Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 11–18

THE MAXIMAL LENGTH OF A CHAIN IN THE BRUHAT ORDER FOR A CLASS OF BINARY MATRICES

ALESSANDRO CONFLITTI, C. M. DA FONSECA AND RICARDO MAMEDE

Abstract: We answer to a question by Brualdi and Deaett about the maximal length of a chain in the Bruhat order for an interesting combinatorial class of binary matrices.

Keywords: Bruhat order, row and column sum vectors, $(0,1)$–matrices, interchange, minimal matrix, maximal matrix.
AMS Subject Classification (2010): 05B20, 06A07, 15A36.

1. Overview and definitions

Let m and n be two positive integers and let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ be positive integral vectors. The class of all m-by-n $(0,1)$–matrices with row sum vector R and column sum vector S is denoted by $\mathscr{A}(R,S)$. Combinatorial properties of such class of matrices have been thoroughly explored over the years (cf. e.g. [2, 3, 4, 5, 6, 7, 8, 10, 11, 15, 17] and references therein). One of the most remarkable results for $\mathscr{A}(R,S)$ known as the Gale–Ryser Theorem, independently due to David Gale [12] and Herbert J. Ryser [17], states that a necessary and sufficient condition for $\mathscr{A}(R,S)$ to be nonempty is $S \trianglelefteq R^*$, where R^* is the conjugate of R and $\trianglelefteq$ is the standard majorization order. We are assuming that R and S are both non–increasing vectors. An important case in which nonemptiness is assured occurs when $m = n$, k is a positive integer number such that $0 \leqslant k \leqslant n$, and $R = S = (k, \ldots, k)$ is the constant vector having each component equal to k. In this case we write $\mathscr{A}(n,k)$ instead of $\mathscr{A}(R,S)$.

In [8] a Bruhat partial order $\preccurlyeq$ on a nonempty class $\mathscr{A}(R,S)$ was defined using a characterization of the Bruhat order on S_n, the symmetric group of n elements, seen as the set of permutation matrices $\mathscr{A}(n,1)$. Precisely, for an $m \times n$ matrix $A = (a_{ij})$, let $\Sigma_A = (\sigma_{ij}(A))$ be the $m \times n$ matrix defined

Received May 30, 2011.
This work is supported by CMUC - Centro de Matemática da Universidade de Coimbra.

by

$$\sigma_{ij}(A) = \sum_{k=1}^{i} \sum_{\ell=1}^{j} a_{k\ell}\,, \qquad \text{for} \qquad 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\,.$$

If $A_1, A_2 \in \mathscr{A}(R,S)$, then $A_1 \preccurlyeq A_2$ if and only if $\Sigma_{A_1} \geqslant \Sigma_{A_2}$ in the entrywise order, i.e., $\sigma_{ij}(A_1) \geqslant \sigma_{ij}(A_2)$, for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

In [7], Brualdi and Deaett characterize all families of the class $\mathscr{A}(n,k)$ with a unique minimal element.

Theorem 1.1 ([7, Theorem 5.1])**.** *Let $n \geqslant 1$ and k be integers with $0 \leqslant k \leqslant n$. Then the class $\mathscr{A}(n,k)$ has a unique minimal element in the Bruhat order if and only if $k \in \{0, 1, n-1, n\}$ or $n = 2k$. Moreover the minimal matrix in $\mathscr{A}(2k,k)$ is*

$$P_k = J_k \oplus J_k = \begin{pmatrix} J_k & O_k \\ O_k & J_k \end{pmatrix}\,,$$

where J_k is the matrix of all 1's of order k and O_k is the zero matrix also of order k.

Since $\mathscr{A}(n,k) \simeq \mathscr{A}(n,n-k)$ (the map $A \longmapsto J_n - A$ does the job), $|\mathscr{A}(n,0)| = 1$, and $\mathscr{A}(n,1) \simeq S_n$, the most interesting case is in fact $\mathscr{A}(2k,k)$.

We remark that $|\mathscr{A}(2k,k)|$ is the sequence A058527 in the *The On-Line Encyclopedia of Integer Sequences* [18]. For instances of values of such sequence the reader is referred to [19]. We observe also that computing a closed formula for such sequence is an open problem which looks quite hard (see e.g. [1, 9, 13, 14, 16] and the references therein for some asymptotic results).

In [7, Section 6] an example is provided to show that Bruhat order $\preccurlyeq$ is not graded, and it is wondered if the maximal length of a chain in the Bruhat order in the class $\mathscr{A}(2k,k)$ from the minimal element P_k to the maximal element

$$Q_k = \begin{pmatrix} O_k & J_k \\ J_k & O_k \end{pmatrix}$$

is $4k^2$.

In this paper we prove that the above conjecture is false and that the correct answer is a much larger number: namely k^4.

2. The main result

For any $A, B \in \mathscr{A}(R,S)$ such that $A \preccurlyeq B$, as an immediate consequence of the definition of Bruhat order, an upper bound for the length of any admissible chain between A and B is clearly given by

$$\varphi(A,B) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\sigma_{ij}(A) - \sigma_{ij}(B)\right] .$$

Since the poset $(\mathscr{A}(2k,k), \preccurlyeq)$ admits a minimum P_k and a maximum Q_k, any chain between two elements pairwise comparable can be extended to a chain between P_k and Q_k.

After some lengthy but rather straightforward computations we get

$$\sigma_{ij}(P_k) = \begin{cases} ij & \text{if } i,j \leqslant k \\ ik & \text{if } i \leqslant k \leqslant j \\ jk & \text{if } i \geq k \geq j \\ ij - k(i+j-2k) & \text{if } i,j \geq k \end{cases},$$

$$\sigma_{ij}(Q_k) = \begin{cases} 0 & \text{if } i,j \leqslant k \\ i(j-k) & \text{if } i \leqslant k \leqslant j \\ j(i-k) & \text{if } i \geq k \geq j \\ k(i+j-2k) & \text{if } i,j \geq k \end{cases},$$

and $\varphi(P_k, Q_k) = k^4$.

Hence it suffices to present an instance of a chain between P_k and Q_k having exactly such length. We do that in an algorithmic way, presenting a procedure to generate an order preserving path in the Hasse diagram of $\mathscr{A}(2k,k)$.

Procedure (Switch(t,r))**.** $1 \leqslant t, r \leqslant 2k-1$.

Input*: $A = (a_{ij}) \in \mathscr{A}(2k,k)$ such that the submatrix*

$$\begin{pmatrix} a_{t,r} & a_{t,r+1} \\ a_{t+1,r} & a_{t+1,r+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Output*: $B = (b_{ij}) \in \mathscr{A}(2k,k)$ such that $b_{ij} = a_{ij}$ if $1 \leqslant i,j \leqslant 2k$ and $(i,j) \notin \{(t,r),(t,r+1),(t+1,r),(t+1,r+1)\}$, and*

$$\begin{pmatrix} b_{t,r} & b_{t,r+1} \\ b_{t+1,r} & b_{t+1,r+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that executing procedure Switch(t, r) the output covers the input in the Bruhat order for any choice of parameters. Our chain will be made by repeated applications of the procedure Switch(t, r).

Procedure (Switch–rows(t))**.** $1 \leqslant t \leqslant 2k-1$.
Input: $A = (a_{ij}) \in \mathscr{A}(2k, k)$ *such that rows* $t, t+1$ *equal*

$$\begin{pmatrix} 1, \dots, 1, 0, \dots, 0 \\ 0, \dots, 0, 1, \dots, 1 \end{pmatrix}.$$

For $\alpha = k$ *down to* 1 *do*
Begin
For $\beta = \alpha$ *to* $\alpha + k - 1$ *do Switch*(t, β).
End.
Output: $B = (b_{ij}) \in \mathscr{A}(2k, k)$ *such that* $b_{ij} = a_{ij}$ *for any* $1 \leqslant i, j \leqslant 2k$ *such that* $i \neq t, t+1$, *and rows* $t, t+1$ *equal*

$$\begin{pmatrix} 0, \dots, 0, 1, \dots, 1 \\ 1, \dots, 1, 0, \dots, 0 \end{pmatrix}.$$

Algorithm 2.1 (Chain(k))**.** $0 \neq k \in \mathbb{N}$.
Input: P_k.
For $\alpha = k$ *down to* 1 *do*
Begin
For $\beta = \alpha$ *to* $\alpha + k - 1$ *do Switch–rows*(β).
End.
Output: Q_k.

We can see that, for any choice of parameters, the procedure "Switch" is invoked k^2 times by procedure "Switch–rows", and that algorithm "Chain" recalls procedure "Switch–rows" k^2 times as well, so there are k^4 application of procedure "Switch". Since, as already remarked, each time that procedure "Switch" is recalled we are moving up (by one cover relation) in the Hasse diagram of the poset $(\mathscr{A}(2k, k), \preccurlyeq)$, all the constructed elements are pairwise distinct members of the desired chain. As a consequence we obtain our result.

Theorem 2.2. *For any positive integer* k, *the maximal length of a chain in the Bruhat order in* $\mathscr{A}(2k, k)$ *equals* k^4.

For the sake of clarity, we present in detail our construction of the chain for the case $k = 2$.

Example 2.1. $k = 2$.

$$
\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix} \longmapsto \begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&1\end{pmatrix} \longmapsto \begin{pmatrix}1&1&0&0\\1&0&0&1\\0&1&1&0\\0&0&1&1\end{pmatrix}
$$

$$
\longmapsto \begin{pmatrix}1&1&0&0\\0&1&0&1\\1&0&1&0\\0&0&1&1\end{pmatrix} \longmapsto \begin{pmatrix}1&1&0&0\\0&0&1&1\\1&1&0&0\\0&0&1&1\end{pmatrix} \longmapsto \begin{pmatrix}1&1&0&0\\0&0&1&1\\1&0&1&0\\0&1&0&1\end{pmatrix}
$$

$$
\longmapsto \begin{pmatrix}1&1&0&0\\0&0&1&1\\1&0&0&1\\0&1&1&0\end{pmatrix} \longmapsto \begin{pmatrix}1&1&0&0\\0&0&1&1\\0&1&0&1\\1&0&1&0\end{pmatrix} \longmapsto \begin{pmatrix}1&1&0&0\\0&0&1&1\\0&0&1&1\\1&1&0&0\end{pmatrix}
$$

$$
\longmapsto \begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&1&1\\1&1&0&0\end{pmatrix} \longmapsto \begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&1&1\\1&1&0&0\end{pmatrix} \longmapsto \begin{pmatrix}0&1&0&1\\1&0&1&0\\0&0&1&1\\1&1&0&0\end{pmatrix}
$$

$$
\longmapsto \begin{pmatrix}0&0&1&1\\1&1&0&0\\0&0&1&1\\1&1&0&0\end{pmatrix} \longmapsto \begin{pmatrix}0&0&1&1\\1&0&1&0\\0&1&0&1\\1&1&0&0\end{pmatrix} \longmapsto \begin{pmatrix}0&0&1&1\\1&0&0&1\\0&1&1&0\\1&1&0&0\end{pmatrix}
$$

$$
\longmapsto \begin{pmatrix}0&0&1&1\\0&1&0&1\\1&0&1&0\\1&1&0&0\end{pmatrix} \longmapsto \begin{pmatrix}0&0&1&1\\0&0&1&1\\1&1&0&0\\1&1&0&0\end{pmatrix}.
$$

References

[1] A. Barvinok, On the number of matrices and a random matrix with prescribed row and column sums and 01 entries, Adv. Math. 224 (1) (2010), 316-339.

[2] A. Berliner, R. A. Brualdi, L. Deaett, K. P. Kiernan, M. Schroeder, Row and column orthogonal $(0, 1)$–matrices, Linear Algebra Appl. 429 (11-12) (2008), 2732-2745.

[3] R. A. Brualdi, Combinatorial Matrix Classes, Encyclopedia of Mathematics and its Applications 108, Cambridge University Press, Cambridge (2006).

[4] R. A. Brualdi, Algorithms for constructing $(0, 1)$–matrices with prescribed row and column sum vectors, Discrete Math. 306 (23) (2006), 3054-3062.

[5] R. A. Brualdi, Matrices of zeros and ones with fixed row and column sum vectors, Linear Algebra Appl. 33 (1980), 159-231.

[6] R. A. Brualdi, G. Dahl, The Bruhat shadow of a permutation matrix, Mathematical papers in honour of E. Marques de Sá, 25-38, Textos Mat. Sér. B, 39, Univ. Coimbra, Coimbra (2006).
[7] R. A. Brualdi, L. Deaett, More on the Bruhat order for $(0,1)$–matrices, Linear Algebra Appl. 421 (2-3) (2007), 219-232.
[8] R. A. Brualdi, S.-G. Hwang, A Bruhat order for the class of $(0,1)$–matrices with row sum vector R and column sum vector S, Electron. J. Linear Algebra 12 (2004/05), 6-16.
[9] E. R. Canfield, B. D. McKay, Asymptotic enumeration of dense 0-1 matrices with equal row sums and equal column sums, Electron. J. Combin. 12 (2005), Research Paper 29, 31 pp. (electronic).
[10] J. A. Dias da Silva, A. Fonseca, Constructing integral matrices with given line sums, Linear Algebra Appl. 431 (9) (2009), 1553-1563.
[11] C. M. da Fonseca, R. Mamede, On $(0,1)$–matrices with prescribed row and column sum vectors, Discrete Math. 309 (8) (2009), 2519-2527.
[12] D. Gale, A theorem on flows in networks, Pacific. J. Math. 7 (1957), 1073-1082.
[13] C. Greenhill, B. D. McKay, Asymptotic enumeration of sparse nonnegative integer matrices with specified row and column sums, Adv. in Appl. Math. 41 (4) (2008), 459481.
[14] C. Greenhill, B. D. McKay, X. Wang, Asymptotic enumeration of sparse 0-1 matrices with irregular row and column sums, J. Combin. Theory Ser. A 113 (2) (2006), 291-324.
[15] M. Krause, A simple proof of the Gale-Ryser theorem, Amer. Math. Monthly 103 (1996), 335-337.
[16] B. D. McKay, X. Wang, Asymptotic enumeration of 0-1 matrices with equal row sums and equal column sums, Linear Algebra Appl. 373 (2003), 273-287.
[17] H. J. Ryser, Combinatorial properties of matrices of zeros and ones, Canad. J. Math. 9 (1957), 371-377.
[18] N. J. A. Sloane, The on-line encyclopedia of integer sequences, Notices Amer. Math. Soc. 50 (8) (2003), 912-915.
[19] `http://oeis.org/A058527`

Alessandro Conflitti
CMUC, Centre for Mathematics, University of Coimbra, Apartado 3008, 3001–454 Coimbra, Portugal

E-mail address: `conflitt@mat.uc.pt`

C. M. da Fonseca
CMUC, Department of Mathematics, University of Coimbra, 3001–454 Coimbra, Portugal

E-mail address: `cmf@mat.uc.pt`

Ricardo Mamede
CMUC, Department of Mathematics, University of Coimbra, 3001–454 Coimbra, Portugal

E-mail address: `mamede@mat.uc.pt`