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INTRINSIC SCHREIER-MAC LANE EXTENSION THEOREM II: THE CASE OF ACTION ACCESSIBLE CATEGORIES

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Abstract: In [11], the Schreier-Mac Lane extension theorem in the category *Gp* of groups, which describes a simply transitive action on a certain indexed class of non-abelian extensions, was extended to any exact pointed protomodular category with split extension classifiers. We show here that the same scheme of proofs allows us to extend it to any exact action accessible category in the sense of [13], which includes the case of any category of interest in the sense of [21].

Keywords: extension with non-abelian kernel, protomodular and action accessible category, split extension classifier, profunctor.
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1. Introduction

Any extension of groups:

$$1 \longrightarrow K \overset{k}{\rightarrowtail} X \overset{f}{\twoheadrightarrow} Y \longrightarrow 1$$

determines, via conjugation in X, a group homomorphism $\phi : Y \to AutK/IntK$ called the *abstract kernel* of this extension. It must be considered as an *indexation* of this extension: if we denote by $Ext_\phi(Y, K)$ the set of all isomorphic classes of extensions with abstract kernel ϕ, the Schreier-Mac Lane extension theorem [19] shows that there is on $Ext_\phi(Y, K)$ a simply transitive action of the abelian group $Ext_{\bar{\phi}}(Y, ZK)$, where ZK is the center of K and $\bar{\phi}$ is induced by ϕ. This allows us evidently to recover the whole set $Ext_\phi(Y, K)$ from the only original extension f.

In [11], the first author showed that this same kind of simply transitive action still holds in any *action representative* category ([5], [6], [4]), namely pointed protomodular category in which, for any object K, there is a split

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extension:

$$\begin{tikzcd} K \arrow[r, tail, "\gamma"] & D_1K \arrow[r, shift left, two heads, "d_0"] & DK \arrow[l, shift left, tail, "s_0"] \end{tikzcd}$$

which is universal (i.e. *a split extension classifier*) in the sense that any other split extension with kernel K:

$$\begin{tikzcd} K \arrow[r, tail, "k"] & H \arrow[r, shift left, two heads, "f"] & G \arrow[l, shift left, tail, "s"] \end{tikzcd}$$

determines a unique pair of morphisms (χ_0, χ_1) such that the following diagram commutes:

$$\begin{tikzcd} K \arrow[r, tail, "k"] \arrow[d, equal] & H \arrow[r, shift left, two heads, "f"] \arrow[d, "\chi_1"'] & G \arrow[l, shift left, tail, "s"] \arrow[d, "\chi_0"] \\ K \arrow[r, tail, "\gamma"'] & D_1K \arrow[r, shift left, two heads, "d_0"] & DK \arrow[l, shift left, tail, "s_0"] \end{tikzcd}$$

The main examples of action representative categories are the category Gp of groups, where $DK = AutK$, and R-Lie of R-Lie algebras, where $DK = DerK$. Many other examples are given in [6], [4], [3]. Starting from our initial extension, the following split extension, where $R[f]$ denote the kernel equivalence relation associated with f:

$$\begin{tikzcd} K \arrow[r, tail] & R[f] \arrow[r, shift left, two heads, "p_0"] & X \arrow[l, shift left, tail, "s_0"] \end{tikzcd}$$

determines a map $\chi : X \to DK$ which allows to make explicit the indexation ϕ:

$$\begin{tikzcd} K \arrow[r, tail, "k"] \arrow[d, equal] & X \arrow[r, two heads, "f"] \arrow[d, "\chi"] & Y \arrow[d, "\phi"] \\ K \arrow[r, "j_K"'] & DK \arrow[r, two heads, "q_K"'] & Q_K \end{tikzcd}$$

Once obtained this indexation, the main observation to obtain the simply transitive action was then to exhibit a meaningful connection between extensions and a certain class of profunctors [2] called *pretorsors*; a connection which, by the way, actually holds in the much larger context of exact Mal'cev categories.

On the other hand, the notion of *action accessible* category was recently introduced in [13]: first was defined the notion of *faithful* split extension, namely such that there is at most one morphism of split extensions with same kernel into it; then the action accessible categories were defined as those pointed protomodular categories which have *enough* faithful split extensions: any split extension has a morphism towards a faithful split extension with same kernel. Clearly split extension classifiers are faithful, and action representative categories are action accessible.

The categories Rg of non-unitary rings and R-Alg of associative R-algebras are examples of action accessible categories which are not action representative, see [13]. The second author showed in [20] that any category of interest in the sense of [21] is action accessible, which includes some of the new notions of algebras introduced by Loday, as the Leibniz algebras [16] and the associative dialgebras [17] and trialgebras [18].

The aim of this article is to show that, in any action accessible category, it is still possible to associate an indexation with any extension and that, modulo the way of producing indexations which is evidently more diversified in the action accessible context, the same method and scheme of proofs allow to extend easily the Schreier-Mac Lane extension theorem from the action representative categories to the much larger field of action accessible categories. As a collateral benefit, this larger context led us to a more lucid proof of this theorem, see Section 4.1 which deals with the canonical simply transitive action on $\underline{Z}_1$-torsors when $\underline{Z}_1$ is an aspherical abelian groupoid: this is, in the Mal'cev context, a natural generalization of the well known abelian group structure on A-torsors when A is an abelian group.

2. Action accessible and groupoid accessible categories

2.1. Faithful split extensions. Let $\mathbb{D}$ be a pointed protomodular category [8]. Let K be an object in $\mathbb{D}$, and consider a split extension:

$$\begin{tikzcd} K \arrow[r, tail, "k"] & H \arrow[r, shift left, two heads, "f"] & G \arrow[l, shift left, tail, "s"] \end{tikzcd}$$

Recall from [13] the following:

Definition 2.1. *This split extension is said to be faithful when, given any other split extension with kernel K as in the upper line below:*

$$\begin{tikzcd} K \arrow[r, tail, "\bar{k}"] \arrow[d, equal] & M \arrow[r, shift left, two heads, "g"] \arrow[d, dotted, "\chi_1"'] & N \arrow[l, shift left, tail, "t"] \arrow[d, dotted, "\chi_0"] \\ K \arrow[r, tail, "k"] & H \arrow[r, shift left, two heads, "f"] & G \arrow[l, shift left, tail, "s"] \end{tikzcd}$$

there is at most one morphism (χ_0, χ_1) *of split extensions.*

The right hand side square above is necessarily a pullback and, the pair (k, t) being jointly strongly epic by definition of protomodularity, the map χ_1 when it exists is entirely determined by the map χ_0. In the category Gp of groups a split extension is faithful if and only if the associated action $G \to AutK$ is faithful. In the category R-Lie of R-Lie algebras a split extension is faithful if and only if the associated morphism $G \to DerK$ is a monomorphism.

Definition 2.2. *The category* $\mathbb{D}$ *is said to be action accessible when there is enough faithful split extensions; namely, given any split extension:*

$$K \overset{\bar{k}}{\rightarrowtail} M \underset{t}{\overset{g}{\rightleftarrows}} N$$

there exists a faithful split extension:

$$K \overset{k}{\rightarrowtail} H \underset{s}{\overset{f}{\rightleftarrows}} G$$

and a morphism into it as above.

As recalled in the introduction: any split extension classifier is faithful and any action representative category is action accessible; the category Rg of non-unitary rings is action accessible and not action representative, see [13]; the second author showed in [20] that any category of interest in the sense of [21] is action accessible. This allows, in particular, to include the new notions of algebras introduced by Loday, as the Leibniz algebras [16] and the associative dialgebras [17] and trialgebras [18].

2.2. Normalization functor and X-groupoids. Let $Grd\mathbb{D}$ denote the category of internal groupoids in $\mathbb{D}$. An internal groupoid $\underline{Z}_1$ in $\mathbb{D}$ will be presented (see [7]) as a reflexive graph $Z_1 \rightrightarrows Z_0$ endowed with an operation ζ_2:

$$\begin{tikzcd} R^2[z_0] \arrow[r, shift left=2, "p_2"] \arrow[r, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[r, bend left=60, "R(\zeta_2)"] & R[z_0] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[r, bend left=60, "\zeta_2"] & Z_1 \arrow[r, shift left=2, "z_1"] \arrow[r, shift right=2, "z_0"'] & Z_0 \arrow[l, "s_0"'] \end{tikzcd}$$

making the previous diagram satisfy all the simplicial identities (including the ones involving the degeneracies). By $R[z_0]$ we denote the kernel equivalence relation of the map z_0. In the set theoretical context, this operation ζ_2 associates the composite $\psi.\phi^{-1}$ with any pair (ϕ,ψ) of arrows with same domain.

Recall that an internal functor $\underline{f}_1 : \underline{Z}_1 \to \underline{W}_1$ is a *discrete fibration* between two groupoids whenever any commutative square of the underlying diagram defining the functor $\underline{f}_1$ is a pullback. We denote by $()_0 : Grd\mathbb{D} \to \mathbb{D}$ the forgetful functor associating with any groupoid $\underline{Z}_1$ its "object of objects" Z_0: it is a fibration. Any fibre $Grd_X\mathbb{D}$ above an object X has an initial object ΔX, namely, the discrete equivalence relation on X, and a final object ∇X, namely, the indiscrete equivalence relation on X. Recall the following:

Definition 2.3. *The normalization of the groupoid $\underline{Z}_1$ is the map $n\underline{Z}_1 = z_1.k_0$ where k_0 is the kernel of z_0:*

$$\begin{tikzcd} N\underline{Z}_1 \arrow[r, tail, "k_0"] \arrow[d] & Z_1 \arrow[r, "z_1"] \arrow[d, "z_0"'] & Z_0 \\ 0 \arrow[r, tail, "\alpha_{Z_0}"'] & Z_0 & \end{tikzcd}$$

Given an object X in $\mathbb{D}$, the groupoid $\underline{Z}_1$ is said to be a X-groupoid when $N\underline{Z}_1 = X$. A morphism of X-groupoid is a functor which induces the morphism 1_X on X.

The normalization of an equivalence relation gives rise to a (normal) monomorphism. A X-morphism between two X-groupoids is necessarily a discrete fibration. On the other hand this normalization process defines a left exact functor $N : Grd\mathbb{D} \to \mathbb{D}$ such that $N\Delta X = 0$. We have also $n_1\nabla X = 1_{\nabla X}$, which implies $N\nabla X = X$. The left exact normalization functor N is moreover:
1) such that the image of any discrete fibration is an isomorphism
2) a right adjoint to the fully faithful functor ∇.

Basic facts about the categorical notion of commutator can be found in [22] and [12]; the next definitions and results are borrowed from [13]:

Definition 2.4. *An internal groupoid $\underline{Z}_1$ is called faithful when the following split exact sequence is faithful:*

$$\begin{tikzcd} N\underline{Z}_1 \arrow[r, tail, "k_0"] & Z_1 \arrow[r, shift left, two heads, "z_0"] & Z_0 \arrow[l, shift left, tail, "s_0"] \end{tikzcd}$$

Definition 2.5. *A pointed protomodular category $\mathbb{D}$ is said to be groupoid accessible, if there is enough faithful groupoids; namely if, for any X-groupoid $\underline{T}_1$, there exists a faithful X-groupoid $\underline{Z}_1$ and a X-morphism $\underline{\chi}_1 : \underline{T}_1 \to \underline{Z}_1$ into it. We shall call this morphism an index of $\underline{T}_1$.*

Proposition 2.1. *Suppose $\mathbb{D}$ is homological (i.e. pointed protomodular and regular), then it is groupoid accessible if and only if it is action accessible.*

Theorem 2.1. *Suppose $\mathbb{D}$ is an action accessible homological category. Let R be an equivalence relation on an object X which, as an internal groupoid, is a K-groupoid. Let $\underline{\chi}_1 : R \to \underline{Z}_1$ be any of its index. Then $R[\chi_0]$ is the centralizer ZR of R, i.e. the largest equivalence relation S on X such that $[S, R] = 0$.*

Proposition 2.2. *Suppose $\mathbb{D}$ is an action accessible homological category. Let R and S be two equivalence relations on an object X. Let $NR \rightarrowtail X$ and $NS \rightarrowtail X$ be their respective normalizations. Then we have $[R, S] = 0$ if and only if we have $[\![NR, NS]\!] = 0$, i.e. if and only the two subobjects NR and NS of X commute in $\mathbb{D}$. In particular, when R is an equivalence relation on an object X, it is abelian (i.e. $[R, R] = 0$) if and only if its normalization $NR \rightarrowtail X$ is an abelian subobject of X.*

3. The indexation of an extension

Our main observation, here, is the following:

Proposition 3.1. *Suppose $\mathbb{D}$ is an action accessible homological category. Let R be an equivalence relation on an object X which, as an internal groupoid, is a K-groupoid. Given any pair $(\underline{\chi}_1, \underline{\chi}'_1)$ of its indexes, their regular epimorphic part is the same. Moreover the codomain of this regular epimorphic part is still a faithful K-groupoid.*

Proof: The kernel equivalence relations $R[\chi_0]$ and $R[\chi'_0]$ are the same since they are the centralizer ZR of R. Accordingly the morphisms χ_0 and χ'_0 have the same regular epimorphic part $q_0 : X \twoheadrightarrow Q_0$. Since $\underline{\chi}_1$ and $\underline{\chi}'_1$ are discrete fibrations, we have also $R[\chi_1] = R[\chi'_1]$, and consequently the morphisms χ_1 and χ'_1 have the same regular epimorphic part $q_1 : R \twoheadrightarrow Q_1$. This produces a decomposition of the discrete fibration $\underline{\chi}_1$ in two discrete fibrations:

$$\begin{tikzcd}
R \arrow[r, two heads, "q_1"] \arrow[rr, bend left, "\chi_1"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & Q_1 \arrow[r, tail, "m_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Z_1 \arrow[d, shift right=2, "z_0"'] \arrow[d, shift left=2, "z_1"] \\
X \arrow[u] \arrow[r, two heads, "q_0"'] \arrow[rr, bend right, "\chi_0"'] & Q_0 \arrow[u] \arrow[r, tail, "m_0"'] & Z_0 \arrow[u]
\end{tikzcd}$$

The groupoid $\underline{Q}_1$, as a fibrant subobject of the faithful K-groupoid $\underline{Z}_1$, is itself a faithful K-groupoid. ■

This discrete fibration $\underline{q}_1$ appears as a universal index associated with R which will be called *the index* of R.

3.1. The indexation. In this section, we shall suppose that *the category* $\mathbb{D}$ *is an exact pointed protomodular action accessible category.* Let us start with an extension:

$$1 \longrightarrow K \overset{k}{\rightarrowtail} X \overset{f}{\twoheadrightarrow} Y \longrightarrow 1$$

First recall that, since $\mathbb{D}$ is protomodular, any morphism of extensions between Y and K is necessarily an isomorphism. Then recall that the normalization of the kernel equivalence relation $R[f]$:

$$\begin{tikzcd}
R[f] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] & X \arrow[l] \arrow[r, dotted, "f"] & Y
\end{tikzcd}$$

is the kernel $k : K \rightarrowtail X$ of f, so that $R[f]$ is a K-groupoid. Let us consider its index $\underline{q}_1 : R[f] \to \underline{Q}_1$. Since $\mathbb{D}$ is exact, then the groupoid $\underline{Q}_1$ admits a $\pi_0(\underline{Q}_1)$, namely the coequalizer of the pair (d_0, d_1) below. Whence a factorization $\phi : Y \to \pi_0(\underline{Q}_1)$ which is necessarily a regular epimorphism:

$$\begin{tikzcd}
R[f] \arrow[r, two heads, "q_1"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & Q_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
X \arrow[u] \arrow[r, two heads, "q_0"'] \arrow[d, "f"'] & Q_0 \arrow[u] \arrow[d, "q"] \\
Y \arrow[r, two heads, "\phi"'] & \pi_0(\underline{Q}_1)
\end{tikzcd}$$

Definition 3.1. *We shall call this morphism* ϕ the indexation *of the extension* (f, k) *in question. We shall denote by* $Ext_\phi(Y, K)$ *the set of all isomorphic classes of extensions:*

$$1 \longrightarrow K \overset{k'}{\rightarrowtail} X' \overset{f'}{\twoheadrightarrow} Y \longrightarrow 1$$

such that the kernel equivalence $R[f']$ *has the faithful* K*-groupoid* $\underline{Q}_1$ *as codomain of its index and the regular epimorphism* ϕ *as indexation.*

The aim of this article is to show that the set $Ext_\phi(Y, K)$ is canonically endowed with a simply transitive action of a certain abelian group, let say B_ϕ. For that, consider the following diagram where the right hand side part is made of pullbacks:

$$\begin{tikzcd}
R[f] \arrow[r, dotted, "f_{1\phi}"] \arrow[rr, bend left, "q_1"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & D_1\phi \arrow[r, two heads, "d_{1\phi}"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & Q_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
X \arrow[r, dotted, "f_\phi"] \arrow[rr, bend left, "q_0"] \arrow[dr, "f"'] & D_0\phi \arrow[r, two heads, "d_{0\phi}"] \arrow[d, "q_\phi"] & Q_0 \arrow[d, "q"] \\
 & Y \arrow[r, two heads, "\phi"'] & \pi_0(\underline{Q}_1)
\end{tikzcd}$$

In other words, consider the image of the K-groupoid $\underline{Q}_1$ along the change of base functor ϕ^*. This produces a K-groupoid $\underline{D}_{1\phi}$ such that $\pi_0(\underline{D}_{1\phi}) = Y$. Moreover the two upper internal functors $\underline{f}_{1\phi}$ and $\underline{d}_{1\phi}$ are discrete fibrations as is the functor $\underline{q}_1$.

Proposition 3.2. *The morphism* f_ϕ *is a regular epimorphism.*

Proof: It is a consequence of Lemma A.1 in [11]. ■

The next section will show how to extract the desired abelian group B_ϕ from the internal groupoid $\underline{D}_{1\phi}$.

4. $\underline{Z}_1$-torsors

Let $\mathbb{E}$ be now any finitely complete regular [1] category and let be given an *aspherical* internal groupoid $\underline{Z}_1$, namely such that the object Z_0 has a global support in $\mathbb{E}$ and the groupoid $\underline{Z}_1$ has a global support in its fibre $Grd_{Z_0}\mathbb{E}$, i.e. that the morphism $(z_0, z_1) : Z_1 \to Z_0 \times Z_0$ is a regular epimorphism.

Definition 4.1. *Let $\underline{Z}_1$ be an aspherical groupoid. A $\underline{Z}_1$-torsor is an object U in $\mathbb{E}$ with global support together with a discrete fibration $\nabla U \to \underline{Z}_1$:*

$$\begin{tikzcd} U \times U \arrow[r, "\tau_1"] \arrow[d, shift right=3, "p_0"'] \arrow[d, shift left=3, "p_1"] & Z_1 \arrow[d, shift right=3, "z_0"'] \arrow[d, shift left=3, "z_1"] \\ U \arrow[u, dotted] \arrow[r, "\tau"'] & Z_0 \arrow[u, dotted] \end{tikzcd}$$

When the groupoid $\underline{Z}_1$ is a group G, we recover the classical notion of G-torsor [1]. What is meaningful for us is that the groupoid $\underline{D}_{1\phi}$ is aspherical in the slice category $\mathbb{D}/Y$ and that the functor $\underline{f}_{1\phi} : R[f] \twoheadrightarrow \underline{D}_{1\phi}$ above is a $\underline{D}_{1\phi}$-torsor in this category.

Proposition 4.1. *A morphism of $\underline{Z}_1$-torsors is necessarily an isomorphism.*

Proof: The proof is the same as the proof of Proposition 1.3 in [11]. ■

We shall denote by $Tors\underline{Z}_1$ the set of isomorphic classes of $\underline{Z}_1$-torsors. The construction given at the end of the last section produces a mapping which is clearly injective:

$$\Theta : Ext_\phi(Y, K) \to Tors\underline{D}_{1\phi}$$

Theorem 4.1. *The mapping Θ is bijective.*

Proof: Let $\underline{g}_1 : R[f'] \twoheadrightarrow \underline{D}_{1\phi}$ be a $\underline{D}_{1\phi}$-torsor in $\mathbb{D}/Y$. Then the following diagram:

$$\begin{tikzcd} R[f'] \arrow[r, two heads, "g_1"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & D_1\phi \arrow[r, two heads, "d_{1\phi}"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Q_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\ X' \arrow[r, two heads, "g_0"] \arrow[dr, "f'"'] & D_0\phi \arrow[r, two heads, "d_{0\phi}"] \arrow[d, "q_\phi"] & Q_0 \arrow[d, "q"] \\ & Y \arrow[r, two heads, "\phi"'] & \pi_0(\underline{Q}_1) \end{tikzcd}$$

shows that the discrete fibration $\underline{d}_{1\phi}.\underline{g}_1$ is the index of $R[f']$ since the groupoid $\underline{Q}_1$ is faithful by construction. Accordingly $R[f']$ is a K-groupoid in the same way as $\underline{Q}_1$, so that the kernel of f' is K. Moreover the indexation of the extension (f', k') is then the factorization ϕ. ■

From this bijection, the Schreier-Mac Lane theorem will be the consequence of a very general phenomenon which we are now going to describe.

4.1. The canonical action on $Tors\underline{Z}_1$. We shall suppose in this section that we work in a finitely complete exact Mal'cev ([14], [15]) category $\mathbb{D}$. Recall that any protomodular category is a Mal'cev category.

Now, let $\underline{Z}_1$ be an aspherical groupoid. The category $\mathbb{D}$ being a Mal'cev category, the groupoid $\underline{Z}_1$ is necessarily abelian, where the word abelian means that $\underline{Z}_1$ is an abelian object in the protomodular fibre $Grd_{Z_0}\mathbb{D}$, see [9].

Since $\mathbb{D}$ is exact, then the aspherical abelian groupoid $\underline{Z}_1$ has a *direction* [9], namely there exists an abelian group A in $\mathbb{D}$ which makes pullbacks the following upper squares:

$$\begin{tikzcd}
R[(z_0,z_1)] \arrow[r, two heads, "\nu_{\underline{Z}_1}"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & A \arrow[d, shift right] \\
Z_1 \arrow[u] \arrow[r, two heads] \arrow[d, "(z_0,z_1)"'] & 1 \arrow[u, shift right, "e"'] \\
Z_0 \times Z_0 &
\end{tikzcd}$$

In the set theoretical context, a groupoid $\underline{Z}_1$ is abelian when, for any of its object z, the group Aut_z of endomaps at z is abelian. The groupoid $\underline{Z}_1$ is apsherical, when it is non empty and connected. So, when the groupoid $\underline{Z}_1$ is aspherical and abelian, all the abelian groups Aut_z are isomorphic. Moreover, what is remarkable is that, given any map $\tau : z \to z'$ in $\underline{Z}_1$, the induced group homomorphism $Aut_z \to Aut_{z'}$ is independent of the choice of the map τ. The direction A of $\underline{Z}_1$ is then any of these abelian groups.

Suppose that we have $Z_0 = 1$, namely that the asperical abelian groupoid $\underline{Z}_1$ is actually an abelian group A. It is well known that the set $TorsA$ is canonically endowed with an abelian group structure, see [1] for instance. In the Mal'cev context, we are now going to give a natural generalization of this observation.

Let us begin by some notations. When G is an internal group, it can be obviously understood as an internal groupoid with the terminal object 1 as object of objects; we shall denote it by $\underline{K}_1G$. When moreover $G = A$ is abelian, the groupoid $\underline{K}_1A$ is naturally endowed with a structure of abelian group in $Grd\mathbb{D}$. Furthermore the previous pullback diagram becomes a pullback diagram in $Grd\mathbb{D}$:

$$\begin{tikzcd}
\underline{Z}_1 \times_0 \underline{Z}_1 \arrow[r, two heads, "\underline{\nu}_1\underline{Z}_1"] \arrow[d, shift right=2, "\underline{p_0}_1"'] \arrow[d, shift left=2, "\underline{p_1}_1"] & \underline{K}_1 A \arrow[d, shift right] \\
\underline{Z}_1 \arrow[r, two heads] \arrow[u] \arrow[d] & 1 \arrow[u, shift right] \\
\underline{\nabla}_1 Z_0 &
\end{tikzcd}$$

where the upper left hand side object is nothing but the product in the fibre $Grd_{Z_0}\mathbb{D}$. Eventually it is underlying a discrete fibration inside $Grd\mathbb{D}$:

$$\begin{tikzcd}
\underline{Z}_1 \times_0 \underline{Z}_1 \times_0 \underline{Z}_1 \arrow[r, two heads, "\underline{\overline{\nu}}_1\underline{Z}_1"] \arrow[d, shift right=3, "\underline{p_0}_1"'] \arrow[d, "\underline{p_1}_1"] \arrow[d, shift left=3, "\underline{p_2}_1"] & \underline{K}_1 A \times \underline{K}_1 A \arrow[d, shift right=3, "\underline{p_0}_1"'] \arrow[d, "+"] \arrow[d, shift left=3, "\underline{p_1}_1"] \\
\underline{Z}_1 \times_0 \underline{Z}_1 \arrow[r, two heads, "\underline{\nu}_1\underline{Z}_1"] \arrow[d, shift right=2, "\underline{p_0}_1"'] \arrow[d, shift left=2, "\underline{p_1}_1"] & \underline{K}_1 A \arrow[d, shift right] \\
\underline{Z}_1 \arrow[r, two heads] \arrow[u] & 1 \arrow[u, shift right]
\end{tikzcd}$$

Theorem 4.2. *Let $\mathbb{D}$ be an exact Mal'cev category. Let $\underline{Z}_1$ be an aspherical groupoid with direction A. Then there is a canonical simply transitive action of the abelian group $TorsA$ on the set $Tors\underline{Z}_1$.*

Proof: To have a simply transitive action of an abelian group A on a set S is equivalent to have a mapping: $\Phi : S \times S \longrightarrow A$, such that 1) $\Phi(u,v) + \Phi(v,w) = \Phi(u,w)$, 2) $\Phi(u,v) = 0$ if and only if $u = v$, and 3) for any pair $(u,a) \in S \times A$, there is a $v \in S$ such that $\Phi(u,v) = a$.

Let be given any pair $(\underline{\tau}_1, \underline{\theta}_1)$ of $\underline{Z}_1$-torsors. We shall first consider the following upper horizontal factorization $\underline{\pi}_1(U,V)$ in $Grd\mathbb{D}$, where the downward quadrangle is a pullback and where $\underline{p_0}_1.\underline{\pi}_1(U,V) = \underline{\tau}_1.\underline{\nabla}_1 p_U$ and $\underline{p_1}_1.\underline{\pi}_1(U,V) = \underline{\theta}_1.\underline{\nabla}_1 p_V$:

$$\begin{tikzcd}
\underline{\nabla}_1(U \times_0 V) \arrow[rr, dashed, "\underline{\pi}_1(U,V)"] \arrow[d, "\underline{\nabla}_1 p_U"'] \arrow[dr, "\underline{\nabla}_1 p_V"] & & \underline{Z}_1 \times_0 \underline{Z}_1 \arrow[r, two heads, "\underline{\nu}_1\underline{Z}_1"] \arrow[d, shift right, "\underline{p_0}_1"'] \arrow[d, shift left, "\underline{p_1}_1"] & \underline{K}_1 A \\
\underline{\nabla}_1 U \arrow[rr, bend left=15, "\underline{\tau}_1"] \arrow[dr, "\underline{\nabla}_1 \tau_0"'] & \underline{\nabla}_1 V \arrow[r, "\underline{\theta}_1"] \arrow[d] & \underline{Z}_1 \arrow[dl] & \\
& \underline{\nabla}_1 Z_0 & &
\end{tikzcd}$$

The object $U \times_0 V$ being defined by the following pullback in $\mathbb{D}$:

$$\begin{tikzcd}
U \times_0 V \arrow[r, two heads, "p_V"] \arrow[d, "p_U"'] & V \arrow[d, "\theta_0"] \\
U \arrow[r, two heads, "\tau_0"'] & Z_0
\end{tikzcd}$$

the object $U \times_0 V$ has global support since U and V have global support. Then we define the A-torsors $\Phi(\underline{\tau}_1, \underline{\theta}_1)$ as the universal factorization $\underline{\psi}_1$ of the functor $\underline{\nu}_1\underline{Z}_1.\underline{\pi}_1(U,V) : \underline{\nabla}_1(U \times_0 V) \to \underline{K}_1 A$ through a discrete fibration (let say $\underline{\psi}_1$):

$$\underline{\nabla}_1(U \times_0 V) \xrightarrow{\underline{\nabla}_1 h} \underline{\nabla}_1 F \xrightarrow{\underline{\psi}_1} \underline{K}_1 A$$

This discrete fibration $\underline{\psi}_1$ does exist in $Grd\mathbb{D}$ since $\mathbb{D}$ is exact, see [7]. The object $U \times_0 V$ having global support, the object F has global support and, consequently, $\underline{\psi}_1$ defines a A-torsor. It remains to show that this Φ satisfies the previous three axioms.

For that let us denote by $\underline{\nabla}_1/\underline{Z}_1$ the category whose objects are the functors $\underline{\nabla}_1 T \to \underline{Z}_1$ where T has global support and by $Fib/\underline{Z}_1$ its full subcategory $Fib/\underline{Z}_1 \hookrightarrow \underline{\nabla}_1/\underline{Z}_1$ whose objects are the discrete fibration $\underline{\nabla}_1 U \to \underline{Z}_1$ where U has global support, i.e. the $\underline{Z}_1$-torsors. The universal factorization through discrete fibrations mentioned above determines an adjoint functor to this inclusion, which gives us an isomorphism between the set $Tors\underline{Z}_1$ and the set $\pi_0(\underline{\nabla}_1/\underline{Z}_1)$ of connected components of the category $\underline{\nabla}_1/\underline{Z}_1$.

1) Let be given $(\underline{\tau}_1, \underline{\theta}_1, \underline{\chi}_1)$ any triple of $\underline{Z}_1$-torsors. Then consider the following diagram in $Grd\mathbb{D}$ where every square commutes:

$$\begin{tikzcd}
\underline{\nabla}_1(U \times_0 V \times_0 V \times_0 W) \arrow[r, "\underline{\pi}_1(U,V) \times_0 \underline{\pi}_1(V,W)"] & \underline{Z}_1 \times_0 \underline{Z}_1 \times_0 \underline{Z}_1 \times_0 \underline{Z}_1 \arrow[dr, "\underline{\nu}_1\underline{Z}_1 \times_0 \underline{\nu}_1\underline{Z}_1"] & \\
\underline{\nabla}_1(U \times_0 V \times_0 W) \arrow[u, "\underline{\nabla}_1(1 \times_0 s_0 \times_0 1)"] \arrow[r, "\underline{\pi}_1(U,V,W)"] \arrow[d, "\underline{\nabla}_1 p(U,W)"'] & \underline{Z}_1 \times_0 \underline{Z}_1 \times_0 \underline{Z}_1 \arrow[u, "1 \times_0 s_0 \times_0 1"'] \arrow[r, "\overline{\nu}_1\underline{Z}_1"] \arrow[d, "\underline{p_1}_1"] & \underline{K}_1 A \times \underline{K}_1 A \arrow[d, "+"] \\
\underline{\nabla}_1(U \times_0 W) \arrow[r, "\underline{\pi}_1(U,W)"'] & \underline{Z}_1 \times_0 \underline{Z}_1 \arrow[r, "\underline{\nu}_1\underline{Z}_1"'] & \underline{K}_1 A
\end{tikzcd}$$

Now $\Phi(\underline{\tau}_1, \underline{\chi}_1)$ is the universal discrete fibration associated with the lower horizontal line, while $\Phi(\underline{\tau}_1, \underline{\theta}_1) + \Phi(\underline{\theta}_1, \underline{\chi}_1)$ is the universal discrete fibration associated with the upper composite $\underline{\nabla}_1(U \times_0 V \times_0 V \times_0 W) \to \underline{K}_1 A$. Accordingly the middle line, which shows that the lower horizontal line and

the upper composite are connected in $\underline{\nabla}_1/\underline{K}_1A$, assures that these discrete fibrations are the same up to isomorphism.
2) The commutation of the following diagram:

$$\begin{tikzcd}
\underline{\nabla}_1(U\times_0 U) \arrow[r, "\underline{\pi}_1(U,U)"] & \underline{Z}_1\times_0 \underline{Z}_1 \arrow[r, "\underline{\nu}_1\underline{Z}_1"] & \underline{K}_1A \\
\underline{\nabla}_1(U) \arrow[u, "\underline{\nabla}_1 s_0"] \arrow[r, "\underline{\tau}_1"'] & \underline{Z}_1 \arrow[u, "s_0"] \arrow[r] & 1 \arrow[u, "0"']
\end{tikzcd}$$

shows that $\Phi(\underline{\tau}_1,\underline{\tau}_1)=0$. Conversely, when $\Phi(\underline{\tau}_1,\underline{\theta}_1)=0$, this means that $\underline{\nu}_1\underline{Z}_1.\underline{\pi}_1(U,V)$ and $0:1\to\underline{K}_1A$ are connected in $\nabla_1/\underline{K}_1A$, let us say by a pair $U\times_0 V \stackrel{d}{\leftarrow} D\to 1$; so consider the following commutative diagram in $Grd\mathbb{D}$ where the outer rectangle commutes:

$$\begin{tikzcd}
\underline{\nabla}_1(U\times_0 V) \arrow[r, "\underline{\pi}_1(U,V)"] & \underline{Z}_1\times_0 \underline{Z}_1 \arrow[r, "\underline{\nu}_1\underline{Z}_1"] & \underline{K}_1A \\
\underline{\nabla}_1 D \arrow[u, "\underline{\nabla}_1 d"] \arrow[r, dotted, "t"] \arrow[rr, bend right] & \underline{Z}_1 \arrow[u, "s_0"] \arrow[r] & 1 \arrow[u, "0"']
\end{tikzcd}$$

The right hand side square being a pullback, it produces the dotted factorization t which makes commute the left hand side square which, in turn, produces a commutative diagram in $Grd\mathbb{D}$:

$$\begin{tikzcd}
\underline{\nabla}_1 D \arrow[r, "\underline{\nabla}_1(d_V)"] \arrow[d, "\underline{\nabla}_1(d_U)"'] & \underline{\nabla}_1 V \arrow[d, "\underline{\theta}_1"] \\
\underline{\nabla}_1 U \arrow[r, "\underline{\tau}_1"'] & \underline{Z}_1
\end{tikzcd}$$

This implies that $\underline{\tau}_1$ and $\underline{\theta}_1$ are connected in $\underline{\nabla}_1/\underline{Z}_1$ and consequently, up to isomorphism, the same $\underline{Z}_1$-torsor.
3) Finally, let be given any pair $(\underline{\tau}_1,\underline{\psi}_1)$ of a $\underline{Z}_1$-torsor and a A-torsor. Let us consider the following diagram:

$$\begin{tikzcd}
\underline{\nabla}_1(U\times E) \arrow[rr, "\underline{\nabla}_1 p_E"] \arrow[dr, dotted, "\underline{\zeta}_1"] \arrow[dd, "\underline{\nabla}_1 p_U"'] & & \underline{\nabla}_1 E \arrow[d, "\underline{\psi}_1"] \\
& \underline{Z}_1\times_0 \underline{Z}_1 \arrow[r, "\underline{\nu}_1\underline{Z}_1"] \arrow[d, shift right, "\underline{p_0}_1"'] \arrow[d, dotted, shift left=3, "\underline{p_1}_1"] & \underline{K}_1A \arrow[d, shift right] \\
\underline{\nabla}_1 U \arrow[r, "\underline{\tau}_1"'] & \underline{Z}_1 \arrow[u, shift left] \arrow[r] & 1 \arrow[u, shift right]
\end{tikzcd}$$

The lower right hand side square being a pullback, there is a factorisation $\underline{\zeta}_1$ making the quadrangles commute. The universal factorization of the functor ${\underline{p_1}}_1.\underline{\zeta}_1$ through a discrete fibration (let say $\underline{\theta}_1$):

$$\underline{\nabla}_1(U \times E) \overset{\underline{\nabla}_1 h}{\to} \underline{\nabla}_1 V \overset{\underline{\theta}_1}{\to} \underline{Z}_1$$

gives us the $\underline{Z}_1$-torsor $\underline{\theta}_1$ such that $\underline{\theta}_1.\underline{\nabla}_1 h = {\underline{p_1}}_1.\underline{\zeta}_1$. Now let us consider the following diagram:

$$\begin{tikzcd}
\underline{\nabla}_1(U \times E) \arrow[rr, "\underline{\nabla}_1 p_E"] \arrow[d, dotted, "\underline{\nabla}_1 l"] \arrow[drr, "\underline{\zeta}_1"] \arrow[dd, bend right, "\underline{\nabla}_1 p_U^E"'] & & \underline{\nabla}_1 E \arrow[dr, "\underline{\psi}_1"] & \\
\underline{\nabla}_1(U \times_0 V) \arrow[rr, "\underline{\pi}_1(U,V)"] \arrow[d, "\underline{\nabla}_1 p_U^V"'] \arrow[dr, "\underline{\nabla}_1 p_V"] & & \underline{Z}_1 \times_0 \underline{Z}_1 \arrow[r, two heads, "\underline{\nu}_1 \underline{Z}_1"] \arrow[d, shift right, "{\underline{p_0}}_1"'] \arrow[d, shift left, "{\underline{p_1}}_1"] & \underline{K}_1 A \\
\underline{\nabla}_1 U \arrow[rr, "\underline{\tau}_1"] \arrow[dr, "\underline{\nabla}_1 \tau_0"'] & \underline{\nabla}_1 V \arrow[r, "\underline{\theta}_1"] \arrow[d] & \underline{Z}_1 \arrow[dl] & \\
& \underline{\nabla}_1 Z_0 & &
\end{tikzcd}$$

From the equalities $\underline{\tau}_1.\underline{\nabla}_1 p_U = {\underline{p_0}}_1.\underline{\zeta}_1$ and $\underline{\theta}_1.\underline{\nabla}_1 h = {\underline{p_1}}_1.\underline{\zeta}_1$, we get a factorization $\underline{\nabla}_1 l : \underline{\nabla}_1(U \times E) \to \underline{\nabla}_1(U \times_0 V)$ such that $\underline{\nabla}_1 p_U^V.\underline{\nabla}_1 l = \underline{\nabla}_1 p_U^E$ and $\underline{\nabla}_1 p_V.\underline{\nabla}_1 l = \underline{\nabla}_1 h$. We then get $\underline{\pi}_1(U,V).\underline{\nabla}_1 l = \underline{\zeta}_1$ by composition with the pair $({\underline{p_0}}_1, {\underline{p_1}}_1)$, whence the following commutative square in $\underline{\nabla}_1/\underline{Z}_1$:

$$\begin{tikzcd}
\underline{\nabla}_1(U \times E) \arrow[rr, "\underline{\nabla}_1 p_E"] \arrow[d, "\underline{\nabla}_1 l"'] & & \underline{\nabla}_1 E \arrow[d, "\underline{\psi}_1"] \\
\underline{\nabla}_1(U \times_0 V) \arrow[rr, "\underline{\nu}_1 \underline{Z}_1.\underline{\pi}_1(U,V)"'] \arrow[dr] & & \underline{K}_1 A \\
& \underline{\nabla}_1 F \arrow[ur, "\Phi(\underline{\tau}_1, \underline{\theta}_1)"'] &
\end{tikzcd}$$

which asserts that the discrete fibrations $\Phi(\underline{\tau}_1, \underline{\theta}_1)$ and $\underline{\psi}_1$ are connected in $\underline{\nabla}_1/\underline{K}_1 A$, and consequently isomorphic. ■

This proof is different from the one given in [11] which was a bit lax about the definition of what was called the domain and codomain of a pretorsor which, actually, were only defined *up to isomorphism*.

5. The Schreier-Mac Lane extension theorem

Let us go back to our initial exact pointed protomodular action accessible category $\mathbb{D}$ and to an extension with indexation ϕ as above:

$$1 \longrightarrow K \overset{k}{\rightarrowtail} X \overset{f}{\twoheadrightarrow} Y \longrightarrow 1$$

We get the bijection $\Theta : Ext_\phi(Y,K) \to Tors\underline{D}_{1\phi}$ by Theorem 4.1 in terms of extension. We have now to translate Theorem 4.2. For that, we shall work now in the slice category $\mathbb{D}/Y$, where the groupoid $\underline{D}_{1\phi}$ becomes aspherical. It is abelian, since the category $\mathbb{D}/Y$ is protomodular. Let us denote the direction of $Tors\underline{D}_{1\phi}$, which is an abelian group in $\mathbb{D}/Y$, by the following split epimorphism:

$$\begin{tikzcd} A \arrow[r, tail, dotted, "k_\phi"] & E\phi \arrow[r, shift left, "e_\phi"] & Y \arrow[l, tail, shift left, "s_\phi"] \end{tikzcd}$$

The kernel A of the map e_ϕ in $\mathbb{D}$ is then necessarily an abelian object in $\mathbb{D}$. Let us begin by the following precision:

Proposition 5.1. *The kernel A of the split epimorphism e_ϕ is the center ZK of K.*

Proof: The direction of the groupoid $\underline{D}_{1\phi}$ is clearly the pullback along the map ϕ of the direction of the K-groupoid $\underline{Q}_1$; so that the kernel A of e_ϕ is also the kernel of the direction of $\underline{Q}_1$ which is determined by the following upper pullbacks:

$$\begin{tikzcd} R[(d_0,d_1)] \arrow[r, two heads, "\nu_{\underline{Q}_1}"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & E_{\underline{Q}_1} \arrow[d, shift right, "e_{\underline{Q}_1}"'] \\ Q_1 \arrow[u] \arrow[r, two heads] \arrow[d, "(d_0,d_1)"'] & \pi_0(\underline{Q}_1) \arrow[u, shift right] \\ Q_0 \times Q_0 & \end{tikzcd}$$

Accordingly the kernel of the split epimorphism $e_{\underline{Q}_1}$ is also the kernel of $(d_0,d_1) : Q_1 \to Q_0 \times Q_0$, namely $Kerd_0 \cap Kerd_1$, or in other words the kernel of the normalization $d_1.k_0 : K \to Q_0$ of the groupoid $\underline{Q}_1$. Let us consider now the following diagram:

$$\begin{tikzcd}
K \times K \arrow[r, "\tilde{k}_0"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R[d_0] \arrow[r, "d_2"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & Q_1 \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\
K \arrow[r, "k_0"'] \arrow[d] & Q_1 \arrow[r, "d_1"'] \arrow[d, "d_0"'] & Q_0 \\
1 \arrow[r, tail, "\alpha_{Q_0}"'] & Q_0 &
\end{tikzcd}$$

The upper part of this diagram determines a discrete fibration $\underline{\nabla}_1 K \to \underline{Q}_1$. Since the groupoid $\underline{Q}_1$ is faithful, this discrete fibration is an index of the equivalence relation ∇K, which implies that the kernel equivalence relation $R[d_1.k_0]$ is the largest central equivalence relation of K, according to Proposition 2.2. According to this same proposition, the kernel of $d_1.k_0$ is consequently the center ZK of K. ■

A $E\phi$-torsor in $\mathbb{D}/Y$ is then nothing but an extension in $\mathbb{D}$:

$$1 \longrightarrow ZK \rightarrowtail E \xrightarrow{e} \!\!\!\!\to Y \longrightarrow 1$$

having the abelian group $E\phi$ in $\mathbb{D}/Y$ as *direction*, namely such that the following square is a pullback:

$$\begin{tikzcd}
R[e] \arrow[r, two heads] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & E\phi \arrow[d, shift right, "e_\phi"'] \\
E \arrow[u] \arrow[r, two heads, "e"'] \arrow[d, "e"'] & Y \arrow[u, shift right, "s_\phi"'] \\
Y &
\end{tikzcd}$$

Now, the abelian group $TorsE\phi$ of $E\phi$-torsors in $\mathbb{D}/Y$, according to the present characterization of $E\phi$-torsors in terms of extension and to Section *Baer sums* in [10], is nothing but the abelian group $Ext_{e_\phi}(Y, ZK)$ of extensions with abelian kernel ZK having the abelian group $E\phi$ in $\mathbb{D}/Y$ as direction, when $Ext_{e_\phi}(Y, ZK)$ is endowed with the classical "Baer sum" operation. Finally, through this translation of the ingredients of Theorem 4.2, we get what we were aiming to:

Theorem 5.1. *Suppose $\mathbb{D}$ is an exact pointed protomodular action accessible category. Let*

$$1 \longrightarrow K \overset{k}{\rightarrowtail} X \xrightarrow{f} \!\!\!\!\to Y \longrightarrow 1$$

be any extension with indexation ϕ. *There is on the set* $Ext_\phi(Y, K)$ *a canonical simply transitive action of the abelian group* $B_\phi = Ext_{e_\phi}(Y, ZK)$.

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