Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 11–28

3×3 LEMMA FOR STAR-EXACT SEQUENCES

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Abstract: A regular category is said to be *normal* when it is pointed and every regular epimorphism in it is a normal epimorphism. Any abelian category is normal, and in a normal category one can define short exact sequences in a similar way as in an abelian category. Then, the corresponding 3×3 lemma is equivalent to the so-called subtractivity, which in universal algebra is also known as congruence 0-permutability. In the context of non-pointed regular categories, short exact sequences can be replaced with "exact forks" and then, the corresponding 3×3 lemma is equivalent, in the universal algebraic terminology, to congruence 3-permutability; equivalently, regular categories satisfying the 3×3 lemma are precisely the Goursat categories. We show how these two seemingly independent results can be unified in the context of star-regular categories recently introduced in a joint work of A. Ursini and the first two authors.

1. Introduction

In an abelian category, the 3×3 lemma states that, given a commutative diagram

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

where all three columns and the second row are short exact sequences, the top row is a short exact sequence if and only if so is the bottom row. This can be split up into *upper* and *lower* 3×3 lemmas, where the upper 3×3 lemma states only that the short exactness of the top row follows from the short exactness of the bottom one, and the lower 3×3 lemma states the converse implication. There is also a *middle* 3×3 lemma, which states that if the composite of the two morphisms in the middle row is null and if the

Received October 9th, 2011.
M. Gran supported by F.N.R.S. grant *Crédit aux chercheurs* 1.5.016.10F.
Z. Janelidze partially supported by South African National Research Foundation and Georgian National Science Foundation (GNSF/ST09_730_3-105).
D. Rodelo supported by Centro de Matemática da Universidade de Coimbra.
The authors gratefully acknowledge the kind hospitality of the Université catholique de Louvain.

top and the bottom rows are short exact, then the middle row is also short exact. It was recently proved by the second author [16] that, in any normal category (i.e. a pointed regular category where every regular epimorphism is a normal epimorphism), the upper and the lower 3×3 lemmas are equivalent, and they hold precisely when the normal category is subtractive [15]. The middle 3×3 lemma turns out to be stronger than the other two, and it is equivalent to protomodularity [2].

The "denormalized 3×3 lemma", studied in [4], replaces short exact sequences with exact forks, i.e. kernel pairs of regular epimorphisms. It was shown by S. Lack in [17] that the denormalized 3×3 lemma holds in any Goursat category and, more recently, it was proved by the first and third authors [12] that in a regular category, the denormalized 3×3 lemma is actually equivalent to the category being a Goursat category [6, 5]. Moreover, just as in the normalized case, the upper and the lower denormalized 3×3 lemmas are equivalent.

These two independent works are now brought together in the present article where we revisit them in the categorical context of a *star-regular category* proposed in [11], where it becomes possible to treat the normalized and the denormalized 3×3 lemmas simultaneously. The notion of a star-regular category is in some sense a merger of two notions: that of a *regular category* [1] and that of a category equipped with an *ideal* $\mathcal{N}$ of morphisms [8]. In [11], in a category with an ideal $\mathcal{N}$, we defined a *star* to be an ordered pair of parallel morphisms $[k_1, k_2] : K \rightrightarrows X$, where the first morphism in the pair belongs to $\mathcal{N}$. The *star-kernel* of a morphism $f : X \to Y$ is defined as a universal star such that $fk_1 = fk_2$. Then *star-regularity* refers to the property that every regular epimorphism is a coequalizer of its star-kernel. In the case when $\mathcal{N}$ is the class of null morphisms in a pointed category, which we call the *pointed context*, star-regular categories become precisely the normal categories, since there the notion of a star-kernel reduces to the usual notion of a kernel. In the case when $\mathcal{N}$ is the class of all morphisms, which we call the *total context*, star-kernels are precisely the kernel pairs, and so star-regular categories are the same as regular categories. Background material regarding stars, star-kernels and star-regularity is presented in Section 2 below.

Replacing kernel pairs with star-kernels, we extend from regular categories to star-regular categories the equivalence of the denormalized 3×3 lemma and the Goursat property (see [6, 5]) that composition of kernel pairs is 3-permutable. We achieve this under additional axioms on a star-regular category, which do hold true both in the total and pointed contexts. Applying our result to the pointed context we get precisely the equivalence of subtractivity and the normalized 3×3 lemma in normal categories.

We also extend from normal categories to star-regular categories, the equivalence of the short five lemma and the middle 3×3 lemma (Section 5). Applying this result in the total context we see that the denormalized middle 3×3 lemma holds true in any regular category. This reveals an interesting "conceptual duality" between the pointed and total contexts, where in the total context the upper/lower 3×3 lemmas are stronger than the middle 3×3 lemma, whereas in the pointed context they are weaker.

2. Stars, constellations and star-regular categories

In this section we give the main notions and properties concerning stars in a category with finite limits; we refer to [11] for further details.

Let $\mathbb{C}$ denote a category with finite limits, and $\mathcal{N}$ a distinguished class of morphisms that forms an *ideal*, i.e. for any composable pair of morphisms g, f, if either g or f belongs to $\mathcal{N}$, then the composite gf belongs to $\mathcal{N}$. An *$\mathcal{N}$-kernel* of a morphism $f : X \to Y$ is defined as a morphism $k : K \to X$ such that $fk \in \mathcal{N}$ and k is universal with this property (note that such k is automatically a monomorphism). A pair of morphisms, denoted by $\sigma = [\sigma_1, \sigma_2] : S \rightrightarrows X$ with $\sigma_1 \in \mathcal{N}$ is called a *star*; it is called a *monic* star when the pair (σ_1, σ_2) is jointly monomorphic. A star $\sigma = [\sigma_1, \sigma_2]$ with both $\sigma_1, \sigma_2 \in \mathcal{N}$ is said to be a *bi-star*.

A commutative diagram

$$\begin{tikzcd} S \arrow[r, shift left, "\sigma"] \arrow[r, shift right] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r, shift left] \arrow[r, shift right, "\tau"'] & Y \end{tikzcd}$$

of stars and morphisms (here $f\sigma = \tau g$ means that $f\sigma_1 = \tau_1 g$ and $f\sigma_2 = \tau_2 g$) is called a *star-pullback* when given another such commutative (outer) diagram

$$
\begin{tikzcd}
S' \arrow[drr, bend left, shift left, "\sigma'"] \arrow[drr, bend left, shift right] \arrow[dr, dotted, "h"] \arrow[ddr, bend right, "g'"'] & & \\
& S \arrow[r, shift left] \arrow[r, shift right, "\sigma"'] \arrow[d, "g"'] & X \arrow[d, "f"] \\
& T \arrow[r, shift left] \arrow[r, shift right, "\tau"'] & Y
\end{tikzcd}
$$

there exists a unique morphism $h : S' \to S$ such that $gh = g'$ and $\sigma h = \sigma'$.

A commutative square of stars

$$
\begin{tikzcd}
H \arrow[r, shift left, "\beta"] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right, "\alpha"'] & E \arrow[d, shift left, "\varepsilon"] \arrow[d, shift right] \\
F \arrow[r, shift left] \arrow[r, shift right, "\varphi"'] & X
\end{tikzcd}
$$

is said to be a *constellation*; the commutativity $\varepsilon\beta = \varphi\alpha$ means that the following diagram commutes:

$$
\begin{tikzcd}
X & E \arrow[l, "\varepsilon_1"'] \arrow[r, "\varepsilon_2"] & X \\
F \arrow[u, "\varphi_1"] \arrow[d, "\varphi_2"'] & H \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_2"] \arrow[u, "\beta_1"] \arrow[d, "\beta_2"'] & F \arrow[u, "\varphi_1"'] \arrow[d, "\varphi_2"] \\
X & E \arrow[l, "\varepsilon_1"] \arrow[r, "\varepsilon_2"'] & X.
\end{tikzcd}
$$

A *universal constellation* (over the stars φ and ε) is a constellation as above such that for any other (outer) constellation, there exists a unique morphism $h : H' \to H$:

$$
\begin{tikzcd}
H' \arrow[drr, bend left, shift left, "\beta'"] \arrow[drr, bend left, shift right] \arrow[dr, dotted, "h"] \arrow[ddr, bend right, shift left] \arrow[ddr, bend right, shift right, "\alpha'"'] & & \\
& H \arrow[r, shift left] \arrow[r, shift right, "\beta"'] \arrow[d, shift left, "\alpha"] \arrow[d, shift right] & E \arrow[d, shift left, "\varepsilon"] \arrow[d, shift right] \\
& F \arrow[r, shift left] \arrow[r, shift right, "\varphi"'] & X
\end{tikzcd}
$$

such that $\alpha h = \alpha'$ and $\beta h = \beta'$.

Example 2.1. (The total context) A double equivalence relation is an (internal) equivalence relation in the category of equivalence relations, pictured as a commutative diagram:

$$\begin{tikzcd} H \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & E \arrow[d, shift left] \arrow[d, shift right] \\ F \arrow[r, shift left] \arrow[r, shift right] & X. \end{tikzcd}$$

In particular, it gives a constellation of (monic) stars. Then, the universal constellation is given by the following classical construction (see [7], [14]):

$$H = F\Box E = \{(x, y, t, z) \in X^4 \mid xFy \wedge tFz \wedge xEt \wedge yEz\}.$$

Remark 2.2. (The pointed context) A constellation in the pointed context simply amounts to a commutative square of morphisms

$$\begin{tikzcd} H \arrow[r, "\beta_2"] \arrow[d, "\alpha_2"'] & E \arrow[d, "\varepsilon_2"] \\ F \arrow[r, "\varphi_2"'] & X. \end{tikzcd}$$

Such a constellation is universal exactly when it is a pullback.

Given a relation $\varrho = (\varrho_1, \varrho_2) : R \rightrightarrows X$ on an object X, we denote by ϱ^* the biggest subrelation of ϱ which is a (monic) star; it can be constructed by setting $\varrho^* = [\varrho_1 k, \varrho_2 k]$, where k is the $\mathcal{N}$-kernel of ϱ_1. In particular, if we denote the discrete (equivalence) relation on an object X by $\Delta_X = (1_X, 1_X) : X \rightrightarrows X$, then $\Delta_X^* = [k_X, k_X]$, where k_X denotes the $\mathcal{N}$-kernel of 1_X.

The *star-kernel* of a morphism $f : X \to Y$ is a universal star $\kappa = [\kappa_1, \kappa_2] : K \rightrightarrows X$ with the property $f\kappa_1 = f\kappa_2$ (such a star is then automatically a monic star); it is easy to see that the star-kernel of f coincides with κ_f^*, where κ_f is the kernel pair of f.

Throughout the paper, we omit proofs of those technical observations which closely mimic corresponding results in the total or pointed context that are usually well known. Below is one such result:

Lemma 2.3. [11] *Consider a commutative diagram*

$$\begin{tikzcd} P \arrow[r, Rightarrow, "\lambda"] \arrow[d, "e"'] & X \arrow[r, "c"] \arrow[d, "f"] & C \arrow[d, "m"] \\ K \arrow[r, Rightarrow, "\kappa"'] & Y \arrow[r, "d"'] & Q \end{tikzcd}$$

of morphisms and stars.

(a) *Suppose κ is a star-kernel of d and m is a monomorphism. Then, the left square is a star-pullback if and only if λ is a star-kernel of c.*
(b) *Suppose c is a coequalizer of λ and e is an epimorphism. Then, the right square is a pushout if and only if d is a coequalizer of κ.*

Convention 2.4. Throughout the rest of the paper, we work in a regular category $\mathbb{C}$ equipped with an ideal $\mathcal{N}$ such that every morphism admits an $\mathcal{N}$-kernel. Following the terminology used in [11], such category will be called a *regular multi-pointed category with kernels.*

Definition 2.5. [11] A regular multi-pointed category $\mathbb{C}$ with kernels is said to be *star-regular* when every regular epimorphism in $\mathbb{C}$ is a coequalizer of a star.

In the total context, a star-regular category is precisely a regular category. In the pointed context, a star-regular category is the same as a normal category [16], i.e. a regular category in which any regular epimorphism is a normal epimorphism.

The following lemma, in the total context, appears as Proposition 1.1 in [4]:

Lemma 2.6. *In a star-regular category, consider a commutative diagram*

$$\begin{tikzcd} P \arrow[r, Rightarrow, "\lambda"] \arrow[d, "e"'] & X \arrow[r, "c"] \arrow[d, "f"] & C \arrow[d, "m"] \\ K \arrow[r, Rightarrow, "\kappa"'] & Y \arrow[r, "d"'] & Q \end{tikzcd}$$

of stars and morphisms, where κ is a star-kernel of d and c is a regular epimorphism. Any two of the following conditions imply the third one:

(a) *m is a monomorphism;*
(b) *λ is a star-kernel of c;*
(c) *the left hand side square is a star-pullback.*

3. Star-exact sequences and the 3×3 lemma

In a star-regular category, a *(short) star-exact sequence* is a diagram

$$K \overset{\kappa}{\rightrightarrows} X \xrightarrow{f} Y$$

where $\kappa = [\kappa_1, \kappa_2]$ is a star-kernel of f and f is a coequalizer of κ_1 and κ_2 (which, by star-regularity, is the same as to say that f is a regular epimorphism). In the total context, the notion of a star-kernel of a morphism becomes the notion of a kernel pair of a morphism and a star-exact sequence is just an exact fork, while in the pointed context they represent a kernel of a morphism and a short exact sequence, respectively.

In this section we formulate a 3×3 lemma for star-exact sequences. Its diagrammatic shape, therefore, resembles the denormalized 3×3 lemma [4], although it captures both the denormalized 3×3 lemma for the total context [4] as well as the 3×3 lemma in the pointed context [3], which in the case of abelian categories gives the classical 3×3 lemma.

In a star-regular category, our 3×3 lemma concerns a commutative diagram

$$\begin{tikzcd}
H \arrow[r, shift left, "\beta"] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right, "\alpha"'] \arrow[dr, phantom, "1"] & E \arrow[r, "b"] \arrow[d, shift left] \arrow[d, shift right, "\varepsilon"'] & G \arrow[d, shift left, "\gamma"] \arrow[d, shift right] \\
F \arrow[r, shift left, "\varphi"] \arrow[r, shift right] \arrow[d, "a"'] & X \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "2"] & Y \arrow[d, "g"] \\
D \arrow[r, shift left] \arrow[r, shift right, "\delta"'] & W \arrow[r, "d"'] & Z
\end{tikzcd} \tag{1}$$

of stars and morphisms. In particular, the commutativity of $\boxed{1}$ means that $\boxed{1}$ is a constellation. A commutative diagram (1) will be called a 3×3 *diagram* when all columns are star-exact sequences. The *upper* 3×3 *lemma* states that in a 3×3 diagram (1), if the second and third rows are star-exact sequences, then so is the first row; the *lower* 3×3 *lemma* states that, if the first and second rows are star-exact sequences, then so is the third row. The *middle* 3×3 *lemma* states that, in a 3×3 diagram (1) where $f\varphi_1 = f\varphi_2$ and the first and third rows are star-exact sequences, the second row is a star-exact sequence. In the total context, the upper and the lower 3×3 lemmas are equivalent, and they hold in a regular category precisely when it is a Goursat category [12]. In the pointed context, they are also equivalent, and they hold precisely when the normal category is subtractive [16]. In the total context the middle 3×3 lemma always holds true and, consequently, the denormalized 3×3 lemma for regular categories is usually stated only in the upper and lower formulations. However, in the pointed context, the middle 3×3 lemma is very meaningful since it characterizes protomodular categories [3].

In the rest of this section we study conditions which lead to a technical simplification of our upper and lower 3×3 lemmas (see Section 4). This study is essentially a direct generalization of the corresponding study in the total context carried out in [4]. For the generalization to work, we need additional assumptions on the star-regular category, which always hold true in the total and pointed contexts. While our approach to the 3×3 lemma for star-exact sequences is very similar to the one adopted in [4] for the denormalized 3×3 lemma, it is quite different from the approach to the normalized 3×3 lemma used in [16] which is based on the classical diagram chasing method (a direct "denormalization" of the approach used in [16] seems to fail).

An object X is said to be $\mathcal{N}$*-trivial* when $1_X \in \mathcal{N}$; equivalently, X is $\mathcal{N}$-trivial when any morphism whose domain or codomain is X belongs to $\mathcal{N}$. Because of the presence of $\mathcal{N}$-kernels, we have that if a composite fg belongs to $\mathcal{N}$ and g is a regular epimorphism, then also f belongs to $\mathcal{N}$. This easily implies that $\mathcal{N}$-trivial objects are closed under quotients.

Definition 3.1. We say that there are *enough trivial objects* in $\mathbb{C}$ when $\mathcal{N}$ is a closed ideal [13], i.e. any morphism in $\mathcal{N}$ factors through an $\mathcal{N}$-trivial object, and moreover, the class of $\mathcal{N}$-trivial objects, apart from being closed under quotients, is also closed under subobjects and squares (the latter meaning that, for any $\mathcal{N}$-trivial object X, the object $X^2 = X \times X$ is $\mathcal{N}$-trivial).

A sufficient condition for the presence of enough trivial objects is when $\mathcal{N}$ is a closed ideal and $\mathcal{N}$-trivial objects do not have either proper subobjects or proper quotients. This is a corollary of the following

Proposition 3.2. *Suppose that $\mathcal{N}$ is a closed ideal and that $\mathcal{N}$-trivial objects do not have proper subobjects (i.e. every monomorphism with $\mathcal{N}$-trivial codomain is an isomorphism). Then the following conditions are equivalent:*

(a) *There are enough trivial objects in $\mathbb{C}$.*
(b) *Every morphism $W \to 1$ from an $\mathcal{N}$-trivial object W to the terminal object is a monomorphism.*
(c) *Every morphism $W \to X$ whose domain is an $\mathcal{N}$-trivial object, is a monomorphism.*

(d) *$\mathcal{N}$-trivial objects do not have proper quotients (i.e. any regular epimorphism with $\mathcal{N}$-trivial domain is an isomorphism).*

Proof: (a)⇔(b): $W \to 1$ is a monomorphism if and only if in the pullback

$$\begin{tikzcd} W \times W \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & W \arrow[d] \\ W \arrow[r] & 1 \end{tikzcd}$$

π_1 is an isomorphism. Suppose W is $\mathcal{N}$-trivial. If (a) holds, then $W \times W$ is also $\mathcal{N}$-trivial and hence the diagonal $(1_W, 1_W) : W \to W \times W$ is an isomorphism. Since π_1 is its right inverse, it follows that π_1 is also an isomorphism. Conversely, if (b) holds, then π_1 is an isomorphism and hence $W \times W$ is $\mathcal{N}$-trivial. This implies that (a) holds, since $\mathcal{N}$-trivial objects have no non-proper subobjects, and hence their subobjects are *trivially* $\mathcal{N}$-trivial.

(b)⇔(c): This follows from the fact that if in the commutative triangle

$$\begin{tikzcd} W \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & 1 & \end{tikzcd}$$

the bottom left arrow is a monomorphism, then so is the top one.

(c)⇔(d) is straightforward (for (d)⇒(c) use the fact any morphism decomposes as a regular epimorphism followed by a monomorphism). ■

According to the following proposition, the case when $\mathcal{N}$ is a closed ideal such that $\mathcal{N}$-trivial objects do not have proper subobjects gives precisely the proto-pointed context in the sense of [11].

Proposition 3.3. *For any regular category $\mathbb{C}$ there exists at most one closed ideal $\mathcal{N}$ for which $\mathbb{C}$ has $\mathcal{N}$-kernels and such that $\mathcal{N}$-trivial objects do not have proper subobjects. Moreover, such an $\mathcal{N}$ exists if and only if every object has the least subobject. Then, $\mathcal{N}$ consists of those morphisms f whose regular image is the least subobject of the codomain of f.*

Proof: First, suppose such an $\mathcal{N}$ exists. Then, consider the $\mathcal{N}$-kernel $k_X : K_X \to X$ of an identity morphism $1_X : X \to X$. Then $k_X \in \mathcal{N}$ and since $\mathcal{N}$ is a closed ideal, it factors through an $\mathcal{N}$-trivial object. The fact that k_X is a monomorphism implies that K_X is a subobject of the same trivial object, which in turn implies that K_X is itself trivial. Now, we prove that k_X is the least subobject of X. Let $m : M \to X$ be any other subobject. Pulling back m along k_X must result in an isomorphism, since, the object K_X, being $\mathcal{N}$-trivial, does not have proper subobjects. But then, k_X factors through m. This shows that when $\mathcal{N}$ exists, every object has a least subobject. The fact that the least subobject was obtained as the $\mathcal{N}$-kernel of the identity morphism shows the last statement of the proposition. Indeed, if a regular image of a morphism $f : W \to X$ is the least subobject $k_X : K_X \to X$ of X, then, since $k_X \in \mathcal{N}$, it follows that $f \in \mathcal{N}$. Conversely, if $f \in \mathcal{N}$ then it must factor through k_X. This implies that the regular image of f factors through k_X, and hence is forced to coincide with k_X, since k_X is the least subobject of X.

Now, suppose every object X has a least subobject $k_X : K_X \to X$. Define the class $\mathcal{N}$ to consist of those morphisms $f : W \to X$ whose regular image is k_X. It is easy to verify that $\mathcal{N}$ has all desired properties. ∎

Obviously, both in the total and pointed contexts, there are enough trivial objects. In the proto-pointed context in the sense of [11], which, as noted above, is precisely the one described in Proposition 3.3, there need not be enough trivial objects. Indeed, consider the category **Rng** of unitary rings, where we take $\mathcal{N}$ to be the class of trivial homomorphisms (i.e. those ring homomorphisms $Q \to R$ whose image is the least subring of R). Then, the ring $\mathbb{Z}$ of integers is $\mathcal{N}$-trivial, but $\mathbb{Z} \times \mathbb{Z}$ is not. According to Proposition 3.2, another reason why **Rng** does not have enough trivial objects is because the the map $\mathbb{Z} \to \{1\}$ is not injective. In fact, in the proto-pointed context of any variety of universal algebras, the presence of enough trivial objects splits up in two cases: when the variety is pointed (i.e. its theory has a unique nullary term), and when it contains the empty algebra (i.e. the algebraic theory of the variety does not contain any nullary terms). In the second case, star-regularity forces the identity $x = y$ to hold true in the variety, and so this shows that the only "interesting" example of star-regularity in a varietal proto-pointed context with enough trivial objects is the star-regularity in the varietal pointed context, which is exactly the context studied in [9] (which, in modern terminology, is the context of pointed 0-regular varieties).

Propositions 3.3 and 3.2 together show that a proto-pointed context where there are enough trivial objects is very similar to the context of so called *quasi-pointed categories* introduced in [3].

Next, we give several equivalent conditions that characterize the presence of enough trivial objects.

Proposition 3.4. *Let $\mathbb{C}$ be a regular multi-pointed category with kernels. The following conditions are equivalent:*

(a) *If a relation $\varrho : R \rightrightarrows X$ is a bi-star, then R is an $\mathcal{N}$-trivial object.*
(b) *If $(s_1, s_2) : S \rightrightarrows X$ is a relation such that $s_1 n, s_2 n \in \mathcal{N}$, then $n \in \mathcal{N}$.*
(c) *In a diagram*

$$\begin{tikzcd} H \arrow[r, shift left, "\beta"] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right, "\alpha"'] & E \arrow[d, shift left, "\varepsilon"] \arrow[d, shift right] \\ F \arrow[r, shift left] \arrow[r, shift right, "\varphi"'] & X \end{tikzcd}$$

with the usual commutativity conditions, if ε is a monic star and φ is a star, then β is a star.
(d) *$\mathbb{C}$ has enough trivial objects.*

Proof: The less trivial part of the proof is the implication (d)⇒(a). We begin by considering $k_X : K_X \to X$, the $\mathcal{N}$-kernel of 1_X. Since k_X belongs to $\mathcal{N}$, then it factors through an $\mathcal{N}$-trivial object T. So K_X is a subobject of T, hence it is also an $\mathcal{N}$-trivial object. Now, if ρ is a bi-star, then both ρ_1 and ρ_2 factor through k_X; say $\rho_1 = k_X \lambda_1$ and $\rho_2 = k_X \lambda_2$. We get a monomorphism $(\lambda_1, \lambda_2) : R \to K_X \times K_X$ and, consequently, R is an $\mathcal{N}$-trivial object. ■

In a star-regular category, when all columns and rows in a given diagram (1) are star-exact sequences, then certain properties concerning $\boxed{1}$ and $\boxed{2}$ must hold (for the first part, we require the existence of enough trivial objects). In the total context, $\boxed{1}$ necessarily represents the double equivalence relation $F \square E$ and $\boxed{2}$ is a pushout (Proposition 2.1 in [4]) and in the pointed context it is easy to see that $\boxed{1}$ must be a pullback and $\boxed{2}$ a pushout. For the general context, these conditions translate into: $\boxed{1}$ is a universal constellation and $\boxed{2}$ is a pushout. We can get the condition on the pushout from the following proposition, which is an immediate consequence of Lemma 2.3 (b):

Proposition 3.5. *In any star-regular category, let* (1) *be a* 3×3 *diagram with a star-exact middle row. The square* $\boxed{2}$ *is a pushout if and only if d is a coequalizer of δ.*

We get the condition on the universal constellation from the following theorem, which will be proved throughout the rest of this section:

Theorem 3.6. *In a star-regular category with enough trivial objects, consider a commutative diagram of stars and morphisms* (1), *where the first column* *is a star-exact sequence, ε is a star-kernel of e, φ is a star-kernel of f and γ is monic. Then the following conditions are equivalent:*

(a) *δ is a monic star;*
(b) *β is a star-kernel of b;*
(c) $\boxed{1}$ *is a universal constellation.*

The above theorem extends Theorem 2.2 of [4] to our star-regular context. Moreover, our proof of the above theorem follows, step-by-step, the proof given in [4]. The technical observations contained in this proof, as well as the theorem itself, are used in Section 4 to establish the equivalence of the upper and lower 3×3 lemmas in star-regular categories with enough trivial objects.

We begin by observing that under the presence of enough trivial objects, we have a stability property for star-kernels with respect to products:

Lemma 3.7. *Suppose $\mathbb{C}$ has enough trivial objects. Then, a pair $\varepsilon = [\varepsilon_1, \varepsilon_2] : E \rightrightarrows X$ is a star if and only if so is the pair $\varepsilon \times \varepsilon = [\varepsilon_1 \times \varepsilon_1, \varepsilon_2 \times \varepsilon_2] : E \times E \rightrightarrows X \times X$. Moreover, ε is a star-kernel of $e : X \to W$ if and only if $\varepsilon \times \varepsilon$ is a star-kernel of $e \times e : X \times X \to W \times W$.*

Proof: The non-trivial part of the proof is to show that $\varepsilon \times \varepsilon$ is a star whenever ε is. This follows by applying Proposition 3.4(c) to the diagram

$$\begin{tikzcd} E \times E \arrow[r, shift left, "\varepsilon\times\varepsilon"] \arrow[r, shift right] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & X \times X \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\ E \arrow[r, shift left] \arrow[r, shift right, "\varepsilon"'] & X \end{tikzcd}$$

■

The following proposition characterizes universal constellations involving a star φ and a monic star ε. The requirement that ε below is a monic star can be dropped in the total context, in which case the result below becomes precisely Remark 2.2 of [4].

Proposition 3.8. *Consider a constellation*

$$\begin{tikzcd} H \arrow[r, shift left, "\beta"] \arrow[r, shift right] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left] & E \arrow[d, shift right] \arrow[d, shift left, "\varepsilon"] \\ F \arrow[r, shift left] \arrow[r, shift right, "\varphi"'] & X \end{tikzcd} \qquad (2)$$

in $\mathbb{C}$, where ε is a monic star. If $\mathbb{C}$ has enough trivial objects, then the following conditions are equivalent:

(a) *The constellation (2) is universal.*
(b) *The commutative diagram*

$$\begin{tikzcd} H \arrow[r, "(\beta_1,\beta_2)"] \arrow[d, Rightarrow, "\alpha"'] & E \times E \arrow[d, Rightarrow, "\varepsilon\times\varepsilon"] \\ F \arrow[r, "(\varphi_1,\varphi_2)"'] & X \times X \end{tikzcd} \tag{3}$$

is a star-pullback.

Proof: First, note that diagram (2) commutes if and only if diagram (3) commutes. Lemma 3.7 guarantees that $\varepsilon \times \varepsilon$ is also a star. It is easy to see that the universal property of the constellation is the same as the universal property of the star-pullback. ■

The following propositions characterize universal constellations which are part of diagrams involving stars and morphisms.

Proposition 3.9. *In a star-regular category with enough trivial objects, consider a commutative diagram of stars and morphisms*

$$\begin{tikzcd} H \arrow[r, Rightarrow, "\beta"] \arrow[d, Rightarrow, "\alpha"'] \arrow[dr, phantom, "1"] & E \arrow[d, Rightarrow, "\varepsilon"] \\ F \arrow[r, Rightarrow, "\varphi"'] \arrow[d, "a"'] & X \arrow[d, "e"] \\ D \arrow[r, Rightarrow, "\delta"'] & W \end{tikzcd}$$

where the left column is a star-exact sequence and ε is the star-kernel of e. Then $\boxed{1}$ is a universal constellation if and only if δ is monic.

Proof: Use Proposition 3.8, Lemma 3.7 and apply Lemma 2.6 to the following diagram:

$$\begin{tikzcd}
H \arrow[r, "(\beta_1,\beta_2)"] \arrow[d, shift left, "\alpha"'] \arrow[d, shift right] & E \times E \arrow[d, shift left, "\varepsilon\times\varepsilon"] \arrow[d, shift right] \\
F \arrow[r, "(\varphi_1,\varphi_2)"'] \arrow[d, "a"'] & X \times X \arrow[d, "e\times e"] \\
D \arrow[r, "(\delta_1,\delta_2)"'] & W \times W
\end{tikzcd}$$

■

Using a similar argument as in the proof of the above proposition, we have:

Proposition 3.10. *In a category with enough trivial objects, consider a commutative diagram of stars and morphisms*

$$\begin{tikzcd}
H \arrow[r, shift left, "\beta"] \arrow[r, shift right] \arrow[d, shift left, "\alpha"'] \arrow[d, shift right] \arrow[dr, phantom, "1"] & E \arrow[r, "b"] \arrow[d, shift left, "\varepsilon"] \arrow[d, shift right] & G \arrow[d, shift left, "\gamma"] \arrow[d, shift right] \\
F \arrow[r, shift left] \arrow[r, shift right, "\varphi"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

where φ is a star-kernel of f and γ is monic. Then $\boxed{1}$ *is a universal constellation if and only if β is a star-kernel of b.*

Proof: Use Proposition 3.8, Lemma 3.7 and apply Lemma 2.3(a) to the following diagram:

$$\begin{tikzcd}
H \arrow[r, shift left, "\beta"] \arrow[r, shift right] \arrow[d, "(\alpha_1,\alpha_2)"'] & E \arrow[r, "b"] \arrow[d, "(\varepsilon_1,\varepsilon_2)"] & G \arrow[d, "(\gamma_1,\gamma_2)"] \\
F \times F \arrow[r, shift left] \arrow[r, shift right, "\varphi\times\varphi"'] & X \times X \arrow[r, "f\times f"'] & Y \times Y
\end{tikzcd}$$

■

Altogether, this proves Theorem 3.6.

4. The equivalence of the upper and lower 3×3 lemmas

Thanks to Theorem 3.6, the upper 3×3 lemma can be equivalently reformulated as follows: in a 3×3 diagram (1), if the second and third rows are star-exact sequences, then b is a regular epimorphism. Similarly, the lower 3×3 lemma becomes: in a 3×3 diagram (1), if the first and second rows are star-exact sequences, then δ (which becomes a relation) is a star-kernel of d. In this section we shall investigate each of these lemmas separately.

We begin by recalling some terminology from [10]. By a *diamond* we mean a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "e"'] \arrow[dr, "f"] & \\ W \arrow[dr, "d"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd} \tag{4}$$

(Note that this use of the term "diamond" is different from the one in [14].) We say that the diamond (4) is

- *left saturated* if the direct image $e\langle\kappa_f^*\rangle$ along e of the star-kernel κ_f^* of f is the star-kernel of d:
$$e\langle\kappa_f^*\rangle = \kappa_d^*;$$
- *right saturated* if, symmetrically, $f\langle\kappa_e^*\rangle = \kappa_g^*$;
- *saturated* if it is both left and right saturated;
- *a regular diamond* if all morphisms in the diamond are regular epimorphisms.

Definition 4.1. [10] $\mathbb{C}$ is said to have *symmetric saturation property* if the following equivalent conditions hold:

(a) Any left saturated regular diamond is right saturated.
(b) Any right saturated regular diamond is left saturated.
(c) Left/right saturated regular diamonds are the same as the saturated ones.

Theorem 4.2. *In a star-regular category $\mathbb{C}$ with enough trivial objects, the following conditions are equivalent:*

(a) *The upper 3×3 lemma holds in $\mathbb{C}$.*
(b) *The lower 3×3 lemma holds in $\mathbb{C}$.*
(c) *$\mathbb{C}$ has symmetric saturation property.*

Proof: (a)⇔(c): Suppose first the upper 3×3 lemma holds true. From a left saturated regular diamond (4) build a diagram (1) by attaching to the diamond star-kernels of its edges, and the induced factorizations, and completing the top left square in (1) with a universal constellation. By Proposition 3.10 and by the left saturation of the diamond, the first column in (1) is a star-exact sequence. Then (1) is a 3×3 diagram. Applying the upper 3×3 lemma we get that the diamond is right saturated.

The converse is trivial: to get the upper 3×3 lemma, apply the saturation assumption to the diamond that appears as the bottom right square in a 3×3 diagram.

(b)⇔(c): Suppose first the lower 3×3 lemma holds true. From a right saturated regular diamond (4) build a diagram (1) by attaching to the diamond star-kernels of its edges e, f, g, and the induced factorization from the star-kernel of e to the star-kernel of g, which is a regular epimorphism since the diamond is right saturated. Then, take the top left square of (1) to be the universal constellation. Complete the bottom left square of (1) via the regular image of the star-kernel of f along e. By Proposition 3.10, the first column is a star-exact sequence. Now, by Proposition 3.10 again the first row is a star-exact sequence. Applying the lower 3×3 lemma we get precisely the left saturation of the diamond.

Conversely, consider a 3×3 diagram (1) where the first two rows are exact. By Theorem 3.6, δ is monic, and since a is a regular epimorphism, we get that δ is the image of φ under e. Now, the bottom right square is right saturated, and therefore left saturated, which gives the exactness of the bottom row. ■

As shown in [10], in the total context the symmetric saturation property is equivalent to the Goursat property, while in the pointed context, it is equivalent to subtractivity. So the above result unifies Proposition 1 of [12] (which is then exactly Theorem 4.2 in the total context) with Theorem 5.4 of [16] (which is the same as Theorem 4.2 in the pointed context).

The following corollary of the above theorem partially refines the theorem:

Corollary 4.3. *In each of the pointed, proto-pointed and total contexts, for a star-regular category* $\mathbb{C}$ *the following conditions are equivalent:*

(a) *The upper* 3×3 *lemma holds in* $\mathbb{C}$.
(b) *The lower* 3×3 *lemma holds in* $\mathbb{C}$.
(c) $\mathbb{C}$ *has enough trivial objects and has symmetric saturation property.*

Proof: In the pointed and total contexts this becomes precisely Theorem 4.2, since in these contexts there always are enough trivial objects. Also, after Theorem 4.2, to prove the equivalence of the above conditions in the proto-pointed context, it suffices to prove that the upper and the lower 3×3 lemmas each imply the presence of enough trivial objects. According to Proposition 3.2, showing the presence of enough trivial objects is equivalent to showing that trivial objects do not have proper quotients. For this, the following fact, which follows directly from Proposition 3.3, will be needed: for any object X, a morphism $W \to X \times X$ from the class $\mathcal{N}$ always factors through the diagonal $X \to X \times X$. Let Z be a trivial object and let $q : Z \to Q$ be a regular epimorphism. Then the object Q, being a regular quotient of a trivial object is itself trivial. Now, using the fact mentioned above, the following 3×3 diagram can be constructed, with the middle and the bottom rows being star-exact:

$$\begin{tikzcd}
Z \arrow[r, shift left, "{[1_Z,1_Z]}"] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & Z \arrow[r, "q"] \arrow[d, shift left] \arrow[d, shift right, "{[1_Z,1_Z]}"'] & Q \arrow[d, shift left, "{[1_Q,1_Q]}"] \arrow[d, shift right] \\
Z \times_Q Z \arrow[r, shift left, "\kappa_q"] \arrow[r, shift right] \arrow[d, "1_{Z \times_Q Z}"'] & Z \arrow[r, "q"] \arrow[d, "1_Z"'] & Q \arrow[d, "1_Q"] \\
Z \times_Q Z \arrow[r, shift left] \arrow[r, shift right, "\kappa_q"'] & Z \arrow[r, "q"'] & Q
\end{tikzcd} \tag{5}$$

The upper 3×3 lemma implies that the top row is star-exact, and hence, by star-regularity, q is a coequalizer of the pair $1_Z, 1_Z : Z \rightrightarrows Z$, which shows that q is an isomorphism. To deduce that q is an isomorphism from the lower

3×3 lemma, we should construct the following 3×3 diagram:

$$
\begin{tikzcd}
Z \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right, "{[1_Z,1_Z]}"'] & Z \times_Q Z \arrow[r, "1_{Z \times_Q Z}"] \arrow[d, shift left] \arrow[d, shift right, "\kappa_q"'] & Z \times_Q Z \arrow[d, shift left, "\kappa_q"] \arrow[d, shift right] \\
Z \arrow[r, shift left, "{[1_Z,1_Z]}"] \arrow[r, shift right] \arrow[d, "1_Z"'] & Z \arrow[r, "1_Z"] \arrow[d, "q"'] & Z \arrow[d, "q"] \\
Z \arrow[r, shift left] \arrow[r, shift right, "{[q,q]}"'] & Q \arrow[r, "1_Q"'] & Q
\end{tikzcd}
\tag{6}
$$

Since the top and the middle rows are star-exact, the lower 3×3 lemma implies that the lower row is star-exact. In particular, this gives that the star $[q, q] : Z \rightrightarrows Q$ is monic, and hence q is a monomorphism. Since q is at the same time a regular epimorphism, it follows that q is an isomorphism. ■

Remark 4.4. The above corollary can be used to deduce that our upper and lower 3×3 lemmas fail in the proto-pointed context of the category **Rng** of unitary rings. This can be also seen directly, by choosing Z in diagrams (5) and (6) to be the ring $\mathbb{Z}$ of integers, and Q its any proper quotient. Since **Rng** has a good theory of ideals (see [18]), this shows that in general, in a category with a good theory of ideals, both the upper and the lower 3×3 lemmas may fail.

5. The middle 3×3 lemma

In the context of star-regular categories, the *short five lemma* states that, given a commutative diagram of horizontal star-exact sequences

$$
\begin{tikzcd}
F \arrow[r, shift left, "\varphi"] \arrow[r, shift right] \arrow[d, "a"'] & X \arrow[r, "f"] \arrow[d, "e"] & Y \arrow[d, "g"] \\
D \arrow[r, shift left] \arrow[r, shift right, "\delta"'] & W \arrow[r, "d"'] & Z,
\end{tikzcd}
\tag{7}
$$

if a and g are isomorphisms then e is an isomorphism.

In the pointed context, this becomes the classical short five lemma, while in the total context this lemma (which can be called the "denormalized short five lemma") always holds true. Indeed, the fact that φ and δ are both reflexive relations, together with a being an isomorphism, imply that e is both a monomorphism and a split epimorphism, thus an isomorphism.

Lemma 5.1. *In a star-regular category where the short five lemma holds, given a commutative diagram (7) of horizontal star-exact sequences, e is a regular epimorphism whenever a and g are regular epimorphisms (we say in this case that the short five lemma for regular epimorphisms holds).*

Proof: Suppose the short five lemma holds. We would like to show that in the diagram

$$\begin{tikzcd} K_f \arrow[r, Rightarrow, "\kappa_f^*"] \arrow[d, "a"'] & X \arrow[r, twoheadrightarrow, "f"] \arrow[d, "e"] & Y \arrow[d, "g"] \\ K_d \arrow[r, Rightarrow, "\kappa_d^*"'] & W \arrow[r, twoheadrightarrow, "d"'] & Z \end{tikzcd}$$

with a and g regular epimorphisms, e is also a regular epimorphism. This diagram can be decomposed as follows, where $ip = e$ is the factorization of e as a regular epimorphism p followed by a monomorphism i:

$$\begin{tikzcd} K_f \arrow[r, Rightarrow, "\kappa_f^*"] \arrow[d, "a"'] & X \arrow[r, twoheadrightarrow, "f"] \arrow[d, "p"] & Y \arrow[d, "g"] \\ K_d \arrow[r, Rightarrow, "\sigma"] \arrow[d, "1_{K_d}"'] & I \arrow[r, twoheadrightarrow, "di"] \arrow[d, "i"] & Z \arrow[d, "1_Z"] \\ K_d \arrow[r, Rightarrow, "\kappa_d^*"'] & W \arrow[r, twoheadrightarrow, "d"'] & Z \end{tikzcd}$$

In the above diagram we get the induced star $\sigma : K_d \rightrightarrows I$ due to the fact that a is a regular epimorphism. Since the left hand side of the bottom part of the above diagram is a star-pullback, applying Lemma 2.3 we get that σ is the star-kernel of di. Now, the short five lemma implies that i is an isomorphism, and hence e is a regular epimorphism, as desired. ■

Theorem 5.2. *In a star-regular category with enough trivial objects, the short five lemma holds if and only if the short five lemma for regular epimorphisms holds.*

Proof: The "only if" part is given by Lemma 5.1. To prove the "if" part, consider a commutative diagram of horizontal star-exact sequences (7), with a and g being isomorphisms. Then, e is a regular epimorphism. Since the category is star-regular, to prove that e is an isomorphism it suffices to show that the star-kernel κ_e^* of e factors through the star-kernel $\Delta_X^* = [k_X, k_X]$ of 1_X, which is the same as to show that κ_e^* has a diagonal form $\kappa_e^* = [e', e']$. First, observe that κ_e^* factors through φ:

$$\begin{tikzcd}
& K_e \arrow[dl, bend right, "b"'] \arrow[d, shift left] \arrow[d, shift right, "\kappa_e^*"'] & \\
F \arrow[r, shift left, "\varphi"] \arrow[r, shift right] \arrow[d, "a"'] & X \arrow[r, "f"] \arrow[d, "e"] & Y \arrow[d, "g"] \\
D \arrow[r, shift left] \arrow[r, shift right, "\delta"'] & W \arrow[r, "d"'] & Z,
\end{tikzcd}$$

Since the composite $e\kappa_e^* = \delta ab$ is a bi-star, from Proposition 3.4(b) we get $b \in \mathcal{N}$. This implies that κ_e^* is a bi-star. Then, κ_e^* is of the from $\kappa_e^* = [k_X c_1, k_X c_2]$, where k_X denotes the $\mathcal{N}$-kernel of $1_X : X \to X$. We want to show that $c_1 = c_2$. Notice that Δ_X^* factors through φ via some monomorphism m. Using the fact that a is an isomorphism and δ is monic we easily get $mc_1 = b = mc_2$. This implies that $c_1 = c_2$. ∎

Proposition 5.3. *In a star-regular category with enough trivial objects, if the short five lemma holds then the middle* 3×3 *lemma holds.*

Proof: Let (1) be a 3×3 diagram such that first and third rows are star-exact sequences and $f\varphi_1 = f\varphi_2$. Then f is a regular epimorphism by Lemma 5.1. We can form the following commutative diagram: let $\kappa_f^* : K_f \rightrightarrows X$ be the star-kernel of f and $\overline{e} : K_f \to D$ and $w : F \to K_f$ the induced morphisms such that $\delta\overline{e} = e\kappa_f^*$ and $\kappa_f^* w = \varphi$. Then, by taking the star-kernel $\kappa_{\overline{e}}^* : K_{\overline{e}} \rightrightarrows K_f$ of the regular epimorphism $\overline{e}$ we can generate a commutative diagram

$$\begin{tikzcd}
H \arrow[r, "\overline{w}"] \arrow[d, shift left] \arrow[d, shift right, "\alpha"'] & K_{\overline{e}} \arrow[r, shift left, "\overline{\kappa}"] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right, "\kappa_{\overline{e}}^*"'] & E \arrow[r, "b"] \arrow[d, shift left, "\varepsilon"] \arrow[d, shift right] & G \arrow[d, shift left, "\gamma"] \arrow[d, shift right] \\
F \arrow[r, "w"'] \arrow[d, "a"'] & K_f \arrow[r, shift left] \arrow[r, shift right, "\kappa_f^*"'] \arrow[d, "\overline{e}"'] & X \arrow[r, "f"'] \arrow[d, "e"] & Y \arrow[d, "g"] \\
D \arrow[r, "1_D"'] & D \arrow[r, shift left] \arrow[r, shift right, "\delta"'] & W \arrow[r, "d"'] & Z.
\end{tikzcd}$$

where $\overline{\kappa}\,\overline{w} = \beta$. By Proposition 3.4(c), $\overline{\kappa}$ is a star. Since δ is monic, applying Theorem 3.6 to the right hand side 3×3 diagram above, we can conclude that $\overline{\kappa}$ is the star-kernel of b. Consequently, $\overline{w}$ is an isomorphism. To finish, we just apply the short five lemma to the left part of the above diagram to conclude that w is an isomorphism. This proves that the middle row in diagram (1) is a star-exact sequence. ■

Definition 5.4. [10] A morphism $f : X \to Y$ is said to be *saturating* if the diamond

$$\begin{tikzcd}
 & X \arrow[dl, "1_X"'] \arrow[dr, "f"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "1_Y"] \\
 & Y &
\end{tikzcd}$$

is right saturated.

Theorem 5.5. *Let $\mathbb{C}$ be a star-regular category with enough trivial objects and saturating regular epimorphisms. Then the following conditions are equivalent:*

(a) *The middle 3×3 lemma holds in $\mathbb{C}$.*
(b) *The short five lemma holds in $\mathbb{C}$.*
(c) *The short five lemma for regular epimorphisms holds in $\mathbb{C}$.*

Proof: (a)⇒(b): Consider a commutative diagram of horizontal star-exact sequences

$$\begin{tikzcd}
F \arrow[r, shift left, "\varphi"] \arrow[r, shift right] \arrow[d, "1_F"'] & X \arrow[r, "f"] \arrow[d, "e"] & Y \arrow[d, "1_Y"] \\
F \arrow[r, shift left] \arrow[r, shift right, "\delta"'] & W \arrow[r, "d"'] & Y
\end{tikzcd}$$

From the top row of the above diagram, construct a commutative diagram

$$\begin{tikzcd}
K_F \arrow[r, shift left, "\beta"] \arrow[r, shift right] \arrow[d, shift right, "\Delta_F^*"'] \arrow[d, shift left] & K_X \arrow[r, "b"] \arrow[d, shift right] \arrow[d, shift left, "\Delta_X^*"] & K_Y \arrow[d, shift right] \arrow[d, shift left, "\Delta_Y^*"] \\
F \arrow[r, shift left, "\varphi"] \arrow[r, shift right] \arrow[d, "1_F"'] & X \arrow[r, "f"] \arrow[d, "1_X"] & Y \arrow[d, "1_Y"] \\
F \arrow[r, shift left] \arrow[r, shift right, "\varphi"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

where, by Proposition 3.4(c), β is a star. Then, by Theorem 3.6, β is a star-kernel of b and since f is saturating, b is a regular epimorphism. Therefore, the top row is star-exact. Now, in the above diagram, replace the bottom part with the diagram we had in the beginning:

$$\begin{tikzcd}
K_F \arrow[r, shift left, "\beta"] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right, "\Delta_F^*"'] & K_X \arrow[r, "b"] \arrow[d, shift left, "\Delta_X^*"] \arrow[d, shift right] & K_Y \arrow[d, shift left, "\Delta_Y^*"] \arrow[d, shift right] \\
F \arrow[r, shift left, "\varphi"] \arrow[r, shift right] \arrow[d, "1_F"'] & X \arrow[r, "f"] \arrow[d, "e"] & Y \arrow[d, "1_Y"] \\
F \arrow[r, shift left] \arrow[r, shift right, "\delta"'] & W \arrow[r, "d"'] & Y
\end{tikzcd}$$

By the middle 3×3 lemma (with the role of rows and columns switched), the middle column is a star-exact sequence. Consequently, e is the coequalizer of $\Delta_X^* = [k_X, k_X]$ and thus it is an isomorphism.

(b)$\Rightarrow$(a) by Proposition 5.3, and (b)$\Leftrightarrow$(c) by Theorem 5.2. ■

Combining the above result with Theorem 4.2, we obtain:

Corollary 5.6. *Let $\mathbb{C}$ be a star-regular category with enough trivial objects and saturating regular epimorphisms. Then the following conditions are equivalent:*

(a) *The complete 3×3 lemma holds in $\mathbb{C}$ (i.e. the lower, upper and middle 3×3 lemmas hold in $\mathbb{C}$).*
(b) *Any left saturated diamond (4) with regular epimorphic edges f, g, d, is both regular and saturated.*

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