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# CONTRACTION OF GABOR FRAMES TO THE INTERVAL (-1,1)

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Abstract: In this paper we introduce frames for the space $L^2(-r,r)$, $r \geq 1$ that formally approach Gabor frames in $L^2(\mathbb{R})$ as $r \to \infty$. This is done using an integral transform which can be seen, in a certain sense, as a contraction of the Bargmann transform. The transform and the associated frames can be related to Bergman spaces in the unit disc, thus allowing a complete characterization by means of Seip´s geometric description of the sampling sequences in the Bergman space.

Keywords: Frames, Bergman spaces, Bargmann-Fock transform, Gegenbauer and Hermite functions.

## 1. Introduction

In this paper we introduce frames for the space $L^2(-r,r)$, $r \geq 1$, in particular for $L^2(-1,1)$, that formally approach Gabor frames in $L^2(\mathbb{R})$ when $r \to \infty$. Our discussion is motivated by the limit

$$\lim_{r\to\infty}\left(1-\frac{\mu(t,\xi)}{r}\right)^r = e^{-\mu(t,\xi)}. \tag{1}$$

Because of the fundamental role played by exponential functions in the theory of Gabor frames of $L^2(\mathbb{R})$, we determine whether functions of the form

$$(1-\mu(t,\xi))^\alpha$$

can be used in the construction of frames in $L^2(-1,1)$. For instance, it follows from a special case of our results that, defining $\lambda_{k,n} = \left(1-2^{1-n}\right)e^{\frac{2\pi i k}{n}}$, the following sequence of functions is a frame of $L^2(-1,1)$:

$$\left\{\frac{2^{-\frac{2}{3}n}(1-t^2)^{\frac{1}{4}}}{(1-2t\lambda_{k,n}+\lambda_{k,n}^2)}\right\}_{n\in\mathbb{N},\quad 0\leq k<n}$$

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This provides a new method for stable representation of functions in $L^2(-1,1)$. In the proofs, generating functions for Gegenbauer polynomials are combined with deep results from the theory of Bergman spaces. Our main result is a complete description of the sequence of points that can be used to construct a certain family of frames for $L^2(-1,1)$. It uses the geometric description of sets of sampling in the Bergman space obtained by Kristian Seip [22] just as corresponding results in the Fock space [17], [20] provide a complete description of the sequences yielding Gabor frames with Gaussian windows. To provide a link to Seip´s results we introduce an integral transform maping $L^2(-1,1)$ to some spaces in the unit disc which are equivalent to weighted Bergman spaces. This integral transform plays a role in the analysis of $L^2(-1,1)$ analogous to the Bargmann-Fock transform for $L^2(\mathbb{R})$ and it approaches the latter in the sense of (1).

The outline of the paper is as follows. We introduce the $L^2(-1,1)$ frames and the associated integral transform in the next section. This will allow us to state the main results of the paper, which will be proved after recalling some well known material about special functions and Bergman spaces in a "Tools" section. The central section of the paper is the fourth, where the density theorem is proved, using a general principle which seems to be applicable in several situations. The remainder of the paper is devoted to a more detailed study of the mapping properties of our integral transform, which can be summarized in a commutative diagram. The last section contains informal remarks related to other work as well as historical remarks about the Bargmann transform.

## 2. Results

**2.1. A contracted Gabor system.** Let $\mathbb{D}$ be the unit disc in the complex plane. For a sequence $\Lambda = \{\lambda\} \subset \mathbb{D}$ define the sequence of functions

$$F_\lambda^\alpha(t) \; = \; \frac{(1-t^2)^{\frac{\alpha}{2}-\frac{1}{4}}(1-|\lambda|^2)^{\alpha+\frac{1}{2}}}{(1-2t\lambda+\lambda^2)^\alpha}$$

on $(-1,1)$. The set $\{F_\lambda^\alpha\}_{\lambda\in\Lambda}$ can be thought of as a *contracted Gabor system* in the sense that

$$\lim_{r\to\infty} F_{\lambda/r}^{\pi r^2}\left(\frac{t}{r}\right) = e^{2\pi t\lambda-\pi\lambda^2-\pi t^2/2-\pi|z|^2} = e^{-\frac{\pi}{2}(t-2x)^2-2\pi i\xi t}e^{2\pi i x\xi}, \qquad (2)$$

setting $\lambda = x - i\xi$. This limit can also be written as

$$\lim_{r\to\infty} F^{\pi r^2}_{\lambda/r}\left(\frac{t}{r}\right) = e^{2\pi i x\xi} M_\xi T_{2x}\varphi(t), \qquad \varphi(t) = e^{-\frac{\pi}{2}t^2},$$

where $T_x$ and $M_\xi$ denote translation and modulation

$$T_x f(t) = f(t-x) \qquad\qquad M_\xi f(t) = e^{-2\pi i \xi t}.$$

Thus, up to the factor $e^{2\pi i x\xi}$, limit (2.1) is a Gabor system with gaussian window.

**2.2. A Beurling density theorem.** A sequence of functions $\{f_\lambda\}_{\lambda\in I}$ is a frame of a Hilbert space $\mathcal{H}$ if there exist positive constants $A, B$ such that, for every $f \in \mathcal{H}$, the inequality

$$A\,\|f\|^2_{\mathcal{H}} \le \sum_{i\in I} |\langle f, f_i\rangle_{\mathcal{H}}|^2 \le B\,\|f\|^2_{\mathcal{H}}\,,$$

holds. We will characterize the sequences $\Lambda \subset \mathbb{D}$ such that $\{F_\lambda(t)\}_{\lambda\in\Lambda}$ is a frame of $L^2(-1,1)$. This requires some aditional terminology and we will follow [23]. Let

$$\rho(z,w) = \left|\frac{z-w}{1-z\overline{w}}\right|, \quad z, w \in \mathbb{D},$$

be the pseudo-hyperbolic metric in the unit disc. A sequence $\Lambda \subset \mathbb{D}$ is then said to be *uniformly separated* if there exists $\delta > 0$ such that $\rho(\lambda_1, \lambda_2) > \delta$ for every $\lambda_1, \lambda_2 \in \Lambda$. Given such a sequence, its lower Beurling-Seip density is defined by

$$D^-(\Lambda) = \limsup_{r\to 1}\sup_z \frac{\sum_{\rho(\lambda_j,z)<r}(1-\rho(\lambda_j,z))}{\log 1/(1-r)}.$$

For example, when $\omega_n = e^{2\pi i/n}$ is the primitive $n^{\text{th}}$ root of unity and $\Lambda$ is the family

$$\lambda_{k,n} = \left(1 - \frac{\gamma}{2^n}\right)(\omega_n)^k, \qquad n \in \mathbb{N}, \quad 0 \le k < n, \tag{3}$$

then $D^-(\Lambda) = \gamma/\log 2$.

We will use the characterization of sampling sequences in the Bergman space ([22]) to establish our main result.

**Theorem 1.** *Let $\Lambda$ be a uniformly separated sequence in the unit disc. Then the set of functions $\{F_\lambda^\alpha(t)\}_{\lambda\in\Lambda}$, $\alpha > 1/2$, is a frame for $L^2(-1,1)$ if and only if $D^-(\Lambda) > \alpha + 1/2$.*

**Corollary 1.** *When $\Lambda = \{\lambda_{k,n}\}$ is the family defined in (2.2), $\{F_\lambda^\alpha(t)\}_{\lambda\in\Lambda}$ is a frame in $L^2(-1,1)$ if and only if $\gamma > (\alpha + 1/2)\log 2$.*

The example in the introduction follows by choosing $\alpha = 1$ and $\gamma = 2$.

**2.3. A Bargmann-type transform.** Denote by $B_\alpha$ the integral transform

$$(B_\alpha f)(z) = \int_{-1}^{1} \frac{f(t)}{(1-2tz+z^2)^\alpha}\, d\omega_\alpha(t), \qquad f \in L^2(-1,1),$$

where

$$d\omega_\alpha(t) = \sqrt{\frac{\Gamma(\alpha)}{\pi^{\frac{1}{2}}\Gamma(\alpha+1/2)}}(1-t^2)^{\frac{\alpha}{2}-\frac{1}{4}}dt.$$

We will show that $B_\alpha$ maps $L^2(-1,1)$ to the space $a_\alpha(\mathbb{D})$ of analytic functions in the unit disc with norm

$$\|F\|^2_{a_\alpha(\mathbb{D})} = \int_{\mathbb{D}} \left[z\frac{dF}{dz}(z) + \alpha F(z)\right]\overline{F(z)}(1-|z|^2)^{2\alpha-2}dz.$$

As a side remark, one should notice that this space is reminiscent of one of the spaces appearing in the disc model of the metapletic representation [8]. The other space which occurs in [8] is the standard Bergman space, with norm

$$\|F\|^2_{A_\alpha(\mathbb{D})} = \int_{\mathbb{D}} |f(z)|^2\,(1-|z|^2)^\alpha dz.$$

Denoting by $M_\alpha$ the mapping defined on the basis elements $\{u_n\}_{n=0}^\infty$ of a Hilbert space by

$$M_\alpha u_n = \sqrt{n+\alpha}\,u_n, \tag{4}$$

the mapping properties of $B_\alpha$ can be expressed in the following commutative diagram:

$$\begin{array}{ccc} L^2(-1,1) & \overset{B_\alpha}{\rightarrow} & a_\alpha(\mathbb{D}) \\ M_\alpha \downarrow & & \downarrow M_\alpha \\ & & \\ H_\alpha(-1,1) & \underset{B_\alpha}{\rightarrow} & A_{2\alpha-2}(\mathbb{D}) \end{array} \tag{5}$$

The diagram provides an indication on why the restriction $\alpha > \frac{1}{2}$ is necessary in Theorem 1: the limit case $\alpha \to \frac{1}{2}^{+}$ of $A_{2\alpha-2}(\mathbb{D})$ is the Hardy space, where sampling sequences do not exist [23]. The space $H_\alpha(-1,1)$ has some interesting properties, and we will see that it is the most appropriate space in which to consider the following formal limit: setting $\alpha = \pi r^2$, replacing $z$ by $z/r$ and taking limits in $B_\alpha$ in the sense of (1), we obtain the Bargmann transform:

$$(Bf)(z) = \int_{\mathbb{R}} f(t) e^{2\pi tz - \pi t^2 - \pi z^2/2} dt.$$

# 3. Background

**3.1. Special functions.** We will now list some properties of the special functions that will be used in the remaining. Detailed expositions concerning such special functions can be found in [1] and [13].

**3.1.1.** *The Gamma function.* The Gamma function is defined, for $\mathcal{R}z > 0$, by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

The shifted factorial is defined as

$$\begin{aligned} (a)_0 &= 1 \\ (a)_n &= a(a+1)...(a+n-1) \end{aligned}$$

We will use the following relation between the Gamma function and the shifted factorial:

$$(a)_n = \Gamma(a+n)/\Gamma(a) \tag{6}$$

as well as Stirling´s formula

$$\Gamma(x) \sim \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x},$$

as $\mathcal{R}x \to \infty$.

**3.1.2.** *The Hermite functions.* The Hermite polynomials are defined by the recurrence relation

$$H_{n+1}(t) = 2tH_{n+1}(t) - 2nH_{n+1}(t).$$

with $H_0(t) = 1$ and $H_1(t) = t$. They satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} H_n(t) H_m(t) e^{-t^2} dt = 2^n n! \sqrt{\pi} \delta_{n,m}$$

and have the generating function

$$e^{2tz-z^2} = \sum_{n=0}^{\infty} H_n(t)\frac{z^n}{n!}. \tag{7}$$

We will also consider the orthonormal Hermite functions defined as

$$h_n(t) = (2^n n!)^{-\frac{1}{2}} H_n(t\sqrt{2\pi})e^{-\pi t^2}.$$

Hermite functions are very important in the context of the Bargmann transform, because they are a basis of the space $L^2(\mathbb{R})$ that is mapped into the basis

$$(Bh_n)(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} z^n$$

of the Bargmann-Fock space by the Bargmann transform.

**3.1.3.** *The Gegenbauer functions.* The Gegenbauer polynomials are defined by the recurrence relation

$$2(n+\alpha)tC_n^{\alpha}(t) = (n+1)C_{n+1}^{\alpha}(t) + (n+2\alpha-1)C_{n-1}^{\alpha}(t)$$

with $C_0^{\alpha}(t) = 1$ and $C_1^{\alpha}(t) = 2\alpha t$. They satisfy the orthogonality

$$\int_{-1}^{1} C_n^{\alpha}(t)C_m^{\alpha}(t)(1-t^2)^{\alpha-\frac{1}{2}}dt = \frac{(2\alpha)_n\sqrt{\pi}\Gamma(\alpha+1/2)}{n!(n+\alpha)\Gamma(\alpha)}\delta_{n,m} \tag{8}$$

and have the generating function

$$(1-2tz+z^2)^{-\alpha} = \sum_{n=0}^{\infty} C_n^{\alpha}(t)z^n, \tag{9}$$

where the sum is uniformly convergent on $\mathbb{R}\times K$, for every compact subset $K$ of the unit disk. The normalized Gegenbauer functions are

$$c_n^{\alpha}(t) = (1-t^2)^{\alpha/2-\frac{1}{4}}\sqrt{\frac{n!(n+\alpha)\Gamma(\alpha)}{(2\alpha)_n\sqrt{\pi}\Gamma(\alpha+1/2)}}C_n^{\alpha}(t) \tag{10}$$

they satisfy

$$\int_{-1}^{1} c_n^{\alpha}(t)c_m^{\alpha}(t)dt = \delta_{n,m}$$

and constitute a complete orthonormal basis of the space $L^2(-1,1)$.

**Remark 1.** *From the generating functions (9) and (7) it is easy to see that*

$$\lim_{\alpha\to\infty}\frac{1}{\alpha^{\frac{n}{2}}}C_n^{\alpha}\left(\frac{x}{\sqrt{\alpha}}\right)=\frac{H_n(x)}{n!}. \tag{11}$$

**3.2. Bergman spaces.** There are recent books on Bergman spaces which provide a current account of their structure and of major advances in their study over the last twenty years [11] [5]. We will need some elementary facts about them.

The functions

$$e_n^{\alpha}(z)=\sqrt{\frac{\Gamma(\alpha+2+n)}{n!\Gamma(\alpha+2)}}z^n$$

constitute an orthonormal basis of the Bergman spaces

$$A_{\alpha}(\mathbb{D})=\left\{f \text{ analytic in } \mathbb{D} \text{ such that } \int_{\mathbb{D}}\left|f(z)\right|^2(1-|z|^2)^{\alpha}dz<\infty\right\}.$$

Therefore, a function with a Taylor series expansion

$$f(z)=\sum_{n=0}^{\infty}a_n z^n,$$

has norm

$$\|f\|^2_{A_{\alpha}(\mathbb{D})}=\sum_{n=0}^{\infty}\frac{n!\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)}\left|a_n\right|^2.$$

The Bergman spaces $A_{\alpha}(\mathbb{D})$ are Hilbert spaces with a reproducing kernel given by

$$R_{\alpha}(z,w)=\frac{1}{(1-z\overline{w})^{2+\alpha}}. \tag{12}$$

# 4. Proof of the main result

**4.1. A general principle.** We will use a very simple principle which allows one to construct frames in a given Hilbert space $U$ from sampling sequences in a reproducing kernel Hilbert space $H$. It is related to work of Hille [12] and may be useful in other contexts. Recall that $\Lambda=\{\lambda\}$ is a sampling sequence for a Hilbert space $H$ with reproducing kernel $R(.,.)$, if there exist positive constants $A, B$ (not necessarily the same at each occurrence) such that

$$A\left\|F\right\|_H^2 \leq \sum_{\lambda\in\Lambda}\left|F(\lambda)\right|^2 R(\lambda,\lambda)^{-1} \leq B\left\|F\right\|_H^2. \tag{13}$$

for every $F \in H$.

**Proposition 1.** *If $\{u_n(t)\}_{n\in I}$ is an orthonormal basis of a Hilbert space $U$ and $\{p_n(z)\}_{n\in I}$ an orthonormal basis of a Hilbert space $H$ with reproducing kernel $R(z,w)$, set*

$$K(t,z) = \sum_{n\in I} m_n u_n(t) p_n(z),$$

*Then*

$$\left\{\frac{K(t,\lambda)}{\sqrt{R(\lambda,\lambda)}}\right\}_{\lambda\in\Lambda}$$

*is a frame for $U$ for every sequence $\{m_n\}_{n\in I}$ of real numbers bounded away from zero and infinity if and only if $\{\lambda\}_{\lambda\in\Lambda}$ is a sampling sequence for $H$.*

In many cases the kernel $K(t,z)$ in Proposition 1 can be identified with important functions, but the "custom" multiplier sequence $\{m_n\}_{n\in I}$ is useful in cases like the one we will need to construct frames of $L^2(-1,1)$.

*Proof*: Define a transform $T : U \to H$ by

$$(Tf)(z) = \langle f(.), K(.,z)\rangle_U .$$

Then

$$(Tu_n(.))(z) = \sum_{n\in I} m_n p_n(z)\left\langle u_n(.), u_m(.)\right\rangle_U = m_n p_n(z). \tag{14}$$

Since $\{m_n\}_{n\in I}$ is bounded away from zero and infinity, there exist constants $A, B$, independent of $n$, such that

$$0 < A \leq m_n \leq B < \infty.$$

For $f(t) = \sum_{n\in I} a_n u_n(t)$, (14) gives $\|Tf\|_U^2 = \sum_{n\in I}|m_n a_n|^2$. Consequently, for every $f \in U$,

$$A\left\|f\right\|_U \leq \left\|Tf\right\|_H \leq B\left\|f\right\|_U. \tag{15}$$

In particular, $T : U \to H$ is surjective since the range of $T$ contains a dense subspace of $H$.

Now set $F(z) = (Tf)(z) = \langle f(.), K(., z)\rangle_U$ in (13). Then

$$A \|Tf\|_H^2 \le \sum_{\lambda\in\Lambda} |\langle f(.), K(., \lambda)\rangle|^2 R(\lambda, \lambda)^{-1} \le B \|Tf\|_H^2 .$$

Using the norm equivalence (15), we find constants $C, D$ such that

$$C \|f\|_U^2 \le \sum_{\lambda\in\Lambda} \left| \left\langle f(.), \frac{K(., \lambda)}{\sqrt{R(\lambda, \lambda)}} \right\rangle_U \right|^2 \le D \|f\|_U^2 .$$

Thus $\left\{ \frac{K(t,\lambda)}{\sqrt{R(\lambda,\lambda)}} \right\}_{\lambda\in\Lambda}$ is a frame of $U$.

One can easily reverse this proof since $T$ is invertible. Starting from the frame property the definition of a sampling sequence follows easily using $T^{-1} : H \to U$. This yields the "if" part of the Proposition. ■

The application of the above principle to the transference of the sampling sequences of the Bargmann-Fock space to Gabor frames with a gaussian window is quite simple, since the multiplier sequence $\{m_n\}$ is not required.

**Example 1.** *Let $U = L^2(\mathbb{R})$ and $H = \mathcal{F}(\mathbb{C})$, the Bargmann-Fock space of analytic functions in $\mathbb{C}$ with the norm*

$$\|F\|_{\mathcal{F}(\mathbb{C})}^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dz.$$

*Then we can take in the above theorem $u_n = h_n$, the Hermite functions and $p_n(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} z^n$. In this case the situation is considerably simple, since we can take $m_n = 1$. Using the generating function for the Hermite functions (7), we can sum*

$$K(t, z) = \sum_{n\in\mathbb{N}} h_n(t) \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} z^n = 2^{1/4} e^{2\pi tz - \pi t^2 - \pi z^2/2}.$$

*Since the reproducing kernel of $\mathcal{F}(\mathbb{C})$ is $e^{\pi z\overline{w}}$, Proposition 1 says that $\Lambda \subset \mathbb{C}$ is* a sampling sequence *for $\mathcal{F}(\mathbb{C})$ if and only if* $\left\{ e^{2\pi t\lambda - \pi t^2 - \pi\lambda^2/2 - \pi|\lambda|^2/2} \right\}_{\lambda\in\Lambda}$ *is a frame of $L^2(\mathbb{R})$ or, by writing $\lambda = x + i\xi$, that $\left\{ e^{2\pi i x\xi} M_\xi T_{2x} \varphi(t) \right\}_{\lambda\in\Lambda}$, the collection of modulation and translations of a gaussian window, is a* Gabor frame *of $L^2(\mathbb{R})$ (see* [9, pag. 53], [8, pag. 39] *for more details on this connection). Now let $I$ be a compact set of measure* 1 *in the complex plane* *and let* $n^{-}(r)$ *denote the smallest number of points from* $\Lambda$ *to be found in a translate of* $rI$*. The* lower *Beurling density of* $\Lambda$ *is given as*

$$D^{-}(\Lambda) = \lim_{r\to\infty} \sup \frac{n^{-}(r)}{r^2} \ .$$

*A famous theorem of Lyubarskii-Seip-Wallstén* [17], [20] *says that the sampling sequences (and consequently the frames we have just described) are characterized by the condition* $D^{-}(\Lambda) > 1$*.*

**4.2. Proof of Theorem 1.** We use Proposition 1. Let $U = L^2(-1,1)$, $H = A_{2\alpha-1}(\mathbb{D})$ and

$$u_n(t) = c_n^{\alpha}(t),$$

where $c_n^{\alpha}(t)$ are the Gegenbauer functions defined in (10). Consider also

$$p_n(z) = e_n^{2\alpha-1}(z).$$

and

$$m_n = m_n^{\alpha} = \frac{2\alpha+n+1}{(n+\alpha)(2\alpha-1)}.$$

The multiplier $m_n^{\alpha}$ is bounded above and below for $\alpha > \frac{1}{2}$. Indeed,

$$\frac{1}{2\alpha-1} < \frac{2\alpha+n+1}{(n+\alpha)(2\alpha-1)} < \frac{4}{2\alpha-1}$$

holds for all $n \geq 0$ whenever $2\alpha > 1$.The function $K(t,z)$ can be evaluated explicitly, using the generating function (9) of the Gegenbauer polynomials and properties of the Gamma function:

$$\begin{aligned} K_{\alpha}(t,z) &= \sum_{n\in\mathbb{N}} m_n^{\alpha} c_n^{\alpha}(t) e_n^{2\alpha-1}(z) \\ &= \sum_{n\in\mathbb{N}} c_n^{\alpha}(t) \sqrt{\frac{\Gamma(2\alpha+n)}{(n+\alpha)n!\Gamma(2\alpha)}} z^n \\ &= k_{\alpha}\left[\sum_{n\in\mathbb{N}} C_n^{\alpha}(t) z^n\right](1-t^2)^{\frac{\alpha}{2}-\frac{1}{4}} \\ &= k_{\alpha}(1-2tz+z^2)^{-\alpha}(1-t^2)^{\frac{\alpha}{2}-\frac{1}{4}}. \end{aligned}$$

with $k_{\alpha} = \sqrt{\Gamma(\alpha)/\left(\sqrt{\pi}\Gamma(\alpha+1/2)\right)}$. The reproducing kernel formula (12) gives

$$R_{2\alpha-1}(z,z) = (1-|z|^2)^{-2\alpha-1}.$$

Thus we can use Proposition 1 to conclude that $\Lambda = \{\lambda\}$ is a sampling sequence in $A_{2\alpha-1}(\mathbb{D})$ if and only if

$$F_\lambda^\alpha(t) = \frac{(1-t^2)^{\frac{\alpha}{2}-\frac{1}{4}}(1-|\lambda|^2)^{\alpha+\frac{1}{2}}}{(1-2t\lambda+\lambda^2)^\alpha}$$

is a frame in $L^2(-1,1)$. Theorem 1 then follows from the result of Seip :

**Theorem** [22]**:** *Let $\Lambda$ be a uniformly separated sequence in the unit disc. Then $\Lambda$ is a sampling sequence for $A_{2\alpha-1}(\mathbb{D})$ if and only if $D^-(\Lambda) > \alpha+1/2$.*

## 5. Mapping properties of $B_\alpha$

**5.1. The unitary transform** $B_\alpha : L^2(-1,1) \to a_\alpha(\mathbb{D})$**.** In this section we will prove the following result.

**Theorem 2.** *The transform $B_\alpha$ is an unitary isomorphism*

$$B_\alpha : L^2(-1,1) \to a_\alpha(\mathbb{D}).$$

*Proof*: In the previous section we saw that

$$\sum_{n\in\mathbb{N}} m_n^\alpha c_n^\alpha(t) e_n^{2\alpha-1}(z) = k_\alpha \frac{(1-t^2)^{\frac{\alpha}{2}-\frac{1}{4}}}{(1-2tz+z^2)^\alpha} \tag{16}$$

Taking into account that

$$(B_\alpha c_n^\alpha)(z) = \int_{-1}^{1} \frac{c_n^\alpha(t)}{(1-2tz+z^2)^\alpha}\, d\omega_\alpha(t) = k_\alpha \int_{-1}^{1} c_n^\alpha(t) \frac{(1-t^2)^{\frac{\alpha}{2}-\frac{1}{4}}}{(1-2tz+z^2)^\alpha}\, dt,$$

we can use (16); since the sum is uniformly convergent on $\mathbb{R}\times K$, for every compact subset $K$ of the unit disk, it can be interchanged with the integral. The result is

$$(B_\alpha c_n^\alpha)(z) = m_n^\alpha e_n^{2\alpha-1}(z) = \frac{1}{\sqrt{(n+\alpha)}} e_n^{2\alpha-2}(z),$$

Thus, $B_\alpha$ is an isometry if and only if $B_\alpha\left[L^2(-1,1)\right]$ is the space of functions analytic in the unit disk having

$$u_n(z) = \frac{1}{\sqrt{(n+\alpha)}} e_n^{2\alpha-2}(z) \tag{17}$$

as an orthonormal basis. It remains to show that this space is indeed $a_\alpha(\mathbb{D})$. First observe that

$$\|F\|^2_{a_\alpha(\mathbb{D})} = \langle T_\alpha F, F\rangle_{A_{2\alpha-2}(\mathbb{D})} \tag{18}$$

where $T_\alpha$ is the operator defined as

$$T_\alpha F = z\frac{dF}{dz} + \alpha F. \tag{19}$$

By definition $B_\alpha\left[L^2(-1,1)\right] \subset A_{2\alpha-2}(\mathbb{D})$ and we can expand $F \in B_\alpha\left[L^2(-1,1)\right]$ in terms of the basis $e_n^{2\alpha-2}(z)$ of $A_{2\alpha-2}(\mathbb{D})$:

$$F(z) = \sum_{n=0}^{\infty} \left\langle F, e_n^{2\alpha-2}\right\rangle_{A_{2\alpha-2}(\mathbb{D})} e_n^{2\alpha-2}(z). \tag{20}$$

On the one hand, this gives

$$\left\langle T_\alpha F, F\right\rangle_{A_{2\alpha-2}(\mathbb{D})} = \sum_{n=0}^{\infty}(n+\alpha)\left|\left\langle F, e_n^{2\alpha-2}\right\rangle_{A_{2\alpha-2}(\mathbb{D})}\right|^2; \tag{21}$$

on the other hand, $F$ can also be written in terms of the basis $u_n(z)$ of $B_\alpha\left[L^2(-1,1)\right]$ as

$$F(z) = \sum_{n=0}^{\infty} \left\langle F, u_n\right\rangle_{B_\alpha[L^2(-1,1)]} \frac{1}{\sqrt{n+\alpha}} e_n^{2\alpha-2}(z).$$

Thus by (20) and the completeness of $e_n^{2\alpha-2}(z)$,

$$\left\langle F, u_n\right\rangle_{B_\alpha[L^2(-1,1)]} = \sqrt{n+\alpha}\left\langle F, e_n^{2\alpha-2}\right\rangle_{A_{2\alpha-2}(\mathbb{D})}.$$

Finally,

$$\begin{aligned}
\|F\|^2_{B_\alpha[L^2(-1,1)]} &= \sum_{n=0}^{\infty}\left|\left\langle F, u_n\right\rangle_{B_\alpha[L^2(-1,1)]}\right|^2 \\
&= \sum_{n=0}^{\infty}(n+\alpha)\left|\left\langle F, e_n^{2\alpha-2}\right\rangle_{A_{2\alpha-2}(\mathbb{D})}\right|^2 \\
&= \left\langle T_\alpha F, F\right\rangle_{A_{2\alpha-2}(\mathbb{D})} \\
&= \|F\|^2_{a_\alpha(\mathbb{D})},
\end{aligned}$$

using (21) and (18). ∎

**5.2. The inverse transform.** In order to compute the inverse transform $B_\alpha^{-1} : a_\alpha(\mathbb{D}) \to L^2(-1,1)$, observe that, since $\{e_n^{2\alpha-2}(z)\}$ is an orthonormal basis of $A_{2\alpha-2}(\mathbb{D})$, we have

$$\int_{\mathbb{D}} z^n \overline{z}^m (1-|z|^2)^{2\alpha-2} dz = \frac{n!\Gamma(2\alpha)}{\Gamma(2\alpha+n)} \delta_{nm}. \tag{22}$$

**Theorem 3.** *The transform*

$$B_\alpha^{-1} : a_\alpha(\mathbb{D}) \to L^2(-1,1)$$

*defined by*

$$(B_\alpha^{-1}F)(t) = C_\alpha \int_{\mathbb{D}} F(z) B_\alpha(\overline{z},t)(1-|z|^2)^{2\alpha-2} dz$$

*is an inverse of* $B_\alpha$*.*

*Proof*: Using the generating function (9) and the orthogonality (22) gives

$$\left( B_\alpha{}^{-1} \left[ \frac{e_n^{2\alpha-2}}{\sqrt{n+\alpha}} \right] \right)(t) = c_n^\alpha(t).$$

Thus $B_\alpha^{-1}$ maps the orthonormal basis of $a_\alpha(\mathbb{D})$ into the orthonormal basis of $L^2(-1,1)$ and this is enough to prove the result. ∎

**Remark 2.** *An alternative proof can be given based on the fact that we already know that* $B_\alpha$ *is unitary: we have, for* $F \in A_{2\alpha-2}(\mathbb{D})$ *and* $g \in L^2(-1,1)$*,*

$$\left\langle B_\alpha^{-1}F, g \right\rangle_{L^2(-1,1)} = \left\langle F, B_\alpha g \right\rangle_{a_\alpha(\mathbb{D})}.$$

*Then, writing the integrals explicitly, we can interchange the order if the integrals are absolutely convergent, which is the case for polynomials. It is then possible to adapt the argument in* [8, pag. 45] *and extend the result to arbitrary functions in* $L^2(-1,1)$*.*

## 6. Further properties of $B_\alpha$

In this section we will observe that there is a Hilbert space of functions $H(-1,1)$ defined in $(-1,1)$ where the transform $B_\alpha$ acts very naturally.

**6.1. A natural domain.** We want to find a Hilbert space in $(-1,1)$ which can formally be deformed such that it approaches $L^2(-\infty,\infty)$. For this purpose, the space $L^2(-1,1)$ is too small and the norms of the deformed Gegenbauer functions would blow during the deformation process. Indeed, although $\|c_n^\alpha\|_{L^2(-1,1)} = 1$, the values $\left\|c_n^\alpha\left(\frac{\cdot}{\sqrt{\alpha}}\right)\right\|_{L^2(-\alpha,\alpha)}$ don´t stay close to one during the limit process where $\|\|_{L^2(-\infty,\infty)}$ is approached, because, from the orthogonality relation of the Gegenbauer polynomials, a calculation employing Stirling´s formula shows that, as $\alpha \to \infty$,

$$\left\|c_n^\alpha\left(\frac{\cdot}{\sqrt{\alpha}}\right)\right\|_{L^2(-\alpha,\alpha)}^2 \sim (n+\alpha).$$

This observation suggests defining a Hilbert space space in terms of Gegenbauer expansions. Let $H_\alpha(-1,1)$ be the function space constituted by all the functions of the form

$$f(t) = \sum_{n=0}^{\infty} a_n c_n^\alpha(t) \tag{23}$$

If $f, g \in H_\alpha(-1,1)$ with $f$ as in (23) and

$$g(x) = \sum_{n=0}^{\infty} b_n c_n^\alpha(t),$$

we define the norm

$$\|f\|_{H_\alpha(-1,1)}^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+\alpha)}$$

and the inner product

$$\langle f, g\rangle_{H_\alpha(-1,1)} = \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{(n+\alpha)}.$$

This provides $H_\alpha(-1,1)$ with a Hilbert space structure. Now, as $\alpha \to \infty$,

$$\left\|c_n^\alpha\left(\frac{\cdot}{\sqrt{\alpha}}\right)\right\|_{H^\alpha(-\alpha,\alpha)}^2 \sim 1$$

Thus, the norm remains constant while $H^\alpha(-\alpha,\alpha)$ approaches $L^2(\mathbb{R})$ for big $\alpha$.

**6.2. The unitary property.** Clearly the space $H_\alpha(-1,1)$ contains $L^2(-1,1)$, since $L^2(-1,1)$ is constituted by functions $f$ with an expansion of the form (23) where

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Moreover,

$$\langle c_n^\alpha, c_m^\alpha \rangle_{H_\alpha(-1,1)} = \frac{\delta_{nm}}{n+\alpha}$$

and therefore an orthonormal basis of $H_\alpha(-1,1)$ is

$$\left\{ \sqrt{n+\alpha} c_n^\alpha(t) \right\}.$$

**Proposition 2.** *The transform $B_\alpha$ is an isometric isomorphism*

$$B_\alpha : H_\alpha(-1,1) \to A_{2\alpha-2}(\mathbb{D}).$$

*Proof*: The proof follows immediately from the unitarity $B_\alpha : L^2(-1,1) \to a_\alpha(\mathbb{D})$, since, following definition (4),

$$H_\alpha(-1,1) = M_\alpha \left( L^2(-1,1) \right)$$

and

$$A_{2\alpha-2}(\mathbb{D}) = M_\alpha \left( a_\alpha(\mathbb{D}) \right),$$

providing the commutative diagram (5). ■

**6.3. The transform $B_\alpha$ as a multiplier.** Let

$$\eta_n^\alpha = \sqrt{\frac{\Gamma(2\alpha+n)}{(n+\alpha)n!\Gamma(2\alpha)}}$$

In the previous sections we have seen that, on the basis elements of $L^2(-1,1)$,

$$(B_\alpha c_n^\alpha)(z) = \frac{1}{\sqrt{n+\alpha}} e_n^{2\alpha-2}(z) = \eta_n^\alpha z^n.$$

Thus, in general, given a separable Hilbert space $U$ with orthonormal basis $\{u_n^\alpha\}$, one can define a transform $B_\alpha$ mapping each $f \in U$ written in the form

$$f = \sum a_n u_n^\alpha$$

to

$$\left( B_\alpha f \right)(z) = \sum a_n \eta_n^\alpha z^n.$$

We have already investigated the cases when $u_n^\alpha = c_n^\alpha$ and $u_n^\alpha = \sqrt{(n+\alpha)} c_n^\alpha$, using the explicit integral. With this more general formulation we can take $U = H^2(\mathbb{D})$ and $u_n^\alpha = z^n$. This leads us to the following result.

**Theorem 4.** *The map $B_\alpha : H^2(\mathbb{D}) \to A_\alpha(\mathbb{D})$ is a unitary isomorphism.*

*Proof*: Let $F, G \in H^2(\mathbb{D})$ such that

$$F(z) = \sum a_n z^n \qquad G(z) = \sum b_n z^n.$$

A direct calculation shows that

$$\begin{aligned}
\langle B_\alpha F, B_\alpha G\rangle_{a_\alpha(\mathbb{D})} &= \frac{2\alpha-1}{\pi}\int_0^1\int_0^{2\pi}\left(\sum_{k=0}^{\infty}(\eta_n^\alpha)^2\, a_k\overline{b_k}(k+\alpha)r^{2k}(1-|r|)^{2\alpha-2}rdrd\theta\right) \\
&= \frac{2\alpha-1}{\pi}\sum_{k=0}^{\infty}(\eta_n^\alpha)^2\,(k+\alpha)a_k\overline{b_k}\int_0^1\int_0^{2\pi} r^{2k}(1-|r|^{2\alpha-2})rdrd\theta \\
&= \sum_{k=0}^{\infty} a_k\overline{b_k} \\
&= \langle F, G\rangle_{H^2(\mathbb{D})}.
\end{aligned}$$

■

**6.4. The formal limit transitions.** The formal limit relations between our transform and the Bargmann transform for the Bargmann-Fock space follow from a simple limit. Set $\alpha = \pi r^2$. Replacing $z$ by $\frac{z}{r}$ and $t$ by $\frac{t}{r}$ allows one to consider the transform

$$(B^{\pi r^2} f)(\frac{z}{r}) = C_{\pi r^2}\int_{-r}^{r} f(\frac{t}{r})B_{\pi r^2}(\frac{z}{r},\frac{t}{r})rdt,$$

mapping functions in the dilated interval $(-r, r)$ to the dilated disk $\mathbb{D}_r$. Now, as $r \to \infty$, the kernel

$$B_{\pi r^2}(\frac{z}{r},\frac{t}{r}) = (1-\frac{2tz-z^2}{r^2})^{-\pi r^2}(1-\frac{t^2}{r^2})^{\pi r^2/2-\frac{1}{4}}$$

approaches $e^{2\pi tz-\pi z^2-\pi t^2/2}$. Stirling formula shows that, as $r \to \infty$,

$$C_{\pi r^2} = \sqrt{\frac{\Gamma(\pi r^2)}{\pi^{\frac{1}{4}}\Gamma(\pi r^2+1/2)}} \sim r^{\frac{1}{2}}.$$

We already remarked that $A_{2\pi r^2-2}(\mathbb{D}_r)$ approaches $\mathcal{F}(\mathbb{C})$, and the $H^{\pi r^2}(-r, r)$ was defined such that it approaches $L^2(\mathbb{R})$. We conclude that

$$\lim_{r\to\infty} 2^{\frac{1}{2}} r^{-\frac{3}{4}}\left(B_{\pi r^2} f\right)\left(\frac{z}{r}\right) = (Bf)(z).$$

A similar argument shows that

$$\lim_{r\to\infty} 2^{\frac{1}{2}} r^{-\frac{3}{4}} \left(B_{\pi r^2}^{-1} F\right)\left(\frac{t}{r}\right) = (B^{-1}F)(t).$$

## 7. Discussion and connections to previous work

The ideas behind the study of integral transforms with kernels having bilinear expansions in terms of special functions can be traced back to Hille [12]. It is quite remarkable that in the last section of [12] it is suggested the study of the transform which has been fully explored by Bargmann in [3]. An interesting understanding of these and related transforms can be found in [4].

In his paper [3], Bargmann points out that his construction can be applied to other integral transforms using generating functions of orthogonal polynomials other than the Hermite polynomials. He gives the example of a transform onto the Bergman space in the unit disk, involving the generating function of Laguerre polynomials, which is essentially (24), up to a Fourier transformation and a Cayley transform between the disk and the half plane. Fock and Bergman spaces have been associated to expansions in signal analysis. A natural approach is by means of integrable group representations [6], [7], [15]. The Bargmann-Fock spaces are associated with the Fock representation of the Heisenberg group and the Bergman spaces are associated with the $ax+b$ group seen as a subgroup of $SL_2(\mathbb{R})$. The study of Bergman spaces provided an answer to a question concerning the existent of a Nyquist density for certain Wavelet frames [22]. Up to a transform from the disk to the upper half plane, the connection to wavelet theory is done by the following unitary mapping between the Hardy space and the Bergman space

$$Ber^{\alpha}\, f(z) = \int_0^{\infty} t^{\alpha} e^{izt}\, \widehat{f}(t) dt, \tag{24}$$

which is a special case of the continuous wavelet transform (on the Fourier side) when one takes as analyzing wavelet the *Poisson wavelet* $\psi_\alpha(t) = \left(\frac{1}{1-it}\right)^{\alpha+1}$. The transforms $B$ and $Ber^{\alpha}$ map the actions of groups in spaces of analytic functions into actions on their domain spaces. The same happens with the transform investigated in this paper, although we were not able to derive an explicit closed formula for the resulting action on $L^2(-1,1)$. Thus Bargmann's suggestion turned out to reveal the following "classical orthogonal polynomial viewpoint of time-frequency analysis":

- Gaussian time frequency in $L^2(\mathbb{R})$ is associated with the generating function of Hermite polynomials.
- Poisson frequency-scale in $L^2(0,\infty)$ is associated with the generating function of Laguerre polynomials.
- "Contracted gaussian time-frequency" in $L^2(-1,1)$ is associated with the generating function of Gegenbauer polynomials.

There are also examples of other transforms which are related to the one we study in this paper. For instance, in [24] the author studies a transform involving the kernel $\alpha(1-z^2)(1-2tz+z^2)^{-\alpha-1}$, which can be obtained from $(1-2tz+z^2)^{-\alpha}$ by applying the operator $z\frac{d}{dz}+\alpha$. The range space is identified as the space of analytic functions in the unit disk with weight:

$$\rho_\alpha(|z|^2) = \frac{1}{\Gamma(2\alpha-1)}\,|z|^{2\alpha-2}\int_{|z|^2}^{1} s^{-\alpha}(1-s)^{2\alpha-2}ds$$

(we assume $\alpha > 1/2$ because there is another expression for $0 < \alpha < 1/2$). Another example was considered in [4], where the kernel $(1-zw)(1-2tz+z^2)^{-\alpha-1}$, obtained from $(1-2tz+z^2)^{-\alpha}$ by applying the operator $z\frac{d}{dz}+2\alpha$ is considered as an example of a broader theory of "Frame transforms". The resulting transform is unitary between $L^2(-1,1)$ and a space of entire functions in the unit disk with a norm equivalent to the Bergman norm.

The idea of viewing the Bergman spaces as "Gaussian deformations", as we did here, was used in [16] to construct a model for a deformed hyperbolic phase space that approaches the Fock space. A group theoretical approach is used by looking at the discrete series of $SU(1,1)$ deformed by a parameter, more precisely, the discrete series of $ASU(1,1)A^{-1}$, with $A$ defined as

$$A = \begin{bmatrix} \sqrt{r} & 0 \\ 0 & \frac{1}{\sqrt{r}} \end{bmatrix}.$$

We also remark that a theory for expansions of functions in $H^2(\mathbb{D})$, related to the discrete series of $SU(1,1)$, has been recently developed in [18]

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