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SUPRA-SUPERCONVERGENT METHODS FOR QUASILINEAR COUPLED PROBLEMS

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ABSTRACT: The aim of this paper is the study of finite difference methods for quasilinear coupled problems of partial differential equations with unexpected convergence rate - two. The partial differential system for the pressure and for the concentration usually used to model a miscible displacement of one incompressible fluid by another in a porous medium is a particularization of the problem considered here. Thus the methods introduced in this paper allow us to compute superconvergent approximations for the pressure, velocity and concentration. As the finite difference methods studied in this paper can be seen as a fully discrete piecewise linear finite element method, we conclude that such piecewise linear finite element approximation for the pressure, velocity and concentration are second order accurate.

Key words: Piecewise linear finite element method, finite difference methods, supraconvergence, superconvergence, pressure, concentration, porous media.

Mathematics Subject Classification (2000): 65M06, 65M20, 65M15

1. Introduction

In this paper we study a fully discrete method for the coupled system

$$-\big(a(c)p_x\big)_x = q_1 \text{ in } (0,1)\times(0,T], \tag{1}$$

$$c_t + \big(b(c,p_x)c_x\big)_x - \big(d(c,p_x)c_x\big)_x = q_2 \text{ in } (0,1)\times(0,T], \tag{2}$$

with the following boundary conditions

$$p(0,t) = p_\ell(t),\ p(1,t) = p_r(t), t\in(0,T], \tag{3}$$

$$c(0,t) = c_\ell(t),\ c(1,t) = c_r(t), t\in(0,T], \tag{4}$$

and initial conditions

$$c(x,0) = c_0(x), x\in(0,1), p(x,0) = p_0(x), x\in(0,1). \tag{5}$$

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The initial boundary value problem (IBVP) (1)-(5) can be used to describe miscible displacement of one incompressible fluid (resident fluid) by another (injected fluid) in one dimensional porous media. In this case,

$$a(c) = K\mu(c)^{-1}, \quad b(c, p_x) = \frac{1}{\phi}v, \quad d(c, p_x) = D_m + D_d\frac{1}{\phi}|v|, \tag{6}$$

where $v = -K\mu(c)^{-1}p_x$ denotes the Darcy velocity of the fluid mixture, p the pressure of the fluid mixture, c the concentration of the injected fluid, K the permeability of the medium, D_m the molecular diffusion coefficient, D_d the dispersion coefficient and ϕ represents the porosity. The viscosity of the mixture $\mu(c)$ is determined by the commonly used rule $\mu(c) = \mu_0((1-c) + M^{\frac{1}{4}}c)^{-4}$, where M denotes the mobility ratio and μ_0 represents the viscosity of the resident fluid. The two-dimensional or three dimensional versions of this problem with Dirichlet boundary conditions or with Neumann or Robin boundary conditions were largely considered in the literature to study the miscible displacement of one incompressible fluid by another in a porous medium (see for instance [10], [16], [17], [19]).

Piecewise linear finite element method for (1) leads to a first order approximation for the space derivative of p in the L^2-norm. This accuracy deteriorates the numerical approximation for c obtained from (2) if the same method is considered. Several approaches have been considered in the literature to increase the convergence order of the numerical approximation for the velocity. Without be exhaustive we mention the use of cell centered schemes ([20]), mixed finite element methods ([2], [5], [12], [18]), gradient recovery technique ([7] and [15]) and mimetic finite difference approximations which can be seen as a mixed finite element methods with convenient quadrature rules ([4]).

Finite difference methods that can be seen as fully discrete piecewise linear Galerkin methods that allow to obtain a second order approximation for the gradient of the solution of elliptic problems have been studied in [3], [8], [9], [13] and [14].

In the present paper we introduce for the IBVP (1)-(5) methods belonging to the class of methods analysed in the last mentioned works that enable us to compute second order approximations for the pressure, for its gradient and for the concentration. As such finite difference scheme can be seen as a fully discrete Galerkin method based on piecewise linear approximation and convenient quadrature rules, our results can be also seen as a superconvecon-vergent results.

The paper is organized as follows. In Section 2 we introduce the semi-discretization of problem (1)-(5). Its stability is established in Section 3. The convergence analysis is presented in Section 4. In the main result of this paper - Theorem 1-presented in this section we establish that the semi-discrete approximations introduced for the pressure, velocity and for the concentration are second order accurate. An implicit-explicit discrete scheme is studied in Section 5. Its stability and convergence are analyzed and a numerical simulation illustrating the convergence rate obtained for the pressure, velocity and concentration is included. Finally in Section 6 we draw some conclusions.

2. The semi-discrete approximation

In what follows we introduce the variational formulation of the IBVP (1)-(5). To simplify we assume homogeneous boundary conditions. By $L^2(0,1)$, $H^1(0,1)$ and $H^1_0(0,1)$ we denote the usual Sobolev spaces where we consider the usual inner products $(.,.)_0$ and $(.,.)_1$. Let V be a Banach space. By $L^2(0,T;V)$ we denote the space of functions $v:(0,T)\to V$ such that

$$\|v\|_{L^2(0,T;V)}=\Big(\int_0^T\|v(t)\|_V^2\,dt\Big)^{1/2}$$

is finite. By $L^\infty(0,T;V)$ we represent the space of functions $v:(0,T)\to V$ such that

$$\|v\|_{L^\infty(0,T;V)}:=\operatorname*{ess\,sup}_{[0,T]}\|v(t)\|_V<\infty.$$

The space os function $v:(0,T)\to V$ such that $v':(0,T)\to V$ defined in distributional sense is such that

$$\sum_{j=0}^{1}\operatorname*{ess\,sup}_{[0,T]}\|v^{(j)}(t)\|_V<\infty,$$

is denoted by $W^{1,\infty}(0,T;V)$ where we consider the norm

$$\|v\|_{W^{1,\infty}(0,T;V)}:=\sum_{j=0}^{1}\operatorname*{ess\,sup}_{[0,T]}\|v^{(j)}(t)\|_V<\infty.$$

We replace the IBVP (1)-(5) by the following variational problem: find $p\in L^\infty(0,T;H^1(0,1))$, $c\in L^2(0,T;H^1(0,1))\cap W^{1,\infty}(0,T;L^2(0,1))$ such that conditions (3), (4) hold and

$$(a(c(t))p_x(t), w')_0 = (q_1(t), w)_0 \text{ a.e. in } (0,T), \forall w \in H_0^1(0,1), \tag{7}$$

$$\begin{aligned} (c'(t), w)_0 \quad & (d(c(t), p_x(t))c_x(t), w')_0 - (b(c(t), p_x(t))c(t), w')_0 \\ & = (q_2(t), w)_0 \text{ a.e. in } (0,T), \forall w \in H_0^1(0,1). \end{aligned} \tag{8}$$

Let H be a sequence of vectors $h = (h_1, \ldots, h_N)$ such that $\sum_{i=1}^{N} h_i = 1$ and $h_{max} = \max_i h_i \to 0$. Let $\mathbb{I}_h = \{x_i, i = 0, \ldots, N, x_0 = 0, x_N = 1, x_i - x_{i-1} = h_i, i = 1, \ldots, N\}$ be a nonuniform partition of $[0,1]$. By $\mathbb{W}_h$ we represent the space of grid functions defined on $\mathbb{I}_h$ and by $\mathbb{W}_{h,0}$ we represent the subspace of $\mathbb{W}_h$ of functions null on the boundary points. Let $\mathbb{P}_h u_h$ be the piecewise linear interpolator of a grid function $u_h \in \mathbb{W}_h$. The space of piecewise linear functions induced by the partition $\mathbb{I}_h$ is denoted by S_h.

The piecewise linear approximations for the pressure and for the concentration are solutions of the finite dimensional coupled variational problem: find $\mathbb{P}_h p_h \in L^\infty(0,T;S_h)$ and $\mathbb{P}_h c_h \in L^2(0,T;S_h) \cap W^{1,\infty}(0,T;S_h)$ satisfying the boundary conditions (3), (4) and such that

$$(a(\mathbb{P}_h c_h(t))(\mathbb{P}_h p_h)_x(t), \mathbb{P}_h w_h')_0 = (q_1(t), \mathbb{P}_h w_h)_0 \text{ a.e. in } (0,T), \forall w_h \in \mathbb{W}_{h,0}, \tag{9}$$

$$\begin{aligned} &((\mathbb{P}_h c_h)_t(t), \mathbb{P}_h w_h)_0 + (d(\mathbb{P}_h c_h(t), (\mathbb{P}_h p_h)_x(t))(\mathbb{P}_h c_h)_x(t), \mathbb{P}_h w_h')_0 \\ &\quad -(b(\mathbb{P}_h c_h(t), (\mathbb{P}_h p_h)_x(t))\mathbb{P}_h c_h(t), \mathbb{P}_h w_h')_0 \\ &\quad = (q_2(t), \mathbb{P}_h w_h)_0 \text{ a.e. in } (0,T), \forall w_h \in \mathbb{W}_{h,0}. \end{aligned} \tag{10}$$

In the space $\mathbb{W}_h$ we consider the norm

$$\|u_h\|_{1,h}^2 = \|u_h\|_h^2 + \|D_{-x} u_h\|_{h,+}^2, \tag{11}$$

where D_{-x} denotes the backward finite difference operator with respect to the space variable, $\|.\|_h$ is the norm induced by the inner product

$$(w_h, v_h)_h = \sum_{i=1}^{N} \frac{h_i}{2} \Big(w_h(x_{i-1})v_h(x_{i-1}) + w_h(x_i)v_h(x_i) \Big), \ w_h, v_h \in \mathbb{W}_h, \tag{12}$$

and

$$\|w_h\|_{h,+} = \Big(\sum_{i=1}^{N} h_i w_h(x_i)^2\Big)^{1/2}.$$

In what follows we use the notation

$$(w_h, v_h)_{h,+} = \sum_{i=1}^{N} h_i w_h(x_i) v_h(x_i),\ \ w_h, v_h \in \mathbb{W}_h.$$

Then the fully discrete (in space) approximations for the pressure and for the concentration are solutions of the following coupled variational problem: find $p_h \in L^\infty(0,T;\mathbb{W}_h)$, $c_h \in L^2(0,T;\mathbb{W}_h) \cap W^{1,\infty}(0,T;\mathbb{W}_h)$ such that

$$(a_h(t)D_{-x}p_h(t), D_{-x}w_h)_{h,+} = (q_{1,h}(t), w_h)_h \text{ a.e. in } (0,T), \forall w_h \in \mathbb{W}_{h,0}, \quad (13)$$

$$\begin{aligned}(c_h'(t), w_h)_h &+ (d_h(t)D_{-x}c_h(t), D_{-x}w_h)_{h,+} - (M_h(b_h(t)c_h(t)), D_{-x}w_h)_{h,+}\\ &= (q_{2,h}(t), w_h)_h \text{ a.e. in } (0,T), \forall w_h \in \mathbb{W}_{h,0},\end{aligned} \quad (14)$$

and

$$p_h(x_0,t) = p_\ell(t), p_h(x_N,t) = p_r(t) \text{ a.e. in } (0,T), \quad (15)$$

$$c_h(x_0,t) = c_\ell(t), c_h(x_N,t) = c_r(t) \text{ a.e. in } (0,T), \quad (16)$$

$$c_h(x_i,0) = c_{0,h}(x_i), p_h(x_i,0) = p_{0,h}(x_i),\ i = 1,\dots,N-1. \quad (17)$$

In (13), (14) the following notations were used:

$$q_{\ell,h}(x_i,t) = \frac{1}{h_{i+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}} q_\ell(x,t)\,dx, i = 1,\dots,N-1, \ell = 1,2, \quad (18)$$

$h_{i+1/2} = \frac{1}{2}(h_i + h_{i+1})$, $M_h(w_h)(x_i) = \dfrac{1}{2}(w_h(x_{i-1}) + w_h(x_i)), i = 1,\dots,N$. The coefficient functions $a_h(t)$ and $d_h(t)$ are defined by

$$a_h(x_i,t) = a(M_h(c_h(t))(x_i)), \quad (19)$$

$$d_h(x_i,t) = d(M_h(c_h(t))(x_i), D_{-x}p_h(x_i,t)) \quad (20)$$

and the grid function $b_h(t)$ is given by

$$b_h(x_i,t) = \begin{cases} b(c_h(x_0,t), D_x p_h(x_0,t)), i = 0,\\[2ex] b(c_h(x_i,t), D_h p_h(x_i,t)), i = 1,\dots,N-1,\\[2ex] b(c_h(x_N,t), D_{-x}p_h(x_N,t))), i = N,\end{cases} \quad (21)$$

with

$$D_h p_h(x_i,t) = \frac{1}{h_i + h_{i+1}}\big(h_i D_{-x} p_h(x_{i+1},t) + h_{i+1} D_{-x} p_h(x_i,t)\big). \tag{22}$$

In what follows we establish an ordinary differential algebraic coupled system equivalent to the variational problem (13)-(17). In order to do that we introduce the following finite difference operators

$$(D_c w_h)_i = \frac{w_{i+1} - w_{i-1}}{h_i + h_{i+1}},$$

$$(D_x w_h)_{i+1/2} = \frac{w_{i+1} - w_i}{h_{i+1}},$$

$$(D_x^{1/2} w_h)_i = \frac{w_{i+1/2} - w_{i-1/2}}{h_{i+1/2}},$$

where $w_j := w_h(x_j)$ and $w_{j\pm 1/2}$ are used as far as it makes sense. In order to simplify the presentation we also consider that $a_h(x_{i\pm 1/2},t) = a_h(x_{i\pm 1},t)$, $d_h(x_{i\pm 1/2},t) = d_h(x_{i\pm 1},t)$.

It can be shown that the approximations $p_h(t)$ and $c_h(t)$ are solutions of the following discrete problem:

$$-D_x^{1/2}(a_h(t) D_x p_h(t)) = q_{1,h}(t) \text{ in } \mathbb{I}_h - \{0,1\}, \tag{23}$$

$$c_h'(t) - D_x^{1/2}(d_h(t)\, D_x p_h(t)) + D_c(b_h(t) c_h(t)) = q_{2,h}(t) \text{ in } \mathbb{I}_h - \{0,1\}, \tag{24}$$

with the conditions (15), (16) and (17).

3. Stability of pressure and concentration

We establish now the stability of the coupled variational problem (13), (14) or equivalently the stability of the coupled finite difference problem (23), (24) under homogeneous Dirichlet boundary conditions, that is, $p_\ell(t) = p_r(t) = c_\ell(t) = c_r(t) = 0$.

We require some smoothness on the solution of the variational problem (13), (14), namely, we assume that $p_h \in C^0(0,T;W_{H,0})$, that is, $p_h : [0,T] \to \mathbb{W}_{h,0}$ is continuous and $c_h \in C^1(0,T;W_{H,0})$, that is, $c_h, c_h' : [0,T] \to \mathbb{W}_{h,0}$ are continuous when we consider the norm $\|.\|_h$ in $\mathbb{W}_{h,0}$.

Proposition 1. *If $0 < a_0 \le a$ then there exists a positive constant C_p, h independent, such that*

$$\|p_h(t)\|_{1,h} \le C_p \|q_{1,h}(t)\|_h, t \in [0,T]. \tag{25}$$

Proof: Taking in (13) $w_h = p_h(t)$ and considering the Poincaré-Friedrich's inequality $\|w_h\|_h \leq \|D_{-x}w_h\|^2_{h,+}$ for $w_h \in \mathbb{W}_{h,0}$, we conclude (25).

□

If

$$\|q_1(t)\|_0 \leq C_{q_1}, t \in [0,T], \tag{26}$$

then the sequence $\|p_h(t)\|_{1,h}, h \in H$, satisfies

$$\|p_h(t)\|_{1,h} \leq C_p,\ t \in [0,T],\ h \in H, \tag{27}$$

for some positive constant C_p.
As

$$|p_h(x_i)| \leq \|p_h(t)\|_{1,h},$$

we get

$$\max_{i=1\dots,N-1} |p_h(x_i,t)| \leq C_q,$$

that is

$$\|p_h(t)\|_\infty \leq C_p.$$

Moreover, as holds the following

$$\begin{aligned} a(M_h(c_h(t))(x_{i+1}))D_{-x}p_h(x_{i+1},t) &= \sum_{j=1}^{i} h_{j+1/2}D_x^{(1/2)}(a_h(t)D_{-x}p_h(t))(x_j) \\ &+a(M_h(c_h(t))(x_1))D_{-x}p_h(x_1,t), \\ &= -\sum_{j=1}^{i} h_{j+1/2}q_{1,h}(x_j,t) \\ &+a(M_h(c_h(t))(x_1))D_{-x}p_h(x_1,t), \end{aligned}$$

for $i = 1,\dots,N-1$, we conclude

$$\max_{i=2,\dots,N} |a(M_h(c_h(t))(x_i))D_{-x}p_h(x_i,t)| \leq C_p + |a(M_h(c_h(t))(x_1))|D_{-x}p_h(x_1,t)|,$$

provided that $q_1 \in L^\infty(0,T;L^2(0,1))$. It is then plausible to admit that, for $0 < a_0 \leq a$ and $h \in H$ with h_{max} small enough, we have

$$\max_{i=1,\dots,N} |D_{-x}p_h(x_i,t)| \leq C_p, \tag{28}$$

for some positive constant C_p. If we replace the Dirichlet boundary conditions for the pressure by Neumann boundary conditions $p_x(0,t) = p_x(1,t) = 0$

that are discretized by $D_{-x}p_h(x_1,t) = D_{-x}p_h(x_N,t) = 0$, then condition (28) holds.

Proposition 2. *If* $0 < a_0 \leq a$, $0 < d_0 \leq d$, *(28) holds,*

$$|b(x,y)| \leq C_b|y|, (x,y) \in \mathbb{R}^2, \tag{29}$$

then

$$\begin{aligned}\|c_h(t)\|_h^2 + \int_0^t \|D_{-x}c_h(s)\|_{h,+}^2\,ds \ &\leq \frac{1}{\min\{1, 2(d_0-\epsilon^2)\}} e^{\left(\frac{1}{2\epsilon^2}C_b^2C_p^2+2\eta^2\right)t} \\ &\Big(\|c_h(0)\|_h^2 + \frac{1}{2\eta^2}\int_0^t \|q_{2,h}(s)\|_h^2\,ds\Big),\ t\in[0,T],\end{aligned} \tag{30}$$

$\eta \neq 0$ *is an arbitrary constant and* $\epsilon \neq 0$ *is such that*

$$d_0 - \epsilon^2 > 0. \tag{31}$$

Proof: Taking in (14) w_h replaced by $c_h(t)$, we easily deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|c_h(t)\|_h^2 + d_0\|D_{-x}c_h(t)\|_{h,+}^2 - (M_h(b_h(t)c_h(t)), D_{-x}c_h(t))_{h,+} \\ &\leq \frac{1}{4\eta^2}\|q_{2,h}(t)\|_h^2 + \eta^2\|c_h(t)\|_h^2,\end{aligned} \tag{32}$$

for arbitrary $\eta \neq 0$.
As under the assumptions (28) and (29), we have successively

$$|(M_h(b_h(t)c_h(t)), D_{-x}c_h(t))_{h,+}| \leq C_bC_p\|c_h(t)\|_h\|D_{-x}c_h(t)\|_{h,+}, \tag{33}$$

consequently, from (32), we obtain

$$\begin{aligned}\frac{d}{dt}\|c_h(t)\|_h^2 + 2(d_0-\epsilon^2)\|D_{-x}c_h(t)\|_{h,+}^2 \leq \Big(\frac{1}{2\epsilon^2}C_b^2C_p^2 + 2\eta^2\Big)\|c_h(t)\|_h^2 \\ + \frac{1}{2\eta^2}\|q_{2,h}(t)\|_h^2,\end{aligned}$$

where ϵ, η are nonzero constants.This inequality leads to

$$\begin{aligned}&\|c_h(t)\|_h^2 + 2(d_0-\epsilon^2)\int_0^t \|D_{-x}c_h(s)\|_{h,+}^2\,ds \leq \|c_h(0)\|_h^2 \\ &+\Big(\frac{1}{2\epsilon^2}C_b^2C_p^2 + 2\eta^2\Big)\int_0^t \|c_h(s)\|_h^2\,ds + +\frac{1}{2\eta^2}\int_0^t \|q_{2,h}(s)\|_h^2\,ds.\end{aligned} \tag{34}$$

Finally inequality (30) easily follows from inequality (34).

□

Remark 1. *Considering in (32) and (33) the discrete Poincaré-Friedrich's inequality* $\|c_h(t)\|_h \leq \|D_{-x}c_h(t)\|_{h,+}$, *we deduce*

$$\frac{d}{dt}\|c_h(t)\|_h^2 + 2(d_0 - \eta^2 - C_bC_p)\|D_{-x}c_h(t)\|_{h,+}^2 \leq \frac{1}{2\eta^2}\|q_{2,h}(t)\|_h^2$$

that leads to

$$\|c_h(t)\|_h^2 \leq \|c_h(0)\|_h^2 + \frac{1}{2\eta^2}\int_0^t \|q_{2,h}(s)\|_h^2\, ds,\ t \in [0,T], \tag{35}$$

provided that d_0, C_b, C_p *and* $\eta \neq 0$ *satisfy*

$$d_0 - \eta^2 - C_bC_p > 0.$$

As a consequence of Propositions 1 and 2 we conclude the stability of the solution of the variational problems (13), (14) or, equivalently, the stability of the coupled finite difference problems (23), (24) under Dirichlet boundary conditions.

4. Supraconvergent result

4.1. Auxiliary results. We start by introducing two auxiliary problems. We assume that $a \in W^{1,\infty}(\mathbb{R})$, $d \in W^{1,\infty}(\mathbb{R}^2)$ and $b \in W^{2,\infty}(\mathbb{R}^2)$. Let $\tilde{p}_h(t), \tilde{c}_h(t) \in \mathbb{W}_{h,0}$ be solutions of the discrete variational problems

$$(\tilde{a}_h(t)D_{-x}\tilde{p}_h(t), D_{-x}w_h)_{h,+} = (q_{1,h}(t), w_h)_h,\ w_h \in \mathbb{W}_{h,0}, \tag{36}$$

$$\begin{aligned}(\tilde{d}_h(t)D_{-x}\tilde{c}_h(t), D_{-x}w_h)_{h,+} &- (M_h(\tilde{b}_h(t)\tilde{c}_h(t)), D_{-x}w_h)_{h,+}\\ &= (\tilde{q}_{2,h}(t), w_h)_h\,, w_h \in \mathbb{W}_{h,0},\end{aligned} \tag{37}$$

with $\tilde{q}_{2,h}(t)$ defined by (18) with $q_2(t)$ replaced by $q_2(t) - c'(t)$. In (36) and (37) the coefficient functions $\tilde{a}_h$ and $\tilde{d}_h$ are defined by

$$\tilde{a}_h(x_i,t) = a(c(x_{i-1/2},t)), i = 1,\ldots,N,$$

$$\tilde{d}_h(x_i,t) = d(c(x_{i-1/2},t), p_x(x_{i-1/2},t)), i = 1,\ldots,N,$$

and

$$\begin{aligned}&\tilde{b}_h(x_i,t)\tilde{c}_h(x_i,t) = b(c(x_i,t), p_x(x_i,t))\tilde{c}_h(x_i,t), \quad i = 1,\ldots,N-1,\\ &\tilde{b}_h(x_i,t)\tilde{c}_h(x_i,t) = 0, \quad i = 0, N.\end{aligned}$$

It can be shown that $\tilde{p}_h(t)$ and $\tilde{c}_h(t)$ are solutions of a coupled finite difference problem analogous to (23), (24).

An error bound for $\tilde{p}_h$ is established now considering Theorem 3.1 of [3]. By R_h we denote the restriction operator $R_h : C[0,1] \to \mathbb{W}_h$, $R_h v(x) = v(x)$, $x \in \mathbb{I}_h$.

Proposition 3. *If $0 < a_0 \leq a$ then, for $\tilde{p}_h(t)$ defined by (36) and for $h \in H$ with h_{max} small enough, holds the following error estimate*

$$\|\mathbb{P}_h\big(\tilde{p}_h(t) - R_h p(t)\big)\|_1^2 \leq C_{\tilde{p}} \sum_{i=1}^{N} h_i^{2s} \|p(t)\|^2_{H^{s+1}(I_i)} \tag{38}$$

provided that $p(t) \in H^{s+1}(0,1) \cap H_0^1(0,1)$, $s \in \{1,2\}$. In (38) $I_i = (x_{i-1}, x_i)$ and $C_{\tilde{p}}$ denotes a positive constant which does not depend on h.

As a consequence of this result, we conclude that, for $h \in H$ with h_{max} small enough, we have

$$\max_{i=1,\dots,N} |D_{-x}\tilde{p}_h(x_i,t)| \leq C_{\tilde{p}}, \tag{39}$$

for some positive constant $C_{\tilde{p}}$. In fact, from (38) we obtain

$$|D_{-x}(\tilde{p}(x_i,t) - p(x_i,t))| \leq C h_{max}^{s-\frac{1}{2}},$$

for some positive constant C. Then

$$\begin{aligned} |D_{-x}\tilde{p}_h(x_i,t)| &\leq |D_{-x}(\tilde{p}(x_i,t) - p(x_i,t))| + |\frac{1}{h_j}\int_{x_{j-1}}^{x_j} p_x(x,t)\,dx| \\ &\leq C h_{max}^{s-\frac{1}{2}} + \|p_x(t)\|_\infty, \end{aligned}$$

that leads to (39) provided that $p \in L^\infty(0,T; H^{s+1}(0,1) \cap H_0^1(0,1))$, $s \in \{1,2\}$.

In order to obtain an upper bound for the error of $\tilde{c}_h(t)$ we need to guarantee the stability of the bilinear form

$$a_{\tilde{c}_h}(v_h, w_h) = (\tilde{d}_h(t) D_{-x}v_h, D_{-x}w_h)_{h,+} - (M_h(\tilde{b}_h(t)v_h), D_{-x}w_h)_{h,+}, v_h, w_h \in \mathbb{W}_{h,0}.$$

In the next proposition we specify the condition that allow us to conclude such stability (see Proposition 3.1 of [3]).

Proposition 4. *Let $\tilde{d}(t)$ and $\tilde{b}(t)$ be defined by $\tilde{d}(t) = d(c(t), p_x(t))$, $\tilde{b}(t) = b(c(t), p_x(t))$, where p, c are the solutions of the coupled variational problem (7), (8) with homogeneous Dirichlet boundary conditions. If the variational problem: find $u \in H_0^1(0,1)$ such that $(\tilde{d}(t)v', w')_0 - (\tilde{b}(t)v, w')_0 = 0$ for $w \in$*

$H_0^1(0,1)$*, has only the null solution, then there exists a positive constant* $\alpha_{e,c}$ *which does not depend on* h *such that, for* $h \in H$ *with* h_{max} *small enough, holds the following stability inequality*

$$\|\mathbb{P}_h v_h\|_1 \leq \alpha_{e,c} \sup_{0\neq w_h \in \mathbb{W}_{h,0}} \frac{|a_{\tilde{c}_h}(v_h, w_h)|}{\|\mathbb{P}_h w_h\|_1},\ v_h \in \mathbb{W}_{h,0}. \tag{40}$$

□

Using now Theorem 3.1 of [3] we can state the error estimate for $\tilde{c}_h$. Considering this result, it suffices to estimate

$$T_d = \sum_{i=1}^{N} h_i d_{i-1/2}\Big(D_{-x}c(x_i,t) - c_x(x_{i-1/2},t)\Big) D_{-x}w_h(x_i), \tag{41}$$

$$T_b = \sum_{i=1}^{N} h_i \Big(b(x_{i-1/2},t) - \frac{b(x_{i-1},t) + b(x_i,t)}{2}\Big) D_{-x}w_h(x_j) \tag{42}$$

with

$$d_{i-1/2} = (c(x_{i-1/2},t), p_x(x_{i-1/2},t))$$

and

$$b(x_\ell,t) = b(c(x_\ell,t), p_x(x_\ell,t)),\ \ell = i-1, i-1/2, i.$$

Using Bramble-Hilbert lemma in T_d we get

$$|T_d| \leq C\|d(c(t),p_x(t))\|_\infty \Big(\sum_{i=1}^{N} h_i^{2s}\|c(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2} \|D_{-x}w_h\|_{h,+}, \tag{43}$$

provided that $c(t) \in H^{s+1}(0,1) \cap H_0^1(0,1)$, for $s \in \{1,2\}$.

To estimate T_b we apply Bramble-Hilbert lemma again. In this case we obtain, for $s \in \{1,2\}$,

$$|T_b| \leq C\Big(\sum_{i=1}^{N} h_i^{2s}|b(c(t),p_x(t))c(t)|^2_{H^s(I_i)}\Big)^{1/2} \|D_{-x}w_h\|_{h,+} \tag{44}$$

As the imbedding of $H^{j+1}(0,1)$ into $C_B^j(0,1)$ is continuous, where $C_B^j(0,1)$ denotes the space of functions having bounded, continuous derivatives up to order j on $(0,1)$ (Theorem 4.12 of [1]), we deduce for $s=1$

$$|T_b| \leq C\Big(\sum_{i=1}^{N} h_i^2\|c(t)\|^2_\infty\Big(\|c(t)\|^2_{H^1(I_i)} + \|p(t)\|^2_{H^2(I_i)}\Big)\Big)^{1/2} \|D_{-x}w_h\|_{h,+} \tag{45}$$

and for $s = 2$

$$
\begin{aligned}
|T_b| \ \leq C\Big(\sum_{i=1}^{N} h_j^4\Big(&\|c_x(t)\|_\infty^2\big(\|c(t)\|_\infty^2+1\big)\big(\|c_x(t)\|^2_{L^2(I_i)^2}+\|p_{x^2}(t)\|^2_{L^2(I_i)}\big)\\
&+\|c(t)\|_\infty^2\big(\|p_{x^2}(t)\|_\infty^2\|p_{x^2}\|^2_{L^2(I_i)}+\|p_{x^3}\|^2_{L^2(I_i)}\big)\\
&+\|c_{x^2}\|^2_{L^2(I_i)}\Big)\Big)^{1/2}\|D_{-x}w_h\|_{h,+}.
\end{aligned}
\tag{46}
$$

We summarize the previous error estimates in the following proposition.

Proposition 5. *Under the assumptions of Proposition 4, for $\tilde{c}_h(t)$ defined by (37) and for $h \in H$ with h_{max} small enough, holds the following error estimate*

$$
\|\mathbb{P}_h(\tilde{c}_h(t)-R_hc(t))\|_1^2 \leq C_{\tilde{c}}\sum_{i=1}^{N} h_i^{2s}\Big(\|c(t)\|^2_{H^{s+1}(I_i)}+\|p(t)\|^2_{H^{s+1}(I_i)}\Big), \tag{47}
$$

provided that $c(t), p(t) \in H^{s+1}(0,1)\cap H_0^1(0,1)$. In (47), $s \in \{1,2\}$ and $C_{\tilde{c}}$ denotes a positive constant which does not depend on h.

Under the assumptions of Proposition 4, it is clear that

$$
\|\tilde{c}_h(t)\|_{1,h} \leq C_{\tilde{c}},
$$

for some positive $C_{\tilde{c}}$, which implies that

$$
\|\tilde{c}_h(t)\|_\infty \leq C_{\tilde{c}}, \tag{48}
$$

provided that $c, p \in L^\infty(0,T; H^2(0,1)\cap H_0^1(0,1))$, for some positive constant $C_{\tilde{c}}$ and for $h \in H$ with h_{max} small enough.

As for $\tilde{p}_h(t)$, it can be shown that, for $h \in H$ with h_{max} small enough, we have

$$
\max_{i=1,\dots,N}|D_{-x}\tilde{c}_h(x_i,t)| \leq C_{\tilde{c}}. \tag{49}
$$

In the next proposition we establish an upper bound for $\|\mathbb{P}_h(p_h(t)-\tilde{p}_h(t))\|_1$.

Proposition 6. *If $0 < a_0 \leq a$, then,for $h \in H$ with h_{max} small enough,*

$$
\begin{aligned}
\|\mathbb{P}_h(p_h(t)-\tilde{p}_h(t))\|_1 \leq \ & C_{p,\tilde{p}}\Big(\|c_h(t)-R_hc(t)\|_h\\
&+\Big(\sum_{i=1}^{N}h_i^{2s}\|c(t)\|^2_{H^s(I_i)}\Big)^{1/2}\Big),
\end{aligned}
\tag{50}
$$

provided that $c(t) \in H^s(0,1) \cap H_0^1(0,1)$. *In (50),* $s \in \{1,2\}$ *and* $C_{p,\tilde{p}}$ *denotes a positive constant which does not depend on* h.

Proof: From (13) and (36) it can be shown that, for $w_h \in \mathbb{W}_{h,0}$, holds the following

$$\begin{aligned}
&(a_h(t)D_{-x}(p_h(t) - \tilde{p}_h(t)), D_{-x}w_h)_{h,+} \\
&\quad = ((\tilde{a}_h(t) - a_h^*(t))D_{-x}\tilde{p}_h(t), D_{-x}w_h)_{h,+} \\
&\quad + ((a_h^*(t) - a_h(t))D_{-x}\tilde{p}_h(t), D_{-x}w_h)_{h,+},
\end{aligned} \tag{51}$$

where $a_h^*(t)$ is defined as $a_h(t)$ but with $c_h(t)$ replaced by $R_h c(t)$.

For the second term of the second member of (51) we have

$$|((a_h^*(t) - a_h(t))D_{-x}\tilde{p}_h(t), D_{-x}w_h)_{h,+}| \ \leq C\|c_h(t) - R_h c(t)\|_h \|D_{-x}w_h\|_{h,+}, \tag{52}$$

for $w_h \in \mathbb{W}_{h,0}$, where $\|.\|_{1,\infty}$ denotes the usual norm in $W^{1,\infty}(0,1)$. Considering now the Bramble-Hilbert lemma in the first term of the second member of (51) we deduce

$$|((\tilde{a}_h(t) - a_h^*(t))D_{-x}\tilde{p}_h(t), D_{-x}w_h)_{h,+}| \leq C\Big(\sum_{i=1}^N h_i^{2s}\|c(t)\|^2_{H^s(I_i)}\Big)^{1/2}\|D_{-x}w_h\|_{h,+}. \tag{53}$$

for $w_h \in \mathbb{W}_{h,0}$. Considering (52) and (53) in (51), we conclude the proof of (50) choosing $w_h = p_h(t) - \tilde{p}_h(t)$.

□

Corollary 1. *If* $0 < a_0 \leq a$, *then for* $p_h(t)$ *and* $c_h(t)$ *defined by (13), (14) and for* $h \in H$ *with* h_{max} *small enough, holds the following*

$$\begin{aligned}
\|\mathbb{P}_h(p_h(t) - R_h p(t))\|_1 \ &\leq C\Big(\|c_h(t) - R_h c(t)\|_h + \Big(\sum_{i=1}^N h_i^{2s}\|c(t)\|^2_{H^s(I_i)}\Big)^{1/2} \\
&\quad + \Big(\sum_{i=1}^N h_i^{2s}\|p(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2}\Big),
\end{aligned} \tag{54}$$

provided that $c(t) \in H^s(0,1) \cap H_0^1(0,1)$, $p(t) \in H^{s+1}(0,1) \cap H_0^1(0,1)$, $s \in \{1,2\}$.

Lemma 1. *Let* $\tilde{c}_h(t)$ *be defined by (37) and* $p(t), c(t) \in H^{s+1}(0,1) \cap H_0^1(0,1)$, $s \in \{1,2\}$. *Under the assumptions of Proposition 4 and Corollary 1, for the* *functional*

$$\tau_d(t,w_h) = (\tilde{d}_h(t)D_{-x}\tilde{c}_h(t), D_{-x}w_h)_{h,+} - (d_h(t)D_{-x}c_h(t), D_{-x}w_h)_{h,+},$$

defined on $\mathbb{W}_{h,0}$ and for $h \in H$ with h_{max} small enough, holds the following

$$\tau_d(t,w_h) = (d_h(t)D_{-x}(R_hc(t) - c_h(t)), D_{-x}w_h)_{h,+} + \tau_{d,h}(t,w_h), \tag{55}$$

where

$$\begin{aligned}|\tau_{d,h}(t,w_h)| \ &\leq C_d\Big(\|c_h(t) - R_hc(t)\|_h + \Big(\sum_{i=1}^N h_i^{2s}\|p(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2} \\ &\qquad + \Big(\sum_{i=1}^N h_i^{2s}\|c(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2}\Big)\|D_{-x}w_h\|_{h,+}, \ \ w_h \in \mathbb{W}_{h,0}.\end{aligned} \tag{56}$$

Proof: For $\tau_d(t,w_h)$ holds the representation (55) with $\tau_{d,h}(t,w_h)$ given by

$$\tau_{d,h}(t,w_h) = \tau^{(1)}_{d,h}(t,w_h) + \tau^{(2)}_{d,h}(t,w_h) + \tau^{(3)}_{d,h}(t,w_h) \tag{57}$$

where

$$\tau^{(1)}_{d,h}(t,w_h) = ((\tilde{d}_h(t) - d^*_h(t))D_{-x}\tilde{c}_h(t), D_{-x}w_h)_{h,+},$$

$$\tau^{(2)}_{d,h}(t,w_h) = ((d^*_h(t) - d_h(t))D_{-x}\tilde{c}_h(t), D_{-x}w_h)_{h,+},$$

$$\tau^{(3)}_{d,h}(t,w_h) = (d_h(t)D_{-x}\big(\tilde{c}_h(t) - R_hc(t)\big), D_{-x}w_h)_{h,+},$$

and d^*_h is defined as d_h with c_h and p_h replaced by R_hc and R_hp, respectively. Using the Bramble-Hilbert lemma it can be shown that for $\tau^{(1)}_{d,h}(t,w_h)$ and for $h \in H$ with h_{max} small enough, holds the following

$$|\tau^{(1)}_{d,h}(t,w_h)| \ \leq C\Big(\Big(\sum_{i=1}^N h_i^{2s}\|c(t)\|^2_{H^s(I_i)}\Big)^{1/2} + \Big(\sum_{i=1}^N h_i^{2s}\|p(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2}\Big)$$

$$\|D_{-x}w_h\|_{h,+}, \ \ w_h \in \mathbb{W}_{h,0}.$$

For $\tau^{(2)}_{d,h}(t,w_h)$ we have, for $w_h \in \mathbb{W}_{h,0}$,

$$|\tau^{(2)}_{d,h}(t,w_h)| \leq \Big(\|R_hc(t) - c_h(t)\|_h + \|D_{-x}(p_h(t) - R_hp(t))\|_{h,+}\Big)\|D_{-x}w_h\|_{h,+}.$$

Considering Corollary 1 we get

$$|\tau^{(2)}_{d,h}(t,w_h)| \leq C\Big(\|c_h(t)-R_hc(t)\|_h + \Big(\sum_{i=1}^N h_i^{2s}\|c(t)\|^2_{H^s(I_i)}\Big)^{1/2} + \Big(\sum_{i=1}^N h_i^{2s}\|p(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2}\Big)\|D_{-x}w_h\|_{h,+},\ \ w_h\in\mathbb{W}_{h,0}.$$

Taking into account Proposition 5, for $\tau^{(3)}_{d,h}(t,w_h)$ we deduce, for $h\in H$ with h_{max} small enough,

$$|\tau^{(3)}_{d,h}(t,w_h)| \ \leq C\Big(\Big(\sum_{i=1}^N h_i^{2s}\|c(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2} + \Big(\sum_{i=1}^N h_i^{2s}\|p(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2}\Big) \|D_{-x}w_h\|_{h,+},\ \ w_h\in\mathbb{W}_{h,0}.$$

From the estimates established for $\tau^{(\ell)}_{d,h}(t,w_h), \ell=1,2,3$, we conclude (56).

□

Lemma 2. *Let $\tilde{c}_h(t)$ be defined by (37) and $c(t),p(t)\in H^{s+1}(0,1)\cap H^1_0(0,1), s\in\{1,2\}$. If $0<a_0\leq a$, condition (28) holds and the coefficient function b satisfies (29) then, under the assumptions of Proposition 4, for the functional*

$$\tau_b(t,w_h) = (M_h(b_h(t)c_h(t)),D_{-x}w_h)_{h,+} - (M_h(\tilde{b}_h(t)\tilde{c}_h(t)),D_{-x}w_h)_{h,+},,$$

defined on $\mathbb{W}_{h,0}$ and for $h\in H$ with h_{max} small enough, holds the following

$$\tau_b(t,w_h) = (M_h(b_h(t)(c_h(t)-R_hc(t))),D_{-x}w_h)_{h,+} + \tau_{b,h}(t,w_h), \tag{58}$$

where

$$|\tau_{b,h}(t,w_h)| \ \leq C_{b,2}\Big(\|c_h(t)-R_hc(t)\|_h + \Big(\sum_{i=1}^N h_i^{2s}\|c(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2} + \Big(\sum_{i=1}^N h_i^{2s}\|p(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2}\Big)\|D_{-x}w_h\|_{h,+},\ \ w_h\in\mathbb{W}_{h,0}, \tag{59}$$

for $h\in H$ with h_{max} small enough.

□

Proof: For $\tau_b(t, w_h)$ holds the representation (58) with

$$\tau_{b,h}(t, w_h) = \tau_{b,h}^{(1)}(t, w_h) + \tau_{b,h}^{(2)}(t, w_h) + \tau_{b,h}^{(3)}(t, w_h),$$

$$\tau_{b,h}^{(1)}(t, w_h) = (M_h(b_h(t)(R_h c(t) - \tilde{c}_h(t))), D_{-x}w_h)_{h,+},$$

$$\tau_{b,h}^{(2)}(t, w_h) = (M_h((b_h(t) - b_h^*(t))\tilde{c}_h(t)), D_{-x}w_h)_{h,+},$$

$$\tau_{b,h}^{(3)}(t, w_h) = (M_h((b_h^*(t) - \tilde{b}_h(t))\tilde{c}_h(t)), D_{-x}w_h)_{h,+},$$

being b_h^* defined as b_h with c_h and p_h replaced by $R_h c$ and $R_h p$, respectively. Considering Proposition 5 and condition (28), under the assumptions (29) for b it can be shown that for $\tau_{b,h}^{(1)}(t, w_h)$ and for $h \in H$ with h_{max} small enough, holds the following

$$|\tau_{b,h}^{(1)}(t, w_h)| \leq \; C\Big(\Big(\sum_{i=1}^{N} h_i^{2s}\|c(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2} + \Big(\sum_{i=1}^{N} h_i^{2s}\big(\|p(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2}\Big)$$

$$\|D_{-x}w_h\|_{h,+},$$

provided that $c(t), p(t) \in H^{s+1}(0,1) \cap H_0^1(0,1)$, for $s \in \{1, 2\}$.

As $\tilde{c}_h(t)$ satisfies (48), we can establish for $\tau_{b,h}^{(2)}(t, w_h)$ the upper bound

$$|\tau_{b,h}^{(2)}(t, w_h)| \leq C\Big(\|c_h - R_h c\|_h + \|D_{-x}(p_h(t) - R_h p(t))\|_{h,+}\Big)\|D_{-x}w_h\|_{h,+}.$$

Considering now Corollary 1, for $h \in H$ with h_{max} small enough, we conclude

$$|\tau_{b,h}^{(2)}(t, w_h)| \leq C\Big(\|c_h(t) - R_h c(t)\|_h + \Big(\sum_{i=1}^{N} h_i^{2s}\|c(t)\|^2_{H^s(I_i)}\Big)^{1/2}$$

$$+ \Big(\sum_{i=1}^{N} h_i^{2s}\|p(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2}\Big)\|D_{-x}w_h\|_{h,+},$$

provided that $c(t) \in H^s(0,1) \cap H_0^1(0,1), p(t) \in H_0^{s+1}(0,1) \cap H_0^1(0,1)$, $s \in \{1, 2\}$.

To estimate $\tau_{b,h}^{(3)}(t, w_h)$ we start by remarking that

$$p_x(x_i, t) - D_h p(x_i, t) = \frac{1}{h_i + h_{i+1}}\lambda(v),$$

with

$$\lambda(v) = v_\xi(\rho) - \hat{\rho}(v(1) - v(\rho)) - \frac{1}{\hat{\rho}}(v(\rho) - v(0),$$

and

$$v(\xi) = p(x_{i-1} + \xi(h_i + h_{i+1}, t)),\ \rho = \frac{h_i}{h_i + h_{i+1}},\ \hat{\rho} = \frac{h_i}{h_{i+1}}.$$

Applying Bramble-Hilbert lemma to $\lambda(v)$ we obtain, for $s \in \{1, 2\}$,

$$\begin{aligned} |\lambda(v)| &\leq C \int_0^1 |v_{\xi^s}(\xi)|\, dx \\ &\leq C(h_i + h_{i+1})^{s-1} \int_{x_{i-1/2}}^{x_{i+1/2}} |p_{x^s}(x,t)|\, dx. \end{aligned}$$

Then, for $h \in H$ with h_{max} small enough, we have

$$|\tau_{b,h}^{(3)}(t, w_h)| \leq C\Big(\sum_{i=1}^{N} h_i^{2s} \|p(t)\|^2_{H^{s+1}(I_i)}\Big)^{1/2} \|D_x w_h\|_{h,+},$$

provided that $p(t) \in H^{s+1}(0,1) \cap H_0^1(0,1)$, for $s \in \{1, 2\}$.

From the upper bounds obtained for $\tau_{b,h}^{(\ell)}(t, w_h)$, $\ell = 1, 2, 3$, we conclude the proof.

□

The following result was proved in [3] and has an important role in the proof of the main result of this paper - Theorem 1.

Lemma 3. *If $g \in H^2(0,1)$ and g_h is defined by (18) with q_ℓ replaced by g, then there exits a positive constant C_{in} which does not depend on h such that*

$$|(g_h - R_h g, w_h)_h| \leq C_{in}\Big(\sum_{i=1}^{N} h_i^4 \|g\|^2_{H^2(I_i)}\Big)^{1/2} \|w_h\|_{1,h},\ w_h \in W_{h,0}, \tag{60}$$

for $h \in H$ with H_{max} small enough.

□

4.2. Main convergence result. Let $e_{c,h}(t) = c_h(t) - R_h c(t)$ $e_{p,h}(t) = p_h(t) - R_h p(t)$be the semi-discretization error induced by the discretization (13), (14), (15) and (16). An estimate for $\|\mathbb{P}_h e_{p,h}(t)\|_1$ depending on $\|e_{c,h}(t)\|_h$ was established in Corollary 1. In the next result we establish an estimate for $\|e_{c,h}(t)\|_h$ that allow us to obtain with Corollary 1 an estimate for $\|\mathbb{P}_h e_{p,h}(t)\|_1$.

Theorem 1. *Let c and p be the solutions of the coupled quasi-linear problem (7), (8), $c \in L^2(0,T;H^{s+1}(0,1) \cap H_0^1(0,1)) \cap H^1(0,T;H^2(0,1))$, $p \in L^\infty(0,T;H^{s+1}(0,1) \cap H_0^1(0,1))$, $s \in \{1,2\}$, and let c_h and p_h be their approximations defined by (13), (14). We assume that the variational problem: find $v \in H_0^1(0,1)$ such that $(\tilde{d}(t)v', w')_0 - (\tilde{b}(t)v, w')_0 = 0$ for $w \in H_0^1(0,1)$, has only the null solution, where $\tilde{d}(t) = d(c(t), p_x(t))$ and $\tilde{b}(t) = b(c(t), p_x(t))$.*

If $0 < a_0 \leq a, 0 < d_0 \leq d$, b satisfies (29), then, under the assumption (28), there exists positive constant C_e such that, for $h \in H$ with h_{max} small enough, holds the following

$$
\begin{aligned}
&\|e_{c,h}(t)\|_h^2 + \int_0^t \|D_{-x}e_{c,h}(\mu)\|_{h,+}^2\, d\mu \leq \frac{1}{\min\{1, 2(d_0 - 4\epsilon^2)\}} e^{\omega t}\Big(\|e_{c,h}(0)\|_h^2 \\
&\qquad + C_e \sum_{i=1}^N \int_0^t \Big(h_i^{2s}\big(\|p(\mu)\|_{H^{s+1}(I_i)}^2 + \|c(\mu)\|_{H^{s+1}(I_i)}^2\big) + h_i^4 \|c'(\mu)\|_{H^2(I_i)}^2\Big) d\mu\Big) \\
&\leq \frac{1}{\min\{1, 2(d_0 - 4\epsilon^2)\}} e^{\omega t}\Big(\|e_{c,h}(0)\|_h^2 + C_e\Big(h_{max}^{2s}\big(\|c\|_{L^2(0,T;H^{s+1}(0,1))}^2 \\
&\qquad + \|p\|_{L^2(0,T;H^{s+1}(0,1))}^2\big) + h_{max}^4 \|c\|_{H^1(0,T;H^2(0,1))}^2\Big)\Big),
\end{aligned}
\tag{61}
$$

where ϵ is nonzero constant such that $d_0 - 4\epsilon^2 > 0$, ω is given by

$$
\omega = \frac{1}{\epsilon^2}\Big(C_d^2 + C_{b,2}^2 + \frac{1}{2}C_b^2 C_p^2\Big) + 2\epsilon^2 \tag{62}
$$

and $C_d, C_b, C_{b,2}, C_{in}$ were introduced before.

Proof: It can be shown that $e_{c,h}(t)$ is solution of the variational problem

$$
\begin{aligned}
(e'_{c,h}(t), w_h)_h &= -(d_h(t)D_{-x}c_h(t), D_{-x}w_h)_{h,+} + (M_h(b_h(t)c_h(t)), D_{-x}w_h)_{h,+} \\
&\quad + (q_{2,h}(t), w_h)_h - (R_h c'(t), v_h)_h.
\end{aligned}
$$

As $\tilde{c}_h(t)$ satisfies (37) we obtain

$$
\begin{aligned}
(e'_{c,h}(t), w_h)_h &= (\tilde{d}_h(t)D_{-x}\tilde{c}_h(t), D_{-x}w_h)_{h,+} - (d_h(t)D_{-x}c_h(t), D_{-x}w_h)_{h,+} \\
&\quad + (M_h(b_h(t)c_h(t)), D_{-x}w_h)_{h,+} - (M_h(\tilde{b}_h(t)\tilde{c}_h(t)), D_{-x}w_h)_{h,+} \\
&\quad + (\hat{c'}_h(t), w_h)_h - (R_h c'(t), w_h)_h,
\end{aligned}
\tag{63}
$$

where $\hat{c'}_h(t)$ is given by (18) with q_ℓ replaced by $c'(t)$.
Taking into account Lemmas 1 and 2 we deduce, from (63) with $w_h = e_{c,h}(t)$, the inequality

$$\begin{aligned}(e'_{c,h}(t), w_h)_h &\leq -(d_h(t)D_{-x}(c_h(t) - R_h c(t)), D_{-x}e_{c,h}(t))_{h,+}\\ &+(M_h(b_h(t)(c_h(t) - R_h c(t))), D_{-x}e_{c,h}(t))_{h,+}\\ &+(\hat{c'}_h(t) - R_h c'(t), e_{c,h}(t))_h + \tau_{d,h}(t, e_{c,h}(t)) + \tau_{b,h}(t, e_{c,h}(t)).\end{aligned} \tag{64}$$

We estimate in what follows the quantities $(\hat{c}_{t h}(t)-R_h c_t(t), e_{c,h}(t))_h$, $\tau_{d,h}(t, e_{c,h}(t))$ and $\tau_{b,h}(t, e_{c,h}(t))$:
From Lemma 3 we have

$$|(\hat{c'}_h(t) - R_h c'(t), e_{c,h}(t))_h| \leq \frac{1}{4\sigma^2} C_{in}^2 \sum_{i=1}^{N} h_i^4 \|c'(t)\|^2_{H^2(I_i)} + \sigma^2 \|e_{c,h}(t)\|^2_{1,h}, \tag{65}$$

provided that $c'(t) \in H^2(0,1)$. In the previous inequality $\sigma \neq 0$ is an arbitrary constant.
We remark that for $\tau_{d,h}(t, e_{c,h}(t))$ and $\tau_{b,h}(t, e_{c,h}(t))$ hold the estimates (56) and (59), respectively. Consequently

$$\begin{aligned}|\tau_{d,h}(t, e_{c,h}(t))| \;&\leq \frac{1}{2\epsilon^2} C_d^2 \|e_{c,h}(t)\|^2_h + \epsilon^2 \|D_{-x}e_{c,h}(t)\|^2_{h,+}\\ &+\frac{1}{2\epsilon^2} C_d^2 \sum_{i=1}^{N} h_i^{2s} \Big(\|p(t)\|^2_{H^{s+1}(I_i)} + \|c(t)\|^2_{H^{s+1}(I_i)}\Big),\end{aligned} \tag{66}$$

and

$$\begin{aligned}|\tau_{b,h}(t, e_{c,h}(t))| \;&\leq \frac{1}{2\eta^2} C_{b,2}^2 \|e_{c,h}(t)\|^2_h + \eta^2 \|D_{-x}e_{c,h}(t)\|^2_{h,+}\\ &+\frac{1}{2\eta^2} C_{b,2}^2 \sum_{i=1}^{N} h_i^{2s} \Big(\|p(t)\|^2_{H^{s+1}(I_i)} + \|c(t)\|^2_{H^{s+1}(I_i)}\Big),\end{aligned} \tag{67}$$

where $\epsilon \neq 0, \eta \neq 0$ are arbitrary constants.

Considering estimates (65), (66) and (67) in (64) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e_{c,h}(t)\|_h^2&+(d_h(t)D_{-x}e_{c,h}(t),D_{-x}e_{c,h}(t))_{h,+}\\&-(M_h(b_h(t)e_{c,h}(t)),D_{-x}e_{c,h}(t))_{h,+}-\Big(\frac{1}{2\epsilon^2}C_d^2+\frac{1}{2\eta^2}C_{b,2}^2+\sigma^2\Big)\|e_{c,h}(t)\|_h^2\\&-(\epsilon^2+\eta^2+\sigma^2)\|D_{-x}e_{c,h}(t)\|_{h,+}^2\le\tau_h(t)^2,\end{aligned}\tag{68}$$

where

$$\begin{aligned}\tau_h(t)^2\ &\le\Big(\frac{1}{2\epsilon^2}C_d^2+\frac{1}{2\eta^2}C_{b,2}^2\Big)\Big(\sum_{i=1}^N h_i^{2s}\big(\|p(t)\|_{H^{s+1}(I_i)}^2+\|c(t)\|_{H^{s+1}(I_i)}^2\big)\Big)\\&\quad+\frac{1}{4\sigma^2}C_{in}^2\sum_{i=1}^N h_i^4\|c'(t)\|_{H^2(I_i)}^2.\end{aligned}$$

In what concerns

$$(d_h(t)(D_{-x}e_{c,h}(t),D_{-x}e_{c,h}(t))_{h,+}$$

and

$$(M_h(b_h(t)e_{c,h}(t)),D_{-x}e_{c,h}(t))_{h,+}$$

we have

$$(d_h(t)(D_{-x}e_{c,h}(t),D_{-x}e_{c,h}(t))_{h,+}\ge d_0\|D_{-x}e_{c,h}(t)\|_{h,+}^2,\tag{69}$$

and, as (33) holds with $c_h(t)$ replaced by $e_{c,h}(t)$, we also have

$$|(M_h(b_h(t)e_{c,h}(t)),D_{-x}e_{c,h}(t))_{h,+}|\le\frac{1}{4\gamma^2}C_b^2C_p^2\|e_{c,h}(t)\|_h^2+\gamma^2\|D_{-x}e_{c,h}(t)\|_{h,+}^2,\tag{70}$$

where $\gamma\neq 0$ is an arbitrary constants.

Considering now in (68) the estimates (69) and (70) for $\epsilon=\eta=\gamma=\sigma$, we conclude

$$\frac{d}{dt}\|e_{c,h}(t)\|_h^2+2(d_0-4\epsilon^2)\|D_{-x}e_{c,h}(t)\|_{h,+}\le\omega\|e_{c,h}(t)\|_h^2+\tau_h(t)^2\tag{71}$$

with ω defined by (62).

Inequality (71) implies

$$\|e_{c,h}(t)\|_h^2 + 2(d_0 - 4\epsilon^2)\int_0^t \|D_{-x}e_{c,h}(s)\|_{h,+}^2\, ds \le \|e_{c,h}(0)\|_h^2$$

$$+\omega\int_0^t \|e_{c,h}(\mu)\|_h^2\, d\mu + \int_0^t \tau_h(\mu)^2\, d\mu$$

that leads to (61). □

Theorem 1 and Corollary 1 imply the error estimate for the pressure.

Corollary 2. *Under the assumption of Theorem 1, for the pressure we have*

$$\begin{aligned}\|\mathbb{P}_h e_{p,h}(t)\|_1^2 &\le C_{p,n}\Big(\|c_h(0)-c(0)\|_h^2 + C_e\sum_{i=1}^N\int_0^t \Big(h_i^{2s}\big(\|p(\mu)\|_{H^{s+1}(I_i)}^2 \\ &\quad + \|c(\mu)\|_{H^{s+1}(I_i)}^2\big) + h_i^4\|c'(\mu)\|_{H^2(I_i)}^2\Big)d\mu\Big) \\ &\le C_{p,n}\Big(\|c_h(0)-c(0)\|_h^2 + C_e\Big(h_{max}^{2s}\big(\|c\|_{L^2(0,T;H^{s+1}(0,1))}^2 \\ &\quad + \|p\|_{L^2(0,T;H^{s+1}(0,1))}^2\big) + h_{max}^4\|c\|_{H^1(0,T;H^2(0,1))}^2\Big), \end{aligned} \tag{72}$$

for some positive constants $C_{p,n}$ and C_e which do not depend on h and for $h\in H$ with h_{max} small enough.

□

5. An IMEX method

In $[0,T]$ we introduce a uniform grid $\{t_n\}$ with $t_0 = 0, t_M = T$ and $t_j - t_{j-1} = \Delta t$. By D_{-t} we denote the backward finite difference operator with respect to t. Let us suppose that the numerical approximations $p_h^n(x_i)$ and $c_h^n(x_i)$ for $p(x_i,t_n)$ and $c(x_i,t_n)$, respectively, are known. Let $p_h^{n+1}(x_i)$ and $c_h^{n+1}(x_i)$ be the numerical approximations for $p(x_i,t_{n+1})$ and $c(x_i,t_{n+1})$, respectively, defined by the following system

$$(a_h^n D_{-x}p_h^{n+1}, D_{-x}w_h)_{h,+} = (q_{1,h}^{n+1}, w_h)_h,\ w_h\in\mathbb{W}_{h,0}, \tag{73}$$

$$\begin{aligned}(D_{-t}c_h^{n+1}, w_h)_h + (d_h^{n,n+1}D_{-x}c_h^{n+1}, D_{-x}w_h)_{h,+} - (M_h(b_h^{n,n+1}c_h^{n+1}), D_{-x}w_h)_{h,+} \\ = (q_{2,h}^{n+1}, w_h)_h,\ w_h\in\mathbb{W}_{h,0},\end{aligned} \tag{74}$$

with the boundary conditions

$$p_h^{n+1}(x_0) = p_\ell(t_{n+1}),\ p_h^{n+1}(x_N) = p_r(t_{n+1}), \tag{75}$$

$$c_h^{n+1}(x_0) = c_\ell(t_{n+1}),\ c_h^{n+1}(x_N) = c_r(t_{n+1}), \tag{76}$$

and with the initial conditions

$$c_h^0(x_i) = c_{0,h}(x_i), p_h^0(x_i) = p_{0,h}(x_i),\ i = 1, \dots, N-1. \tag{77}$$

In (73) and (74), $q_{\ell,h}^{n+1}$ is obtained from $q_{\ell,h}(t)$ taking $t = t_{n+1}$, $(\ell = 1, 2)$, the coefficient a_h^n is obtained from $a_h(t)$ replacing $c_h(t)$ by c_h^n, $d_h^{n,n+1}$ and $b_h^{n,n+1}$ are obtained from $d_h(t)$ and $b_h(t)$, respectively, replacing $c_h(t)$ and $p_h(t)$ by c_h^n and p_h^{n+1}, respectively.

We establish in what follows the stability of the numerical approximations c_h^{n+1}, p_h^{n+1} defined by (73) and (74). In order to do that we assume that $p_\ell = p_r = c_\ell = c_r = 0$.

Proposition 7. *Under the assumption of Proposition 1, there exists a positive constant C_p which does not depend on h such that*

$$\|p_h^{n+1}\|_{1,h} \leq C_p \|q_{1,h}^{n+1}\|_h, n = 0, \dots, M-1. \tag{78}$$

□

If q_1 satisfies (26) then the sequence $\|p_h^n\|_{1,h}, h \in H$, satisfies

$$\|p_h^n\|_{1,h} \leq C_p,\ n = 1, \dots, M-1,\ h \in H, \tag{79}$$

for some positive constant C_p.

As in the semi-discrete case, from (79) we conclude

$$\|p_h^n\|_\infty \leq C_p,\ n = 1, \dots, M,$$

and, as in the semi-discrete case, it is reasonable to assume

$$\max_{i=1,\dots,N} |D_{-x} p_h^n(x_i)| \leq C_p,\ n = 1, \dots, M. \tag{80}$$

In the proof of Proposition 8 we use the following discrete Gronwall Lemma:

Lemma 4. *(Discrete Gronwall Lemma (Lemma 4.3 of* [6]*)) Let $\{\eta_n\}$ be a sequence of nonnegative real numbers satisfying*

$$\eta_n \leq \sum_{j=0}^{n-1} \omega_j \eta_j + \beta_n \quad \textit{for } n \geq 1,$$

where $\omega_j \geq 0$ *and* $\{\beta_n\}$ *is a nondecreasing sequence of nonnegative numbers. Then*

$$\eta_n \leq \beta_n \exp\big(\sum_{j=0}^{n-1} \omega_j\big) \quad \text{for } n \geq 1. \tag{81}$$

Proposition 8. *If* $0 < a_0 \leq a$, $0 < d_0 \leq d$, *(29) and (80), then* c_h^n *defined by (73),(74) with homogeneous boundary conditions satisfies*

$$\|c_h^n\|_h^2 + \Delta t \sum_{j=0}^{n} \|D_{-x} c_h^j\|_{h,+}^2 \leq \frac{1}{\min\{1-\theta\Delta t, 2(d_0-\epsilon^2)\}} e^{\frac{\theta n \Delta t}{\min\{1-\theta\Delta t, 2(d_0-\epsilon^2)\}}}$$
$$\Big((1-\theta\Delta t)\|c_h(0)\|_h^2 + 2(d_0-\epsilon^2)\|D_{-x}c_h^0\|_{h,+}^2 + \frac{1}{2\eta^2}\Delta t \sum_{m=1}^{n} \|q_{2,h}^m\|_h^2 \, ds\Big), \tag{82}$$

where

$$\theta = \frac{1}{2\epsilon^2} C_b^2 C_p^2 + 2\eta^2, \tag{83}$$

$\eta \neq 0$ *is an arbitrary constant,* $\epsilon \neq 0$ *is fixed by (31) and* Δt *satisfies*

$$1 - \theta\Delta t > 0. \tag{84}$$

Proof: Taking in (74) n and w_h replaced by m and c_h^{m+1}, respectively, and following the proof of Proposition 2, it can be shown that

$$\|c_h^{m+1}\|_h^2 + 2(d_0-\epsilon^2)\Delta t \|D_{-x}c_h^{m+1}\|_{h,+}^2 \leq \|c_h^m\|_h^2 + \Big(\frac{1}{2\epsilon^2}C_b^2 C_p^2 + 2\eta^2\Big)\Delta t \|c_h^{m+1}\|_h^2$$
$$+ \frac{1}{2\eta^2}\Delta t \|q_{2,h}^{m+1}\|_h^2, \tag{85}$$

where ϵ, η are nonzero constants. Summing (85) for $m = 0, \ldots, n-1$, we obtain

$$\|c_h^n\|_h^2 + 2(d_0-\epsilon^2)\Delta t \sum_{m=1}^{n} \|D_{-x}c_h^m\|_{h,+}^2 \leq \|c_h^0\|_h^2 + \theta\Delta t \sum_{m=1}^{n} \|c_h^m\|_h^2$$
$$+ \frac{1}{2\eta^2}\Delta t \sum_{m=1}^{n} \|q_{2,h}^m\|_h^2, \tag{86}$$

with θ defined by (83).

Inequality (86) can be rewritten in the following equivalent form

$$\|c_h^n\|_h^2 + \Delta t \sum_{m=0}^{n} \|D_{-x} c_h^m\|_{h,+}^2 \, ds \leq \frac{\theta \Delta t}{\min\{2(d_0-\epsilon^2), 1-\theta\Delta t\}} \sum_{m=0}^{n-1} \|c_h^m\|_h^2$$
$$+ \frac{1}{\min\{2(d_0-\epsilon^2), 1-\theta\Delta t\}} \Big((1-\theta\Delta t)\|c_h^0\|_h^2 + 2(d_0-\epsilon^2)\Delta t \|D_{-x} c_h^0\|_{h,+}^2$$
$$+ \frac{1}{2\eta^2} \Delta t \sum_{m=1}^{n} \|q_{2,h}^m\|_h^2 \Big), \tag{87}$$

provided that ϵ and Δt satisfy (31) and (84), respectively. Using in (87) Gronwall Lemma we deduce (82).

□

We establish in what follows an upper bound for the errors $e_{c,h}^{n+1} = c_h^{n+1} - R_h c(t_{n+1})$ $e_{p,h}^{n+1} = p_h^{n+1} - R_h p(t_{n+1})$

$$\|e_{p,h}^{n+1}\|_{1,h}\,,\ \|e_{c,h}^{n+1}\|_h^2 + \Delta t \sum_{j=0}^{n+1} \|D_{-x} e_{c,h}^j\|_{h,+}^2.$$

We start by introducing $\tilde{p}_h^{n+1}, \tilde{c}_h^{n+1}$ as the solutions of the auxiliary problems (36), (37) where the source terms and the coefficients are defined taking $t = t_{n+1}$. The estimate (38) holds for $\tilde{p}_h^{n+1}$. Moreover, under the assumptions of Proposition 4 for $t = t_{n+1}$, Proposition 5 enable us to conclude that (47) holds for $\tilde{c}_h^{n+1}$. As in the semi-discrete case, it can be shown that the time discrete version of (39) and (49) holds, that is,

$$\max_{i=1,\dots,N} |D_{-x}\tilde{p}_h^n(x_i)| \leq C_{\tilde{p}},\ n = 1, \dots, M, \tag{88}$$

$$\max_{i=1,\dots,N} |D_{-x}\tilde{c}_h^n(x_i)| \leq C_{\tilde{c}},\ n = 1, \dots, M, \tag{89}$$

for $h \in H$ with h_{max} small enough. We also have

$$\|\tilde{c}_h^n\|_\infty \leq C_{\tilde{c}},\ n = 1, \dots, M.$$

Under the assumptions of Proposition 6 we can prove that

$$\begin{aligned}\|\mathbb{P}_h(p_h^{n+1}-\tilde{p}_h^{n+1})\|_1 \leq C_{p,\tilde{p}}\Big(\|e_{c,h}^n\|_h + \Big(\sum_{i=1}^N h_i^{2s}\|c(t_{n+1})\|^2_{H^s(I_i)}\Big)^{1/2} \\ +\Big(\Delta t^2\|R_hc'(t_n)\|_h^2+\Delta t^3\|R_hc\|^2_{H^2(t_n,t_{n+1},\mathbb{W}_h)}\Big)^{1/2}\Big).\end{aligned} \tag{90}$$

As in Corollary 1, for $\|\mathbb{P}_h e_{p,h}^{n+1}\|_1$ we have

$$\begin{aligned}\|\mathbb{P}_h e_{p,h}^{n+1}\|_1 \leq C\Big(\|e_{c,h}^n\|_h + \Big(\sum_{i=1}^N h_i^{2s}\|c(t_{n+1})\|^2_{H^s(I_i)}\Big)^{1/2} \\ +\Big(\sum_{i=1}^N h_i^{2s}\|p(t_{n+1})\|^2_{H^{s+1}(I_i)}\Big)^{1/2} \\ +\Delta t\Big(\|R_hc'(t_n)\|_h^2+\Delta t\|R_hc\|^2_{H^2(t_n,t_{n+1},\mathbb{W}_h)}\Big)^{1/2}\Big).\end{aligned} \tag{91}$$

We are now in position to establish for $(D_{-t}e_{c,h}^{m+1},w_h)_h$ an estimate similar to the one established in Theorem 1 for $(e_{c,h}(t),w_h)_h$. In fact under the assumptions of this result and using the fact (80) we can prove that holds the following

$$\begin{aligned}(D_{-t}e_{c,h}^{m+1},w_h)_h &= -(d_h^{m,m+1}D_{-x}e_{c,h}^{m+1},D_{-x}w_h)_{h,+} \\ &\quad +(M_h(b_h^{m,m+1}e_{c,h}^{m+1}),D_{-x}w_h)_{h,+}+\tau_h^{m+1}(w_h),\end{aligned} \tag{92}$$

where

$$\tau_h^{m+1}(w_h)=\tau_{d,h}^{m+1}(w_h)+\tau_{b,h}^{m+1}(w_h)+\tau_{c,h}^{m+1}(w_h)$$

with

$$\begin{aligned}|\tau_{d,h}^{m+1}(w_h)| \leq C_{d,d}\Big(\|e_{c,h}^m\|_h+\Delta t\Big(\|R_hc'(t_m)\|_h^2+\Delta t\|R_hc\|^2_{H^2(t_m,t_{m+1},\mathbb{W}_h)}\Big)^{1/2} \\ +\Big(\sum_{i=1}^N h_i^{2s}\|p(t_{m+1})\|^2_{H^{s+1}(I_i)}\Big)^{1/2} \\ +\Big(\sum_{i=1}^N h_i^{2s}\|c(t_{m+1})\|^2_{H^{s+1}(I_i)}\Big)^{1/2}\Big)\|D_{-x}w_h\|_{h,+},\end{aligned}$$

$$|\tau_{b,h}^{m+1}(w_h)| \leq C_{b,d}\Big(\|e_{c,h}^m\|_h + \Delta t\Big(\|R_h c'(t_m)\|_h^2 + \Delta t\|R_h c\|^2_{H^2(t_m,t_{m+1},\mathbb{W}_h)}\Big)^{1/2}$$
$$+\Big(\sum_{i=1}^N h_i^{2s}\|c(t_{m+1})\|^2_{H^{s+1}(I_i)}\Big)^{1/2}$$
$$+\Big(\sum_{i=1}^N h_i^{2s}\|p(t_{m+1})\|^2_{H^{s+1}(I_i)}\Big)^{1/2}\Big)\|D_{-x}w_h\|_{h,+}$$

and

$$|\tau_{c,h}^{m+1}(w_h)| \leq C_{in,d}\Big(\Delta t\|R_h c\|_{W^{2,\infty}(t_m,t_{m+1},\mathbb{W}_h)}$$
$$+\Big(\sum_{i=1}^N h_i^4\|R_h c'(t_{m+1})\|^2_{H^2(I_i)}\Big)^{1/2}\Big)\|D_{-x}w_h\|_{h,+},$$

for some positive constants $C_{d,d}, C_{b,d}$ and $C_{in,d}$ and for $h \in H$ with h_{max} small enough.

Taking in (92) $w_h = e_{c,h}^{m+1}$ and following the proof of Proposition 8 we can prove that

$$\|e_{c,h}^{m+1}\|_h^2 + 2\Delta t(d_0 - 4\epsilon^2)\|D_{-x}e_{c,h}^{m+1}\|^2_{h,+} \leq (1+\theta_2\Delta t)\|e_{c,h}^m\|_h^2 + \theta_1\Delta t\|e_{c,h}^{m+1}\|_h^2 + \Delta t(\tau_r^{m+1})^2 \tag{93}$$

with

$$\theta_1 = \frac{1}{2\epsilon^2}C_p^2 C_b^2,$$
$$\theta_2 = \frac{1}{2\epsilon^2}\Big(C_{d,d}^2 + C_{b,d}^2 + C_{in,d}^2\Big)$$

and

$$(\tau_r^{m+1})^2 \leq \frac{1}{2\epsilon^2}\Big(C_{d,d}^2 + C_{b,d}^2\Big)\Big(\Delta t\|R_h c'(t_m)\|_h + \Delta t\|R_h c\|^2_{H^2(t_m,t_{m+1},\mathbb{W}_h)}$$
$$+\Big(\sum_{i=1}^N h_i^{2s}\|c(t_{m+1})\|^2_{H^{s+1}(I_i)}\Big)^{1/2} + \Big(\sum_{i=1}^N h_i^{2s}\|p(t_{m+1})\|^2_{H^{s+1}(I_i)}\Big)^{1/2}\Big)^2$$
$$+\frac{1}{2\epsilon^2}C_{in,d}^2\Big(\Delta t\|R_h c\|_{W^{2,\infty}(t_m,t_{m+1},\mathbb{W}_h)} + \Big(\sum_{i=1}^N h_i^4\|c'(t_{m+1})\|^2_{H^2(I_i)}\Big)^{1/2}\Big)^2. \tag{94}$$

In (93) and (94) ϵ is a nonzero constants.

Summing (93) for $m = 0, \ldots, n-1$, we obtain

$$(1-\theta_1\Delta t)\|e^n_{c,h}\|^2_h + 2(d_0 - 4\epsilon^2)\Delta t \sum_{m=0}^{n} \|D_{-x}e^m_{c,h}\|^2_{h,+}$$

$$\le (1-\theta_1\Delta t)\|e^0_{c,h}\|^2_h + 2\Delta t(d_0 - 4\epsilon^2)\|D_{-x}e^0_{c,h}\|^2_{h,+}$$
$$+(\theta_1+\theta_2)\Delta t \sum_{m=0}^{n-1} \|e^m_{c,h}\|^2_h + \Delta t \sum_{m=1}^{n} (\tau^m_r)^2$$

which implies

$$\|e^n_{c,h}\|^2_h + \Delta t \sum_{m=0}^{n} \|D_{-x}e^m_{c,h}\|^2_{h,+}$$
$$\le \frac{1}{\min\{1-\theta_1\Delta t, 2(d_0-4\epsilon^2)\}}\Big((1-\theta_1\Delta t)\|e^0_{c,h}\|^2_h + 2\Delta t(d_0 - 4\epsilon^2)\|D_{-x}e^0_{c,h}\|^2_{h,+}\Big)$$
$$+\frac{(\theta_1+\theta_2)\Delta t}{\min\{1-\theta_1\Delta t, 2(d_0-4\epsilon^2)\}} \sum_{m=0}^{n-1} \|e^m_{c,h}\|^2_h + \Delta t \sum_{m=1}^{n} (\tau^m_r)^2, \tag{95}$$

provided that

$$1 - \theta_1\Delta t > 0 \tag{96}$$

and

$$d_0 - 4\epsilon^2 > 0. \tag{97}$$

Applying discrete Gronwall Lemma to (95) we deduce

$$\|e^n_{c,h}\|^2_h + \Delta t \sum_{m=0}^{n} \|D_{-x}e^m_{c,h}\|^2_{h,+} \le \frac{1}{\min\{1-\theta_1\Delta t, 2(d_0-4\epsilon^2)\}} e^{\frac{(\theta_1+\theta_2)n\Delta t}{\min\{1-\theta_1\Delta t, 2(d_0-4\epsilon^2)\}}}$$
$$\Big((1-\theta_1\Delta t)\|e^0_{c,h}\|^2_h + 2(d_0-4\epsilon^2)\Delta t\|D_{-x}e^0_{c,h}\|^2_{h,+} + \Delta t \sum_{m=1}^{n} (\tau^m_r)^2\Big). \tag{98}$$

We remark that the error estimate for the concentration depends on $\|e^0_{c,h}\|_h$ and $\|D_{-x}e^0_{c,h}\|_{h,+}$. Moreover, if c and p are smooth enough, then there exists a positive constant C which does not depend on Δt and h such that

$$\Delta t\sum_{m=1}^{n}(\tau_r^m)^2 \leq C\Big(\Delta t^2\Big(\max_{j=1,\dots,M}\|R_hc'(t_j)\|_h^2+\|R_hc\|^2_{W^{2,\infty}(0,T;\mathbb{W}_h)}\Big)$$
$$+\max_{j=1,\dots,M}\sum_{i=1}^{N}\Big(h_i^{2s}\Big(\|c(t_j)\|^2_{H^{s+1}(I_i)}+\|p(t_j)\|^2_{H^{s+1}(I_i)}\Big)+h_i^4\|c'(t_j)\|^2_{H^2(I_i)}\Big)\Big). \tag{99}$$

Considering $c_h^0 = R_hc_0$ we conclude from (98) and (99) that, for some positive constant C and for Δt and h_{max} small enough, holds the following

$$\|e_{c,h}^n\|_h^2+\Delta t\sum_{m=0}^{n}\|D_{-x}e_{c,h}^m\|^2_{h,+}$$
$$\leq C\Big(\Delta t^2\Big(\max_{j=1,\dots,M}\|R_hc'(t_j)\|_h^2+\|R_hc\|^2_{W^{2,\infty}(0,T;\mathbb{W}_h)}\Big)$$
$$+\max_{j=1,\dots,M}\sum_{i=1}^{N}\Big(h_i^{2s}\Big(\|c(t_j)\|^2_{H^{s+1}(I_i)}+\|p(t_j)\|^2_{H^{s+1}(I_i)}\Big)+h_i^4\|c'(t_j)\|^2_{H^2(I_i)}\Big)\Big). \tag{100}$$

Considering now (100) in the estimate (91) we deduce for the pressure the estimate

$$\|e_{p,h}^{n+1}\|^2_{1,h} \leq C\Big(\Delta t^2\Big(\max_{j=1,\dots,M}\|R_hc'(t_j)\|_h^2+\|R_hc\|^2_{W^{2,\infty}(0,T;\mathbb{W}_h)}\Big)$$
$$+\max_{j=1,\dots,M}\sum_{i=1}^{N}\Big(h_i^{2s}\Big(\|c(t_j)\|^2_{H^{s+1}(I_i)}+\|p(t_j)\|^2_{H^{s+1}(I_i)}\Big)+h_i^4\|c'(t_j)\|^2_{H^2(I_i)}\Big)\Big). \tag{101}$$

Estimates (100) and (101) allow us to conclude, for $s\in\{1,2\}$,

$$\|e_{p,h}^{n+1}\|^2_{1,h} \leq C\Big(\Delta t^2+h_{max}^{2s}\Big), \tag{102}$$

$$\|e_{c,h}^n\|_h^2+\Delta t\sum_{m=0}^{n}\|D_{-x}e_{c,h}^n\|^2_{h,+} \leq C\Big(\Delta t^2+h_{max}^{2s}\Big). \tag{103}$$

We illustrate in what follows the estimates (102) and (103).

Example 1. *Let us consider (1)-(5) with*

$$a(c) = 1+c(x,t),\;\; b(c,p_x) = (c(x,t)p_x(x,t))^2,\;\; d(c,p_x) = c(x,t)+p_x(x,t)+2$$

q_1, q_2, the initial and boundary conditions such that this IBVP has the following solution

$$p(x,t) = e^t x(x-1), \;\; c(x,t) = e^t(1 - cos(2\pi x))sin(x), \;\; x \in [0,1],\, t \in [0,T].$$

The numerical approximations c_h^n and p_h^n were obtained with the IMEX method (73)-(77) with nonuniform grids in $[0,1]$ and with $T = 0.1$ and $\Delta t = 10^{-6}$. The spatial initial grid is arbitrary and the new grid is obtained introducing in $[x_i, x_{i+1}]$ the midpoint. In Table 1 we present the errors

$$Error_c = \max_{n=1,\dots,M} \left(\|e^n_{c,h}\|_h^2 + \Delta t \sum_{j=0}^{n} \|D_{-x} e^j_{c,h})\|^2_{h,+} \right)^{1/2}, \tag{104}$$

$$Error_p = \max_{n=1,\dots,M} \|D_{-x} e^n_{p,h}\|_{h,+} \tag{105}$$

and the rates $Rate_c$, $Rate_p$ that were computed by the formula

$$Rate = \frac{\ln\left(\frac{Error_{h_{max,1}}}{Error_{h_{max,2}}}\right)}{\ln\left(\frac{h_{max,1}}{h_{max,2}}\right)}, \tag{106}$$

where $h_{max,1}$ and $h_{max,2}$ are the maximum step sizes of two consecutive partitions.

h_{max}	$Error_c$	$Error_p$	$Rate_c$	$Rate_p$
1.3174×10^{-1}	5.5435×10^{-2}	1.1099×10^{-2}	*1.9492*	*1.5048*
6.5869×10^{-2}	1.4355×10^{-2}	3.9113×10^{-3}	*2.0010*	*1.5808*
3.2934×10^{-2}	3.5863×10^{-3}	1.3075×10^{-3}	*2.0024*	*1.8337*
1.6467×10^{-2}	8.9511×10^{-4}	3.6682×10^{-4}	*2.0008*	*1.9296*
8.2336×10^{-3}	2.2366×10^{-4}	9.6288×10^{-5}	*2.0029*	*1.9671*
4.1168×10^{-3}	5.5804×10^{-5}	2.4628×10^{-5}	*2.0109*	*1.9866*
2.0584×10^{-3}	1.3846×10^{-5}	6.2144×10^{-6}	*2.0301*	*2.0015*
1.0292×10^{-3}	3.3899×10^{-6}	1.5520×10^{-6}	-	-

Table 1

The numerical results presented in Table 1 show that $Error_p = O(h^2_{max})$ and $Error_c = O(h^2_{max})$.

6. Conclusions

The behavior of the pressure and concentration of an incompressible fluid in a one dimensional porous media is described by an elliptic equation for the pressure and a parabolic equation for the concentration linked by the

Darcy's law for the velocity. Quasilinear coupled problems that have as a particular case the previous problem were considered in this paper.

The use of piecewise linear finite element method for the pressure and concentration of a incompressible fluid in a porous media leads to a first order approximation to the velocity. Consequently, the concentration is of first order in the L^2-norm. This behavior is observed for uniform and nonuniform partitions of the spatial domain. Fully discrete schemes based on the piecewise linear finite element method with special quadrature formulas were studied in this paper. Error estimates for the semi-discrete and fully discrete approximations were established. These error estimates allow us to conclude that the methods studied leads to second order accuracy numerical approximations for the pressure and concentration and for their gradients.

A common approach in the convergence analysis of the spatial discretization of parabolic equations is the split of the semi-discretization error into two terms ([21]) considering the correspondent discretization of an auxiliary elliptic problem. Such approach was largely followed in the literature and implies an increasing in the smoothness requirements of the solution for the parabolic problem. In this paper a different approach was followed that avoids such smoothness requirements.

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References

[1] R. A. Adams, J.J.F. Fournier, Sobolev spaces, Second Edition, Elsevier, 2003.
[2] T. Arbogast, M.F. Wheeler, I. Yotov, Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences, SIAM Journal of Numerical Analysis, 34, 828-852, 1997.
[3] S. Barbeiro, J.A. Ferreira, R. Grigorieff, Supraconvergence of finite difference schemes for solutions in $H^s(0, L)$, IMA Journal of Numerical Analysis, 25, 797-811, 2005.
[4] M. Berndt, K. Lipnikov, M. Sashkov, M.F. Wheeler, I. Yotov, Superconvergence of the velocity in mimetic finite difference methods on quadrilaterals, SIAM Journal of numerical analysis, 43, 1728-1749, 2005
[5] J. Douglas Jr, Superconvergence in the pressure in the simulation of miscible displacement, SIAM Journal of Numerical Analysis, 22, 962-969, 1985.
[6] C. Chuanmiao, S. Tsimin, Finite Element Methods for Integrodifferential Equations, World Scientific Publishers, 1998.
[7] T.F. Dupont, P.T. Keenan, Superconvergence and postprocessing of fluxes from lowest-order mixed methods on triangles and tetrahedra, SIAM Journal of Numerical Analysis, 19, 1322-1332, 1998.

[8] E. Emmrich, and R.D. Grigorieff, Supraconvergence of a finite difference scheme for ellitic boundary value problems of the third kind in fractional order Sobolev spaces, Computational Methods in Applied Mathematics, 6, 154-177, 2006.
[9] E. Emmrich, Supraconvergence and supercloseness of a discretization for ellitic third kind boundary value problems on polygonal domains, Computational Methods in Applied Mathematics, 7, 153-162, 2007.
[10] R. E. Ewing, Problems arising in the modeling of hydrocarbon recovery, in The Mathematics of Reservoir Simulation, SIAM, Philadelphia, 3–34, 1983.
[11] R.E. Ewing, M.F. Wheeler, Galerkin methods for miscible displacement problems in porous media, SIAM Journal of Numerical Analysis, 17, 351-365, 1980.
[12] R.E. Ewing, M. Liu, J. Wang, A new superconvergence for mixed finite element approximation, SIAM Journal of Numerical Analysis, 40, 2133-2150, 2002.
[13] J.A. Ferreira, R. Grigorieff, On the supraconvergence of elliptic finite difference schemes, Applied Numerical Mathematics, 28, 275-292, 1998.
[14] J.A. Ferreira, R. Grigorieff, Supraconvergence and supercloseness of a scheme for elliptic equations on nonuniform grids, Numerical Functional Analysis and Optimization, 27 (5-6), 539-564, 2006.
[15] B.O. Heimsund, X.C. Tai, J. Wang, Superconvergence for the gradient of finite element approximations by L^2 projections, SIAM Journal of Numerical Analysis, 40, 1263-1280, 2002.
[16] D.W. Peaceman, Improved treatment of dispersion in numerical calculation of multidimensional miscible displacement, Society of Petroleum Engineers, 6, 213-216, 1966.
[17] T. F. Russel and M.F, Wheeler, Finite element and finite difference methods for continuous flows in porous media, in The Mathematics of Reservoir Simulation, SIAM, Philadelphia, 35–106, 1983.
[18] T.F. Russel, M.F. Wheeler, I. Yotov, Superconvergence for control-volume mixed finite element methods on rectangular grids, SIAM Journal of Numerical Analysis, 45, 223-235, 2007.
[19] A. Settari, H.S. Price, T. Dupont, Development and application of variational methods for simulation of miscible displacement in porous media, Society of Petroleum Engineers, 17, 228-246, 1977.
[20] A. Weiser, M.F. Wheeler, On convergence of block-centered finite differences for elliptic problems, SIAM Journal of Numerical Analysis, 25, 352-375, 1988.
[21] M.F. Wheeler, 1973, A priori L^2 error estimates for Galerkin approximation to parabolic partial differential equations, SIAM Journal of Numerical Analysis, 10, 723-759.

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