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ON THE PROBABILISTIC STRUCTURE OF POWER THRESHOLD GENERALIZED ARCH STOCHASTIC PROCESSES

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Abstract: The aim of this paper is to develop a probabilistic study on a large and general class of conditionally heteroscedastic models, namely the $\delta-$TGARCH processes. For this class of processes we establish necessary and sufficient conditions of strict stationarity, ergodicity and existence of moments. A discussion on the weak stationarity up to the δ order as well as on the weak stationarity is also presented. Finally, the minimal representation of a $\delta-$ TGARCH process is obtained developing, in a unique way, the corresponding conditional moment of order δ in terms of present and past observations.

Keywords: Power TGARCH models, minimal representation, nonlinear time series, stationarity, ergodicity, stationarity up to order δ.
AMS Subject Classification (2000): 62G20, 62M10.

1. Introduction

Let $X = (X_t, t \in \mathbb{Z})$ be a real stochastic process and, for any $t \in \mathbb{Z}$, let us consider $X_t^+ = X_t \mathbb{I}_{\{X_t \geq 0\}}$, $X_t^- = -X_t \mathbb{I}_{\{X_t < 0\}}$ and $\underline{X}_t$ the $\sigma-$field generated by $(X_{t-i}, i \geq 0)$.

The stochastic process $X = (X_t, t \in \mathbb{Z})$ is said to follow a δ power threshold generalized autoregressive conditional heteroscedastic ($\delta-$TGARCH) model with orders p and q $(p, q \in \mathbb{N})$ if, for every $t \in \mathbb{Z}$, we have

$$\begin{cases} X_t = \sigma_t \varepsilon_t \\ \sigma_t^\delta = \omega + \sum\limits_{i=1}^{p} \left[\alpha_i \left(X_{t-i}^+\right)^\delta + \beta_i \left(X_{t-i}^-\right)^\delta \right] + \sum\limits_{j=1}^{q} \gamma_j \sigma_{t-j}^\delta \end{cases} \tag{1.1}$$

for some constants $\delta > 0$, $\omega > 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$, $i = 1, ..., p$, $\gamma_j \geq 0$, $j = 1, ..., q$, and where $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ is a sequence of independent and identically distributed real random variables such that ε_t is independent of $\underline{X}_{t-1}$, for every $t \in \mathbb{Z}$. The process ε is called the generator process of X.

If $\gamma_j = 0$, $j = 1, ..., q$, the $\delta-$TGARCH(p, q) model is simply denoted $\delta-$TARCH(p).

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Power conditional heteroscedastic models were proposed, among others, by Ding, Granger and Engle [6] arguing, in particular, that the introduction of the exponent δ allows long memory in the shocks of the conditional variance. Following this original idea, we consider here a natural extension of TGARCH processes that allows to take into account both long memory property and the asymmetry in the stochastic volatility. For these processes the TGARCH equation propagates not just conditional standard deviation but, more generally, absolute moments of order δ.

With this general formulation we include the principal conditional heteroscedastic models, namely:

1) GARCH (Engle [7], Bollerslev [3]): considering $\delta = 2$ and $\alpha_i = \beta_i$, $i = 1, ..., p$.

2) TGARCH (Zakoian [17]): considering $\delta = 1$.

3) $\delta-$GARCH, $\delta > 0$ (Mittnik, Paolella and Rachev [14]): considering $\alpha_i = \beta_i$, $i = 1, ..., p$. In fact, $\alpha_i \left[\left(X_{t-i}^{+} \right)^{\delta} + \left(X_{t-i}^{-} \right)^{\delta} \right] = \alpha_i \left[X_{t-i}^{+} + X_{t-i}^{-} \right]^{\delta} = \alpha_i \left| X_{t-i} \right|^{\delta}$.

4) APARCH (Ding, Granger and Engle [6]), considering $\alpha_i = a_i \left(1 - \tau_i\right)^{\delta}$ and $\beta_i = a_i \left(1 + \tau_i\right)^{\delta}$, where $a_i \geq 0$, $|\tau_i| \leq 1$, $i = 1, ..., p$.

This class of power-transformed and threshold GARCH models was introduced by Pan, Wang and Tong [16] with a slightly different parametrization and they called them PTTGARCH models. The parametrization here considered is a more natural one. These authors evaluate, in what concerns the probabilistic behaviour of the process X, the strict stationarity and the existence of moments. Their hypotheses are different from those here presented. In particular, the sufficient condition of strict stationarity and the condition on the moments existence are in our study clearly less demanding. Moreover, we use these results to establish the weak stationarity up to the $\delta-$order and we develop a study on the weak stationarity of a related vectorial model. We also point out that the error process here considered is quite general and not necessarily symmetrically distributed.

In order to simplify the presentation we consider in the following section $m = \max(p, q)$ and introduce $\alpha_i = \beta_i = 0$, $i = p + 1, ..., q$, if $q > p$, and

$\gamma_j = 0, j = q+1, ..., p$, if $q < p$. With this convention we have

$$\sigma_t^\delta = \omega + \sum_{i=1}^{m} \left[\alpha_i \left(X_{t-i}^+ \right)^\delta + \beta_i \left(X_{t-i}^- \right)^\delta + \gamma_i \sigma_{t-i}^\delta \right].$$

To develop the probabilistic study of these models and following the idea present in Mittnik, Paolella and Rachev [14], let us consider the vectorial stochastic process of $\mathbb{R}^m$, $Y = (Y_t, t \in \mathbb{Z})$, whose k-component, $Y_t^{(k)}$, has the following definition

$$\begin{cases} Y_t^{(1)} = \sigma_t^\delta \\ Y_t^{(k)} = \sum\limits_{i=k}^{m} \left[\alpha_i \left(X_{t-i+k-1}^+ \right)^\delta + \beta_i \left(X_{t-i+k-1}^- \right)^\delta + \gamma_i \sigma_{t-i+k-1}^\delta \right], \ k = 2, ..., m. \end{cases} \tag{1.2}$$

This process satisfies the recurrence equation

$$Y_{t+1} = A_t Y_t + B \tag{1.3}$$

where $(A_t, t \in \mathbb{Z})$ is a sequence of random square matrices of order m and B is a determinist vector of $\mathbb{R}^m$ given by

$$A_t = \begin{bmatrix} \sum\limits_{i=1}^{m-1} \left[\alpha_i \left(\varepsilon_t^+ \right)^\delta + \beta_i \left(\varepsilon_t^- \right)^\delta + \gamma_i \right] e_i & I_{m-1} \\ \alpha_m \left(\varepsilon_t^+ \right)^\delta + \beta_m \left(\varepsilon_t^- \right)^\delta + \gamma_m & 0_{m-1} \end{bmatrix},$$

$$B = \begin{bmatrix} \omega e_1 \\ 0 \end{bmatrix},$$

with $e_1, ..., e_{m-1}$ the canonical base of $\mathbb{R}^{m-1}$, I_{m-1} the identity matrix of $m-1$ order and 0_{m-1} the null vector of $\mathbb{R}^{m-1}$.

As we assume that $(\varepsilon_t, t \in \mathbb{Z})$ are independent and identically distributed random variables, the random matrices $(A_t, t \in \mathbb{Z})$ are also independent and identically distributed.

The vectorial representation here obtained is different from that considered in Pan, Wang and Tong [16] and apart its writing simplicity it has the advantage of being valid for every orders and particular cases as those considered in the Corollary of Theorems 1 and 2 in the next Section.

Considering a $\delta-$TGARCH model with general orders p and q we establish, in Section 2, a sufficient condition of strict stationarity and ergodicity. Under such condition, we explicit the unique strict stationary and ergodic solution.

Other sufficient or necessary conditions are also established in general or in some particular cases. In addition, in Section 3, we state a necessary and sufficient condition of weak stationarity up to the order δ which is also a simpler, and useful in practice, condition of strict stationarity. Several examples illustrate our results.

A discussion of the weak stationarity of this model is developed in Section 4. Finally in Section 5 we establish a minimal definition of these models. This canonical definition is a consequence of the representation obtained for σ_t^δ as a sum of functions of present and past observations.

2. $\delta-$TGARCH processes: strict stationarity

Let us consider any norm $\|.\|$ on the set $\mathbb{M}$ of the square matrices of order m and the following hypothesis on the matrices $(A_t, t \in \mathbb{Z})$:

(**H1**): The sequence $\left(\frac{1}{n}\log\|A_0...A_{-n}\|\right)_{n\in\mathbb{N}}$ converges almost surely $(a.s.)$ to a strictly negative constant γ.

The existence of a stationary solution for the $\delta-$TGARCH model is stated in the next theorem.

Theorem 1. *If the sequence* $\left(\frac{1}{n}\log\|A_0...A_{-n}\|\right)_{n\in\mathbb{N}}$ *satisfies the hypothesis* (**H1**)*, there exists a unique strictly stationary and ergodic solution,* $(X_t, t \in \mathbb{Z})$, *of the* $\delta-$*TGARCH model (1.1).*

Proof. Under the hypothesis (**H1**),

1) the vectorial process

$$Z_t = B + \lim_i\ (a.s.) \sum_{n=1}^{i} A_{t-1}...A_{t-n}B, t \in \mathbb{Z}$$

is well defined $(a.s.)$. In fact, it is easy to show that $\lim_n\ (a.s.)\ \|A_{t-1}...A_{t-n}\|^{\frac{1}{n}} = e^\gamma < 1$ which implies the $(a.s.)$ convergence of the series $\sum_{n=1}^{+\infty}\|A_{t-1}...A_{t-n}B\|$.

2) the process $(Z_t,\ t \in \mathbb{Z})$ is a strictly stationary solution of equation (1.3). In fact, $Z_{t+1} = B + A_tZ_t$ and, as Z_t is a measurable function of the strictly stationary and ergodic sequence $(A_t, t \in \mathbb{Z})$, the process $(Z_t,\ t \in \mathbb{Z})$ is strictly stationary and ergodic.

3) the process $(Z_t,\ t \in \mathbb{Z})$ is the unique strictly stationary solution of the equation (1.3). In fact, for another stationary solution U_t of this equation we deduce that

$$U_t = A_{t-1}...A_{t-i-1}U_{t-i-1} + \sum_{n=1}^{i} A_{t-1}...A_{t-n}B + B.$$

So, for every $i \in \mathbb{N}$,

$$\|Z_t - U_t\| \leq \|A_{t-1}...A_{t-i-1}\| \left(\|Z_{t-i-1}\| + \|U_{t-i-1}\|\right)$$

and

$$P\left(\|Z_t - U_t\| > \epsilon\right) \leq P\left(\|A_{t+i}...A_t\| \|Z_t\| > \frac{\epsilon}{2}\right) + P\left(\|A_{t+i}...A_t\| \|U_t\| > \frac{\epsilon}{2}\right)$$

taking into account that the laws of $(A_{t-1}, ..., A_{t-i-1}, Z_{t-i-1})$ and $(A_{t-1}, ..., A_{t-i-1}, U_{t-i-1})$ are independent of t since $Z_{t-i-1} = f\left(A_{t-i-2}, A_{t-i-3}, ...\right)$ and $U_{t-i-1} = g\left(A_{t-i-2}, A_{t-i-3}, ...\right).$

Let us consider a positive real η, arbitrarily fixed, and choose reals m_1 and m_2 such that $P\left(\|Z_t\| > m_1\right) < \frac{\eta}{4}$ and $P\left(\|U_t\| > m_2\right) < \frac{\eta}{4}$. We have

$$\begin{aligned} P\left(\|Z_t - U_t\| > \epsilon\right) \leq P\left(\|A_{t+i}...A_t\| > \frac{\epsilon}{2m_1}\right) + P\left(\|U_t\| > m_1\right) + \\ + P\left(\|A_{t+i}...A_t\| > \frac{\epsilon}{2m_2}\right) + P\left(\|U_t\| > m_2\right). \end{aligned}$$

But under the hypothesis (**H1**) $\lim_i (a.s.) \|A_{t+i}...A_t\| = 0$, which leads to the unique representation

$$U_t = \lim_i (a.s.) \sum_{n=1}^{i} A_{t-1}...A_{t-n}B + B = Z_t.$$

4) Let $Z_t^{(i)}$ denotes the Z_t order i component, $1 \leq i \leq m$. The real stochastic process given by $V_t = \varepsilon_t \left[Z_t^{(1)}\right]^{\frac{1}{\delta}}$ is a solution of the model (1.1). In fact, using

the recurrence equation (1.3) we get

$$\begin{aligned}
Z_t^{(1)} &= \left[\alpha_1\left(\varepsilon_{t-1}^{+}\right)^{\delta}+\beta_1\left(\varepsilon_{t-1}^{-}\right)^{\delta}+\gamma_1\right]Z_{t-1}^{(1)}+Z_{t-1}^{(2)}+\omega\\
Z_{t-1}^{(2)} &= \left[\alpha_2\left(\varepsilon_{t-2}^{+}\right)^{\delta}+\beta_2\left(\varepsilon_{t-2}^{-}\right)^{\delta}+\gamma_2\right]Z_{t-2}^{(1)}+Z_{t-2}^{(3)}\\
&\dots\\
Z_{t-m+2}^{(m-1)} &= \left[\alpha_{m-1}\left(\varepsilon_{t-m+1}^{+}\right)^{\delta}+\beta_{m-1}\left(\varepsilon_{t-m+1}^{-}\right)^{\delta}+\gamma_{m-1}\right]Z_{t-m+1}^{(1)}+Z_{t-m+1}^{(m)}\\
Z_{t-m+1}^{(m)} &= \left[\alpha_m\left(\varepsilon_{t-m}^{+}\right)^{\delta}+\beta_m\left(\varepsilon_{t-m}^{-}\right)^{\delta}+\gamma_m\right]Z_{t-m}^{(1)}.
\end{aligned}$$

So,

$$\begin{aligned}
Z_t^{(1)} &= \omega+\sum_{i=1}^{m}\left[\alpha_i\left(\varepsilon_{t-i}^{+}\right)^{\delta}+\beta_i\left(\varepsilon_{t-i}^{-}\right)^{\delta}+\gamma_i\right]Z_{t-i}^{(1)}\\
&= \omega+\sum_{i=1}^{m}\left\{\alpha_i\left(V_{t-i}^{+}\right)^{\delta}+\beta_i\left(V_{t-i}^{-}\right)^{\delta}\right\}+\sum_{i=1}^{m}\gamma_i Z_{t-i}^{(1)}.
\end{aligned}$$

Moreover, $V_t=\varepsilon_t\left[Z_t^{(1)}\right]^{\frac{1}{\delta}}$ is a measurable function of the strictly stationary process ε and so V is strictly stationary.

The unicity of this strictly stationary solution of the model (1.1), V, follows from the unicity of the strictly stationary solution of equation (1.3).

Finally, as V_t is a measurable function of $(\varepsilon_t,\varepsilon_{t-1},...)$ and as ε is a strictly stationary and ergodic process, we also have the ergodicity of $(V_t, t\in\mathbb{Z})$. ∎

Theorem 2. *Let $E\left(\log^{+}\|A_0\|\right)<+\infty$. If there exists a unique strictly stationary and ergodic solution, $(X_t, t\in\mathbb{Z})$, of the $\delta-$TGARCH model (1.1) then the sequence $\left(\frac{1}{n}\log\|A_0...A_{-n}\|\right)_{n\in\mathbb{N}}$ satisfies the hypothesis* (**H1**).

Proof. Let us now assume that there exists a (unique strictly stationary and ergodic) solution, $(X_t,\ t\in\mathbb{Z})$, of the $\delta-$TGARCH model (1.1). Denote by Y the corresponding solution of equation (1.3). We have

$$Y_0=A_{-1}...A_{-i-1}Y_{-i-1}+\sum_{n=1}^{i}A_{-1}...A_{-n}B+B.$$

As all the coefficients of the vector B and matrices A_t are nonnegative, we can write for every $i \in \mathbb{N}$

$$\sum_{n=1}^{i} A_{-1}...A_{-n}B \leq Y_0,$$

where, for x, y in $\mathbb{R}^m$, $x \leq y$ means $y - x \in (\mathbb{R}^+)^m$.

This shows that the series $\sum\limits_{n=1}^{+\infty} A_{-1}...A_{-n}B$ converges $(a.s.)$. So, $\lim\limits_n \|A_{-1}...A_{-n}B\| = 0$ $(a.s.)$ which implies $\lim\limits_n \|A_{-1}...A_{-n}f_i\| = 0$ $(a.s.)$, $1 \leq i \leq m$, as $B = wf_1, ..., A_{-n}f_i = f_{i-1}, 2 \leq i \leq m-1$ and $A_{-n}f_m = 0$, where $f_1, ..., f_m$ is the canonical basis of $\mathbb{R}^m$.

So, for any vector U of $\mathbb{R}^m$

$$\lim_n A_{-1}...A_{-n}U = 0 \quad (a.s.)$$

and finally $\lim\limits_n \|A_{-1}...A_{-n}\| = 0$ $(a.s.)$ which implies $\gamma < 0$ (Bougerol and Picard [4, Lema 2.1]). ■

We recall that, considering any norm $\|.\|$ on $\mathbb{R}^m$ and defining an operator norm on the set $\mathbb{M}$ of the square matrices of order m by

$$\|M\| = \sup\left\{\|Mx\| / \|x\|, x \in \mathbb{R}^m, x \neq 0\right\}$$

for any M in $\mathbb{M}$, the top Lyapunov exponent associated to a sequence $(A_t,\ t \in \mathbb{Z})$ of independent, identically distributed random matrices and such that $E\left(\log^+ \|A_0\|\right)$ is finite, is defined by

$$\gamma_L = \inf\left\{E\left(\frac{1}{n+1}\log\|A_0A_{-1}...A_{-n}\|\right), n \in \mathbb{N}\right\}.$$

From Kingman [13, Theorem 6] it follows that, almost surely,

$$\gamma_L = \lim_{n \to +\infty} \frac{1}{n}\log\|A_0...A_{-n}\|$$

and, since all the norms are equivalent on $\mathbb{M}$, γ_L is independent of the norm. Moreover, from Bougerol and Picard [4, Lema 2.1] if, almost surely,

$$\lim_{n \to +\infty} \|A_0...A_{-n}\| = 0$$

then the top Lyapunov exponent associated to the sequence $(A_t,\ t \in \mathbb{Z})$ is strictly negative.

So, if the sequence of matrices $(A_t, t \in Z)$ related to the $\delta-$TGARCH model satisfies $E\left(\log^+\|A_0\|\right) < +\infty$, the (**H1**) hypothesis is a necessary and sufficient condition for the existence and unicity of a strictly stationary and ergodic solution of model (1.1).

The next corollary states a necessary and sufficient condition of strict stationarity when we are in presence of a $\delta-$TGARCH model of order m, $m \in \mathbb{N}$, with a particular form for σ_t^δ.

Corollary. *If* $E\left\{\log\left[\alpha_m\left(\varepsilon_t^+\right)^\delta + \beta_m\left(\varepsilon_t^-\right)^\delta + \gamma_m\right]\right\}$ *exists then a necessary and sufficient condition for the existence of a unique strictly stationary solution of the* $\delta-$*TGARCH model of order* m *where* $\sigma_t^\delta = \omega + \alpha_m\left(X_{t-m}^+\right)^\delta + \beta_m\left(X_{t-m}^-\right)^\delta + \gamma_m\sigma_{t-m}^\delta$ *is*

$$E\left\{\log\left[\alpha_m\left(\varepsilon_t^+\right)^\delta + \beta_m\left(\varepsilon_t^-\right)^\delta + \gamma_m\right]\right\} < 0.$$

Proof. Taking into account the particular form of the matrices $(A_t, t \in \mathbb{Z})$ in this case, it can be seen that the product of m consecutive matrices is a diagonal one. For example, $A_0...A_{m-1} = D_{m-1,0}$ with element

$$d_{ii} = \left[\alpha_m\left(\varepsilon_{m-i}^+\right)^\delta + \beta_m\left(\varepsilon_{m-i}^-\right)^\delta + \gamma_m\right],\ i = 1, ..., m.$$

So, taking groups of m consecutive matrices and $k = \left\lfloor \frac{n}{m} \right\rfloor$, with $\lfloor x \rfloor$ denoting the integral part of x, we have

$$\begin{aligned} A_0...A_n &= A_0...A_{m-1}A_m...A_{2m-1}A_{2m}...A_{3m-1}... ...A_{(k-1)m}...A_{km-1}A_{km}...A_n \\ &= D_{m-1,0}D_{2m-1,m}...D_{km-1,(k-1)m}\ \ A_{km}...A_n. \end{aligned}$$

Firstly, we note that $D_{m-1,0}D_{2m-1,m}...D_{km-1,(k-1)m}$ is also a diagonal matrix

$$D = \begin{bmatrix} a_{m-1} & 0 & \ldots & 0 & 0 \\ 0 & a_{m-2} & \ldots & 0 & 0 \\ ... & ... & \ddots & ... & ... \\ 0 & 0 & ... & a_1 & 0 \\ 0 & 0 & ... & 0 & a_0 \end{bmatrix}$$

where

$$a_i = \prod_{j=0}^{k-1}\left[\alpha_m\left(\varepsilon_{i+jm}^+\right)^\delta + \beta_m\left(\varepsilon_{i+jm}^-\right)^\delta + \gamma_m\right],\ i = 0, ..., m-1.$$

So,

$$\begin{aligned}\frac{1}{n}\log\|A_0...A_n\| &= \frac{1}{n}\log\left\|D_{m-1,0}D_{2m-1,m}...D_{km-1,(k-1)m}\quad A_{km}...A_n\right\| \\ &\leq \frac{m}{n}\log\|D\| + \frac{1}{n}\log\|A_{km}...A_n\|.\end{aligned}$$

Considering $\|D\| = \left(\text{maximum proper value of } D^TD\right)^{\frac{1}{2}}$, we have for example

$$\|D\| = \prod_{j=0}^{k-1}\left[\alpha_m\left(\varepsilon^+_{i+jm}\right)^\delta + \beta_m\left(\varepsilon^-_{i+jm}\right)^\delta + \gamma_m\right].$$

So,

$$\frac{m}{n}\log\|D\| = \frac{km}{n}\frac{1}{k}\sum_{j=0}^{k-1}\log\left[\alpha_m\left(\varepsilon^+_{i+jm}\right)^\delta + \beta_m\left(\varepsilon^-_{i+jm}\right)^\delta + \gamma_m\right]$$

which converges $(a.s.)$ to $E\left\{\log\left[\alpha_m\left(\varepsilon^+_{i+jm}\right)^\delta + \beta_m\left(\varepsilon^-_{i+jm}\right)^\delta + \gamma_m\right]\right\}$ as $n \to +\infty$.

Moreover, as $n = km + j$, with $0 \leq j \leq m-1$,

$$\frac{1}{n}\log\|A_{km}...A_n\| \leq \frac{1}{n}\sum_{i=0}^{j}\log\|A_{km+i}\|.$$

So, $\frac{1}{n}\log\|A_{km}...A_n\|$ converges (a.s.) to zero as $n \to +\infty$.

We have proved that if $E\left\{\log\left[\alpha_m\left(\varepsilon^+_{i+jm}\right)^\delta + \beta_m\left(\varepsilon^-_{i+jm}\right)^\delta + \gamma_m\right]\right\} = a < 0$ then $\frac{1}{n}\log\|A_0...A_n\|$ converges (a.s.) to $\gamma < 0$ as $n \to +\infty$. So, the strict stationary solution exists by Theorem 1.

Conversely, it is enough to observe that if $\frac{1}{n}\log\|A_0\ ...\ A_n\|$ converges $(a.s.)$ to $\gamma < 0$ as $n \to +\infty$ then $\frac{1}{km}\log\|A_0...A_{km}\|$ converges $(a.s.)$ to $\gamma < 0$ as $k \to +\infty$. So, by the previous calculations $E\left\{\log\left[\alpha_m\left(\varepsilon^+_{i+jm}\right)^\delta + \beta_m\left(\varepsilon^-_{i+jm}\right)^\delta + \gamma_m\right]\right\} < 0$. ■

In the next examples we illustrate the previous results considering $\delta-$TGARCH models with a generator process ε following the standard Cauchy law, that is, with density $f(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$.

Example 1. Under the conditions of the last Corollary, if we consider $\delta = 2$ we obtain

$$E\left\{\log\left[\alpha_m\left(\varepsilon_t^+\right)^2+\beta_m\left(\varepsilon_t^-\right)^2+\gamma_m\right]\right\}=\log\left[\left(\sqrt{\alpha_m}+\sqrt{\gamma_m}\right)\left(\sqrt{\beta_m}+\sqrt{\gamma_m}\right)\right].$$

So, there exists a unique strictly stationary solution X if and only if $(\alpha_m, \beta_m, \gamma_m)$ belongs to the set

$$\left\{(\alpha,\beta,\gamma)\in\,]0,+\infty[^3:\gamma<\left[\sqrt{\left(\sqrt{\alpha}-\sqrt{\beta}\right)^2+4}-\left(\sqrt{\alpha}+\sqrt{\beta}\right)\right]^2\right\}.$$

The frontier of this region is depicted in the Figure 1.

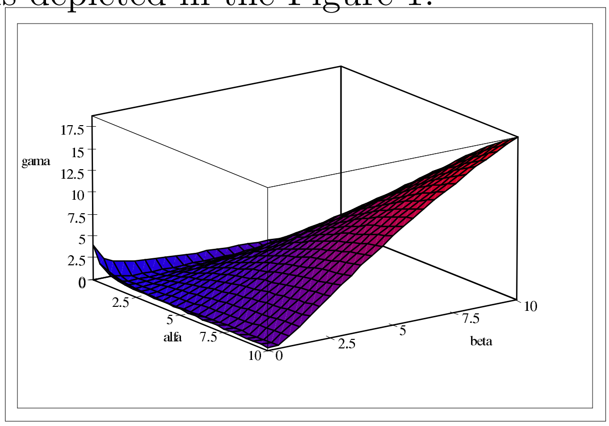

Figure 1. Region of strict stationarity of a Cauchy 2−TGARCH model.

Example 2. Under the conditions of the last Corollary, if we consider now $\gamma_m = 0$, that is a $\delta-$TARCH model, we obtain

$$E\left\{\log\left[\alpha_m\left(\varepsilon_t^+\right)^\delta+\beta_m\left(\varepsilon_t^-\right)^\delta\right]\right\}=\frac{1}{2}\log\left(\alpha_m\beta_m\right)$$

using the fact that $\int\limits_0^{+\infty}\frac{\log x}{1+x^2}dx=0$ (Gradshteyn and Ryzhik [11, p. 564]).

A necessary and sufficient condition of strict stationarity for X is then $\alpha_m\beta_m < 1$, which is independent of the parameter δ. The frontier of this region is depicted in the Figure 2.

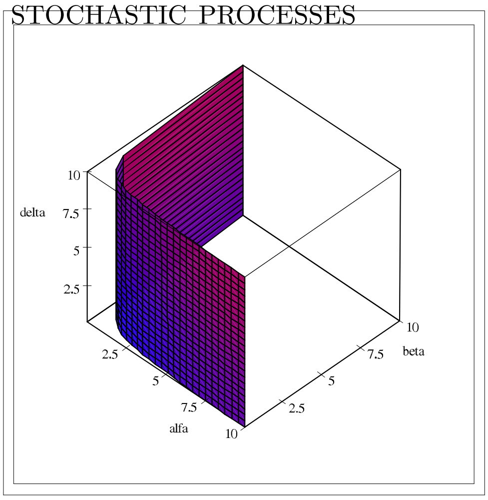

FIGURE 2. Region of strict stationarity of a Cauchy $\delta-$ TARCH model.

A necessary condition for the existence of a strictly stationary solution of model (1.1) is now stated.

Theorem 3. *Let* $E\left(\log^{+}\|A_0\|\right) < +\infty$. *If the* $\delta-$*TGARCH model presented in (1.1) has a strictly stationary solution then*

$$\sum_{j=1}^{q} \gamma_j < 1.$$

Proof. Let A be the matrix which is obtained replacing ε_t^{+} and ε_t^{-} by 0 in the matrix $A_t, t \in \mathbb{Z}$.

Since, for each $k \in \mathbb{Z}$, $A_k \geq A$, we have $A_0 ... A_{-n} \geq A^{n+1}$. So, the Lyapunov exponent associated with the sequence of matrices $(A_t,\ t \in \mathbb{Z})$ satisfies $\gamma_L \geq \log\|A\|$.

As $\|A\| \geq \rho(A)$, where $\rho(A)$ is the spectral radius of matrix A, and $\gamma_L < 0$, we deduce that $\rho(A) < 1$. Moreover, we have

$$\begin{aligned}
\det(zI_m - A) &= \det \begin{bmatrix} z-\gamma_1 & -1 & 0 & \ldots & 0 \\ -\gamma_2 & z & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma_{m-1} & 0 & 0 & \ldots & -1 \\ -\gamma_m & 0 & 0 & \ldots & z \end{bmatrix} \\
&= z^m \left(1 - \sum_{i=1}^{m} \gamma_i z^{-i}\right) = z^m \left(1 - \sum_{i=1}^{q} \gamma_i z^{-i}\right).
\end{aligned}$$

We conclude that the continuous function $f(x) = 1 - \sum_{j=1}^{q} \gamma_j x^j$ has no zeros when $0 \leq x \leq 1$. Since $f(0) = 1$, this implies that $f(1) = 1 - \sum_{j=1}^{q} \gamma_j > 0$, as required. ■

3. $\delta-$TGARCH processes: stationarity up to the δ order and relation with strict stationarity

We analyze now the existence of the order δ moment of the process X that is solution of model (1.1). Let us suppose that

(**H2**): $E\left(|\varepsilon_t|^\delta\right) < +\infty$ and $P\left(\varepsilon_t = 0\right) \neq 1$.

We denote $E\left(|\varepsilon_t|^\delta\right) = \phi_\delta$, $E\left[\left(\varepsilon_t^+\right)^\delta\right] = \phi_{1,\delta}$ and $E\left[\left(\varepsilon_t^-\right)^\delta\right] = \phi_{2,\delta}$. As $\sigma_t > 0$,

$$E\left[\left(X_t^+\right)^\delta |\underline{X}_{t-1}\right] = \sigma_t^\delta E\left[\left(\varepsilon_t^+\right)^\delta |\underline{X}_{t-1}\right] = \sigma_t^\delta \phi_{1,\delta}$$

and formally we have $E\left[\left(X_t^+\right)^\delta\right] = E\left(\sigma_t^\delta\right)\phi_{1,\delta}$. Analogously, $E\left[\left(X_t^-\right)^\delta\right] = E\left(\sigma_t^\delta\right)\phi_{2,\delta}$.

Theorem 4. *Under* (**H2**), $E\left(|X_t|^\delta\right)$ *exists and is independent of t if and only if*

$$\sum_{i=1}^{m} \left(\alpha_i \phi_{1,\delta} + \beta_i \phi_{2,\delta} + \gamma_i\right) < 1. \tag{3.1}$$

Proof. Let us assume that $\sum_{i=1}^{m} \left(\alpha_i \phi_{1,\delta} + \beta_i \phi_{2,\delta} + \gamma_i\right) < 1$. Taking into account the integration properties of positive measurable functions, we get

$$E\left(\sigma_t^\delta\right) = \omega + \sum_{i=1}^{m} \left(\alpha_i \phi_{1,\delta} + \beta_i \phi_{2,\delta} + \gamma_i\right) E\left(\sigma_{t-i}^\delta\right).$$

Considering the polynomial $\alpha(L) = 1 - \sum_{i=1}^{m} \left(\alpha_i \phi_{1,\delta} + \beta_i \phi_{2,\delta} + \gamma_i\right) L^i$ associated to the previous recurrence equation, we conclude that this equation has a solution, $E\left(\sigma_t^{\delta}\right)$, that is independent of t if the roots of $\alpha(L)$ are outside the unit circle, which happens if and only if $\sum_{i=1}^{m}\left(\alpha_i\phi_{1,\delta}+\beta_i\phi_{2,\delta}+\gamma_i\right)<1$.

We assume now that $E\left(|X_t|^{\delta}\right)$ exists and is independent of t. Taking into account the definition of σ_t^{δ} we have, as $E\left(|X_t|^{\delta}\right)=\phi_\delta E\left(\sigma_t^{\delta}\right)$,

$$
\begin{aligned}
E\left(\sigma_t^{\delta}\right) \quad &= \quad \omega+\sum_{i=1}^{m}\left\{\alpha_i E\left[\left(X_{t-i}^{+}\right)^{\delta}\right]+\beta_i E\left[\left(X_{t-i}^{-}\right)^{\delta}+\gamma_i E\left(\sigma_{t-i}^{\delta}\right)\right]\right\}\\
&\iff \; E\left(\sigma_t^{\delta}\right)=\omega+\sum_{i=1}^{m}\left\{\alpha_i E\left(\sigma_{t-i}^{\delta}\right)\phi_{1,\delta}+\beta_i E\left(\sigma_{t-i}^{\delta}\right)\phi_{2,\delta}+\gamma_i E\left(\sigma_{t-i}^{\delta}\right)\right\}\\
&\iff \; \left[1-\sum_{i=1}^{m}\left\{\alpha_i\phi_{1,\delta}+\beta_i\phi_{2,\delta}+\gamma_i\right\}\right]E\left(|X_t|^{\delta}\right)=\omega.
\end{aligned}
$$

The positivity of $E\left(|X_t|^{\delta}\right)$ and ω implies $1-\sum_{i=1}^{m}\left(\alpha_i\phi_{1,\delta}+\beta_i\phi_{2,\delta}+\gamma_i\right)>0$. ■

Corollary. *If ε satisfy the hypothesis* (**H2**) *and* $\sum_{i=1}^{m}\left(\alpha_i\phi_{1,\delta}+\beta_i\phi_{2,\delta}+\gamma_i\right)<1$ *then $E\left(|X_t|^{\delta}\right)$ exists and is independent of t and we have*

$$
E\left(|X_t|^{\delta}\right)=\frac{\omega\ \phi_\delta}{1-\sum_{i=1}^{m}\left(\alpha_i\phi_{1,\delta}+\beta_i\phi_{2,\delta}+\gamma_i\right)}.
$$

We note that for $\delta\in\mathbb{N}$ or $\delta=\frac{a}{2b+1}, a\in\mathbb{N}, b\in\mathbb{N}$, and under the previous conditions, we can write the δ order moment of X_t in terms of those of ε_t^{δ}, $\left(\varepsilon_t^{+}\right)^{\delta}$ and $\left(\varepsilon_t^{-}\right)^{\delta}$:

$$
E\left(X_t^{\delta}\right)=\frac{\omega\ E\left(\varepsilon_t^{\delta}\right)}{1-\sum_{i=1}^{m}\left\{\alpha_i E\left[\left(\varepsilon_t^{+}\right)^{\delta}\right]+\beta_i E\left[\left(\varepsilon_t^{-}\right)^{\delta}\right]+\gamma_i\right\}}.
$$

Let us show now that the condition (3.1) is also a sufficient condition of strict stationarity.

We point out that the hypothesis (**H2**) implies $E\left(\log^+ \|A_0\|\right) < +\infty$. In fact, $E\left(\log^+ \|A_0\|\right) \leq E\left(\|A_0\| I_{\|A_0\|>1}\right) \leq E\left(\|A_0\|\right)$ which is finite if $E\left(|\varepsilon_t|^\delta\right) < +\infty$.

Theorem 5. *The $\delta-$TGARCH process X satisfying the model (1.1), the hypothesis* (**H2**) *and* $\sum\limits_{i=1}^{m} (\alpha_i \phi_{1,\delta} + \beta_i \phi_{2,\delta} + \gamma_i) < 1$ *is strictly stationary.*

Proof. Let us consider the matrix A_1. Denote by $A_1(i)$ the sub-matrix of A_1 with its first i lines and columns. We have

$$\begin{aligned}
&\det\left(zI_m - E\left(A_1(m)\right)\right) \\
&\quad = \det \begin{bmatrix} z - \alpha_1\phi_{1,\delta} - \beta_1\phi_{2,\delta} - \gamma_1 & -1 & 0 & \ldots & 0 \\ -\alpha_2\phi_{1,\delta} - \beta_2\phi_{2,\delta} - \gamma_2 & z & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_{m-1}\phi_{1,\delta} - \beta_{m-1}\phi_{2,\delta} - \gamma_{m-1} & 0 & 0 & \ldots & -1 \\ -\alpha_m\phi_{1,\delta} - \beta_m\phi_{2,\delta} - \gamma_m & 0 & 0 & \ldots & z \end{bmatrix} \\
&\quad = z \det\left(zI_{m-1} - E\left(A_1(m-1)\right)\right) + \left(-\alpha_m\phi_{1,\delta} - \beta_m\phi_{2,\delta} - \gamma_m\right)(-1)^{m+1}(-1)^{m-1} \\
&\quad = z^2 \det\left(zI_{m-2} - E\left(A_1(m-2)\right)\right) - z\left(\alpha_{m-1}\phi_{1,\delta} + \beta_{m-1}\phi_{2,\delta} + \gamma_{m-1}\right) - \\
&\qquad\qquad - \left(\alpha_m\phi_{1,\delta} + \beta_m\phi_{2,\delta} + \gamma_m\right) \\
&\quad = \ldots \\
&\quad = z^m \left(1 - \sum_{i=1}^{m} \left(\alpha_i\phi_{1,\delta} + \beta_i\phi_{2,\delta} + \gamma_i\right) z^{-i}\right)
\end{aligned}$$

Then

$$\left|\det\left(zI_m - E\left(A_1\right)\right)\right| \geq |z^m| \left[1 - \sum_{i=1}^{m} \left(\alpha_i\phi_{1,\delta} + \beta_i\phi_{2,\delta} + \gamma_i\right)\left|z^{-i}\right|\right].$$

We deduce that if $|z| > 1$ then $\left|\det\left(zI_m - E\left(A_1\right)\right)\right| > 1 - \sum\limits_{i=1}^{m} \left(\alpha_i\phi_{1,\delta} + \beta_i\phi_{2,\delta} + \gamma_i\right)$ and, if $|z| = 1$, then $\left|\det\left(zI_m - E\left(A_1\right)\right)\right| \geq 1 - \sum\limits_{i=1}^{m} \left(\alpha_i\phi_{1,\delta} + \beta_i\phi_{2,\delta} + \gamma_i\right).$

As, by hypothesis, $1-\sum_{i=1}^{m}(\alpha_i\phi_{1,\delta}+\beta_i\phi_{2,\delta}+\gamma_i)>0$, there is no proper value of $E(A_1)$, z, such that $|z|\geq 1$. Consequently, the spectral radius ρ of the matrix $E(A_1)$ satisfies $\rho<1$. But, as Kesten and Spitzer [12, (1.4)] pointed out, it is always true that $\gamma_L\leq\log\rho$. So, $\gamma_L<0$ and the $\delta-$TGARCH model presented in (1.1) has a unique strictly stationary and ergodic solution. ■

As a consequence of the previous theorem and Holder inequality, we deduce now the weak stationarity up to the δ order of the process X under the necessary and sufficient condition of existence of its moment of δ order.

We remember that $(X_t,\ t\in\mathbb{Z})$ is weak stationary up to the δ order if all the joint moments of $(X_{t_1},...,\ X_{t_n})$ of order less or equal to δ exist and are equal to the corresponding joint moments of $(X_{t_1+h},...,\ X_{t_n+h})$, $h\in\mathbb{Z}$, that is,

$$E\left[(X_{t_1})^{j_1}...(X_{t_n})^{j_n}\right]=E\left[(X_{t_1+h})^{j_1}...(X_{t_n+h})^{j_n}\right]$$

with $j_1\geq 0,...,j_n\geq 0$, $j_1+...+j_n\leq\delta$, $(t_1,...t_n)\in\mathbb{Z}^n$, $h\in\mathbb{Z}$.

Corollary. *Let X be a stochastic process satisfying a $\delta-TGARCH\ (p,q)$ model with generator process $(\varepsilon_t,t\in\mathbb{Z})$ under condition* (**H2**)*. X is weak stationary up to the order δ if and only if $\sum_{i=1}^{m}(\alpha_i\phi_{1,\delta}+\beta_i\phi_{2,\delta}+\gamma_i)<1$.*

Proof. Under the necessary and sufficient condition of existence of the moment of order δ, the process X is strictly stationary. So, we only have to ensure the existence of those expectations.

The generalized Holder's inequality for positive exponents (*) gives

$$E\left[|X_{t_1}|^{j_1}...|X_{t_n}|^{j_n}\right]=\int\prod_{i=1}^{n}|X_{t_i}|^{j_i}\,dP\leq\prod_{i=1}^{n}\left[\int|X_{t_i}|^{\delta}\,dP\right]^{\frac{j_i}{\delta}}$$

$$=\prod_{i=1}^{n}\left[E\left(|X_{t_i}|^{\delta}\right)\right]^{\frac{j_i}{\delta}}<+\infty$$

*Holder inequality

Let $p_1,...,p_m$ in $]0,+\infty[$ be such that $\sum_{i=1}^{m}\frac{1}{p_i}=1$. Consider $f_i\in L_{p_i}(\Omega,A,\mu)$, $i=1,...,m$.

Then $\prod_{i=1}^{m}f_i\in L(\Omega,A,\mu)$ and $\int\prod_{i=1}^{m}|f_i|\,d\mu\leq\prod_{i=1}^{m}\left(\int|f_i|^{p_i}\,d\mu\right)^{\frac{1}{p_i}}$.

as $E\left(|X_{t_i}|^\delta\right)$ exists. ∎

Example 3. Let us consider again the particular Cauchy $\delta-$TARCH process studied in the example 2. For $\delta < 1$, $E\left(|\varepsilon_t|^\delta\right)$ exists and $E\left(|\varepsilon_t|^\delta\right) = \frac{1}{\sin\left(\frac{\delta+1}{2}\pi\right)}$ (Gradshteyn and Ryzhik [11, p. 340]). So, a necessary and sufficient condition for the existence of $E\left(|X_t|^\delta\right)$ is $(\alpha_m + \beta_m)\frac{1}{2\sin\left(\frac{\delta+1}{2}\pi\right)} < 1$.

The regions of strict and up to the $\delta-$order weak stationarity of the X process are depicted in the Figure 3.

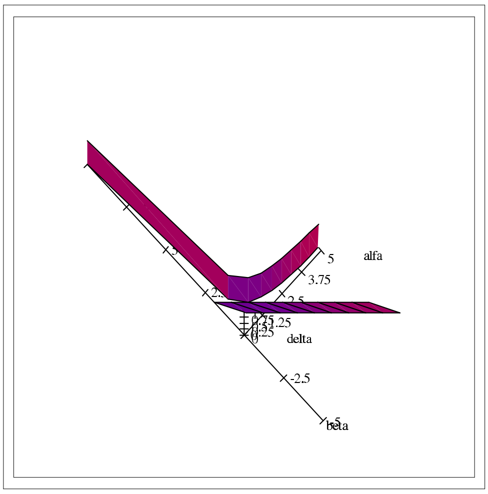

FIGURE 3. Regions of strict and up to $\delta-$ order weak stationarity of a Cauchy $\delta-$TARCH model.

In order to illustrate last results, in the next examples we take again the $\delta-$TARCH model with $\sigma_t^\delta = \omega + \alpha_m \left(X_{t-m}^+\right)^\delta + \beta_m \left(X_{t-m}^-\right)^\delta, m \in \mathbb{N}$, and consider several distributions for the generator process.

Example 4. Let us consider that the generator process ε follows the standard Gaussian law. There is a strictly and stationary solution X if and only if $\log(\alpha_m\beta_m) - \delta(c + \log 2) < 0$, where c is the Euler constant (c≃0.577215). Moreover, a necessary and sufficient condition for the existence of $E\left(|X_t|^\delta\right)$ is $(\alpha_m + \beta_m)\frac{\Gamma(\delta+1)}{2^{\frac{\delta}{2}}(2+\delta)} < 1$.

These results follow easily from the relations $\int\limits_0^{+\infty} \log(x)\exp\left(-\frac{x^2}{2}\right)dx = -\frac{\sqrt{2\pi}}{4}(c+\log 2)$ and $\int\limits_0^{+\infty} x^\delta \exp\left(-\frac{x^2}{2}\right)dx = \Gamma(\delta+1)\frac{2^{\frac{-\delta+1}{2}}}{2+\delta}\sqrt{\pi}$ (Gradshteyn and Ryzhik [11, pp. 602 and 382, resp.]).

Example 5. If the generator process ε follows the Laplace law, that is, with density $f(x) = \frac{1}{2}\exp(-|x|)$, $x \in \mathbb{R}$, then taking into account the relations $\int\limits_0^{+\infty} \log(x)\exp(-x)\,dx = -c$ and $\int\limits_0^{+\infty} x^\delta \exp(-x)\,dx = \Gamma(\delta+1)$ (Gradshteyn and Ryzhik [11, pp. 602, 371 and 1085]), we find that a necessary and sufficient condition of existence of a strictly stationary solution X is $\log(\alpha_m\beta_m) - \delta c < 0$. A necessary and sufficient condition of weak stationarity up to the $\delta-$order of X is $(\alpha_m+\beta_m)\frac{\Gamma(\delta+1)}{2} < 1$.

Example 6. Let us consider now that the generator process ε follows the Uniform law in $]-1,1[$. A necessary and sufficient condition of existence of a strictly stationary solution X is $\log(\alpha_m\beta_m) - 2\delta < 0$ and a necessary and sufficient condition for the existence of $E\left(|X_t|^\delta\right)$ is $(\alpha_m+\beta_m)\frac{1}{2(\delta+1)} < 1$.

4. $\delta-$TGARCH processes: discussion on the weak stationarity

The process $Y_t = B + \lim\limits_n (q.c.)\sum\limits_{k=1}^{n} A_{t-1}...A_{t-k}B$, $t \in \mathbb{Z}$, is the strictly stationary and ergodic solution of the vectorial model (1.3). Let us show that this solution is weakly stationary if and only if

$$(\mathbf{H3}) : E\left(\|A_0...A_m\|^2\right) \text{ exists for every } m \in \mathbb{N}_0 \text{ and } \exists r \in \mathbb{N}_0 : E\left(\|A_0...A_r\|^2\right) < 1. \tag{4.1}$$

This condition is equivalent to

$$\lim_n \; E\left(\|A_0...A_n\|^2\right) = 0. \tag{4.2}$$

In fact, the independence of the sequence of matrices $(A_t, t \in \mathbb{Z})$ gives, with $\lfloor x \rfloor$ representing the integral part of x,

$$\begin{aligned}
& E\left(\|A_0 ... A_n\|^2\right) \\
\leq \; & E\left(\|A_0 ... A_r\|^2 ... \left\|A_{\left(\left\lfloor\frac{n}{r+1}\right\rfloor-1\right)(r+1)} ... A_{\left\lfloor\frac{n}{r+1}\right\rfloor(r+1)-1}\right\|^2 \left\|A_{\left\lfloor\frac{n}{r+1}\right\rfloor(r+1)} ... A_n\right\|^2\right) \\
\leq \; & \left[E\left(\|A_0 ... A_r\|^2\right)\right]^{\left\lfloor\frac{n}{r+1}\right\rfloor}\left(E\|A_0\|^2\right)^k,
\end{aligned}$$

with $k \leq r+1$. Consequently, the convergence follows easily under (4.1).

Conversely, if $\lim\limits_n E\left(\|A_0 ... A_n\|^2\right) = 0$ we have, obviously, the condition (4.1).

We observe that, by Lemma 1 in the Appendix A, the strictly stationary solution $Y = (Y_t, t \in \mathbb{Z})$ is weakly stationary if (4.1) is assumed. Moreover, taking into account the same Lemma, under condition (4.1) the referred strictly stationary solution exists.

On the other hand, Y is also the unique weakly stationary solution of the model (1.3). In fact, if $W = (W_t, t \in \mathbb{Z})$ is another weakly stationary solution of the model (1.3) we get, using recurrence, for every $i \geq 1$,

$$\begin{aligned}
\|W_t - Y_t\|^2 & \leq \|A_{t-1} ... A_{t-i-1}\|^2 \left(\|W_{t-i-1}\| + \|Y_{t-i-1}\|\right)^2 \\
& \leq 2\|A_{t-1} ... A_{t-i-1}\|^2 \left(\|W_{t-i-1}\|^2 + \|Y_{t-i-1}\|^2\right).
\end{aligned}$$

From the independence of the matrices $(A_t, t \in \mathbb{Z})$ we obtain

$$E\left(\|W_t - Y_t\|^2\right) \leq 2E\left(\|A_{t-1} ... A_{t-i-1}\|^2\right) E\left(\|W_{t-i-1}\|^2 + \|Y_{t-i-1}\|^2\right)$$

and so $E\left(\|W_t - Y_t\|^2\right) \leq c$, for every $c > 0$, which implies $W_t = Y_t$ in L^2_m.

If Y is a solution of the model (1.3), we know that the corresponding solution of model (1.1) is the first component of the vectorial process $U = (U_t, t \in \mathbb{Z})$ defined by

$$U_t = \varepsilon_t Y_t^{\frac{1}{\delta}}$$

with $Y_t^{\frac{1}{\delta}} = \left(\left(Y_t^{(1)}\right)^{\frac{1}{\delta}}, ..., \left(Y_t^{(m)}\right)^{\frac{1}{\delta}}\right)$, that is, $U_t^{(1)} = \varepsilon_t \left(Y_t^{(1)}\right)^{\frac{1}{\delta}}$ is a solution of (1.1).

We can now state the following result.

Theorem 6. *There exists a unique weak stationary of order* 2δ *solution of the model (1.1) with generator process* ε *with moments of order* 2δ *if and only if the sequence of matrices* $(A_t, t \in \mathbb{Z})$ *satisfies the hypothesis* (**H3**). *Moreover, this solution is the strictly stationary one.*

Proof. Let Y be the weakly stationary solution of equation (1.3). So, $E\left(\|Y_t\|^2\right) < +\infty$, $t \in \mathbb{Z}$, and then $Y^{(1)} = \left(Y_t^{(1)}, t \in \mathbb{Z}\right)$ is a second order process. As

$$\left|U_t^{(1)}\right|^{\delta} = |\varepsilon_t|^{\delta}\, Y_t^{(1)}$$

we conclude that the process $U^{(1)} = \left(U_t^{(1)}, t \in \mathbb{Z}\right)$ is of order 2δ if ε_t has moment of order 2δ. So, under this condition and hypothesis (4.1), there exists a unique weak stationary of order 2δ solution of the model (1.1) given by

$$U_t^{(1)} = \varepsilon_t \left(Y_t^{(1)}\right)^{\frac{1}{\delta}}$$

where $Y_t^{(1)}$ is the first component of the weak and strict stationary solution of the model (1.3).

Finally, the unicity in L_m^2 of the solution of equation (1.3) implies the unicity of the solution of (1.1) of order 2δ.

Let us now study the necessity of the condition (4.1) for the existence of the weak stationary solution of order 2δ.

Let $X = (X_t, t \in \mathbb{Z})$ be the weakly stationary solution of order 2δ and let us consider the corresponding vectorial process Y such that $Y_{t+1} = A_t Y_t + B$.

Using this equation we can write, for any $r \in \mathbb{N}$,

$$Y_0 = A_{-1} ... A_{-n} Y_{-n-1} + \sum_{n=1}^{r} A_{-1} ... A_{-n} B + B.$$

So, taking into account that $A_{-1} ... A_{-n} Y_{-n-1}$ is a vector with positive coefficients, we have, for any $r \in \mathbb{N}$,

$$Y_0 \geq \sum_{n=1}^{r} A_{-1} ... A_{-n} B.$$

As $X_t^{2\delta} = \sigma_t^{2\delta}\varepsilon_t^{2\delta}$, the existence of $E\left(X_t^{2\delta}\right)$ and $E\left(\varepsilon_t^{2\delta}\right)$ ensure that of $E\left(\sigma_t^{2\delta}\right)$. So, Y is a second order process because the expectation of $\|Y_t\|^2 = \sum\limits_{i=1}^{m}\left[Y_t^{(i)}\right]^2$ is finite.

We can then conclude that, for any $r \in \mathbb{N}$,

$$E\left(\left\|\sum_{n=1}^{r} A_{-1}...A_{-n}B\right\|^2\right) \leq E\left(\|Y_0\|^2\right) < +\infty$$

and, noting the positiveness of the vectors coefficients involved, we can use Beppo-Levi's Theorem to obtain

$$\left(\left\|\sum_{n=1}^{+\infty} A_{-1}...A_{-n}B\right\|^2\right) \leq E\left(\|Y_0\|^2\right) < +\infty.$$

Let us show now that we also have

$$\sum_{n=1}^{+\infty} E\left(\|A_{-1}...A_{-n}B\|^2\right) = \sum_{n=1}^{+\infty} E\left(\|C_nB\|^2\right) < +\infty,$$

where $C_n = A_{-1}...A_{-n}$ has generic element $c_{ij}^{(n)}$. We have

$$\|C_nB\|^2 = \omega^2 \sum_{i=1}^{m}\left(c_{i1}^{(n)}\right)^2$$

and so

$$\sum_{n=1}^{+\infty} E\left(\|C_nB\|^2\right) = E\left[\sum_{i=1}^{m}\sum_{n=1}^{+\infty}\omega^2\left(c_{i1}^{(n)}\right)^2\right] \leq E\left[\left\|\sum_{n=1}^{+\infty} A_{-1}...A_{-n}B\right\|^2\right] < +\infty.$$

We can then conclude that $\lim\limits_{n} E\left(\|C_nB\|^2\right) = 0$.

Let us consider now the canonical basis of $\mathbb{R}^m$, $(f_1, ..., f_m)$. As $E\left(\|C_nB\|^2\right) = \omega^2 E\left(\|C_nf_1\|^2\right)$ we deduce that $\lim\limits_{n} E\left(\|C_nf_1\|^2\right) = 0$.

Moreover, for $i = 2, ..., m-1$, $\|C_nf_i\| = \|C_{n-1}f_{i-1}\| = \|C_{n-2}f_{i-2}\| = ... = \left\|C_{n-(i-1)}f_1\right\|$ and $\|C_nf_m\| = \|C_{n-1}f_{m-1}\| = \left\|C_{n-(m-1)}f_1\right\|$. So,

$$\lim_{n} \quad E\left(\|C_nf_i\|^2\right) = 0, \quad i = 2, ..., m.$$

We are now able to prove that

$$\lim_n E\left(\|C_n\|^2\right) = 0 \tag{4.3}$$

or, equivalently, that the condition (4.1) holds.

To obtain this, we use the operator norm defined by $\|M\| = \sup_{\|x\|\leq 1} \|Mx\|, x \in \mathbb{R}^m$, $M \in \mathbb{M}$, taking into account that all the norms in $\mathbb{M}$ are equivalent.

We have then $E\left(\|C_n\|^2\right) = E\left(\sup_{\|x\|\leq 1} \|C_n x\|^2\right)$. But using the continuity of the operator norm and the compactness of $\{x : \|x\| \leq 1\}$ we have, for every $\omega \in \Omega$, $\sup_{\|x\|\leq 1} \|C_n x\|^2 = \|C_n X_0(\omega)\|^2$ for some $X_0(\omega)$ such that $\|X_0(\omega)\| \leq 1$. So, choosing a measurable version of X_0 we have $\sup_{\|x\|\leq 1} \|C_n x\|^2 = \|C_n X_0\|^2$.

Let us consider then any random vector X verifying $\|X\| \leq 1$. We can write $X = \sum_{i=1}^{m} B_i f_i$ with B_i real random variables such that $B_i^2 \leq 1$. We have then

$$E\left(\|C_n X\|^2\right) \leq E\left(\sum_{i=1}^{m} |B_i| \|C_n f_i\|\right)^2$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} |B_i| |B_j| \left[E\left(\|C_n f_i\|^2\right)\right]^{\frac{1}{2}} \left[E\left(\|C_n f_j\|^2\right)\right]^{\frac{1}{2}}$$

which implies $\lim_n E\left(\|C_n X\|^2\right) = 0$ and, in consequence, the equality (4.3). ■

5. $\delta-$TGARCH processes: minimal representation

In this section we obtain a representation unique for the process $X = (X_t, t \in \mathbb{Z})$ defined in (1.1), in terms of its past, described by X_{t-i}^+ and X_{t-i}^-, $i \geq 1$, and the generator process ε. We also state a necessary and sufficient condition for the existence of a minimal representation of X_t.

We begin by establishing a representation for σ_t in terms of X_{t-i}^+ and X_{t-i}^-, $i \geq 1$.

Let us define the following polynomials, whose coefficients are those present in the definition of σ_t^δ:

$$A(x) = \alpha_1 x + ... + \alpha_p x^p, \quad B(x) = \beta_1 x + ... + \beta_p x^p, \quad G(x) = 1 - \gamma_1 x - ... - \gamma_q x^q.$$

To ensure that the model orders are in fact p and q, we suppose $\gamma_q \neq 0$ and α_p or β_p non zero.

In the following we assume the strict stationarity of the $\delta-$TGARCH process $X = (X_t, t \in \mathbb{Z})$ defined in (1.1) and that the matrices $(A_t, t \in \mathbb{Z})$ satisfy the condition $E\left(\log^+ \|A_0\|\right) < +\infty$. From Theorem 3 we have $\gamma_1 + ... + \gamma_q < 1$ and so all the roots of $G(x) = 0$ are outside the unit circle, which implies

$$\frac{1}{G(x)} = \sum_{j=0}^{+\infty} d_j x^j, \quad |x| \leq 1,$$

where the coefficients d_j decrease exponentially as $j \longrightarrow +\infty$. Obviously,

$$\frac{A(x)}{G(x)} = \sum_{j=1}^{+\infty} c_j x^j, \qquad \frac{B(x)}{G(x)} = \sum_{j=1}^{+\infty} \widetilde{c}_j x^j, \quad |x| \leq 1$$

with

$$c_j = \alpha_1 d_{j-1} + ... + \alpha_p d_{j-p}, \quad \widetilde{c}_j = \beta_1 d_{j-1} + ... + \beta_p d_{j-p}, \quad j \geq 1,$$

and c_j and $\widetilde{c}_j$ decreasing exponentially as $j \longrightarrow +\infty$.

From Lemma 2 in the Appendix A, we conclude that if $E\left(\log^+ \sigma_0\right) < +\infty$ then

$$\sigma_t^\delta = c_0 + \sum_{i=1}^{+\infty} c_i \left(X_{t-i}^+\right)^\delta + \sum_{i=1}^{+\infty} \widetilde{c}_i \left(X_{t-i}^-\right)^\delta,$$

for every t, with probability one, where $c_0 = \frac{\omega}{G(1)} = \omega \sum_{j=0}^{+\infty} d_j$. Moreover, if ε_0^+ and ε_0^- are non-degenerated random variables, this representation of σ_t in terms of past values of X_t^+ and X_t^- is unique.

Using the backward shift operator L, the last result may be written as follows:

$$\begin{aligned} \sigma_t^\delta &= \frac{1}{G(L)} \left[\omega + A(L) \left(X_t^+\right)^\delta + B(L) \left(X_t^-\right)^\delta\right] \\ &= \frac{\omega}{G(1)} + \frac{A(L)}{G(L)} \left(X_t^+\right)^\delta + \frac{B(L)}{G(L)} \left(X_t^-\right)^\delta. \end{aligned} \tag{5.1}$$

From this representation we deduce a unique representation of $X_t = \sigma_t \varepsilon_t$ in terms of its past, for each arbitrarily fixed generator process ε.

Let us now study the minimality of the definition (1.1) of the $\delta-$TGARCH process, in the sense that there is no pair (p^*, q^*), such that $p^* < p$ or $q^* < q$ and

$$\sigma_t^\delta = \omega^* + \sum_{i=1}^{p^*} \alpha_i^* \left(X_{t-i}^+\right)^\delta + \sum_{i=1}^{p^*} \beta_i^* \left(X_{t-i}^-\right)^\delta + \sum_{j=1}^{q^*} \gamma_j^* \sigma_{t-j}^\delta \tag{5.2}$$

for some (not necessarily non negatives) $\omega^*, \alpha_i^*, \beta_i^* \left(i = 1, ..., p^*\right), \gamma_j^* \left(j = 1, ..., q^*\right)$.

Theorem 7. *We suppose that* $E\left(\log^+ \sigma_0\right) < +\infty$ *and that the random variables* ε_0^+ *and* ε_0^- *are non degenerated. The definition (1.1) is minimal if and only if*

$$A(x) \text{ and } G(x) \text{ are coprimes } \textbf{or } B(x) \text{ and } G(x) \text{ are coprimes} \tag{5.3}$$

in the set of the polynomials with real coefficients.

Proof. We suppose that $A(x)$ and $G(x)$ are coprimes or $B(x)$ and $G(x)$ are coprimes and that there are (p^*, q^*), $p^* < p$ or $q^* < q$, and ω^*, α_i^*, $\beta_i^* \left(i = 1, ..., p^*\right)$, $\gamma_j^* \left(j = 1, ..., q^*\right)$ such that (5.2) holds. From the strict stationarity we necessarily have $\sum_{j=1}^{q^*} \gamma_j^* < 1$. We define

$$A^*(x) = \alpha_1^* x + ... + \alpha_{p^*}^* x^{p^*}, \quad B^*(x) = \beta_1^* x + ... + \beta_{p^*}^* x^{p^*}, \quad G^*(x) = 1 - \gamma_1^* x - ... - \gamma_{q^*}^* x^{q^*}.$$

This gives

$$\frac{A^*(x)}{G^*(x)} = \sum_{j=1}^{+\infty} c_j x^j, \qquad \frac{B^*(x)}{G^*(x)} = \sum_{j=1}^{+\infty} \widetilde{c}_j x^j$$

and so, from the unicity of the representation,

$$\frac{A(x)}{G(x)} = \frac{A^*(x)}{G^*(x)}, \qquad \frac{B(x)}{G(x)} = \frac{B^*(x)}{G^*(x)}.$$

If $A(x)$ and $G(x)$ are coprimes, we conclude that there is a polynomial $P(x)$ such that

$$A^*(x) = A(x) P(x), \quad G^*(x) = G(x) P(x),$$

with a similar conclusion if B and G are coprimes. Then $q^* \geq q, p^* \geq p$, which is a contradiction.

Conversely, let us now suppose that the definition (1.1) is minimal but the condition (5.3) fails, that is, neither A and G nor B and G are coprimes.

It is always possible to write

$$G(x) = F_A(x)\, F_B(x)\, G_1(x)$$

where $F_A(x) = \gcd(G(x), A(x))$, $F_B(x) = \gcd(G(x), B(x))$ and $G_1(x)$ is a polynomial with a degree less than or equal to $q-2$. Similarly, we have

$$A(x) = F_A(x)\, A^{\bullet}(x), \quad B(x) = F_B(x)\, B^{\bullet}(x),$$

where degree$(A^{\bullet}(x)) < p$, degree$(B^{\bullet}(x)) < p$.

If we introduce $F(x) = \operatorname{lcm}(F_A(x), F_B(x))$, we have, from (5.1),

$$\begin{aligned}
\sigma_t^{\delta} &= \frac{1}{G(L)}\left[\omega + A(L)\left(X_t^{+}\right)^{\delta} + B(L)\left(X_t^{-}\right)^{\delta}\right] \\
&= \frac{\widetilde{\omega}}{F(1)\, G_1(1)} + \frac{A^{\bullet}(L)}{F_B(L)\, G_1(L)}\left(X_t^{+}\right)^{\delta} + \frac{B^{\bullet}(L)}{F_A(L)\, G_1(L)}\left(X_t^{-}\right)^{\delta} \\
&= \frac{\widetilde{\omega}}{F(1)\, G_1(1)} + \frac{A^{\bullet}(L)\frac{F(L)}{F_B(L)}}{F(L)\, G_1(L)}\left(X_t^{+}\right)^{\delta} + \frac{B^{\bullet}(L)\frac{F(L)}{F_A(L)}}{F(L)\, G_1(L)}\left(X_t^{-}\right)^{\delta}
\end{aligned}$$

where $\widetilde{\omega} = \omega\dfrac{F(1)\, G_1(1)}{G(1)}$.

So, we have

$$\sigma_t^{\delta} = \frac{1}{F(L)\, G_1(L)}\left[\widetilde{\omega} + \widetilde{A}(L)\left(X_t^{+}\right)^{\delta} + \widetilde{B}(L)\left(X_t^{-}\right)^{\delta}\right]$$

where $F(L)\, G_1(L)$, $\widetilde{A}(L) = A^{\bullet}(L)\frac{F(L)}{F_B(L)}$ and $\widetilde{B}(L) = B^{\bullet}(L)\frac{F(L)}{F_A(L)}$ are polynomials whose degrees are less than those of G, A and B, respectively. This fact contradicts the hypothesis of minimal definition. ■

Finally, under the hypotheses of the Theorem 7 and supposing that the polynomials $A(x)$ and $G(x)$ are coprimes or the polynomials $B(x)$ and $G(x)$ are coprimes, there is no

$$\left(\omega^*, \alpha_i^*, \beta_i^*\,(i = 1, ..., p), \gamma_j^*\,(j = 1, ..., q)\right) \neq \left(\omega, \alpha_i, \beta_i\,(i = 1, ..., p), \gamma_j\,(j = 1, ..., q)\right)$$

such that $\sigma_t^{\delta} = \omega^* + \sum\limits_{i=1}^{p}\alpha_i^*\left(X_{t-i}^{+}\right)^{\delta} + \sum\limits_{i=1}^{p}\beta_i^*\left(X_{t-i}^{-}\right)^{\delta} + \sum\limits_{j=1}^{q}\gamma_j^*\sigma_{t-j}^{\delta}$, which assures the unicity of the minimal definition of σ_t^{δ}.

6. Conclusion

The probabilistic analysis developed in this paper for $\delta-$TGARCH models has enormous impact on statistical applications of such models. Indeed we note that we ensure the existence of stationary and ergodic solutions under conditions of great simplicity expressed in terms of the model coefficients.

Moreover, the results stated in Section 5 reinforce this contribution as we obtain a unique representation for σ_t^δ in terms of the present and past observations, which may have a great relevance for estimating and testing methodologies as well as in the forecasting phase. Finally, we remark that the whole study is valid for general generator processes, in particular with laws without classical moments or non-symmetrical ones. So, it may be applied to models generated by processes of stable laws which has great interest in financial applications (Nollan [15], Bellini and Bottolo [1]).

Appendix A

Lema 1. *If* $E\left(\|A_0...A_m\|^2\right)$ *exists for every* $m \in \mathbb{N}_0$ *and* $\exists r \in \mathbb{N}_0 : E\left(\|A_0...A_r\|^2\right) < 1$ *then the strictly stationary solution* $Y = (Y_t, t \in \mathbb{Z})$ *of the vectorial model (1.3) is weakly stationary.*

Proof. We can write

$E\|Y_t\|^2$

$$\leq \|B\|^2 \left[1 + 2\sum_{k=1}^{+\infty} E\left(\|A_{t-1}...A_{t-k}\|\right) + \sum_{k=1}^{+\infty}\sum_{j=1}^{+\infty} E\left(\|A_{t-1}...A_{t-k}\| \|A_{t-1}...A_{t-j}\|\right)\right].$$

Taking into account the independence of the sequence of matrices $(A_t, t \in \mathbb{Z})$ we have

$$\begin{aligned}
\sum_{k=r}^{+\infty} E\left(\|A_{t-1}...A_{t-k}\|\right) &= \sum_{k=r}^{+\infty} E\left(\|A_1...A_k\|\right) \\
&= \sum_{i=0}^{r-1}\sum_{k=1}^{+\infty} E\left(\|A_1...A_{kr+i}\|\right) \\
&\leq \sum_{i=0}^{r-1}\sum_{k=1}^{+\infty} a^{\frac{k}{2}} E\left(\|A_{kr+1}...A_{kr+i}\|\right)
\end{aligned}$$

with $a = E\left(\|A_1...A_r\|^2\right) < 1$ and $E\left(\|A_1...A_i\|^2\right) = 1$ if $i = 0$.

So,

$$\sum_{k=r}^{+\infty} E\left(\|A_{t-1}...A_{t-k}\|\right) \leq \tfrac{\sqrt{a}}{1-\sqrt{a}}\ a_1, \text{ where } a_1 = \begin{cases} r, & \text{if } E\left(\|A_0\|\right) = 1 \\ \frac{1-[E(\|A_0\|)]^r}{1-E(\|A_0\|)}, & \text{if } E\left(\|A_0\|\right) \neq 1. \end{cases}$$

The convergence of the other series results from the convergence of the series $\sum\limits_{k=r}^{+\infty} E\left(\|A_{t-1}...A_{t-k}\|^2\right)$, whose proof is analogous to the previous one, and from that of $\sum\limits_{k=1}^{+\infty}\sum\limits_{j>k} E\left(\|A_{t-1}...A_{t-k}\|\,\|A_{t-1}...A_{t-j}\|\right)$.

To study this last one we observe firstly that

$$\sum_{k=1}^{r-1}\sum_{j=r+1}^{+\infty} E\left(\|A_{t-1}...A_{t-k}\|\,\|A_{t-1}...A_{t-j}\|\right)$$

is a convergent series as it is upper bounded by $\sum\limits_{k=1}^{r-1}\left[E\left(\|A_0\|^2\right)\right]^k \sum\limits_{j=r+1}^{+\infty} E\left(\|A_{k+1}...A_j\|\right)$ and

$$\sum_{k=r}^{+\infty}\sum_{j=k+1}^{+\infty} E\left(\|A_{t-1}...A_{t-k}\|\,\|A_{t-1}...A_{t-j}\|\right)$$

is also convergent since it is upper bounded by

$$\left(a_2 + \tfrac{\sqrt{a}}{1-\sqrt{a}}\ a_1\right)\sum_{k=r}^{+\infty} E\left(\|A_1...A_k\|^2\right)$$

with $a_2 = \sum\limits_{j=k+1}^{k+r-1}\left[E\left(\|A_0\|^2\right)\right]^{j-k}$.

We also point out that the condition (4.1) implies $\lim\limits_{n} \frac{1}{n}\log\|A_0...A_n\| = \gamma < 0$. In fact, let us show firstly that $\left(\frac{1}{n}\log\|A_0...A_n\|\right)^+$ is integrable for every $n \in \mathbb{N}$. We have

$$\left(\frac{1}{n}\log\|A_0...A_n\|\right)^+ \leq \frac{1}{n}\sum_{i=0}^{-n}\left(\log\|A_i\|\right)^+.$$

and then the integrability results from the integrability of $\|A_0\|$.

The result will follow from the Theorem 2.1 of Kingman [13] if we show that $E\left(\frac{1}{n}\log\|A_0...A_{-n}\|\right)$ converges to a strictly negative limit. Using a decomposition in groups of $r+1$ elements and taking into account that the matrices $(A_t, t\in\mathbb{Z})$ are independent and identically distributed, we have

$$E\left(\frac{1}{n}\log\|A_0...A_{-n}\|\right) \leq \frac{\left\lfloor\frac{n}{r+1}\right\rfloor}{n}E\left(\log\|A_0...A_r\|\right)+\frac{1}{n}E\left(\log\left\|A_{\left\lfloor\frac{n}{r+1}\right\rfloor(r+1)}...A_n\right\|\right).$$

Using the Jensen's inequality, the integrability of $\|A_0\|$ and the independence of the sequence $(A_t, t\in\mathbb{Z})$ we have the convergence to zero of the last term.

On the other hand, we have

$$\lim_n \frac{\left\lfloor\frac{n}{r+1}\right\rfloor}{n}E\left(\log\|A_0...A_r\|\right) \leq \frac{1}{r+1}\log\left[E\left(\|A_0...A_r\|\right)^2\right]^{\frac{1}{2}}$$

which establishes the result.

So, the Liapunov exponent associated to the matrices $(A_t, t\in\mathbb{Z})$ is negative which implies the existence of the strictly stationary solution considered.

We conclude that under condition (**H3**) the process Y is a weak and strictly stationary solution of the vectorial model (1.3).

■

Lema 2. *If* $E\left(\log^+\sigma_0\right)<+\infty$ *then*

$$\sigma_t^\delta = c_0+\sum_{i=1}^{+\infty}c_i\left(X_{t-i}^+\right)^\delta+\sum_{i=1}^{+\infty}\widetilde{c}_i\left(X_{t-i}^-\right)^\delta, \quad \textit{for every } t, \textit{ with probability one,} \tag{A.1}$$

with coefficients c_i *and* $\widetilde{c}_i$ *that decrease exponentially. If, in addition,* ε_0^+ *and* ε_0^- *are non-degenerated random variables, the given representation is unique.*

Proof. As $E\left(\log^+\|A_0\|\right) < +\infty$ and $\|A_0\| \geq 1$ we deduce that $E\left\{\log^+\left[\alpha_i\left(\varepsilon_0^+\right)^\delta+\beta_i\left(\varepsilon_0^-\right)^\delta+\gamma_i\right]\right\}$, $i=1,...,m$, is finite. Since the function $\log^+$ is non decreasing then $E\left[\log^+\left(\varepsilon_0^+\right)\right]$ and $E\left[\log^+\left(\varepsilon_0^-\right)\right]$ are finite. Consequently, as $E\left(\log^+\sigma_0\right)<+\infty$, the same occurs to $E\left[\log^+\left(X_0^+\right)\right]$ and to $E\left[\log^+\left(X_0^-\right)\right]$.

Moreover, $\left(X_t^+, t \in \mathbb{Z}\right)$ is a sequence of real random variables, identically distributed, as well as $\left(X_t^-, t \in \mathbb{Z}\right)$, since $(X_t, t \in \mathbb{Z})$ is strictly stationary. In consequence (†), the series

$$\sum_{i=1}^{+\infty} c_j \left(X_{t-i}^+\right)^\delta \quad \text{and} \quad \sum_{i=1}^{+\infty} \widetilde{c}_j \left(X_{t-i}^-\right)^\delta$$

are absolutely convergent with probability 1.

Considering the strictly stationary process $\xi = (\xi_t, t \in \mathbb{Z})$ such that

$$\xi_t = \omega + \sum_{i=1}^{q} \alpha_i \left(X_{t-i}^+\right)^\delta + \sum_{i=1}^{q} \beta_i \left(X_{t-i}^-\right)^\delta,$$

let us show that

$$\sigma_t^\delta = \sum_{m=0}^{+\infty} d_m \xi_{t-m}. \tag{A.2}$$

As $E\left(\log^+ |\xi_0|\right) < +\infty$, this series is absolutely convergent with probability 1, taking into account the exponential decrease of $d_m, m \in \mathbb{N}$.

On the other hand, from

$$\frac{1}{G(x)} = \sum_{j=0}^{+\infty} d_j x^j \Longleftrightarrow 1 = (1 - \gamma_1 x - ... - \gamma_q x^q) \sum_{j=0}^{+\infty} d_j x^j$$

we deduce that $d_0 = 1$, $d_1 = \gamma_1$, $d_2 = d_1\gamma_1 + \gamma_2$, ..., $d_q = d_{q-1}\gamma_1 + ... + d_1\gamma_{q-1} + \gamma_q$ and $d_i = d_{i-1}\gamma_1 + ... + d_{i-q}\gamma_q$, for $i > q$.

For $j \geq p$, the following relation holds

$$\xi_t + d_1\xi_{t-1} + ... + d_j\xi_{t-j} = \sigma_t^\delta - \sum_{i=1}^{q} \left(d_{i+j-q}\gamma_q + ... + d_j\gamma_i\right) \sigma_{t-i-j}^\delta.$$

The left-hand side of this equality converges *a.s.*, when $j \longrightarrow +\infty$, to the right-hand side of (A.2).

---

†If $(\xi_k, 0 \leq k < +\infty)$ is a sequence of real random variables identically distributed such that

$$E\left(\log^+ |\xi_0|\right) < +\infty$$

then the series $\sum_{k=0}^{+\infty} \xi_k z^k$ converges, with probability 1, for any z in the region $|z| < 1$ (Berkes, Horvath and Kokoszka [2, Lemma 2.2]).

Moreover, using the exponential decrease of d_j and the fact that $E\left(\log^+ \sigma_0\right) < +\infty$ we have (Gonçalves and Mendes-Lopes [10])

$$\sum_{j=1}^{+\infty} P\left\{\left|\sum_{i=1}^{q}\left(d_{i+j-q}\gamma_q + ... + d_j\gamma_i\right)\sigma_{t-i-j}^{\delta}\right| > c\right\} < +\infty, \forall c > 0.$$

From Borel-Cantelli Lemma, we obtain

$$\sigma_t^{\delta} = c_0 + \sum_{i=0}^{+\infty} c_i \left(X_{t-i}^{+}\right)^{\delta} + \sum_{i=0}^{+\infty} \widetilde{c}_i \left(X_{t-i}^{-}\right)^{\delta}$$

with probability 1.

Now, let ε_0^+ and ε_0^- be non degenerated variables. To establish the unicity of the representation obtained for σ_t, let us consider, for some t,

$$\sigma_t^{\delta} = c_0 + \sum_{i=1}^{+\infty} c_i \left(X_{t-i}^{+}\right)^{\delta} + \sum_{i=1}^{+\infty} \widetilde{c}_i \left(X_{t-i}^{-}\right)^{\delta}, \ (a.s.)$$

and

$$\sigma_t^{\delta} = f_0 + \sum_{i=1}^{+\infty} f_i \left(X_{t-i}^{+}\right)^{\delta} + \sum_{i=1}^{+\infty} \widetilde{f}_i \left(X_{t-i}^{-}\right)^{\delta}, \ (a.s.)\,.$$

By contradiction, let $m_1 > 0$ and $m_2 > 0$ be the smallest integers such that $c_{m_1} \neq f_{m_1}$ and $\widetilde{c}_{m_2} \neq \widetilde{f}_{m_2}$ (we note that if $c_i = f_i$ and $\widetilde{c}_i = \widetilde{f}_i$, for every $i > 0$, then $c_0 = f_0$).

By the definition of m_1 and m_2, and taking into account that $X_t = \sigma_t \varepsilon_t$, we obtain

$$\begin{aligned}
&\left(f_{m_1} - c_{m_1}\right)\sigma_{t-m_1}\varepsilon_{t-m_1}^{+} + \left(\widetilde{f}_{m_2} - \widetilde{c}_{m_2}\right) X_{t-m_2}^{-} = \\
&= c_0 - f_0 + \sum_{i=m_1+1}^{+\infty}\left(c_i - f_i\right) X_{t-i}^{+} - \sum_{i=m_2+1}^{+\infty}\left(\widetilde{c}_i - \widetilde{f}_i\right) X_{t-i}^{-}.
\end{aligned}$$

If $m_1 \leq m_2$, we get

$$\begin{aligned}
\varepsilon_{t-m_1}^{+} = &\frac{1}{\left(f_{m_1} - c_{m_1}\right)\sigma_{t-m_1}}\left[\left(\widetilde{c}_{m_2} - \widetilde{f}_{m_2}\right) X_{t-m_2}^{-} + \left(c_0 - f_0\right)\right] + \\
&+ \frac{1}{\left(f_{m_1} - c_{m_1}\right)\sigma_{t-m_1}}\left[\sum_{i=m_1+1}^{+\infty}\left(c_i - f_i\right) X_{t-i}^{+} - \sum_{i=m_2+1}^{+\infty}\left(\widetilde{c}_i - \widetilde{f}_i\right) X_{t-i}^{-}\right].
\end{aligned}$$

As $\sigma_{t-m_1} \geq \alpha_0 > 0$, $\varepsilon^+_{t-m_1}$ is well defined. As X^+_j is $\underline{\varepsilon}_j-$measurable, as well as X^-_j ($\underline{\varepsilon}_j$ is the $\sigma-$field generated by $\varepsilon_j, \varepsilon_{j-1}, ...$), the right-hand side of the last relation (and, consequently, $\varepsilon^+_{t-m_1}$) is a real random variable measurable with respect to $\underline{\varepsilon}_{t-m_1-1}$. Taking into account that $\varepsilon_t, t \in \mathbb{Z}$, are independent, we conclude that $\varepsilon^+_{t-m_1}$ is constant ($a.s.$).

The conclusion is analogous if $m_1 > m_2$.

■

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