 1$. For the modified Shewhart chart, given the critical value $c \in \mathbb{R}^+$, X is said to be out-of-control at time t if $|X_t - \mu_0| > c\sigma_Y$ (Schmid 1995). Consequently, we have $ARL = E(N_S)$, where N_S is the run length (RL) of the modified Shewhart chart defined as

$$N_S = \inf \{t \in \mathbb{N} : |X_t - \mu_0| > c\sigma_Y\} \ . \tag{1.1}$$

The ARL is also important to set the design of the control chart, since c is often taken so that the in-control ARL is equal to a specified constant. For example, Severin and Schmid (1999) state that, in financial applications using daily stock market values, the in-control ARL should be 20, 40 or 60, because these values correspond to 1, 2 or 3 months of stock market, respectively, in this way conditioning the value of c.

The primary goal of this paper is to improve or derive theoretical bounds for the in-control ARL of the modified Shewhart chart for two classes of conditionally heteroskedastic models: the ARCH model (Engle 1982) and the threshold ARCH (TARCH) model (Zakoian 1994). Whenever possible we will compare the ARL of these charts with the ARL of classical Shewhart charts, which allows to realize what happens if the process is wrongly assumed independent.

The remainder of this article is organized as follows. Section 2 introduces the two conditionally heteroskedastic models which are taken into account in this study. Section 3 is dedicated to the ARCH model. We start this section with an overview of existing bounds for $P(N_S > n)$, obtained by Severin and Schmid (1999) and Pawlak and Schmid (2001), and derive a new lower bound for this probability following the method present in Gonçalves and Mendes-Lopes (2007). This enables us to calculate the corresponding bounds for the ARL, which we then compare. In section 4, we introduce the modified

Shewhart chart for TARCH processes and determine bounds for $P(N_S > n)$ and, consequently, for ARL. Section 5 provides a simulation study to assess the quality of the bounds calculated for the ARL by comparing them with the estimated ARL. In this paper, we will make use of the following conventions: $\sum\limits_{i=1}^{0}(\cdot) = 0$ and $\prod\limits_{t=2}^{1}(\cdot) = 1$.

2. The ARCH(q) and TARCH(q) models

In conditionally heteroskedastic models, the real stochastic process $Y = (Y_t, t \in \mathbb{Z})$ is set to be, for every $t \in \mathbb{Z}$,

$$Y_t = Z_t \sigma_t \tag{2.1}$$

where $Z = (Z_t, t \in \mathbb{Z})$ is a sequence of independent and identically distributed (iid) real random variables, with zero mean and unit variance, such that Z_t is independent of the σ-field generated by the past of Y, $\underline{Y}_{t-1} = \sigma(Y_{t-1}, Y_{t-2}, ...)$, and where $\sigma_t^2 = V(Y_t | \underline{Y}_{t-1})$ is a measurable function of past observations of Y.

The specification of σ_t determines the model. When Y is an ARCH(q) process, we set

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i Y_{t-i}^2 \tag{2.2}$$

with $\alpha_0 > 0$ and $\alpha_i \geq 0$ $(i = 1, ..., q)$. When Y is a TARCH(q) process, we specify

$$\sigma_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i Y_{t-i}^{+} - \sum_{i=1}^{q} \beta_i Y_{t-i}^{-} \tag{2.3}$$

where $Y_t^+ = Y_t 1\!\!1_{\{Y_t \geq 0\}}$, $Y_t^- = Y_t 1\!\!1_{\{Y_t < 0\}}$ and with $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$ $(i = 1, ..., q)$.

Regarding the specification of σ_t, the main difference between these two models is the ability which only the TARCH model has to take into account different reactions of volatility according to the sign of past values of the process. Nevertheless, both models are well-established among practitioners particularly of the financial field.

Our aim to evaluate the in-control ARL requires that we work with stationary processes. We observe that, under weak stationarity conditions, the conditionally heteroskedastic models are centered and non correlated processes. A necessary and sufficient condition of stationarity for the ARCH(q) model is $\sum_{i=1}^{q} \alpha_i < 1$. Under this condition, the variance of Y, $V(Y_t) = \sigma_Y^2$, exists and is equal to

$$\sigma_Y^2 = \frac{\alpha_0}{1 - \sum_{i=1}^{q} \alpha_i} \ . \tag{2.4}$$

For the generalized TARCH (GTARCH) model, the necessary and sufficient condition of stationarity depends not only on the parameters of the model but also on the moments of Z_t^+ and Z_t^- (Gonçalves and Mendes-Lopes 1994). For the TARCH(1) process, the condition is $E\left[\left(\alpha_1 Z_t^+ - \beta_1 Z_t^-\right)^2\right] < 1$ and the variance of Y, $V(Y_t) = \sigma_Y^2$, exists and is equal to

$$\sigma_Y^2 = \frac{\alpha_0^2 \left[1 + E\left(\alpha_1 Z_t^+ - \beta_1 Z_t^-\right)\right]}{\left[1 - E\left(\alpha_1 Z_t^+ - \beta_1 Z_t^-\right)\right] \left\{1 - E\left[\left(\alpha_1 Z_t^+ - \beta_1 Z_t^-\right)^2\right]\right\}} \ . \tag{2.5}$$

Remark 1. *In both formulations of σ_t, when all parameters, apart from α_0, are equal to zero, the model is no longer conditionally heteroskedastic. In this case Y is an iid process and N_S follows the geometric law $G(\theta)$, with support $\mathbb{N}$ and $\theta = 1 - F_{|Z|}(c) = 1 - F_{Z^2}\left(c^2\right)$, where $F_{|Z|}$ and F_{Z^2} are the distribution functions of $|Z_t|$ and Z_t^2, respectively. Thus, in the in-control state,*

$$P_{iid}(N_S > n) = (1-\theta)^n \ , \ n = 0, 1, 2, ..., \quad and \quad ARL_{iid} = \frac{1}{\theta}$$

with the index "iid" meaning that the target process Y is an iid process.

3. In-control ARL of modified Shewhart chart for the ARCH process

In this section we assume that the target process Y follows a stationary ARCH(q) model and that the observed process X is in-control. So,

$$P(N_S > n) = P(|Y_1| \le c\sigma_Y, ..., |Y_n| \le c\sigma_Y) \ , \ n = 1, 2, ... \ .$$

We note that $P(N_S > 0) = 1$.

3.1. Bounds for $P_n = P(N_S > n)$.

The first bounds for the $P_n = P(N_S > n)$ were derived by Severin and Schmid (1999). Their upper bound,

$$\underset{\text{ARCH}(q)}{UB_1 P_n} = \left[F_{Z^2}\left(\frac{c^2\sigma_Y^2}{\alpha_0}\right)\right]^n, \tag{3.1}$$

valid for every $c \in \mathbb{R}^+$ and obtained by an elementary technique, is simple to use but its performance is not always good. Nevertheless, it is the unique known in the literature until now. In what concerns their lower bound,

$$\underset{\text{ARCH}(q)}{LB_1 P_n} = P(N_S > \min\{q,n\}) \times \left[F_{Z^2}\left(\frac{c^2}{1+(c^2-1)\sum_{i=1}^{q}\alpha_i}\right)\right]^{\max\{0,n-q\}}, \tag{3.2}$$

also valid for every $c \in \mathbb{R}^+$, we remark that it depends on the probability in evaluation. This problem was firstly addressed by Pawlak and Schmid (2001), who obtained the following lower bound depending only on the distribution function of Z_t^2 and the parameters of the model, considering Z_t absolutely continuous with a differentiable density of probability f_Z:

$$\underset{\text{ARCH}(q)}{LB_2 P_n} = \left[\prod_{t=1}^{\min\{q,n\}} F_{Z^2}\left(\frac{c^2}{1+(c^2-1)\sum_{i=1}^{t-1}\alpha_i}\right)\right] \times \left[F_{Z^2}\left(c^2\right)\right]^{\max\{0,n-q\}}, \tag{3.3}$$

valid for every $c \in \mathbb{R}^+$ such that $2f_{Z^2}(x) + xf'_{Z^2}(x) \geq 0$, where $x = \frac{c^2\sigma_Y^2}{\alpha_0+u}$, with $u \geq 0$.

We also consider this matter following the method proposed in Gonçalves and Mendes-Lopes (2007) for generalized TARCH processes. We begin by deriving a lower bound for the laws of finite dimension for the process $|Y|$.

Theorem 1. *Let $Y = (Y_t, t \in \mathbb{Z})$ be an ARCH(q) process such that Z_t is absolutely continuous with a differentiable density of probability f_Z. Then, if Y is stationary, it holds:*

(a) *for $n \leq q$,*

$$P\left(Y_t^2 \leq x_t, t=1,...,n\right) \geq F_{Z^2}\left(\frac{x_1}{\sigma_Y^2}\right)\times$$
$$\times\prod_{t=2}^{n} F_{Z^2}\left(\frac{x_t}{\alpha_0+\sum\limits_{i=1}^{t-1}\alpha_i x_{t-i}+\sigma_Y^2\sum\limits_{i=t}^{q}\alpha_i}\right) \ ,$$

(b) *for $n > q$,*

$$P\left(Y_t^2 \leq x_t, t=1,...,n\right) \geq F_{Z^2}\left(\frac{x_1}{\sigma_Y^2}\right)\times$$
$$\times\prod_{t=2}^{q} F_{Z^2}\left(\frac{x_t}{\alpha_0+\sum\limits_{i=1}^{t-1}\alpha_i x_{t-i}+\sigma_Y^2\sum\limits_{i=t}^{q}\alpha_i}\right)\times\prod_{t=q+1}^{n} F_{Z^2}\left(\frac{x_t}{\alpha_0+\sum\limits_{i=1}^{q}\alpha_i x_{t-i}}\right) \ ,$$

in both cases, for all $(x_1,...,x_n)\in\,]0,+\infty[^n$ such that

$$2f_{Z^2}\left(\frac{x_t}{\alpha_0+u}\right)+\frac{x_t}{\alpha_0+u}f'_{Z^2}\left(\frac{x_t}{\alpha_0+u}\right)\geq 0 \ , \ t=1,...,\min\{q,n\} \ ,$$

with u any non-negative real number.

Proof:

(b) Let $n > q$. We start by observing that, if $Y_1^2 \leq x_1, ..., Y_n^2 \leq x_n$, then:

(i) for $t \in \{2,...,q\}$,

$$\sigma_t^2 = \alpha_0+\sum_{i=1}^{t-1}\alpha_i Y_{t-i}^2+\sum_{i=t}^{q}\alpha_i Y_{t-i}^2 \leq \alpha_0+\sum_{i=1}^{t-1}\alpha_i x_{t-i}+\sum_{i=t}^{q}\alpha_i Y_{t-i}^2 \ ,$$

since, for $i \in \{1,...,t-1\}$, $1 \leq t-i \leq q-1 < n$ and, for $i \in \{t,...,q\}$, $t-i \leq 0$;

(ii) and, for $t \in \{q+1,...,n\}$,

$$\sigma_t^2 = \alpha_0+\sum_{i=1}^{q}\alpha_i Y_{t-i}^2 \leq \alpha_0+\sum_{i=1}^{q}\alpha_i x_{t-i} \ ,$$

as, for $i \in \{1,...,q\}$, $1 \leq t-i \leq n-1$.

So, for every $(x_1, ..., x_n) \in]0, +\infty[^n$,

$$P\left(Y_t^2 \le x_t, t = 1, ..., n\right) =$$

$$= P\left(Z_1^2 \le \frac{x_1}{\sigma_1^2}, Z_t^2 \le \frac{x_t}{\sigma_t^2}, t = 2, ..., q, Z_t^2 \le \frac{x_t}{\sigma_t^2}, t = q+1, ..., n\right)$$

$$\ge P\left(Z_1^2 \le \frac{x_1}{\sigma_1^2}, Z_t^2 \le \frac{x_t}{\alpha_0 + \sum_{i=1}^{t-1} \alpha_i x_{t-i} + \sum_{i=t}^{q} \alpha_i Y_{t-i}^2}, t = 2, ..., q,\right.$$

$$\left. Z_t^2 \le \frac{x_t}{\alpha_0 + \sum_{i=1}^{q} \alpha_i x_{t-i}}, t = q+1, ..., n\right)$$

$$= E\left[P\left(Z_1^2 \le \frac{x_1}{\sigma_1^2}, Z_t^2 \le \frac{x_t}{\alpha_0 + \sum_{i=1}^{t-1} \alpha_i x_{t-i} + \sum_{i=t}^{q} \alpha_i Y_{t-i}^2}, t = 2, ..., q,\right.\right.$$

$$\left.\left. Z_t^2 \le \frac{x_t}{\alpha_0 + \sum_{i=1}^{q} \alpha_i x_{t-i}}, t = q+1, ..., n \,\middle|\, \underline{Y}_0\right)\right] .$$

As Z_t is independent of $\underline{Y}_{t-1}$ and $\underline{Y}_0 \subseteq \underline{Y}_{t-1}$ $(t = 1, 2, ...)$ and $Z_1, ..., Z_n$ are iid, it holds, according to the expectation and conditional expectation properties,

$$P\left(Y_t^2 \le x_t, t = 1, ..., n\right) \ge$$

$$\ge E\left[F_{Z^2}\left(\frac{x_1}{\sigma_1^2}\right) \prod_{t=2}^{q} F_{Z^2}\left(\frac{x_t}{\alpha_0 + \sum_{i=1}^{t-1} \alpha_i x_{t-i} + \sum_{i=t}^{q} \alpha_i Y_{t-i}^2}\right) F_{q,n}\right]$$

$$= E\left[F_{Z^2}\left(\frac{x_1}{\sigma_1^2}\right)\right] \prod_{t=2}^{q} E\left[F_{Z^2}\left(\frac{x_t}{\alpha_0 + \sum_{i=1}^{t-1} \alpha_i x_{t-i} + \sum_{i=t}^{q} \alpha_i Y_{t-i}^2}\right)\right] F_{q,n},$$

where

$$F_{q,n}=\prod_{t=q+1}^{n}F_{Z^2}\left(\frac{x_t}{\alpha_0+\sum\limits_{i=1}^{q}\alpha_i x_{t-i}}\right).$$

For t arbitrarily fixed in $\{1,...,q\}$, we consider the function $R_t:\ [0,+\infty[\ \longrightarrow\ [0,1]$ defined by

$$R_t\left(u_t\right)=F_{Z^2}\left(\frac{x_t}{\alpha_0+\sum\limits_{i=1}^{t-1}\alpha_i x_{t-i}+u_t}\right).$$

Let $m_t=\alpha_0+\sum\limits_{i=1}^{t-1}\alpha_i x_{t-i}+u_t$. Since

$$\frac{dR_t}{du_i}\left(u_t\right)=f_{Z^2}\left(\frac{x_t}{m_t}\right)\cdot\frac{d}{du_t}\left(\frac{x_t}{m_t}\right)=f_{Z^2}\left(\frac{x_t}{m_t}\right)\cdot\frac{-x_t}{m_t^2}\ ,$$

then

$$\begin{aligned}\frac{d^2R_t}{du_t^2}\left(u_t,...,u_q\right) &= f'_{Z^2}\left(\frac{x_t}{m_t}\right)\cdot\frac{x_t^2}{m_t^4}+f_{Z^2}\left(\frac{x_t}{m_t}\right)\cdot\frac{2x_t}{m_t^3}\\ &= \frac{x_t}{m_t^3}\left[2f_{Z^2}\left(\frac{x_t}{m_t}\right)+\frac{x_t}{m_t}f'_{Z^2}\left(\frac{x_t}{m_t}\right)\right]\ ;\end{aligned}$$

hence, if $2f_{Z^2}\left(\frac{x_t}{m_t}\right)+\frac{x_t}{m_t}f'_{Z^2}\left(\frac{x_t}{m_t}\right)\geq 0$, where $m_t=\alpha_0+u$, with $u\geq 0$, R_t is a convex function.

Therefore, if, for each $x_t, t\in\{1,...,q\}$, $\frac{x_t}{\alpha_0+u}f'_{Z^2}\left(\frac{x_t}{\alpha_0+u}\right)+2f_{Z^2}\left(\frac{x_t}{\alpha_0+u}\right)\geq 0$, with $u\geq 0$, we can apply Jensen's inequality and obtain

$$\begin{aligned}&P\left(Y_t^2\leq x_t, t=1,...,n\right)\geq\\ &\geq F_{Z^2}\left(\frac{x_1}{E\left(\sigma_1^2\right)}\right)\prod_{t=2}^{q}F_{Z^2}\left(\frac{x_t}{\alpha_0+\sum\limits_{i=1}^{t-1}\alpha_i x_{t-i}+\sum\limits_{i=t}^{q}\alpha_i E\left(Y_{t-i}^2\right)}\right)F_{q,n}.\end{aligned}$$

So, as, for all $t \in \mathbb{Z}$, $E\left(Y_t^2\right) = E\left(\sigma_1^2\right) = V\left(Y_t\right)$ and $V\left(Y_t\right) = \sigma_Y^2$, we can conclude that

$$P\left(Y_t^2 \leq x_t, t = 1, ..., n\right) \geq$$
$$\geq F_{Z^2}\left(\frac{x_1}{\sigma_Y^2}\right) \prod_{t=2}^{q} F_{Z^2}\left(\frac{x_t}{\alpha_0 + \sum\limits_{i=1}^{t-1} \alpha_i x_{t-i} + \sigma_Y^2 \sum\limits_{i=t}^{q} \alpha_i}\right) F_{q,n}.$$

(a) Let us now turn to the case $n \leq q$. In this case, if $Y_1^2 \leq x_1, ..., Y_n^2 \leq x_n$, then, for $t \in \{2, ..., n\}$,

$$h_t = \alpha_0 + \sum_{i=1}^{t-1} \alpha_i Y_{t-i}^2 + \sum_{i=t}^{q} \alpha_i Y_{t-i}^2 \leq \alpha_0 + \sum_{i=1}^{t-1} \alpha_i x_{t-i} + \sum_{i=t}^{q} \alpha_i Y_{t-i}^2 \ ,$$

as, for $i \in \{1, ..., t-1\}$, $1 \leq t - i \leq n - 1$ and, for $i \in \{t, ..., q\}$, $t - i \leq 0$.

Hence, for every $(x_1, ..., x_n) \in \left]0, +\infty\right[^n$,

$$P\left(Y_t^2 \leq x_t, t = 1, ..., n\right) =$$
$$= P\left(Z_1^2 \leq \frac{x_1}{h_1}, Z_t^2 \leq \frac{x_t}{h_t}, t = 2, ..., n\right)$$
$$\geq P\left(Z_1^2 \leq \frac{x_1}{h_1}, Z_t^2 \leq \frac{x_t}{\alpha_0 + \sum\limits_{i=1}^{t-1} \alpha_i x_{t-i} + \sum\limits_{i=t}^{q} \alpha_i Y_{t-i}^2}, t = 2, ..., n\right) .$$

Following the same line of reasoning of the previous case, we can state that, if, for every $x_t, t \in \{1, ..., n\}$, $\frac{x_t}{\alpha_0+u} f'_{Z^2}\left(\frac{x_t}{\alpha_0+u}\right) + 2 f_{Z^2}\left(\frac{x_t}{\alpha_0+u}\right) \geq 0$, with $u \geq 0$, then

$$P\left(Y_t^2 \leq x_t, t = 1, ..., n\right) \geq F_{Z^2}\left(\frac{x_1}{\sigma_Y^2}\right) \prod_{t=2}^{n} F_{Z^2}\left(\frac{x_t}{\alpha_0 + \sum\limits_{i=1}^{t-1} \alpha_i x_{t-i} + \sigma_Y^2 \sum\limits_{i=t}^{q} \alpha_i}\right).$$

■

Taking into account that, for the in-control state, we have

$$P\left(N_S > n\right) = P\left(Y_t^2 \leq c^2 \sigma_Y^2, t = 1, ..., n\right) \ ,$$

the previous theorem will be useful considering $x_1 = ... = x_n = c^2\sigma_Y^2$. Since $\sigma_Y^2 = \frac{\alpha_0}{1-\sum_{i=1}^{q}\alpha_i}$, it is easy to verify that

$$\frac{c^2\sigma_Y^2}{\alpha_0 + \sum_{i=1}^{t-1}\alpha_i c^2\sigma_Y^2 + \sigma_Y^2\sum_{i=t}^{q}\alpha_i} = \frac{c^2}{1+(c^2-1)\sum_{i=1}^{t-1}\alpha_i}$$

and that

$$\frac{c^2\sigma_Y^2}{\alpha_0 + \sum_{i=1}^{q}\alpha_i c^2\sigma_Y^2} = \frac{c^2}{1+(c^2-1)\sum_{i=1}^{q}\alpha_i} \,.$$

Therefore, a new lower bound for $P\left(N_S > n\right), n \in \mathbb{N}$, is deduced from theorem 1.

Corollary. *Under the conditions of the previous theorem, a lower bound for P_n is*

$$\underset{\text{ARCH}(q)}{LB_3P_n} = \prod_{t=1}^{\min\{q,n\}} F_{Z^2}\left(\frac{c^2}{1+(c^2-1)\sum_{i=1}^{t-1}\alpha_i}\right)\times \qquad (3.4)$$

$$\times\left[F_{Z^2}\left(\frac{c^2}{1+(c^2-1)\sum_{i=1}^{q}\alpha_i}\right)\right]^{\max\{0,n-q\}}$$

which is valid for every $c \in \mathbb{R}^+$ such that $2f_{Z^2}(x) + xf'_{Z^2}(x) \geq 0$, where $x = \frac{c^2\sigma_Y^2}{\alpha_0+u}$, with $u \geq 0$.

We note that, when $\alpha_1 = ... = \alpha_q = 0$, we have

$$\underset{\text{ARCH}(q)}{UB_1P_n} = \underset{\text{ARCH}(q)}{LB_1P_n} = \underset{\text{ARCH}(q)}{LB_2P_n} = \underset{\text{ARCH}(q)}{LB_3P_n} = \left[F_{Z^2}\left(c^2\right)\right]^n = P_{iid}\left(N_S > n\right) \,.$$

3.2. Bounds for the ARL.

The bounds for the ARL are determined using the bounds for $P\left(N_S > n\right)$, considering that

$$ARL = E\left(N_S\right) = \sum_{n=1}^{+\infty} nP\left(N_S = n\right) = \sum_{n=0}^{+\infty} P\left(N_S > n\right) \ .$$

Starting from $\underset{\text{ARCH}(q)}{UB_1 P_n}$ and $\underset{\text{ARCH}(q)}{LB_1 P_n}$, Severin and Schmid (1999) deduced, respectively, the following upper bound for the in-control ARL

$$\underset{\text{ARCH}(q)}{UB_1 ARL} = \left[1 - F_{Z^2}\left(\frac{c^2}{1 - \sum\limits_{i=1}^{q} \alpha_i}\right)\right]^{-1} \tag{3.5}$$

and the lower bound for the in-control ARL

$$\underset{\text{ARCH}(q)}{LB_1 ARL} = 1 + \sum_{n=1}^{q-1} P\left(N_S > n\right) + \frac{P\left(N_S > q\right)}{1 - F_{Z^2}\left(\dfrac{c^2}{1 + \left(c^2 - 1\right)\sum\limits_{i=1}^{q} \alpha_i}\right)} \tag{3.6}$$

with both bounds valid for every $c \in \mathbb{R}^+$.

Using the two other lower bounds for $P\left(N_S > n\right)$, $\underset{\text{ARCH}(q)}{LB_2 P_n}$ and $\underset{\text{ARCH}(q)}{LB_3 P_n}$, we have the following result:

Corollary. *Under the conditions of theorem 1,*

$$\underset{\text{ARCH}(q)}{LB_2 ARL} = 1 + \sum_{n=1}^{q-1} \prod_{t=1}^{n} F_{Z^2}\left(\frac{c^2}{1 + \left(c^2 - 1\right)\sum\limits_{i=1}^{t-1} \alpha_i}\right) + \tag{3.7}$$
$$+ \left[\prod_{t=1}^{q} F_{Z^2}\left(\frac{c^2}{1 + \left(c^2 - 1\right)\sum\limits_{i=1}^{t-1} \alpha_i}\right)\right] \frac{1}{1 - F_{Z^2}\left(c^2\right)}$$

and

$$\underset{\text{ARCH}(q)}{LB_3ARL} = 1 + \sum_{n=1}^{q-1} \prod_{t=1}^{n} F_{Z^2}\left(\frac{c^2}{1+(c^2-1)\sum_{i=1}^{t-1}\alpha_i}\right) +$$
$$+ \left[\prod_{t=1}^{q} F_{Z^2}\left(\frac{c^2}{1+(c^2-1)\sum_{i=1}^{t-1}\alpha_i}\right)\right] \frac{1}{1-F_{Z^2}\left(\frac{c^2}{1+(c^2-1)\sum_{i=1}^{q}\alpha_i}\right)} \tag{3.8}$$

are lower bounds for the ARL, for every $c \in \mathbb{R}^+$ *such that* $2f_{Z^2}(x) + xf'_{Z^2}(x) \geq 0$, *where* $x = \frac{c^2\sigma_Y^2}{\alpha_0+u}$, *with* $u \geq 0$.

In the following, we analyse in some particular cases the regions where these bounds are valid.

Example 1. If Z_t follows the standard normal distribution, then, for $x > 0$,

$$f_{Z^2}(x) = \frac{1}{\sqrt{2\pi}} \times \frac{e^{-x/2}}{x^{1/2}}$$

and

$$2f_{Z^2}(x) + xf'_{Z^2}(x) = f_{Z^2}(x)\left[2 + x\left(-\frac{1}{2} - \frac{1}{2x}\right)\right] .$$

Consequently

$$2f_{Z^2}(x) + xf'_{Z^2}(x) \geq 0 \Leftrightarrow x \leq 3 .$$

As $x = \frac{c^2\sigma_Y^2}{\alpha_0+u}$, with $u \geq 0$, hence $\frac{c^2\sigma_Y^2}{\alpha_0+u} \leq \frac{c^2\sigma_Y^2}{\alpha_0}$. So, if $\frac{c^2\sigma_Y^2}{\alpha_0} \leq 3$, it holds that $x \leq 3$. Then for

$$0 < c \leq \frac{\sqrt{3\alpha_0}}{\sigma_Y} = \sqrt{3\left(1 - \sum_{i=1}^{q}\alpha_i\right)} ,$$

$\underset{\text{ARCH}(q)}{LB_2ARL}$ and $\underset{\text{ARCH}(q)}{LB_3ARL}$ are lower bounds for the in-control ARL.

Example 2. If Z_t follows the unit variance distribution based on the Student's t-distribution with $\alpha > 2$ degrees of freedom, then

$$f_Z(x) = \frac{1}{\sqrt{(\alpha-2)\pi}} \cdot \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \cdot \left(1 + \frac{x^2}{\alpha-2}\right)^{-\frac{\alpha+1}{2}} .$$

So, for $x > 0$, $f_{Z^2}(x) = \frac{1}{2\sqrt{x}}\left[f_Z(\sqrt{x}) + f_Z(-\sqrt{x})\right] = \frac{1}{\sqrt{(\alpha-2)\pi x}} \cdot \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \cdot \left(1 + \frac{x}{\alpha-2}\right)^{-\frac{\alpha+1}{2}}$. Hence

$$2f_{Z^2}(x) + xf'_{Z^2}(x) = f_{Z^2}(x)\left[2 - \frac{\alpha - 2 + 2x + \alpha x}{2(\alpha - 2 + x)}\right] .$$

As $\alpha > 2$, we get

$$2f_{Z^2}(x) + xf'_{Z^2}(x) \geq 0 \Leftrightarrow x \leq 3 .$$

Thus, just like in the previous case, $\underset{\mathrm{ARCH}(q)}{LB_2ARL}$ and $\underset{\mathrm{ARCH}(q)}{LB_3ARL}$ are lower bounds for the in-control ARL, if

$$0 < c \leq \frac{\sqrt{3\alpha_0}}{\sigma_Y} = \sqrt{3\left(1 - \sum_{i=1}^{q}\alpha_i\right)} .$$

Example 3. If Z_t follows the bidirectional Pareto law with parameter $\alpha > 2$, its density function is

$$f_Z(x) = \frac{\alpha}{2}\left(\sqrt{\frac{\alpha-2}{\alpha}}\right)^{\alpha} \frac{1}{|x|^{\alpha+1}} 1\!\!1_{\left]-\infty,-\sqrt{\frac{\alpha-2}{\alpha}}\right[\cup\left]\sqrt{\frac{\alpha-2}{\alpha}},+\infty\right[}(x) .$$

Then $f_{Z^2}(x) = \frac{\alpha}{2}\left(\sqrt{\frac{\alpha-2}{\alpha}}\right)^{\alpha} \frac{1}{x^{\frac{\alpha}{2}+1}} 1\!\!1(x)_{\left]\frac{\alpha-2}{\alpha},+\infty\right[}$. So

$$2f_{Z^2}(x) + xf'_{Z^2}(x) = f_{Z^2}(x)\left[2 + \left(-\frac{\alpha}{2} - 1\right)\right] .$$

Considering the expression of $f_{Z^2}(x)$, it is straightforward that, for $x \in \left]\frac{\alpha-2}{\alpha}, +\infty\right[$, $2f_{Z^2}(x) + xf'_{Z^2}(x) < 0$ and, for $x \in \left]-\infty, \frac{\alpha-2}{\alpha}\right]$, $xf'_{Z^2}(x) + 2f_{Z^2}(x) = 0$.

Thus, bearing in mind that $x = \frac{c^2 \sigma_Y^2}{\alpha_0 + u}$, with $u \geq 0$, it holds that $\underset{\mathrm{ARCH}(q)}{LB_2ARL}$ and $\underset{\mathrm{ARCH}(q)}{LB_3ARL}$ are lower bounds for the in-control ARL, if

$$0 < c \leq \sqrt{\frac{(\alpha-2)\,\alpha_0}{\alpha\sigma_Y^2}} = \sqrt{\frac{(\alpha-2)}{\alpha}\left(1-\sum_{i=1}^{q}\alpha_i\right)}\ .$$

3.3. Comparing the lower bounds for the ARL.

In this section, we show that, for $0 < c \leq 1$ such that $2f_{Z^2}(x) + xf'_{Z^2}(x) \geq 0$, where $x = \frac{c^2 \sigma_Y^2}{\alpha_0 + u}$, with $u \geq 0$, it holds that

$$\underset{\mathrm{ARCH}(q)}{LB_1ARL} \geq \underset{\mathrm{ARCH}(q)}{LB_3ARL} \geq \underset{\mathrm{ARCH}(q)}{LB_2ARL} \geq ARL_{iid}\ .$$

In fact, as $P(N_S > n) \geq \underset{\mathrm{ARCH}(q)}{LB_3P_n}$, for all $n \in \mathbb{N}$, then

$$\underset{\mathrm{ARCH}(q)}{LB_1ARL} \geq 1 + \sum_{n=1}^{q-1} \underset{\mathrm{ARCH}(q)}{LB_3P_n} + \frac{\underset{\mathrm{ARCH}(q)}{LB_3P_q}}{1 - F_{Z^2}\left(\frac{c^2}{1+(c^2-1)\sum\limits_{i=1}^{q}\alpha_i}\right)} = \underset{\mathrm{ARCH}(q)}{LB_3ARL}\ .$$

Since, for $0 < c \leq 1$ we have $\dfrac{c^2}{1+(c^2-1)\sum\limits_{i=1}^{q}\alpha_i} \geq c^2$, thus

$$\frac{1}{1 - F_{Z^2}\left(\frac{c^2}{1+(c^2-1)\sum\limits_{i=1}^{q}\alpha_i}\right)} \geq \frac{1}{1-F_{Z^2}(c^2)} \Leftrightarrow \underset{\mathrm{ARCH}(q)}{LB_3ARL} \geq \underset{\mathrm{ARCH}(q)}{LB_2ARL}\ .$$

Finally, for $0 < c \leq 1$, it holds

$$\underset{\mathrm{ARCH}(q)}{LB_2ARL} \geq 1 + \sum_{n=1}^{q-1}\prod_{t=1}^{n} F_{Z^2}\left(c^2\right) + \left[\prod_{t=1}^{q} F_{Z^2}\left(c^2\right)\right]\frac{1}{1-F_{Z^2}(c^2)} = ARL_{iid}\ .$$

As $\underset{\mathrm{ARCH}(q)}{LB_1ARL}$ depends on the probability in evaluation, we point out that for small critical values ($0 < c \leq 1$), the best lower bound for the in-control ARL

is $\underset{\text{ARCH}(q)}{LB_3 ARL}$. Furthermore, we can state that if the analyst falsely assumes the process to be independent, then, for small values of c, the ARL for an ARCH(q) process is always greater than in the iid case.

4. In-control ARL of modified Shewhart chart for the TARCH process

In this section we assume that the target process Y follows a stationary TARCH(q) model and that the observed process X is in-control.

Bounds for the ARL of a TARCH model are obtained considering the work of Gonçalves and Mendes-Lopes (2007).

4.1. Bounds for $P_n = P\left(N_S > n\right)$.

In what concerns the upper bound of P_n, it is straightforward that

$$\underset{\text{TARCH}(q)}{UB_1 P_n} = \prod_{t=1}^{n} F_{|Z|}\left(\frac{c\sigma_Y}{\alpha_0}\right) = \left[F_{|Z|}\left(\frac{c\sigma_Y}{\alpha_0}\right)\right]^n \tag{4.1}$$

is an upper bound for $P\left(N_S > n\right)$, valid for every $c \in \mathbb{R}^+$.

In order to determine a lower bound for $P\left(N_S > n\right)$, we present a new version of theorem 4 of Gonçalves and Mendes-Lopes (2007), in which a smaller and, therefore, better upper bound for σ_t is considered, introducing $\phi_i = \max\left\{\alpha_i, \beta_i\right\}$, for $i = 1, ..., q$. The proof of the following result is omitted due to the similarity with the proof of theorem 1.

Theorem 2. *Let $Y = (Y_t, t \in \mathbb{Z})$ be a TARCH(q) process such that Z_t is absolutely continuous with a differentiable density probability f_Z. Then, if Y is stationary, it holds:*

(a) *for $n \leq q$,*

$$\begin{aligned} P\left(|Y_t| \leq x_t, t = 1, ..., n\right) \;\; \geq \;\; & F_{|Z|}\left(\frac{x_1}{\sigma_Y}\right) \times \\ & \times \prod_{t=2}^{n} F_{|Z|}\left(\frac{x_t}{\alpha_0 + \sum\limits_{i=1}^{t-1} \phi_i x_{t-i} + \sigma_Y \sum\limits_{i=t}^{q} \phi_i}\right), \end{aligned}$$

(b) *for $n > q$,*

$$P\left(|Y_t| \leq x_t, t=1,...,n\right) \geq F_{|Z|}\left(\frac{x_1}{\sigma_Y}\right) \times$$

$$\times \prod_{t=2}^{q} F_{|Z|}\left(\frac{x_t}{\alpha_0 + \sum_{i=1}^{t-1}\phi_i x_{t-i} + \sigma_Y \sum_{i=t}^{q}\phi_i}\right) \times \prod_{t=q+1}^{n} F_{|Z|}\left(\frac{x_t}{\alpha_0 + \sum_{i=1}^{q}\phi_i x_{t-i}}\right),$$

in both cases, for all $(x_1,...,x_n) \in \left]0,+\infty\right[^n$ such that

$$2f_{|Z|}\left(\frac{x_t}{\alpha_0+u}\right) + \frac{x_t}{\alpha_0+u} f'_{|Z|}\left(\frac{x_t}{\alpha_0+u}\right) \geq 0 \ , \ t=1,...,\min\{q,n\},$$

where u is any non-negative real number.

For the in-control state, $P(N_S > n) = P\left(|Y_t| \leq c\sigma_Y, t=1,...,n\right)$ and using this theorem with $x_1 = ... = x_n = c\sigma_Y$, we have the following lower bound for P_n,

$$\begin{aligned}\underset{\text{TARCH}(q)}{LB_1P_n} &= F_{|Z|}(c) \prod_{t=2}^{\min\{q,n\}} F_{|Z|}\left(\frac{c}{\frac{\alpha_0}{\sigma_Y} + c\sum_{i=1}^{t-1}\phi_i + \sum_{i=t}^{q}\phi_i}\right) \times \\ &\quad \times \left[F_{|Z|}\left(\frac{c}{\frac{\alpha_0}{\sigma_Y} + c\sum_{i=1}^{q}\phi_i}\right)\right]^{\max\{0,n-q\}},\end{aligned} \tag{4.2}$$

valid for every $c \in \mathbb{R}^+$ such that $2f_{|Z|}(x) + xf'_{|Z|}(x) \geq 0$, where $x = \frac{c\sigma_Y}{\alpha_0+u}$, with $u \geq 0$.

We observe that, if $\alpha_1 = ... = \alpha_q = 0$, considering remark 1, then

$$\underset{\text{TARCH}(q)}{UB_1P_n} = \underset{\text{TARCH}(q)}{LB_1P_n} = \left[F_{|Z|}(c)\right]^n = P_{iid}(N_S > n) \ .$$

4.2. Bounds for the ARL.

Considering $\underset{\text{TARCH}(q)}{UB_1P_n}$ and $\underset{\text{TARCH}(q)}{LB_1P_n}$, respectively, we derive the following upper bound for the in-control ARL

$$\underset{\text{TARCH}(q)}{UB_1ARL} = \frac{1}{1 - F_{|Z|}\left(\dfrac{c\sigma_Y}{\alpha_0}\right)} \tag{4.3}$$

valid for every $c \in \mathbb{R}^+$, and lower bound for the in-control ARL

$$\underset{\text{TARCH}(q)}{LB_1ARL} = \; = 1 + F_{|Z|}(c) \sum_{n=1}^{q-1} \prod_{t=2}^{n} F_{|Z|}\left(\frac{c}{\frac{\alpha_0}{\sigma_Y} + c\sum\limits_{i=1}^{t-1}\phi_i + \sum\limits_{i=t}^{q}\phi_i}\right) + \tag{4.4}$$

$$+\frac{F_{|Z|}(c)\prod\limits_{t=2}^{q} F_{|Z|}\left(\frac{c}{\frac{\alpha_0}{\sigma_Y}+c\sum\limits_{i=1}^{t-1}\phi_i+\sum\limits_{i=t}^{q}\phi_i}\right)}{1 - F_{|Z|}\left(\frac{c}{\frac{\alpha_0}{\sigma_Y}+c\sum\limits_{i=1}^{q}\phi_i}\right)}$$

valid for every $c \in \mathbb{R}^+$ such that $2f_{|Z|}(x) + xf'_{|Z|}(x) \geq 0$, where $x = \frac{c\sigma_Y}{\alpha_0+u}$, with $u \geq 0$. For example, for a TARCH(1) model, we have

$$\underset{\text{TARCH}(1)}{LB_1ARL} = 1 + \frac{F_{|Z|}(c)}{1 - F_{|Z|}\left(\frac{c}{\frac{\alpha_0}{\sigma_Y}+c\max\{\alpha_1,\beta_1\}}\right)}.$$

We analyse now the regions where these bounds are valid for the same distributions of Z_t considered in the ARCH case. The results are summarized in the next example.

Example 4.

(a) Considering that Z_t follows the standard normal distribution, then, for $x > 0$,

$$f_{|Z|}(x) = f_Z(x) + f_Z(-x) = \frac{2e^{-x^2/2}}{\sqrt{2\pi}}$$

and

$$f'_{|Z|}(x) = -\frac{2xe^{-x^2/2}}{\sqrt{2\pi}} = -xf_{|Z|}(x) \ .$$

Consequently, for $x > 0$,

$$2f_{|Z|}(x) + xf'_{|Z|}(x) \geq 0 \Leftrightarrow 0 < x \leq \sqrt{2} \ .$$

When $u \geq 0$, then $x = \frac{c\sigma_Y}{\alpha_0 + u} \leq \frac{c\sigma_Y}{\alpha_0}$ and so for $0 < c \leq \frac{\sqrt{2}\alpha_0}{\sigma_Y}$, $\underset{\text{TARCH}(q)}{LB_1ARL}$ is a lower bound for the in-control ARL.

(b) Suppose Z_t follows the unit variance distribution based on the Student's t-distribution with $\alpha > 2$ degrees of freedom as in example 2. So, for $x > 0$, $f_{|Z|}(x) = 2f_Z(x)$ and $f'_{|Z|}(x) = f_{|Z|}(x) \cdot \frac{-(\alpha+1)x}{\alpha - 2 + x^2}$. Hence, for $x > 0$,

$$2f_{|Z|}(x) + xf'_{|Z|}(x) \geq 0 \Leftrightarrow 0 < x \leq \sqrt{\frac{2(\alpha - 2)}{\alpha - 1}} \ .$$

Thus $\underset{\text{TARCH}(q)}{LB_1ARL}$ is a lower bound for the in-control ARL, if

$$0 < c \leq \sqrt{\frac{2(\alpha - 2)}{\alpha - 1}} \cdot \frac{\alpha_0}{\sigma_Y} \ .$$

(c) If Z_t follows the bidirectional Pareto law with parameter $\alpha > 2$ as in example 3, then $f_{|Z|}(x) = \alpha \left(\sqrt{\frac{\alpha-2}{\alpha}}\right)^{\alpha} \frac{1}{x^{\alpha+1}} 1\!\!1(x)_{\left]\sqrt{\frac{\alpha-2}{\alpha}}, +\infty\right[}$ and $f'_{|Z|}(x) = \frac{-\alpha-1}{x} f_{|Z|}(x)$. So, for $x > 0$,

$$2f_{|Z|}(x) + xf'_{|Z|}(x) = f_{|Z|}(x)\left[2 + (-\alpha - 1)\right] \ .$$

Considering the expression of $f_{|Z|}(x)$, it is straightforward that, for $x \in \left]\sqrt{\frac{\alpha-2}{\alpha}}, +\infty\right[$, $2f_{|Z|}(x) + xf'_{|Z|}(x) < 0$ and, for $x \in \left]-\infty, \sqrt{\frac{\alpha-2}{\alpha}}\right]$, $2f_{|Z|}(x) + xf'_{|Z|}(x) = 0$.

Thus, bearing in mind that $x = \frac{c\sigma_Y}{\alpha_0 + u}$, with $u \geq 0$, it holds that $\underset{\text{TARCH}(q)}{LB_1ARL}$ is a lower bound for the in-control ARL, if

$$0 < c \leq \sqrt{\frac{(\alpha - 2)}{\alpha}} \cdot \frac{\alpha_0}{\sigma_Y} \ .$$

In the next remark we analyse the particular case of the TARCH(1) model and find a region for the critical value for which we can ensure that the ARL is always greater than in the iid case.

Remark 2. *Let us consider $q = 1$. In this case, the expression of σ_Y^2 is given in (2.5). Then*

$$\sigma_Y^2 \leq \frac{\alpha_0^2 (1+\phi_1)}{(1-\phi_1)(1-\phi_1^2)} = \frac{\alpha_0^2}{(1-\phi_1)^2} \ ,$$

since $\alpha_1 \leq \phi_1$ and $\beta_1 \leq \phi_1$, so, for every $t \in \mathbb{Z}$, $\alpha_1 Z_t^+ - \beta_1 Z_t^- \leq \phi_1 |Z_t|$ and, by Lyapunov's inequality, $E(|Z_t|) \leq \sqrt{E(Z_t^2)} = 1$.

Therefore, $\frac{\alpha_0}{\sigma_Y} + \phi_1 \geq 1 \Leftrightarrow \frac{1}{\phi_1}\left(1 - \frac{\alpha_0}{\sigma_Y}\right) \leq 1$ and choosing $c \leq \frac{1}{\phi_1}\left(1 - \frac{\alpha_0}{\sigma_Y}\right)$ such that $2f_{|Z|}(x) + x f'_{|Z|}(x) \geq 0$, where $x = \frac{c\sigma_Y}{\alpha_0 + u}$, with $u \geq 0$, we have

$$\frac{\alpha_0}{\sigma_Y} + c\phi_1 \leq 1 \Leftrightarrow \frac{c}{\frac{\alpha_0}{\sigma_Y} + c\phi_1} \geq c \Leftrightarrow F_{|Z|}\left(\frac{c}{\frac{\alpha_0}{\sigma_Y} + c\phi_1}\right) \geq F_{|Z|}(c) \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{1 - F_{|Z|}\left(\frac{c}{\frac{\alpha_0}{\sigma_Y} + c\phi_1}\right)} \geq \frac{1}{1 - F_{|Z|}(c)} \Leftrightarrow \underset{\text{TARCH(1)}}{LB_1 ARL} \geq ARL_{iid} \ .$$

5. Simulation study

The goal of the simulation work, as mentioned, is to evaluate and compare the bounds obtained for the in-control ARL of the modified Shewhart chart for ARCH and TARCH processes. We consider, in both cases, the first order models and present the results in two separate subsections. For Z_t we take, in the case of the ARCH model, the standard normal distribution, $N(0,1)$, and, for the TARCH model, we consider the standard normal distribution and the unit variance distribution based on the Student's t-distribution with 6 degrees of freedom, $t(6)$. The study for the ARCH process follows Severin and Schmid (1999).

5.1. ARCH(1).

We consider the bounds below, already appropriately particularized for the case where the target process, Y, follows a stationary ARCH(1) model, that is $\alpha_1 < 1$:

(i) $\underset{\text{ARCH}(1)}{UB_1ARL} = \dfrac{1}{1 - F_{Z^2}\left(\frac{c^2}{1-\alpha_1}\right)}$ valid for every $c \in \mathbb{R}^+$;

(ii) $\underset{\text{ARCH}(1)}{LB_1ARL} = 1 + \dfrac{P\left(|Y_1| \leq c\sigma_Y\right)}{1 - F_{Z^2}\left(\frac{c^2}{1+(c^2-1)\alpha_1}\right)}$, valid for every $c \in \mathbb{R}^+$;

(iii) $\underset{\text{ARCH}(1)}{LB_2ARL} = 1 + \dfrac{F_{Z^2}\left(c^2\right)}{1 - F_{Z^2}\left(c^2\right)} = \dfrac{1}{1 - F_{Z^2}\left(c^2\right)}$, valid for every $c \in \mathbb{R}^+$ such that $2f_{Z^2}(x) + xf'_{Z^2}(x) \geq 0$, where $x = \frac{c^2\sigma_Y^2}{\alpha_0+u}$, with $u \geq 0$;

(iv) $\underset{\text{ARCH}(1)}{LB_3ARL} = 1 + \dfrac{F_{Z^2}\left(c^2\right)}{1 - F_{Z^2}\left(\frac{c^2}{1+(c^2-1)\alpha_1}\right)}$, valid for every $c \in \mathbb{R}^+$ such that $2f_{Z^2}(x) + xf'_{Z^2}(x) \geq 0$, where $x = \frac{c^2\sigma_Y^2}{\alpha_0+u}$, with $u \geq 0$.

We considered $\alpha_0 = 1$, $\alpha_1 = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8$ and $Z_t \sim N(0,1)$. For each parameterization, we generated 25000 trajectories of 150 observations of the corresponding process Y. They were used to estimate the ARL ($\widehat{ARL}$), $P\left(|Y_1| \leq c\sigma_Y\right)$ and, consequently, $\underset{\text{ARCH}(1)}{LB_1ARL}$ ($\underset{\text{ARCH}(1)}{\widehat{LB_1ARL}}$). The first 50 observations were discarded to eliminate the effect of choosing the value of the first observation.

In table 1 we report the results obtained in this simulation study, for $c = 0.5$, $c = 0.8$ and $c = 1$, namely, the value of the $\underset{\text{ARCH}(1)}{LB_2ARL}$, $\underset{\text{ARCH}(1)}{LB_3ARL}$ and $\underset{\text{ARCH}(1)}{UB_3ARL}$ and the estimated $\underset{\text{ARCH}(1)}{LB_1ARL}$ and ARL. In each table, for the estimated values, in the line immediately under, we give radius of the 95% confidence interval. The values typed in italic are the ones where the validity of the bounds cannot be ensured analytically. As seen on example 1, we have ensured the validity of the lower bounds $\underset{\text{ARCH}(1)}{LB_2ARL}$ and $\underset{\text{ARCH}(1)}{LB_3ARL}$ when $0 < c \leq \sqrt{3\left(1-\alpha_1\right)}$.

We observe that, for $c = 0.5$ and $c = 0.8$, $\widehat{ARL}$, $\underset{\text{ARCH}(1)}{\widehat{LB_1ARL}}$ and $\underset{\text{ARCH}(1)}{LB_3ARL}$ increase with α_1, but $\underset{\text{ARCH}(1)}{LB_2ARL}$ maintains its value. For example, for $c = 0.5$ and $0 < \alpha_1 \leq 0.5$, the deviation of $\underset{\text{ARCH}(1)}{LB_3ARL}$ from $\widehat{ARL}$ is at most 7%. For

Table 1. In-control ARL of the modified Shewhart chart for an ARCH(1) process (with $\alpha_0 = 1$ and $Z_t \sim N(0, 1)$).

c	α_1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.8
0.5	$\underset{\text{ARCH}(1)}{LB_2ARL}$	1.621	1.621	1.621	1.621	1.621	1.621	1.621	1.621
	$\underset{\text{ARCH}(1)}{LB_3ARL}$	1.621	1.635	1.652	1.672	1.696	1.726	1.766	1.892
	$\underset{\text{ARCH}(1)}{\widehat{LB_1ARL}}$	1.615	1.639	1.661	1.705	1.742	1.831	1.915	2.297
	CI 95%	$\pm$0.010	$\pm$0.010	$\pm$0.010	$\pm$0.011	$\pm$0.011	$\pm$0.012	$\pm$0.012	$\pm$0.014
	$\widehat{ARL}$	1.623	1.648	1.669	1.729	1.771	1.884	2.011	2.736
	CI 95%	$\pm$0.013	$\pm$0.013	$\pm$0.013	$\pm$0.015	$\pm$0.015	$\pm$0.017	$\pm$0.019	$\pm$0.031
	$\underset{\text{ARCH}(1)}{UB_1ARL}$	1.621	1.672	1.736	1.818	1.928	2.086	2.330	3.794
0.8	$\underset{\text{ARCH}(1)}{LB_2ARL}$	2.360	2.360	2.360	2.360	2.360	2.360	2.360	*2.360*
	$\underset{\text{ARCH}(1)}{LB_3ARL}$	2.360	2.388	2.418	2.452	2.488	2.529	2.573	*2.680*
	$\underset{\text{ARCH}(1)}{\widehat{LB_1ARL}}$	2.360	2.395	2.433	2.499	2.567	2.691	2.802	3.214
	CI 95%	$\pm$0.014	$\pm$0.015	$\pm$0.015	$\pm$0.015	$\pm$0.016	$\pm$0.016	$\pm$0.016	$\pm$0.015
	$\widehat{ARL}$	2.370	2.455	2.522	2.682	2.846	3.148	3.547	5.903
	CI 95%	$\pm$0.022	$\pm$0.024	$\pm$0.025	$\pm$0.028	$\pm$0.030	$\pm$0.034	$\pm$0.040	$\pm$0.073
	$\underset{\text{ARCH}(1)}{UB_1ARL}$	2.360	2.506	2.695	2.950	3.315	3.877	4.857	13.580
1	$\underset{\text{ARCH}(1)}{LB_2ARL}$	3.151	3.151	3.151	3.151	3.151	3.151	3.151	*3.151*
	$\underset{\text{ARCH}(1)}{LB_3ARL}$	3.151	3.151	3.151	3.151	3.151	3.151	3.151	*3.151*
	$\underset{\text{ARCH}(1)}{\widehat{LB_1ARL}}$	3.154	3.154	3.162	3.206	3.241	3.323	3.387	3.651
	CI 95%	$\pm$0.018	$\pm$0.018	$\pm$0.018	$\pm$0.018	$\pm$0.018	$\pm$0.017	$\pm$0.017	$\pm$0.014
	$\widehat{ARL}$	3.166	3.277	3.440	3.708	4.044	4.553	5.326	9.625
	CI 95%	$\pm$0.032	$\pm$0.034	$\pm$0.037	$\pm$0.041	$\pm$0.046	$\pm$0.052	$\pm$0.063	$\pm$0.119
	$\underset{\text{ARCH}(1)}{UB_1ARL}$	3.151	3.427	3.794	4.310	5.084	6.357	8.784	39.452

$c = 1$, $\underset{\text{ARCH}(1)}{LB_3ARL}$ and $\underset{\text{ARCH}(1)}{LB_2ARL}$ are equal and, as α_1 increases, they remain constant. In this case, only for $0 < \alpha_1 \leq 0.2$ the deviation of $\underset{\text{ARCH}(1)}{LB_3ARL}$ from $\widehat{ARL}$ is at most 9%.

5.2. TARCH(1).

We consider now the bounds indicated next, already properly particularized for the case where the target process Y is a TARCH(1) process:

(i) $\underset{\text{TARCH(1)}}{UB_1ARL} = \dfrac{1}{1 - F_{|Z|}\left(\frac{c\sigma_Y}{\alpha_0}\right)}$, valid for every $c \in \mathbb{R}^+$;

(ii) $\underset{\text{TARCH(1)}}{LB_1ARL} = 1 + \dfrac{F_{|Z|}(c)}{1 - F_{|Z|}\left(\frac{c}{\frac{\alpha_0}{\sigma_Y} + c\max\{\alpha_1, \beta_1\}}\right)}$, valid for every $c \in \mathbb{R}^+$ such that $2f_{|Z|}(x) + xf'_{|Z|}(x) \geq 0$, where $x = \frac{c\sigma_Y}{\alpha_0 + u}$, with $u \geq 0$.

We note that the TARCH(1) model such that Z_t has a symmetrical distribution, as are the cases considered, is stationary if $\alpha_1^2 + \beta_1^2 < 2$.

For each parameterization indicated in tables 2 to 5, we generated 25000 trajectories of 150 observations of the process $Y \sim$ TARCH(1) such that $\alpha_0 = 1$. They were used to estimate the ARL ($\widehat{ARL}$). In each table, the 95% confidence interval of $\widehat{ARL}$ in given in the line immediately under the line of the estimated values. We note that the first 50 observations were discarded to eliminate the effect of choosing the value of the first observation.

Since the distributions considered for Z_t are symmetrical, the ARL and its bounds are symmetrical in α_1 and β_1, meaning that, for example, the cases $(\alpha_1, \beta_1) = (0, 0.1)$ and $(\alpha_1, \beta_1) = (0.1, 0)$ produce the same results. So, we only present one of them, leaving the other one blank in the tables.

In tables 2 and 3 we report the results when $Z_t \sim N(0,1)$. As detailed in example 4 (a), we need to have $0 < c \leq \frac{\sqrt{2}\alpha_0}{\sigma_Y}$ so that we can ensure analytically the validity of $\underset{\text{TARCH(1)}}{LB_1ARL}$. In tables 4 and 5 we report the results when $\sqrt{\frac{3}{2}}Z_t \sim t(6)$. As described in example 4 (b), we have guaranteed analytically the validity of $\underset{\text{TARCH(1)}}{LB_1ARL}$ if $0 < c \leq \frac{\sqrt{1.6}\alpha_0}{\sigma_Y}$. When c exceeds the indicated value, the $\underset{\text{TARCH(1)}}{LB_1ARL}$ is typed in italic. Nevertheless, we observe, in tables 2 to 5, that all of these values are still smaller than the $\widehat{ARL}$. This suggests that there might be a less restrictive condition than $2f_{|Z|}(x) + xf'_{|Z|}(x) \geq 0$.

Moreover, in tables 2 to 5, if we fix α_1, as β_1 increases the $\widehat{ARL}$ and $\underset{\text{TARCH(1)}}{UB_1ARL}$ increase too. But the $\underset{\text{TARCH(1)}}{LB_1ARL}$ does not perform as well, since, for $c = 0.5$, if we fix α_1, as β_1 increases its value is virtually unchanged and, for $c \geq 1$, decreases. However, we can state that the accuracy of the bounds improves as c, α_1 and β_1 are closer to zero, regardless of the distribution considered.

Table 2. In-control ARL of the modified Shewhart chart, with $c = 0.5$, for a TARCH(1) process (with $\alpha_0 = 1$ and $Z_t \sim N\,(0,1)$).

α_1 \ β_1		0	0.1	0.3	0.5	0.7	0.9
0	$LB_1ARL_{\text{TARCH(1)}}$	1.621	1.618	1.615	1.616	1.620	1.628
	$\widehat{ARL}$	1.617	1.635	1.686	1.783	1.936	2.260
	CI 95%	±0.013	±0.013	±0.014	±0.015	±0.017	±0.022
	$UB_1ARL_{\text{TARCH(1)}}$	1.621	1.661	1.773	1.950	2.256	2.897
0.1	$LB_1ARL_{\text{TARCH(1)}}$	—	1.632	1.628	1.628	1.631	1.638
	$\widehat{ARL}$	—	1.649	1.674	1.781	1.941	2.264
	CI 95%	—	±0.013	±0.014	±0.015	±0.018	±0.023
	$UB_1ARL_{\text{TARCH(1)}}$	—	1.706	1.829	2.025	2.372	3.118
0.3	$LB_1ARL_{\text{TARCH(1)}}$	—	—	1.663	1.660	1.661	1.667
	$\widehat{ARL}$	—	—	1.748	1.831	2.000	2.371
	CI 95%	—	—	±0.015	±0.017	±0.019	±0.025
	$UB_1ARL_{\text{TARCH(1)}}$	—	—	1.986	2.248	2.737	3.904
0.5	$LB_1ARL_{\text{TARCH(1)}}$	—	—	—	1.708	1.706	1.710
	$\widehat{ARL}$	—	—	—	1.943	2.147	2.599
	CI 95%	—	—	—	±0.019	±0.022	±0.029
	$UB_1ARL_{\text{TARCH(1)}}$	—	—	—	2.643	3.464	5.862
0.7	$LB_1ARL_{\text{TARCH(1)}}$	—	—	—	—	1.776	*1.778*
	$\widehat{ARL}$	—	—	—	—	2.452	3.217
	CI 95%	—	—	—	—	±0.027	±0.038
	$UB_1ARL_{\text{TARCH(1)}}$	—	—	—	—	5.309	13.877

TABLE 3. In-control ARL of the modified Shewhart chart, with $c = 1$, for a TARCH(1) process (with $\alpha_0 = 1$ and $Z_t \sim N(0,1)$).

α_1 \ β_1		0	0.1	0.3	0.5	0.7
0	LB_1ARL TARCH(1)	3.151	2.980	2.745	2.592	*2.485*
	$\widehat{ARL}$	3.143	3.238	3.473	3.887	4.684
	CI 95%	±0.033	±0.033	±0.037	±0.043	±0.053
	UB_1ARL TARCH(1)	3.151	3.369	4.025	5.245	7.987
0.1	LB_1ARL TARCH(1)	—	3.093	2.815	2.639	*2.519*
	$\widehat{ARL}$	—	3.282	3.475	3.938	4.723
	CI 95%	—	±0.034	±0.037	±0.044	±0.054
	UB_1ARL TARCH(1)	—	3.619	4.383	5.843	9.260
0.3	LB_1ARL TARCH(1)	—	—	3.010	*2.767*	*2.608*
	$\widehat{ARL}$	—	—	3.707	4.126	5.006
	CI 95%	—	—	±0.041	±0.047	±0.059
	UB_1ARL TARCH(1)	—	—	5.529	7.902	14.243
0.5	LB_1ARL TARCH(1)	—	—	—	*2.956*	*2.736*
	$\widehat{ARL}$	—	—	—	4.696	5.584
	CI 95%	—	—	—	±0.055	±0.067
	UB_1ARL TARCH(1)	—	—	—	12.798	29.552
0.7	LB_1ARL TARCH(1)	—	—	—	—	*2.926*
	$\widehat{ARL}$	—	—	—	—	6.886
	CI 95%	—	—	—	—	±0.085
	UB_1ARL TARCH(1)	—	—	—	—	117.450

Table 4. In-control ARL of the modified Shewhart chart, with $c = 0.5$, for a TARCH(1) process (with $\alpha_0 = 1$ and $\sqrt{\frac{3}{2}}Z_t \sim t\,(6)$).

α_1 \ β_1		0	0.1	0.3	0.5	0.7	0.9
0	LB_1ARL TARCH(1)	1.777	1.771	1.765	1.765	1.770	1.781
	$\widehat{ARL}$	1.780	1.789	1.846	1.972	2.170	2.546
	CI 95%	±0.015	±0.015	±0.016	±0.018	±0.021	±0.026
	UB_1ARL TARCH(1)	1.777	1.827	1.964	2.179	2.551	3.307
0.1	LB_1ARL TARCH(1)	—	1.792	1.783	1.781	1.785	1.795
	$\widehat{ARL}$	—	1.793	1.871	1.977	2.201	2.604
	CI 95%	—	±0.015	±0.016	±0.018	±0.021	±0.027
	UB_1ARL TARCH(1)	—	1.881	2.031	2.269	2.685	3.549
0.3	LB_1ARL TARCH(1)	—	—	1.832	1.825	1.826	1.833
	$\widehat{ARL}$	—	—	1.941	2.051	2.288	2.713
	CI 95%	—	—	±0.018	±0.019	±0.023	±0.030
	UB_1ARL TARCH(1)	—	—	2.220	2.532	3.101	4.376
0.5	LB_1ARL TARCH(1)	—	—	—	1.892	1.887	*1.893*
	$\widehat{ARL}$	—	—	—	2.196	2.455	3.051
	CI 95%	—	—	—	±0.022	±0.026	±0.035
	UB_1ARL TARCH(1)	—	—	—	2.988	3.892	6.226
0.7	LB_1ARL TARCH(1)	—	—	—	—	1.983	*1.987*
	$\widehat{ARL}$	—	—	—	—	2.826	3.796
	CI 95%	—	—	—	—	±0.032	±0.046
	UB_1ARL TARCH(1)	—	—	—	—	5.703	11.924

TABLE 5. In-control ARL of the modified Shewhart chart, with $c = 1$, for a TARCH(1) process (with $\alpha_0 = 1$ and $\sqrt{\frac{3}{2}}Z_t \sim t\,(6)$).

α_1 \ β_1		0	0.1	0.3	0.5	0.7
0	LB_1ARL TARCH(1)	3.751	3.510	3.179	*2.964*	*2.816*
	$\widehat{ARL}$	3.731	3.851	4.107	4.621	5.477
	CI 95%	±0.039	±0.041	±0.045	±0.051	±0.063
	UB_1ARL TARCH(1)	3.751	4.007	4.759	6.078	8.718
0.1	LB_1ARL TARCH(1)	—	3.654	3.269	*3.025*	*2.859*
	$\widehat{ARL}$	—	3.887	4.177	4.682	5.499
	CI 95%	—	±0.042	±0.046	±0.053	±0.064
	UB_1ARL TARCH(1)	—	4.296	5.150	6.673	9.782
0.3	LB_1ARL TARCH(1)	—	—	*3.517*	*3.190*	*2.975*
	$\widehat{ARL}$	—	—	4.479	4.957	5.946
	CI 95%	—	—	±0.051	±0.057	±0.070
	UB_1ARL TARCH(1)	—	—	6.344	8.572	13.423
0.5	LB_1ARL TARCH(1)	—	—	—	*3.432*	*3.141*
	$\widehat{ARL}$	—	—	—	5.645	6.637
	CI 95%	—	—	—	±0.068	±0.081
	UB_1ARL TARCH(1)	—	—	—	12.385	21.650
0.7	LB_1ARL TARCH(1)	—	—	—	—	*3.385*
	$\widehat{ARL}$	—	—	—	—	8.102
	CI 95%	—	—	—	—	±0.101
	UB_1ARL TARCH(1)	—	—	—	—	45.744

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