Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 11–35

ON EXPONENTIABILITY OF ÉTALE ALGEBRAIC HOMOMORPHISMS

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ABSTRACT: In this paper we show that Cagliari-Mantovani result stating that, in the category of compact Hausdorff spaces, every étale map is exponentiable can be formulated in a general category $\mathbf{Alg}(T)$ of Eilenberg-Moore T-algebras, for a monad T, and proved in case T satisfies the so-called Beck-Chevalley condition. For that, $\mathbf{Alg}(T)$ is embedded in the (topological) category $\mathbf{RelAlg}(T)$ of relational T-algebras, where a suitable notion of étale morphism can be studied, it is shown that morphisms between T-algebras are exponentiable in $\mathbf{RelAlg}(T)$, and, moreover, these exponentials belong to $\mathbf{Alg}(T)$ whenever the morphisms are étale.

KEYWORDS: exponentiable morphism, Eilenberg-Moore algebra, relational algebra, ultrafilter monad.
AMS SUBJECT CLASSIFICATION (2000): 18C15, 18C20, 18D15, 18B30.

0. Introduction

The existence of "internal function objects" in a category $\mathbf{C}$ with finite products, that is the existence of a right adjoint to the functor $(\ )\times X : \mathbf{C}\to\mathbf{C}$ for any $\mathbf{C}$-object X – calling then $\mathbf{C}$ cartesian closed –, is a widely studied problem. It is in general more interesting in topology than in algebra, since for pointed categories $(\ )\times X : \mathbf{C}\to\mathbf{C}$ has a right adjoint only if X is the zero object. There is, however, an interesting complete characterization of cartesian closed varieties by Johnstone [14], where, moreover, the exponentiable objects in a variety $\mathbf{C}$, that is, the objects X that induce a left adjoint functor $(\ )\times X : \mathbf{C}\to\mathbf{C}$, are characterized. The exponential maps are not studied explicitly in [14], unlike the papers [20, 2, 3, 10], where it is done in topological contexts. In particular, Cagliari-Mantovani Theorem [2], stating that a continuous map in the category of compact Hausdorff spaces is exponentiable if and only if it is a local homeomorphism, or an étale

Received December 4, 2011.
Research partially supported by Centro de Matemática da Universidade de Coimbra, by Centro de Investigação e Desenvolvimento em Matemática e Aplicações da Universidade de Aveiro/FCT, by Fundação para a Ciência e Tecnologia, through European program COMPETE/FEDER, by the project MONDRIAN (under the contract PTDC/EIA-CCO/108302/2008) and by South African NRF.

map, raised the question whether exponentiability in categories of Eilenberg-Moore T-algebras, for a monad T, could be more interesting at the level of morphisms than at the object level. This was the starting point of the work presented here. We remark that, while Johnstone used algebraic theories to study varieties, here we will use monads.

Consider the following table of concepts, depending on a monad T on the category of sets:

	T=identity monad	T=ultrafilter monad
Eilenberg-Moore T-algebras	Sets=discrete preorders	Compact Hausdorff spaces
Relational T-algebras [1] = reflexive & transitive lax algebras [5]	Preorders	Topological spaces
Pseudo-relational T-algebras = reflexive lax algebras [5]	Reflexive relations	Pseudo-topological spaces in the sense of Herrlich

and recall:

A morphism $f : A \to B$, in a category $\mathbf{C}$ with finite products, is said to be exponentiable if the functor $(\ ) \times (A, f) : (C \downarrow B) \to (C \downarrow B)$ has a right adjoint (other well-known equivalent definitions are given in Theorem 3.1 below). The exponentiability problem, which is to describe the exponentiable morphisms in a given category, is highly non-trivial even where it is solved in many cases, and in particular for topological spaces (see S. Niefield [20] and references there). However, it turns out to be easier in other categories of our table above, and in particular:

(a) Every morphism of pseudo-topological spaces is exponentiable, and, more generally, this is true for reflexive lax T-algebras [9] whenever T satisfies the so-called Beck-Chevalley Condition.
(b) The same is true in the case of reflexive relations, as follows from the result of [9] of course, but a direct proof is also easy.
(c) The description known for preorders is a simplified version of the topological one, and at the time is a simplified version of the one, well-known, for categories (in the category of all categories).

(d) The case of compact Hausdorff spaces is again easier than the case of general topological spaces, while the case of sets, where again every morphism is exponentiable, should be considered as trivial.

In the present paper we begin to study the exponentiability problem for the ordinary T-algebras for an arbitrary monad T on the category of sets, assuming, as in [9], that T satisfies the Beck-Chevalley Condition. After recalling all necessary definitions and the exponentiability result mentioned in (a) above, we show that:

- When f is a perfect map of relational T-algebras, its pseudo-relational exponents are relational, making it exponentiable in the category of relational T-algebras (Theorem 4.2). The notion of perfect used there, as well as the notion of étale used later, and related notions of open and proper, is suggested by the topological one as expected (see Section 2).
- Every homomorphism of T-algebras is perfect, and therefore exponentiable in the category of relational T-algebras (Corollary 4.3).
- When a homomorphism of T-algebras is étale, its (pseudo-)relational exponents are T-algebras, making it exponentiable in the category of T-algebras (Theorem 5.5).

In the last section we consider several examples of categories of algebras, namely, of compact Hausdorff spaces, sup-lattices, continuous lattices, monoids, semigroups, and monoid actions. In particular, we point out that, for a monoid M, although every morphism of M-sets is (well-known to be) exponentiable, not every morphism is étale – unless M is a group.

In summary, we show that Cagliari-Mantovani result that étale maps are exponentiable in $\mathbf{CompHaus} = \mathbf{Alg}(U)$, with exponentials built as in $\mathbf{Top} = \mathbf{RelAlg}(U)$, for U the ultrafilter monad, can be generalized for $\mathbf{Alg}(T)$ in case T satisfies the Beck-Chevalley condition. It remains to be shown whether the converse is true. The example of M-**Set** shows that there may be non-étale exponentiable morphisms in $\mathbf{Alg}(T)$, but in this example the exponentials, that exist both in $\mathbf{Alg}(T)$ and in $\mathbf{RelAlg}(T)$, are built differently.

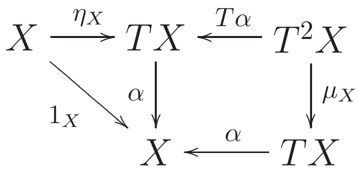
The general problems of describing:

(I) exponentiable morphisms of relational T-algebras;
(II) exponentiable morphisms of T-algebras;
(III) exponentiable morphisms of T-algebras with exponents inherited from the category of relational T-algebras

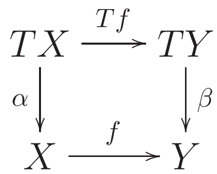
remain open, but it seems that Proposition 4.1 should be helpful in solving of Problem (II) (see also [10]), while étale maps should provide, in many cases, an answer to Problem (III). Our results on Problem (II) that do not use relational algebras, which is a work in progress now, will be published elsewhere. In particular, we will give an easy proof of the fact that a group homomorphism is exponentiable if and only if it is an isomorphism, and that the same is true in any semi-abelian category.

1. Relational algebras

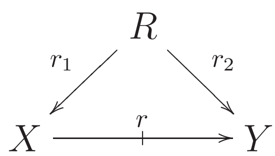
Given a monad $T = (T, \eta, \mu)$ on **Set**, we consider the category $\mathbf{Alg}(T)$ of Eilenberg-Moore T-algebras; recall that an object of $\mathbf{Alg}(T)$ is a pair (X, α), where X is a set and $\alpha : TX \to X$ is a map making the diagram



commute, and a morphism $f : (X, \alpha) \to (Y, \beta)$ is a map $f : X \to Y$ with $f \cdot \alpha = \beta \cdot Tf$:



The monad T can be extended to the category **Rel** of relations (see [1, 6]) as follows: for a relation $r : X \nrightarrow Y$, with $r = r_2 \cdot r_1^\circ$,



where r_1, r_2 are the projections and r_1° is the opposite relation of r_1, let $\overline{T}r = Tr_2 \cdot (Tr_1)^\circ$. Then $\overline{T} : \mathbf{Rel} \to \mathbf{Rel}$ is an op-lax functor, and $\eta : \mathrm{Id}_{\mathbf{Rel}} \to \overline{T}$ and $\mu : \overline{T}^2 \to \overline{T}$ become op-lax natural transformations. The functor $\overline{T} : \mathbf{Rel} \to \mathbf{Rel}$ is a lax – hence a strict – functor if and only if $T : \mathbf{Set} \to \mathbf{Set}$ has the *Beck-Chevalley property* (BC) (in the sense of [5]), as shown in [6]. We recall that a *(BC)-square* is a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{k} & X \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

such that $f^\circ \cdot g = k \cdot h^\circ$, where f° and h° are the opposite relations of f and h, respectively, and that T has (BC) property if it preserves (BC)-squares. This implies, in particular, that T preserves pullbacks along monomorphisms, that is, T is *taut* (see [19]).

Throughout we assume that $T : \mathbf{Set} \to \mathbf{Set}$ *has (BC) property and that the natural transformation* $\mu : TT \to T$ *has (BC) property*, meaning that, for every map $f : X \to Y$, the naturality diagram

$$\begin{array}{ccc} T^2X & \xrightarrow{\mu_X} & TX \\ {\scriptstyle T^2f}\big\downarrow & & \big\downarrow{\scriptstyle Tf} \\ T^2Y & \xrightarrow{\mu_Y} & TY \end{array}$$

is a (BC)-square. We recall that these assumptions mean precisely that $\overline{T} : \mathbf{Rel} \to \mathbf{Rel}$ is a strict functor and that $\mu : \overline{T}^2 \to \overline{T}$ is a strict natural transformation. We also assume that T *is non-trivial*, or, equivalently, T is faithful, or, equivalently, the natural transformation $\eta : \mathrm{Id} \to T$ is pointwise monic (see [15] for details).

As already studied by Barr [1] and studied later by Clementino, Hofmann and Tholen [5, 11, 9, 6, 7] and others, one can relax the conditions above, defining a *relational* T*-algebra*, or a *lax* T*-algebra*, or a $(T, \mathbf{2})$*-category*, as a pair (X, a), where $a : TX \nrightarrow X$ is a relation such that $1_X \leq a \cdot \eta_X$ and $a \cdot Ta \leq a \cdot \mu_X$, that is

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & TX & \xleftarrow{Ta}\!\!\!\!\!\!\!\!+\;\;\; & T^2X \\ & {\scriptstyle 1_X}\searrow \;\; \leq & {\scriptstyle a}\big\downarrow & \leq & \big\downarrow{\scriptstyle \mu_X} \\ & & X & \xleftarrow{a}\!\!\!\!\!\!\!+\;\;\; & TX \end{array}$$

Morphisms $f : (X, a) \to (Y, b)$ between relational T-algebras are maps $f : X \to Y$ with $f \cdot a \leq b \cdot Tf$:

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\big\downarrow\!\!\!\!\!\!\!\;+ & \leq & +\!\!\!\!\!\!\!\;\big\downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{A}$$

We will denote the category of relational T-algebras by $\mathbf{RelAlg}(T)$. Given a relational T-algebra $a : TX {\longrightarrow\!\!\!\!\!\!\!\!\!\;+}\; X$, for $\mathfrak{x} \in TX$ and $x \in X$, we will write $\mathfrak{x} \to x$ whenever $\mathfrak{x}\, a\, x$. Using this notation, a relation $a : TX {\longrightarrow\!\!\!\!\!\!\!\!\!\;+}\; X$ is a relational T-algebra if

(a) $(\forall x \in X)\ \eta_X(x) \to x$,
(b) $(\forall \mathfrak{X} \in T^2X)\ (\forall \mathfrak{x} \in TX)\ (\forall x \in X)\ \mathfrak{X} \to \mathfrak{x}$ and $\mathfrak{x} \to x \ \Rightarrow\ \mu_X(\mathfrak{X}) \to x$

(here $\mathfrak{X} \to \mathfrak{x}$ means $\mathfrak{X}\,(Ta)\,\mathfrak{x}$); a map $f : X \to Y$ is a morphism between the relational T-algebras (X, a) and (Y, b) if

(c) $(\forall \mathfrak{x} \in TX)\ (\forall x \in X)\ \mathfrak{x} \to x \ \Rightarrow\ Tf(\mathfrak{x}) \to f(x)$.

We remark that $\mathbf{Alg}(T)$ is (fully) embedded in $\mathbf{RelAlg}(T)$, since the inequality of diagram (A) becomes an equality whenever a and b are maps. Morphisms in $\mathbf{RelAlg}(T)$ will be called simply homomorphisms, unless we want to identify those between T-algebras, calling them then *algebraic homomorphisms.*

We will make use also of a category containing $\mathbf{RelAlg}(T)$ as a full subcategory. A relation $a : TX {\longrightarrow\!\!\!\!\!\!\!\!\!\;+}\; X$ is said to be a *pseudo-relational T-algebra*, or a *lax reflexive T-algebra*, or a *$(T, \mathbf{2})$-graph*, if $\eta_X(x)\, a\, x$, that is:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ & {\scriptstyle 1_X}\searrow \quad \leq & +\!\!\!\!\!\!\!\;\big\downarrow{\scriptstyle a} \\ & & X \end{array}$$

A morphism $f : (X, a) \to (Y, b)$ between pseudo-relational T-algebras is a map $f : X \to Y$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\big\downarrow\!\!\!\!\!\!\!\;+ & \leq & +\!\!\!\!\!\!\!\;\big\downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Pseudo-relational T-algebras and relational homomorphisms form the category $\mathbf{PsRelAlg}(T)$. The full embeddings

$$\mathbf{Alg}(T) \hookrightarrow \mathbf{RelAlg}(T) \hookrightarrow \mathbf{PsRelAlg}(T).$$

have left adjoints; moreover, while $\mathbf{Alg}(T)$ is monadic over $\mathbf{Set}$, $\mathbf{RelAlg}(T)$ and $\mathbf{PsRelAlg}(T)$ are topological categories over $\mathbf{Set}$ (see [5] for details). In particular, the forgetful functor $\mathbf{Alg}(T) \to \mathbf{Set}$ creates limits, while $\mathbf{RelAlg}(T) \to \mathbf{Set}$ and $\mathbf{PsRelAlg}(T) \to \mathbf{Set}$ preserve limits and colimits.

2. Étale homomorphisms

In case T is the ultrafilter monad, that is the monad induced by the adjunction

$$\mathbf{Set} \underset{\mathbf{Bool}(-,2)}{\overset{\mathbf{Set}(-,2)}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathbf{Bool}^{\mathrm{op}}$$

that assigns to each set X its set of ultrafilters, T-algebras are compact Hausdorff spaces (as shown by Manes [17]), relational T-algebras are topological spaces, homomorphisms between relational T-algebras are continuous maps (as shown by Barr [1]), and pseudo-relational algebras are pseudotopological spaces. This is the example that guides our approach to étale algebraic homomorphisms. Indeed, in order to introduce the notion of étale homomorphism, we recall first the characterization of étale continuous maps obtained in [8].

A continuous map $f : X \to Y$ is *étale*, or a *local homeomorphism*, if it is locally a homeomorphism, that is each $x \in X$ has an open neighbourhood U such that $f(U)$ is open and the map $f_{|U} : U \to f(U)$ is a homeomorphism. It was shown in [4] that if $f : X \to Y$ is étale, then, for each $x \in X$ and each ultrafilter $\mathfrak{y}$ with $y \to f(x)$, there exists a unique ultrafilter $\mathfrak{x} \in X$ such that $\mathfrak{x} \to x$ and $Tf(\mathfrak{x}) = \mathfrak{y}$:

$$\begin{array}{ccc} X & \qquad & \mathfrak{x} \dashrightarrow x \\ {\scriptstyle f}\downarrow & & \downarrow \qquad \downarrow \\ Y & & \mathfrak{y} \longrightarrow f(x); \end{array}$$

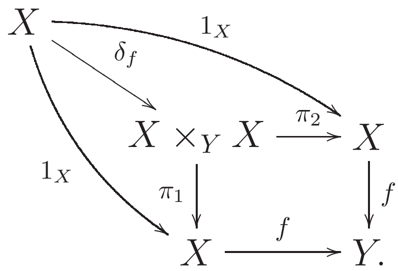
a continuous map with this property is called a *discrete fibration*. As shown in [8], there are discrete fibrations that are not étale.

We point out that, if X and Y are compact Hausdorff spaces, with T-algebra structures α and β respectively, a continuous map $f : X \to Y$ is a discrete fibration if and only if the diagram

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback.

Theorem 2.1. [8] *For a continuous map $f : X \to Y$, consider the commutative diagram:*



(B)

The following conditions are equivalent:

(i) *f is étale;*
(ii) *f is a pullback-stable discrete fibration;*
(iii) *both f and $\pi_1 : X \times_Y X \to X$ are discrete fibrations;*
(iv) *both f and δ_f are discrete fibrations.*

Based on this example, in our general setting, a relational T-homomorphism $f : (X, a) \to (Y, b)$ will be said to be a *discrete fibration* if for every $x \in X$ and $\mathfrak{y} \in TY$ with $\mathfrak{y} \to f(x)$ in Y, there exists a unique $\mathfrak{x} \in TX$ with $\mathfrak{x} \to x$ and $Tf(\mathfrak{x}) = \mathfrak{y}$. It will be called *étale* if it is a pullback-stable discrete fibration.

Proposition 2.2. *If $f : (X, \alpha) \to (Y, \beta)$ is an algebraic homomorphism, then f is a discrete fibration if and only if the diagram*

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback.

Theorem 2.3. *If $f : (X, \alpha) \to (Y, \beta)$ is an algebraic discrete fibration, then the following assertions are equivalent:*

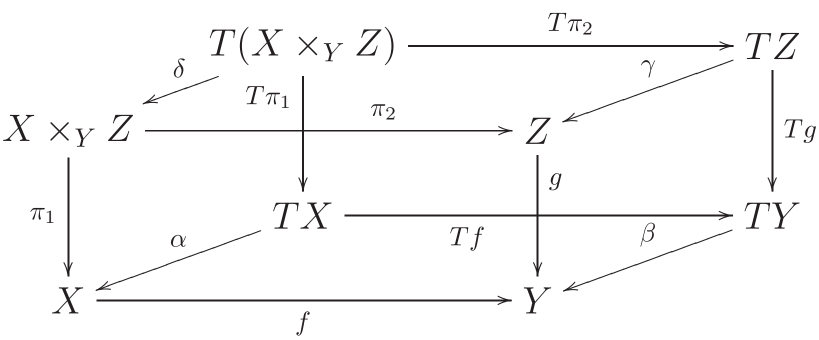
(i) *f is étale;*

(ii) *for any algebraic homomorphism* $g : (Z, \gamma) \to (Y, \beta)$*, the pullback of* g *along* f

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\pi_2} & Z \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} \tag{C}$$

is preserved by T*.*

Proof. In the commutative diagram



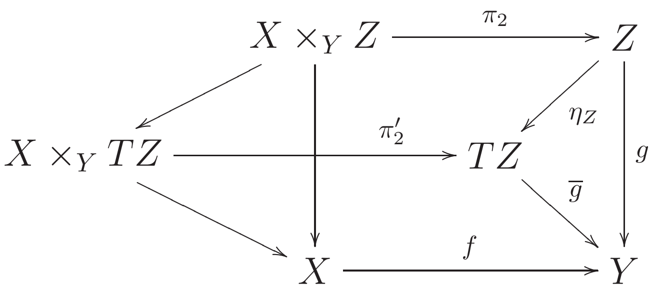
(D)

assume that the front face is a pullback. Closedness under limits of $\mathbf{Alg}(T)$ guarantees that both π_1 and π_2 are algebraic homomorphisms. Since f is a discrete fibration, the bottom square is also a pullback. Hence the top square is a pullback – that is, π_2 is a discrete fibration – if and only if the back square is a pullback, that is T preserves the pullback of g along f. $\square$

Corollary 2.4. *If* T *is non-trivial, then the following conditions are equivalent, for an algebraic discrete fibration* $f : X \to Y$*:*

(i) f *is étale;*
(ii) *for any map* $g : Z \to Y$*, the pullback (C) of* g *along* f *is preserved by* T*.*

Proof. We have to show that (i) ⇒ (ii): Any map $g : Z \to Y$ factors through $\eta_Z : Z \to TZ$ via an algebraic homomorphism $\overline{g} : (TZ, \mu_Z) \to (Y, \beta)$. Hence preservation of pullback (C) by T reduces to preservation of the pullback of $\overline{g}$ along f, which follows from Theorem 2.3, and the preservation, by T, of the pullback of η_Z and π_2':



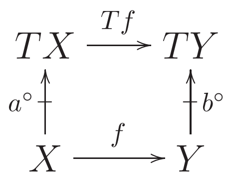
Since η_Z is monic, because T is non-trivial, this follows from (BC) property of T. $\square$

Corollary 2.5. (1) *Every injective algebraic discrete fibration is étale.*
(2) *If the functor T preserves pullbacks, then every algebraic discrete fibration is étale.*

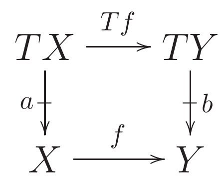
Following the characterization of proper, perfect and open continuous maps stated in [4, Theorem 2.2], in [7, 16] the following notions were studied in the context of relational T-algebras. A morphism $f : (X, a) \to (Y, b)$ is:

(a) *open* if for each $x \in X$ and $\mathfrak{y} \in TY$ such that $\mathfrak{y} \to f(x)$ in Y, there exists $\mathfrak{x} \in TX$ such that $\mathfrak{x} \to x$ in X and $Tf(\mathfrak{x}) = \mathfrak{y}$.
(b) *proper* (*perfect*) if for each $\mathfrak{x} \in TX$ and $y \in Y$ such that $Tf(\mathfrak{x}) \to y$ in Y, there exists (a unique) $x \in X$ such that $\mathfrak{x} \to x$ and $f(x) = y$.

We remark that $f : (X, a) \to (Y, b)$ is open if and only if the diagram



is commutative, while f is proper if and only if the diagram



is commutative. Therefore, every algebraic homomorphism is proper. It is in fact perfect because unicity of x in (b) follows from the fact that a is a map, so that $x = a(\mathfrak{x})$.

Finally we introduce some categories that will be used in the sequel. It is well known that, for a topological space X,

$$\begin{aligned} X \text{ is Hausdorff} &\iff \text{every ultrafilter on } X \text{ converges to at most one point} \\ &\iff \text{the diagonal map } \delta_X : X \to X \times X \text{ is proper,} \end{aligned}$$

$$\begin{aligned} X \text{ is compact} &\iff \text{every ultrafilter converges to at least one point} \\ &\iff \text{the unique map } X \to 1 \text{ is proper.} \end{aligned}$$

In our general setting we will say that a pseudo-relational T-algebra (X, a) is *Hausdorff* if $a : TX \to\!\!\!\!\!\!\mapsto X$ is a partial map, that is in its relation span

$$\begin{array}{ccc} & R & \\ {}^{a_1}\swarrow & & \searrow^{a_2} \\ TX & & X \end{array}$$

a_1 is injective. If a_1 is surjective, (X, a) is said to be *compact*. It is easy to check that:

Proposition 2.6. *If (X, a) is a pseudo-relational T-algebra, then:*
(1) *(X, a) is Hausdorff if and only if δ_X is proper;*
(2) *(X, a) is compact if and only if $!_X : X \to 1$ is proper.*

We denote by $\mathbf{Haus}(T)$ and $\mathbf{Comp}(T)$ the (full) subcategories of $\mathbf{RelAlg}(T)$ of Hausdorff and compact relational T-algebras, respectively. We point out that $\mathbf{Alg}(T)$ is exactly the category of compact and Hausdorff relational T-algebras. It is a reflective subcategory of $\mathbf{RelAlg}(T)$, with the reflection constructed via the appropriate Stone-Čech compactification (see [18, 5] for details).

3. Pseudo-relational algebras form a quasitopos

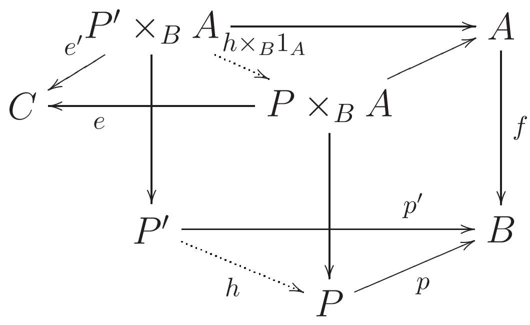
We recall that a category $\mathbf{C}$ is said to be:

(1) *cartesian closed* if it has finite products and every object C in $\mathbf{C}$ is *exponentiable*, that is the functor $(-) \times C : \mathbf{C} \to \mathbf{C}$ has a right adjoint;
(2) *locally cartesian closed* if, for every object B of $\mathbf{C}$, the comma category $(\mathbf{C} \downarrow B)$ is cartesian closed, and $\mathbf{C}$ has a terminal object (and therefore all finite limits);
(3) a *quasitopos* if it is locally cartesian closed and it has a strong subobject classifier and all finite colimits.

When $\mathbf{C}$ is locally cartesian closed, and $f : A \to B$ is a morphism in $\mathbf{C}$, the right adjoint of the functor $(-) \times (A, f) : (\mathbf{C} \downarrow B) \to (\mathbf{C} \downarrow B)$ will be written as $(-)^{(A,f)}$. In particular, for every object C in $\mathbf{C}$, we have

$$(B \times C, \mathrm{pr}_1)^{(A,f)} = \text{ the partial product of } f \text{ and } C.$$

The original definition of (categorical) partial product, due to R. Dyckhoff and W. Tholen [13], is formulated as follows: the *partial product of a morphism $f : A \to B$ and an object C* in a finitely complete category $\mathbf{C}$ is a pair $(p : P \to B, e : P \times_B A \to C)$ such that, given any pair of the form $(p' : P' \to B, e' : P' \times_B A \to C)$, there exists a unique $h : P' \to P$ over B with $e' = e \cdot (h \times_B 1_A)$ in the diagram



In fact $(P, p) = (B \times C, \mathrm{pr}_1)^{(A,f)}$, and e determines the universal arrow

$$(-) \times (A, f) \to (C \times B, \mathrm{pr}_2)$$

as the pair $((P, p), \langle e, p \cdot \pi_1\rangle : (P, p) \times (A, f) \to (C \times B, \mathrm{pr}_2))$, where $\pi_1 : P \times_B A \to C$ is the pullback projection and of course $(P, p) \times (A, f) = (P \times_B A, p \cdot \pi_1)$. Note that, conversely, the existence of such universal arrows for all C in $\mathbf{C}$ implies the existence of the right adjoint for $(-) \times (A, f)$, since every object in $(\mathbf{C} \downarrow B)$ can be presented as the equalizer of two parallel morphisms of the form $(C \times B, \mathrm{pr}_2) \to (C' \times B, \mathrm{pr}_2)$. Moreover, we have:

Theorem 3.1. [13, 20] *For a morphism $f : A \to B$ in a finitely complete category $\mathbf{C}$, the following conditions are equivalent:*

(i) *f is exponentiable, i.e. $(-) \times (A, f) : (\mathbf{C} \downarrow B) \to (\mathbf{C} \downarrow B)$ has a right adjoint;*

(ii) *the 'change-of-base' functor $f^* : (\mathbf{C} \downarrow B) \to (\mathbf{C} \downarrow A)$ has a right adjoint;*

(iii) *the composite $(\mathbf{C} \downarrow B) \to \mathbf{C}$ of f^* and the forgetful functor $(\mathbf{C} \downarrow A) \to \mathbf{C}$ has a right adjoint;*

(iv) *$\mathbf{C}$ has partial products of each of its objects with f.*

In $\mathbf{PsRelAlg}(T)$ the pullback of $f : (X, a) \to (Y, b)$ and $g : (Z, c) \to (Y, b)$

$$\begin{array}{ccc} (Z,c) \times_{(Y,b)} (X,a) & \xrightarrow{\pi_2} & (X,a) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ (Z,c) & \xrightarrow{\quad g \quad} & (Y,b) \end{array}$$

is built as in **Set**, that is, $Z \times_Y X = \{(z, x) \,|\, g(z) = f(x)\}$, with $\mathfrak{w} \to (z, x)$, for $\mathfrak{w} \in T(Z \times_Y X)$ and $(z, x) \in Z \times_Y X$, if $T\pi_1(\mathfrak{w}) \to z$ and $T\pi_2(\mathfrak{w}) \to x$ (see [9] for details). In [9] it was shown that, for a morphism $f : (X, a) \to (Y, b)$ between pseudo-relational T-algebras and any pseudo-relational T-algebra (Z, c), the *partial product*

$$\begin{array}{ccccc} (Z,c) & \xleftarrow{\text{ev}} & (P,d) \times_{(Y,b)} (X,a) & \xrightarrow{\pi_2} & (X,a) \\ & & {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ & & (P,d) & \xrightarrow{\quad p \quad} & (Y,b) \end{array} \tag{E}$$

of f *and* Z can be constructed as $P = Z^{(X,f)} = \{(s, y) \,|\, y \in Y,\ s : (X_y, a_y) \to (Z, c)\}$, where (X_y, a_y) is defined as the pullback

$$\begin{array}{ccc} (X_y,a_y) & \longrightarrow & (X,a) \\ \downarrow & & \downarrow{\scriptstyle f} \\ (1,\eta_1^\circ) & \xrightarrow{\ y\ } & (Y,b) \end{array}$$

and in particular X_y can be identified with $f^{-1}(y)$; the structure d on P, that is the relation $d : TP \,{\to\!\!\!\!\!\!\mapsto}\, P$, is defined, for $\mathfrak{p} \in TP$ and $(s, y) \in P$, by:

$$\mathfrak{p} \to (s,y) \quad \text{if} \quad \begin{cases} Tp(\mathfrak{p}) \to y \text{ and} \\ T\text{ev}(\mathfrak{w}) \to \text{ev}((s,y),x) = s(x), \\ \ \text{whenever } \mathfrak{w} \in T(P \times_Y X) \text{ and } x \in X \\ \ \text{are such that } T\pi_1(\mathfrak{w}) = \mathfrak{p},\ f(x) = y \text{ and } T\pi_2(\mathfrak{w}) \to x \end{cases}$$

Moreover, it was shown that:

Theorem 3.2. [9] *The category* $\mathbf{PsRelAlg}(T)$ *of pseudo-relational* T*-algebras is a quasitopos.*

4. Algebraic homomorphisms are exponentiable, topologically

In this section we will prove that the partial product of an algebraic homomorphism and a relational T-algebra is a relational T-algebra, showing that algebraic homomorphisms are exponentiable in $\mathbf{RelAlg}(T)$. First we show that perfect homomorphisms, hence in particular algebraic homomorphisms, have an interpolation property which is sufficient for exponentiability (similarly to the proof for the ultrafilter monad done in [10]).

Proposition 4.1. *Let $f : (X,a) \to (Y,b)$ be a perfect morphism in $\mathbf{RelAlg}(T)$, and let the diagram*

$$\begin{array}{ccc} (Z \times_Y X, d) & \xrightarrow{\pi_2} & (X,a) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ (Z,c) & \xrightarrow{g} & (Y,b) \end{array}$$

be a pullback in $\mathbf{PsRelAlg}(T)$. Then:

(1) *π_1 is a proper map.*
(2) *If $\mathfrak{W} \in T^2(Z \times_Y X)$, $(z,x) \in Z \times_Y X$ and $\mathfrak{Z} \in TZ$ are such that*
 (a) $\mu_X(T^2\pi_2(\mathfrak{W})) \to x$,
 (b) $T^2\pi_1(\mathfrak{W}) \to \mathfrak{Z} \to z$,
 then there exists $\mathfrak{w} \in T(Z \times_Y X)$ such that $T\pi_1(\mathfrak{w}) = \mathfrak{Z}$ and

$$\mathfrak{W} \to \mathfrak{w} \to (z,x).$$

Proof. It was proved in [9] that proper maps are pullback-stable in $\mathbf{PsRelAlg}(T)$, hence π_1 is proper, that is $\pi_1 \cdot d = c \cdot T\pi_1$. Therefore, also $T\pi_1 \cdot Td = Tc \cdot T^2\pi_1$. Hence, if $T^2\pi_1(\mathfrak{W}) \to \mathfrak{Z}$, there exists $\mathfrak{w} \in T(Z \times_Y X)$ such that $T\pi_1(\mathfrak{w}) = \mathfrak{Z}$ and $\mathfrak{W} \to \mathfrak{w}$. Using now the fact that π_1 is proper and $T\pi_1(\mathfrak{w}) = \mathfrak{Z} \to z$, there exists $(z,x') \in Z \times_Y X$ such that $\mathfrak{w} \to (z,x')$. The chain $\mathfrak{W} \to \mathfrak{w} \to (z,x')$ gives rise to a chain in X, $T^2\pi_2(\mathfrak{W}) \to T\pi_2(\mathfrak{w}) \to x'$, which implies that $\mu_X(T^2\pi_2(\mathfrak{W})) \to x'$. Since $f(x) = f(x')$, perfectness of f gives $x = x'$. □

Theorem 4.2. *Perfect maps are exponentiable in $\mathbf{RelAlg}(T)$.*

Proof. For $f : (X,a) \to (Y,b)$ perfect in $\mathbf{RelAlg}(T)$ and $(Z,c) \in \mathbf{RelAlg}(T)$, form the partial product $(p : (P,d) \to (Y,b), e : (P,d) \times_{(Y,b)} (X,a) \to (Z,c))$

of f and (Z, c) in $\mathbf{PsRelAlg}(T)$ as in (E). To check that $(P, d) \in \mathbf{RelAlg}(T)$, we need to check that, for

$$\mathfrak{P} \to \mathfrak{p} \to (s, y) \text{ in } P$$

we have $\mu_P(\mathfrak{P}) \to (s, y)$ in P. The first of the two conditions defining $\mu(\mathfrak{P}) \to (s, y)$ easily follows from the fact that (Y, b) belongs to $\mathbf{RelAlg}(T)$, and we will check only the second one. Suppose there exist $\tilde{\mathfrak{w}} \in T(P \times_Y X)$ and $x \in X$ such that $T\pi_1(\tilde{\mathfrak{w}}) = \mu_P(\mathfrak{P})$, $f(x) = y$ and $T\pi_2(\tilde{\mathfrak{w}}) \to x$. By (BC) property of the diagram

$$\begin{array}{ccc} T^2(P \times_Y X) & \xrightarrow{\mu_{P\times_Y X}} & T(P \times_Y X) \\ {\scriptstyle T^2\pi_1}\downarrow & & \downarrow{\scriptstyle T\pi_1} \\ T^2P & \xrightarrow{\quad\mu_P\quad} & TP \end{array}$$

there exists $\mathfrak{W} \in T^2(P \times_Y X)$ such that

$$\mu_{P\times_Y X}(\mathfrak{W}) = \tilde{\mathfrak{w}} \text{ and } T^2\pi_1(\mathfrak{W}) = \mathfrak{P}.$$

The lemma above guarantees the existence of $\mathfrak{w} \in T(P \times_Y X)$ such that $T\pi_1(\mathfrak{w}) = \mathfrak{p}$ and

$$\mathfrak{W} \to \mathfrak{w} \to ((s, y), x).$$

Therefore $T^2\mathrm{ev}(\mathfrak{W}) \to T\mathrm{ev}(\mathfrak{w}) \to s(x)$, hence

$$T\mathrm{ev}(\tilde{\mathfrak{w}}) = T\mathrm{ev}(\mu_{P\times_Y X}(\mathfrak{W})) = \mu_Z(T^2\mathrm{ev}(\mathfrak{W})) \to s(x),$$

and so we can conclude that $\mu_P(\mathfrak{P}) \to (s, y)$ in P by the definition of the structure on P. □

Corollary 4.3. *Algebraic homomorphisms are exponentiable in* $\mathbf{RelAlg}(T)$.

5. Étale algebraic homomorphisms are exponentiable, algebraically

In this section we will show that étale algebraic homomorphisms are exponentiable in $\mathbf{Alg}(T)$, generalizing the corresponding result for topological spaces obtained by Cagliari and Mantovani [2].

Proposition 5.1. *If (X, a), (Y, b) and (Z, c) are pseudo-relational T-algebras, $f : (X, a) \to (Y, b)$ is open, and (Y, b) and (Z, c) are Hausdorff, then the partial product of f and (Z, c) in* $\mathbf{PsRelAlg}(T)$ *is Hausdorff too.*

Proof. Consider the partial product (E) and assume that $\mathfrak{p} \in TP$ is such that $\mathfrak{p} \to (s, y)$ and $\mathfrak{p} \to (s', y')$ in P. Hausdorffness of Y guarantees that $y = y'$. If $X_y = f^{-1}(y)$ is empty, then necessarily $s = s'$. Otherwise, let $x \in f^{-1}(y)$. Since $Tp(\mathfrak{p}) \to f(x)$, openness of f gives $\mathfrak{x} \in Tf^{-1}(Tp(\mathfrak{p}))$ with $\mathfrak{x} \to x$. By (BC) property of T there exists $\mathfrak{w} \in T(P \times_Y X)$ such that $T\pi_1(\mathfrak{w}) = \mathfrak{p}$ and $T\pi_2(\mathfrak{w}) = \mathfrak{x}$. So, in the pullback structure, $\mathfrak{w}$ is in relation with both $((s, y), x)$ and $((s', y), x)$, and, consequently, $T\mathrm{ev}(\mathfrak{w})$ is related to both $s(x)$ and $s'(x)$, which implies $s(x) = s'(x)$ because Z is Hausdorff. □

Corollary 5.2. *If $f : (X, a) \to (Y, b)$ is open and exponentiable in* **RelAlg**(T)*, then it is exponentiable in* **Haus**(T)*.*

We remark that this result implies the corresponding result for topological spaces, due to Cagliari and Mantovani [3]. We do not know whether, as in **Top**, openness of f is essential for its exponentiability in **Haus**(T).

Proposition 5.3. *Let the diagram*

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{\pi_2} & X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

be a pullback in **PsRelAlg**(T)*, with π_1 a discrete fibration, X and Z compact, and Y Hausdorff. Then the functor T preserves the underlying pullback of sets.*

Proof. Applying T to the diagram and forming the pullback

$$\begin{array}{ccccc} T(Z \times_Y X) & \xrightarrow{\quad T\pi_2 \quad} & & & TX \\ & \searrow^{\mathrm{can}} & & \nearrow_{\pi_2'} & \\ {\scriptstyle T\pi_1}\downarrow & & TZ \times_{TY} TX & & \downarrow{\scriptstyle Tf} \\ & \swarrow_{\pi_1'} & & & \\ TZ & \xrightarrow{\quad Tg \quad} & & & TY \end{array}$$

we have to show that the comparison map $\mathrm{can} : T(Z \times_Y X) \to TZ \times_{TY} TX$ is injective. Suppose $\mathfrak{w}, \mathfrak{w}' \in T(Z \times_Y X)$ have $\mathrm{can}(\mathfrak{w}) = \mathrm{can}(\mathfrak{w}')$. Denote $T\pi_1(\mathfrak{w}) = T\pi_1(\mathfrak{w}')$ by $\mathfrak{z}$, and $T\pi_2(\mathfrak{w}) = T\pi_2(\mathfrak{w}')$ by $\mathfrak{x}$. By compactness of Z and X, we can write $\mathfrak{z} \to z$ and $\mathfrak{x} \to x$. Then Hausdorffness of Y gives

$g(z) = f(x)$. Hence both $\mathfrak{w}$ and $\mathfrak{w}'$ are in relation with (z, x) and are mapped, by $T\pi_1$, to $\mathfrak{z}$. Since π_1 is a discrete fibration, $\mathfrak{w} = \mathfrak{w}'$. $\square$

This result assures that, if f is an étale algebraic homomorphism and $Z \in \mathbf{Alg}(T)$, preservation of the pullback of the partial product (E) in $\mathbf{PsRelAlg}(T)$ is a necessary condition for f to be exponentiable in $\mathbf{Alg}(T)$ with its exponentials built as in $\mathbf{RelAlg}(T)$. The next result shows that this pullback-preservation property is also sufficient.

Proposition 5.4. *Let $f : (X, \alpha) \to (Y, \beta)$ be an étale algebraic homomorphism and (Z, γ) a T-algebra. The domain (P, d) of the partial product of f and (Z, c) in $\mathbf{RelAlg}(T)$ (or in $\mathbf{PsRelAlg}(T)$) is compact if and only if T preserves the pullback*

$$\begin{CD} P \times_Y X @>{\pi_2}>> X \\ @V{\pi_1}VV @VV{f}V \\ P @>{p}>> Y \end{CD}$$

Proof. Consider again the partial product (E), assume that T preserves its underlying pullback of sets, and let $\mathfrak{p} \in TP$. We need to find $(s, y) \in P$ such that $\mathfrak{p} \to (s, y)$, with $y \in Y$ and $s : X_y \to Z$, where $X_y = f^{-1}(y)$ is to be seen as

$$\begin{CD} X_y @>{i_y}>> X \\ @V{!}VV @VV{f}V \\ 1 @>{y}>> Y \end{CD}$$

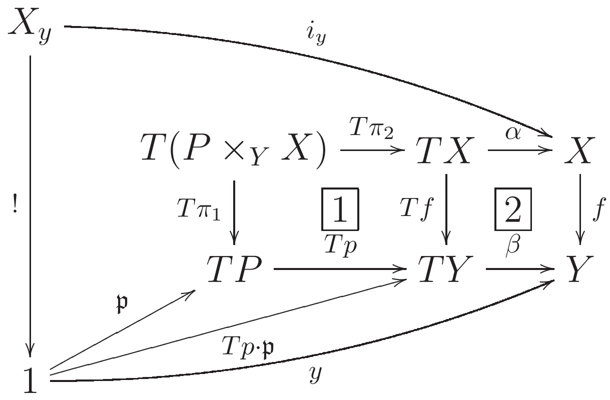
Since Y is a T-algebra, there is a unique $y \in Y$ such that $Tp(\mathfrak{p}) \to y$. Hence, we are left with the construction of $s : X_y \to Z$.

Since f is étale, the diagram

$$\begin{CD} TX @>{Tf}>> TY \\ @V{\alpha}VV @VV{\beta}V \\ X @>{f}>> Y \end{CD}$$

is a pullback in $\mathbf{Set}$, and so it determines a pullback in $\mathbf{Alg}(T)$ (where TX and TY have the free algebra structures) and therefore in $\mathbf{RelAlg}(T)$. Hence we can consider the following commutative diagram, where both squares $\boxed{1}$

and $\boxed{2}$ are pullbacks.



Since the diagram $\boxed{1}\boxed{2}$ is a pullback, there exists a (unique) map $\kappa : X_y \to T(P \times_Y X)$ with $T\pi_1(\kappa(x)) = \mathfrak{p}$ and $\alpha(T\pi_2(\kappa(x))) = x$, for each $x \in X_y$. Using this map, we define s as the composite

$$X_y \xrightarrow{\kappa} T(P \times_Y X) \xrightarrow{T\mathrm{ev}} TZ \xrightarrow{\gamma} Z.$$

To prove that $\mathfrak{p} \to (s, y)$, we need to show that $T\mathrm{ev}(\mathfrak{w}) \to s(x)$ whenever $\mathfrak{w} \in T(P \times_Y X)$ and $x \in X$ have $T\pi_1(\mathfrak{w}) = \mathfrak{p}$, $f(x) = y$, and $T\pi_2(\mathfrak{w}) \to x$. Since (X, α) is a T-algebra, $T\pi_2(\mathfrak{w}) \to x$ means $\alpha(T\pi_2(\mathfrak{w})) = x$, and since $\boxed{1}\boxed{2}$ is a pullback, this equality, together with $T\pi_1(\mathfrak{w}) = \mathfrak{p}$, imply $\mathfrak{w} = \kappa(x)$. This gives $s(x) = \gamma(T\mathrm{ev}(\kappa(x))) = \gamma(T\mathrm{ev}(\mathfrak{w}))$, and so $T\mathrm{ev}(\mathfrak{w}) \to s(x)$ in the T-algebra (Z, γ), as desired. □

Theorem 5.5. *Every étale algebraic homomorphism is exponentiable in* $\mathbf{Alg}(T)$*, with the exponentials built as in* $\mathbf{RelAlg}(T)$*.*

Proof. It follows directly from Theorem 2.3, and Propositions 5.1 and 5.4. □

6. Examples

6.1 Compact Hausdorff spaces. In case $T = (T, \eta, \mu)$ is the ultrafilter monad, both T and μ satisfy (BC) property. Hence Theorem 5.5 says that *an étale continuous map between compact Hausdorff spaces is exponentiable.* Cagliari and Mantovani [2] showed that also the reverse implication holds, that is, *an exponentiable continuous map in* **CompHaus** *is étale*. Moreover, the exponentials coincide with the exponentials built in **Top**.

6.2 Sup-lattices. When $T = (P, \eta, \mu)$ is the powerset monad, again both P and μ satisfy (BC) property. $\mathbf{Alg}(P)$ is the category **Sup** of complete lattices and sup-preserving maps. Theorem 5.5 says that étale homomorphisms are exponentiable in **Sup**. In fact étale and exponentiable homomorphisms coincide, as shown next.

Proposition. *For a morphism $f : X \to Y$ in* **Sup***, the following conditions are equivalent:*

(i) *f is exponentiable;*
(ii) *the diagram*

$$\begin{array}{ccc} X + X & \xrightarrow{f+f} & Y + Y \\ {\scriptstyle \nabla_X}\downarrow & & \downarrow{\scriptstyle \nabla_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback;
(iii) *for all $x \in X$ and $y' \in Y$ with $y' \leq y = f(x)$, there is a unique $x' \in X$ with $x' \leq x$ and $f(x') = y'$;*
(iv) *f is a down-closed embedding;*
(v) *f is étale.*

Proof. We know that (v) $\Rightarrow$ (i), by Theorem 5.5, and that (i) $\Rightarrow$ (ii) since pulling back along an exponentiable morphism preserves colimits.

(ii) $\Rightarrow$ (iii): Since **Sup** admits an enrichment in the category of commutative monoids via the $\vee$ operation, its finite coproducts are canonically isomorphic to products, and the codiagonal $\nabla_X : X + X \to X$ is given by $\nabla_X(x, x') = x \vee x'$. Let $x \in X$ and $y' \in Y$ be such that $y' \leq y = f(x)$. Then $y = y' \vee y$, hence there is a (unique) pair (x', x'') with $f(x') = y'$, $f(x'') = y$ and $x' \vee x'' = x$, hence $x' \leq x$. By uniqueness, $x'' = x$ and x' is indeed unique.

To conclude that (iii) $\Rightarrow$ (iv) we only have to check that f is injective, which follows easily from the uniqueness of x' above.

(iv) $\Rightarrow$ (v): Since down-closed embeddings are pullback-stable, we only have to show that every down-closed embedding is a discrete fibration. A morphism $f : X \to Y$ in **Sup** is a discrete fibration if and only if

$$\begin{array}{ccc} PX & \xrightarrow{Pf} & PY \\ {\scriptstyle \sup_X}\downarrow & & \downarrow{\scriptstyle \sup_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback. If f is a down-closed embedding, and $x = \sup S$, for $x \in X$ and $S \subseteq Y$, then $S \subseteq X$ and the diagram is a pullback. □

The category **Inf** of complete lattices and inf-preserving maps is isomorphic to **Sup**, via

$$f : X \to Y \mapsto f^{\rm op} : X^{\rm op} \to Y^{\rm op}.$$

Hence we can conclude that, in **Inf**, *a morphism $f : X \to Y$ is exponentiable if and only if it is an up-closed embedding.*

6.3 Continuous lattices. For the filter monad $F = (F, \eta, \mu)$, $\mathbf{Alg}(F)$ is the category **ContLat** of continuous lattices and monotone maps preserving infima and directed suprema (see [12]). Since F and μ satisfy (BC) property, Theorem 5.5 applies, that is every étale homomorphism is exponentiable in **ContLat**.

Lemma. *For a continuous lattice Y and $y \in Y$, the following conditions are equivalent:*

(i) *the embedding $\uparrow y \to Y$ is étale;*
(ii) *y is compact.*

Proof. (i) ⇒ (ii): Let $\mathfrak{f} = \langle\{\uparrow x \,|\, x \ll y\}\rangle$. Then $y = \sup_{B\in\mathfrak{f}} \inf B$. Hence, $\uparrow y \in \mathfrak{f}$, that is, $\uparrow y \supseteq\uparrow x$ for some $x \ll x$. But this implies $x \geq y$, hence $x = y$ and $y \ll y$.

(ii) ⇒ (i): Let $\mathfrak{g}$ be a filter on Y with $y = \sup_{B\in\mathfrak{g}} \inf B$. Since $y \ll y$, $y \leq \inf B$ for some $B \in \mathfrak{g}$, hence $B \subseteq\uparrow y$ and therefore $\uparrow y \in \mathfrak{g}$, which, together with the preservation of infima and directed suprema by the inclusion $\uparrow y \to Y$, gives that this morphism is étale. □

The arguments used in the proof of Proposition 6 can be used here, since **ContLat** is also enriched in the category of commutative monids via the $\wedge$ operation. Hence:

(1) *every exponentiable morphism in* **ContLat** *is an up-closed embedding, that is, it is (up to isomorphism) an inclusion* $\uparrow y \to Y$*, for some* $y \in Y$*;*
(2) *a morphism is étale if and only if it is, up to isomorphism, an inclusion* $\uparrow y \to Y$ *with* y *compact.*

Therefore one has:

Proposition. *For a morphism* $f : X \to Y$ *in* **ContLat***, each of the conditions below implies the following one:*

(i) *there is a compact element* y *of* Y *such that* f *is, up to isomorphism, the inclusion* $\uparrow y \to Y$*;*
(ii) f *is exponentiable;*
(iii) *there is an element* y *of* Y *such that* f *is, up to isomorphism, the inclusion* $\uparrow y \to Y$*.*

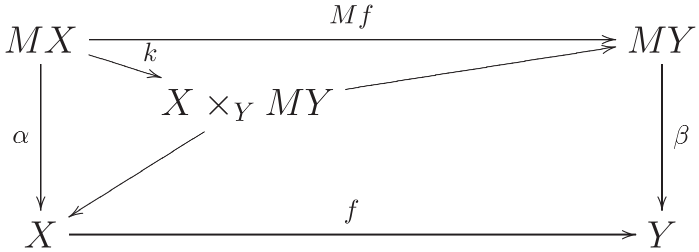
We do not know whether exponentiability of $\uparrow y \to Y$ implies compactness of y.

6.4 Monoids. Let $M = (M, \eta, \mu)$ be the free-monoid monad on **Set**, with MX the set of words in the alphabet X (of length ≥ 0), η_X the insertion of X into MX as one-letter words, and concatenation μ_X. It is well known that **Alg**(M) is the category **Mon** of monoids and monoid homomorphisms. This monad is cartesian, hence in particular both M and μ satisfy (BC) property.

We will show next that *in* **Mon** *étale homomorphisms are exactly the exponentiable ones.*

It follows from Theorem 5.5 that étale homomorphisms are exponentiable since the free-monoid monad satisfies (BC) property.

To prove the converse, let $f : (X, \alpha) \to (Y, \beta)$ be a homomorphism in **Mon**, and consider the commutative diagram



We will show that the canonical map k is injective and surjective.

Injectivity. First we observe that exponentiability of f implies preservation of the initial object by the functor f^*, which means that the kernel Ker(f)

of f is a trivial monoid. Now, suppose $k(x_1, \cdots, x_n) = k(x'_1, \cdots, x'_{n'})$. This means

$$x_1 \cdots x_n = x'_1 \cdots x'_{n'} \text{ in } X, \text{ and } (f(x_1), \cdots, f(x_n)) = (f(x'_1), \cdots, f(x'_{n'})) \text{ in } MY,$$

which implies $n = n'$, and is equivalent to

$$x_1 \cdots x_n = x'_1 \cdots x'_n \text{ in } X, \text{ and } f(x_i) = f(x'_i) \text{ in } Y \text{ for each } i = 1, \cdots, n. \tag{F}$$

If $f(x_i) = f(x'_i) = 1$ for some i, then $x_i = x'_i = 1$ since $\mathrm{Ker}(f) = \{1\}$, and simultaneously removing x_i from the sequence $(x_1, \cdots, x_n)$ and x'_i from the sequence $(x'_1, \cdots, x'_n)$ will not change anything in (F) except decreasing n. Therefore, without loss of generality, we can assume that $f(x_i) \neq 1 \neq f(x'_i)$, and so $x_i \neq 1 \neq x'_i$ for each i. Let $\iota_1, \iota_2 : X \to X + X$ and $\kappa_1, \kappa_2 : Y \to Y + Y$ be the coproduct injections. Consider the elements

$$t = \iota_{\varepsilon_1}(x_1) \cdots \iota_{\varepsilon_n}(x_n) \text{ and } t' = \iota_{\varepsilon_1}(x'_1) \cdots \iota_{\varepsilon_n}(x'_n) \in X + X,$$

where $\varepsilon_i = 1$ for even i and $\varepsilon_i = 2$ for odd i; their images in $Y + Y$ under $f + f$ are

$$(f+f)(t) = \kappa_{\varepsilon_1}(f(x_1)) \cdots \kappa_{\varepsilon_n}(f(x_n)) \text{ and } (f+f)(t') = \kappa_{\varepsilon_1}(f(x'_1)) \cdots \kappa_{\varepsilon_n}(f(x'_n)), \tag{G}$$

respectively. Next, consider the diagram

$$\begin{array}{ccc} X + X & \xrightarrow{f+f} & Y + Y \\ {\scriptstyle \nabla_X}\downarrow & & \downarrow{\scriptstyle \nabla_Y} \\ X & \xrightarrow{f} & Y \end{array} \tag{H}$$

as in Proposition 6.2, which is a pullback again since f is exponentiable. In this diagram, as follows from (F) and (G), the elements t and t' have the same images in X and in $Y + Y$. Therefore $t = t'$, and since $x_i \neq 1 \neq x'_i$ for each i, this implies $(x_1, \cdots, x_n) = (x'_1, \cdots, x'_{n'})$. That is, k is injective.

Surjectivity. We have to prove that, for every $x \in X$ and every $(y_1, \cdots, y_n) \in MY$ with $f(x) = y_1 \cdots y_n$, there exist $(x_1, \cdots, x_n) \in MX$ with $x_1 \cdots x_n = x$ in X and $f(x_i) = y_i$ for each i. Again, if $y_i = 1$ for some i, removing it will not change the equality $f(x) = y_1 \cdots y_n$, and so we can assume that $y_i \neq 1$ for each i. Next, in the diagram (H) we have $f(x) = \nabla_Y(\kappa_{\varepsilon_1}(y_1) \cdots \kappa_{\varepsilon_n}(y_n))$ in the notation above. Since (H) is a pullback diagram, and having in mind the construction of coproducts of monoids and the fact that $y_i \neq 1$ for each

i, we conclude that there a sequence $(x_1, \cdots, x_n)$ of elements in X with the desired properties.

That is, f is exponentiable if and only if it is étale.

Let us make some further remarks about monoids. As mentioned above, whenever f is exponentiable, we have $\mathrm{Ker}(f) = \{1\}$. More generally, the same is true whenever f is open. This immediately follows from the fact that $\mathrm{Ker}(Mf) = \{1\}$ for any f. However, even if f étale, it does not have to be injective. Note that, for every map f from a set X to a set Y, since M is a cartesian monad, the diagram

$$\begin{array}{ccc} M^2X & \xrightarrow{M^2f} & M^2Y \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_Y} \\ MX & \xrightarrow{Mf} & MY \end{array}$$

is a pullback, and so $Mf : (MX, \mu_X) \to (MY, \mu_Y)$ is étale. On the other hand, injectivity of an algebraic homomorphism is not sufficient for being étale, and not even for being open. For example, for an additive monoid $\mathbb{N}$ of natural numbers, the inclusion homomorphism $\mathbb{N} \setminus \{1\} \to \mathbb{N}$ is not open simply because $1 + 1 = 2 \in \mathbb{N} \setminus \{1\}$, while $1 \notin \mathbb{N} \setminus \{1\}$.

6.5 Semigroups. If we take instead the monad M' of non-empty words, $\mathbf{Alg}(M')$ is the category **SGrp** of semigroups and semigroup homomorphisms. An argument analogous to the previous one, used for monoids, shows that *exponentiable homomorphisms in* **SGrp** *are exactly the étale homomorphisms.*

6.6 M-Sets. For a monoid $M = (M, \cdot, e)$, consider the free M-set monad $T = (M\times -, \eta, \mu)$, where $\eta_X = \langle e, 1_X\rangle : X \to M\times X$, and $\mu_X : M\times M\times X \to M\times X$ is given by $\mu_X(m, n, x) = (m\cdot n, x)$. Then $\mathbf{Alg}(T)$ is the category M-**Set** of M-sets, that is, sets X equipped with an M-action, and equivariant maps. Again, since the monad is cartesian, $M\times -$ and μ have (BC) property. The extension $\overline{T} : \mathbf{Rel} \to \mathbf{Rel}$ is defined, for a relation $r : X \relbar\joinrel\mapsto Y$, by

$$(m, x)\,\overline{T}r\,(n, y) \;\Leftrightarrow\; m = n \;\&\; x\,r\,y.$$

To describe a relational T-algebra $a : M \times X \relbar\joinrel\mapsto X$, we write $x \xrightarrow{n} y$ if $(n, x)\,a\,y$. Then $a : M \times X \relbar\joinrel\mapsto X$ is a relational algebra if and only if, for all $x, y, z \in X$:

(a) $x \xrightarrow{e} x$, and

(b) $x \xrightarrow{n} y \;\&\; y \xrightarrow{m} z \;\Rightarrow\; x \xrightarrow{m\cdot n} z$.

A relational homomorphism $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that, for all $x, x' \in X$,

(c) $x \xrightarrow{n} x' \;\Rightarrow\; f(x) \xrightarrow{n} f(x')$.

Therefore relational T-algebras can be seen as *M-labeled ordered sets* and relational homomorphisms as *monotone maps.*

The category M-**Set** is a topos, hence it is locally cartesian closed, so that *every homomorphism is exponentiable in M-**Set***. However, in general *there are homomorphisms in M-**Set** which are not étale.* Consider for instance $M = (\mathbb{N}, \times, 1)$ and the action of $\mathbb{N}$ on $\mathbb{Z}$ and $\mathbb{Q}$ via multiplication. The inclusion

$$\begin{array}{ccc} \mathbb{N}\times\mathbb{Z} & \xrightarrow{1\times f} & \mathbb{N}\times\mathbb{Q} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathbb{Z} & \xrightarrow{\quad f \quad} & \mathbb{Q} \end{array}$$

is not open, since $f(1) = 1 = \beta(2, \frac{1}{2})$ and there is no element of $\mathbb{N}\times\mathbb{Z}$ mapped by $1 \times f$ into $(2, \frac{1}{2})$.

The case of M-sets is worth to study in detail. Theorem 5.5 states that étale maps are exponentiable in M-**Set**, with exponentials built as in the category of M-labeled sets and monotone maps. In fact, although any map is exponentiable in M-**Set**, the exponentials are built as in M-labeled ordered sets if and only if the map is étale, as we show next. Note that $f : (X, \alpha) \to (Y, \beta)$ *in M-**Set** is étale if and only if*

$$(\forall x \in X)\,(\forall m \in M)\,(\forall y \in Y)\; f(x) = my \;\Rightarrow\; (\exists! x' \in X)\, mx' = x \text{ and } f(x') = y.$$

Now, we are going to check that the partial product of $f : (X, \alpha) \to (Y, \beta) \in M$-**Set** and $(Z, \gamma) \in M$-**Set** calculated in $\mathbf{RelAlg}(T)$ (as described in Section 3) is an M-set only if f is étale. First we remark that, for each $y \in Y$, the relational structure α_y on X_y, obtained by pulling back $y : (1, \eta_1^\circ) \to (Y, \beta)$ along f, is discrete, that is $(m, x) \to x'$, if and only if $m = 1$ and $x = x'$. Hence any map $s : X_y \to Z$ becomes a relational homomorphism $s : (X_y, \alpha_y) \to (Z, \gamma)$; that is,

$$P = Z^{(X,f)} = \{(s, y) \mid y \in Y,\; s : X_y \to Z \text{ a map}\}.$$

The relational T-algebra structure on P has

$$(m,(s,y)) \to (s',y') \Leftrightarrow \begin{cases} my = y', \text{ and} \\ m(s(x)) = s'(x'), \text{ whenever } f(x') = y' \text{ and } mx = x', \end{cases}$$

which is the direct translation of its general description in Section 3 to the present case. It shows that, whenever Z has more than one element, the uniqueness of (s', y') satisfying $(m,(s,y)) \to (s',y')$ is equivalent to the existence and uniqueness of $x \in X$ satisfying $f(x) = y$ and $mx = x'$ for each $y \in Y$ and $x' \in X$ with $f(x') = my$. That is, the partial product of every T-algebra with f is a T-algebra if and only if f is étale.

6.7 G-Sets. In the previous example, if M is a group G, then every morphism in G-**Set** is étale. Indeed, for every homomorphism $f : (X, \alpha) \to (Y, \beta)$, given $x \in X$ and $(g, y) \in G \times Y$ such that $f(x) = gy$, put $x' = g^{-1}x$. Then (g, x') is the unique element of $G \times X$ such that $gx' = x$ and $(1_G \times f)(g, x') = (g, g^{-1}f(x)) = (g, y)$. Hence, *exponentiable and étale homomorphisms coincide.*

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