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HIGHER CENTRAL EXTENSIONS VIA COMMUTATORS

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ABSTRACT: We prove that all semi-abelian categories with the *Smith is Huq* property satisfy the *Commutator Condition* (CC): higher central extensions may be characterised in terms of binary (Huq or Smith) commutators. In fact, even Higgins commutators suffice. As a consequence, in the presence of enough projectives we obtain explicit Hopf formulae for homology with coefficients in the abelianisation functor, and an interpretation of cohomology with coefficients in an abelian object in terms of equivalence classes of higher central extensions. We also give a counterexample against (CC) in the semi-abelian category of (commutative) loops.

KEYWORDS: Higgins, Huq, Smith commutator; higher central extension; Hopf formula; semi-abelian, exact Mal'tsev category; (co)homology.
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Introduction

The concept of *higher centrality* is a cornerstone in the recent approach to homology and cohomology of non-abelian algebraic structures based on categorical Galois theory [5, 33]. Through higher central extensions, the Brown–Ellis–Hopf formulae [11, 14] which express homology objects as a quotient of commutators have been made categorical [18, 19, 20], which greatly extends their scope while simplifying the study of concrete cases (see, for instance, [13, 15]). Higher central extensions are also essential in the study of relative commutators [22, 23] and are classified by cohomology groups [46].

To take full advantage of these results, sufficiently explicit characterisations of higher centrality are essential. On the one hand, the higher Hopf formulae are valid in any semi-abelian category [37] with enough projectives, but these formulae only become concrete once the relevant concept of higher centrality is appropriately characterised, ideally in terms of classical binary commutators. Indeed, the main result of [20] says that in a semi-abelian monadic category $\mathcal{A}$,

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for any n-presentation F of Z,

$$\mathrm{H}_{n+1}(Z, \mathsf{ab}) \cong \frac{[F_n, F_n] \wedge \bigwedge_{i \in n} \mathrm{Ker}(f_i)}{L_n[F]}. \qquad (\mathbf{A})$$

Coefficients are chosen in the abelianisation functor $\mathsf{ab}\colon \mathcal{A} \to \mathsf{Ab}(\mathcal{A})$. Here F_n is the initial object of F and the f_i are the initial arrows. The object $[F_n, F_n]$ is the Huq commutator of F_n with itself, which makes the numerator entirely explicit. But the denominator is not; rather, $L_n[F]$ is the smallest normal subobject of F_n which, when divided out, makes F *central with respect to* $\mathsf{Ab}(\mathcal{A})$ in the sense of categorical Galois theory. Nevertheless, in all known examples also this object may be expressed in terms of commutators.

On the other hand, given an object Z in a semi-abelian category, its cohomology with coefficients in an abelian object A classifies the higher central extensions of Z by A, provided those higher central extensions admit a characterisation in terms of Huq commutators. Thus far, such precise characterisations of higher central extensions were only available in concrete cases.

A semi-abelian category $\mathcal{A}$ satisfies the **Smith is Huq Condition (SH)** when two equivalence relations (Smith) commute if and only if their normalisations (Huq) commute. Under (SH) we may remedy the lack of characterisation mentioned above. We prove that an n-fold extension in a semi-abelian category $\mathcal{A}$ with (SH) is central with respect to the abelian objects in $\mathcal{A}$ if and only if a certain join of binary Huq commutators vanishes. This gives us following the refined version of the main theorem of [46].

Theorem. Let Z be an object and A an abelian object in a semi-abelian category with (SH). Then for every $n \geqslant 1$ we have an isomorphism $\mathrm{H}^{n+1}(Z, A) \cong \mathrm{Centr}^n(Z, A)$. ∎

Examples of semi-abelian categories with (SH) are all *action representative* semi-abelian categories [6, 4] and all *action accessible* ones [10], in particular in all *strongly semi-abelian* categories [7], all *Moore categories* [24, 44], all *categories of interest* [42, 41], but not all varieties of Ω-groups: the category of *digroups* is a counterexample [3, 7]. Hence our results are valid, e.g., in the categories of groups, Lie and Leibniz algebras, (pre)crossed modules and associative algebras.

The above can be made slightly more precise as follows. We shall say that an n-fold extension F in a semi-abelian category $\mathcal{A}$ is **H-central** when

$$\Big[\bigwedge_{i\in I}\mathrm{Ker}(f_i),\bigwedge_{i\in n\setminus I}\mathrm{Ker}(f_i)\Big]=0$$

for all $I\subseteq n$. Here the f_i are the initial arrows of the n-fold extension F and the commutators are either Huq or Higgins commutators. The category $\mathcal{A}$ satisfies the **Commutator Condition (CC)** when H-centrality is equivalent to centrality with respect to $\mathsf{Ab}(\mathcal{A})$ in the Galois-theory sense. This means that the denominator $L_n[F]$ of $(\mathbf{A})$ may be expressed as the join

$$\bigvee_{I\subseteq n}\Big[\bigwedge_{i\in I}\mathrm{Ker}(f_i),\bigwedge_{i\in n\setminus I}\mathrm{Ker}(f_i)\Big].$$

It follows from results in [8] and [27] that the Commutator Condition holds for (one-fold) extensions (Subsection 1.6). For double extensions, the Commutator Condition holds as soon as the Smith is Huq Condition does (see Subsection 1.7). Our main concern now becomes to find conditions which imply (CC) in all degrees.

In Section 1 we give a more detailed outline of the mathematical context we shall be working in. Section 2 contains the main result of the paper: Theorem 2.8, which says that the Commutator Condition for double extensions implies the Commutator Condition for all higher degrees. Hence the Commutator Condition is weaker than the Smith is Huq Condition.

Even though (SH) is known to be independent of semi-abelianness, thus far we did not have any examples to show that also (CC) is independent. The known counterexamples (in digroups [3, 7] or loops [30]) give an action of an object on an abelian object which is not a module. However, when an action is considered as double extension, it cannot be H-central without being central—see Subsection 4.1—which forces us to find a new counterexample. This is done in Section 3 where we show that the category of loops Loop does not satisfy (CC). In fact this counterexample also works in the category of commutative loops CLoop; it gives a new example of a semi-abelian category in which (SH) does not hold.

There are certain further questions which remain unanswered as yet; we give a short overview in Section 4.

1. Preliminaries

In this paper $\mathcal{A}$ will always denote a semi-abelian category [37].

1.1. The Huq commutator and the Smith commutator. A coterminal pair

$$K \overset{k}{\rightarrowtail} X \overset{l}{\leftarrowtail} L$$

of normal monomorphisms (i.e., kernels) in $\mathcal{A}$ is said to **(Huq-)commute** [9, 32] when there is a (necessarily unique) morphism φ such that the diagram

$$\begin{tikzcd} & K \arrow[dl, "\langle 1_K, 0\rangle"'] \arrow[dr, "k"] & \\ K \times L \arrow[rr, dotted, "\varphi"] & & X \\ & L \arrow[ul, "\langle 0, 1_L\rangle"] \arrow[ur, "l"'] & \end{tikzcd}$$

is commutative. The **Huq commutator** $[k,l]^{\mathrm{Huq}}\colon [K,L]^{\mathrm{Huq}} \to X$ **of** k **and** l [8, 3] is the smallest normal subobject of X which should be divided out to make k and l commute, so that k and l commute if and only if $[K,L]^{\mathrm{Huq}} = 0$. We can define $[K,L]^{\mathrm{Huq}}$ as the kernel of the (normal epi)morphism $X \to Q$, where Q is the colimit of the outer square above.

Given a pair of equivalence relations (R,S) on a common object X

$$\begin{tikzcd} R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "\Delta_R"' description] \arrow[r, "\Delta_S" description] & S, \arrow[l, shift right=2, "s_2"'] \arrow[l, shift left=2, "s_1"] \end{tikzcd}$$

consider the induced pullback of r_2 and s_1:

$$\begin{tikzcd} R \times_X S \arrow[r, "\pi_S"] \arrow[d, "\pi_R"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "s_1"] \\ R \arrow[r, "r_2"'] & X. \end{tikzcd}$$

The congruences R and S **centralise each other** or **(Smith-)commute** [47, 43, 9] when there is a (necessarily unique) morphism θ such that the diagram

$$\begin{tikzcd} & R \arrow[dl, "\langle 1_R, \Delta_S r_2\rangle"'] \arrow[dr, "r_1"] & \\ R \times_X S \arrow[rr, dotted, "\theta"] & & X \\ & S \arrow[ul, "\langle \Delta_R s_1, 1_S\rangle"] \arrow[ur, "s_2"'] & \end{tikzcd}$$

is commutative. Like for the Huq commutator, the **Smith commutator** is the smallest equivalence relation $[R, S]^{\mathrm{S}}$ on X which, divided out of X, makes R and S commute. It can be obtained through a colimit, similarly to the situation above; see Section 3 for a concrete example. Thus R and S commute if and only if $[R, S]^{\mathrm{S}} = \Delta_X$, where Δ_X denotes the smallest equivalence relation on X. We say that R is a **central** equivalence relation when it commutes with ∇_X, the largest equivalence relation on X, so that $[R, \nabla_X]^{\mathrm{S}} = \Delta_X$.

1.2. The Smith is Huq Condition. It is well known, and easily verified, that if the Smith commutator of two equivalence relations is trivial, then the Huq commutator of their normalisations is also trivial [9]. But, in general, the converse is false; in [3, 7] a counterexample is given in the category of digroups, which is a semi-abelian variety, even a variety of Ω-groups [31]. The requirement that the two commutators vanish together is known as the **Smith is Huq Condition (SH)** and it is shown in [40] that, for a semi-abelian category, this condition holds if and only if every star-multiplicative graph is an internal groupoid, which is important in the study of internal crossed modules [35]. Moreover, the Smith is Huq Condition is also known to hold for pointed strongly protomodular categories [9] (in particular, for any Moore category [24, 44]) and in action accessible categories [10] (in particular, for any category of interest [41, 42]).

1.3. Extensions. We write $\mathsf{Arr}^n(\mathcal{A})$ for the category of n-fold arrows in $\mathcal{A}$.

A **zero-fold extension** in $\mathcal{A}$ is an object of $\mathcal{A}$ and a **(one-fold) extension** is a regular epimorphism in $\mathcal{A}$. For $n \geqslant 2$, an **n-fold extension** is an object

(c, f) of $\mathsf{Arr}^n(\mathcal{A})$ (a morphism of $\mathsf{Arr}^{n-1}(\mathcal{A})$) as in

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d, "g"] \\ D \arrow[r, "f"'] & Z, \end{tikzcd}$$

such that the morphisms c, d, f, g and the universally induced comparison morphism $\langle d, c\rangle \colon X \to D \times_Z C$ to the pullback of f with g are $(n-1)$-fold extensions. A two-fold extension is also called a **double extension**. The n-fold extensions determine a full subcategory $\mathsf{Ext}^n(\mathcal{A})$ of $\mathsf{Arr}^n(\mathcal{A})$; we write $\mathsf{Ext}(\mathcal{A}) = \mathsf{Ext}^1(\mathcal{A})$.

An n-fold arrow may be considered as a diagram $2^n \to \mathcal{A}$ in $\mathcal{A}$, a cube of dimension n; in particular, n-fold extensions are pictured as n-cubes. Given such an n-fold extension F, we shall write F_n for its initial object and

$$f_i \colon F_n \to F_{n\backslash\{i\}},$$

$i \in n$, for the initial arrows. The extension property of F implies that for any choice of $i \in n$, the induced square in $\mathcal{A}$

$$\begin{tikzcd} F_n \arrow[r, "f_i"] \arrow[d] & F_{n\backslash\{i\}} \arrow[d] \\ \lim\limits_{\{i\}\subseteq J\subsetneq n} F_J \arrow[r] & \lim\limits_{J\subsetneq n\backslash\{i\}} F_J \end{tikzcd}$$

is still a double extension [46].

1.4. Central extensions. We write $\mathsf{Ab}(\mathcal{A})$ for the full subcategory of $\mathcal{A}$ determined by the abelian objects, that is, those objects which admit an internal abelian group structure. Let $\mathsf{ab} \colon \mathcal{A} \to \mathsf{Ab}(\mathcal{A})$ denote the abelianisation functor, left adjoint to the inclusion of $\mathsf{Ab}(\mathcal{A})$ in $\mathcal{A}$. It sends an object X of $\mathcal{A}$ to the abelian object $\mathsf{ab}(X) = X/[X, X]^{\mathrm{Huq}}$. We define centrality of (higher) extensions with respect to the Birkhoff subcategory $\mathsf{Ab}(\mathcal{A})$ of $\mathcal{A}$ [36, 8].

An extension $f \colon X \to Z$ is called **trivial** when the induced naturality square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Z \arrow[d, "\eta_Z"] \\ \mathsf{ab}(X) \arrow[r, "\mathsf{ab}(f)"'] & \mathsf{ab}(Z) \end{tikzcd}$$

is a pullback, and f is **central** when there exists an extension $g\colon Y \to Z$ such that the pullback of f along g is trivial. In our context we can take $g = f$ so that f is central if and only if either projection of its kernel pair is trivial (central extensions coincide with **normal** extensions).

The full subcategory $\mathsf{CExt}^1_{\mathsf{Ab}(\mathcal{A})}(\mathcal{A})$ of $\mathsf{Ext}^1(\mathcal{A})$ determined by those extensions which are central is again reflective. Inductively, we get a reflective subcategory $\mathsf{CExt}^n_{\mathsf{Ab}(\mathcal{A})}(\mathcal{A})$ of $\mathsf{Ext}^n(\mathcal{A})$ containing the **n-fold central extensions (relative to $\mathsf{Ab}(\mathcal{A})$)** of $\mathcal{A}$, $n \geqslant 1$. Each level gives rise to a notion of central extension which determines the next level—see [20, Theorem 4.6] and [18] where this is worked out in detail. In particular, for every $n \geqslant 1$ we have a reflector, the **centralisation functor**

$$\mathsf{centr}_n\colon \mathsf{Ext}^n(\mathcal{A}) \to \mathsf{CExt}^n_{\mathsf{Ab}(\mathcal{A})}(\mathcal{A}),$$

left adjoint to the inclusion of $\mathsf{CExt}^n_{\mathsf{Ab}(\mathcal{A})}(\mathcal{A})$ in $\mathsf{Ext}^n(\mathcal{A})$.

1.5. The Commutator Condition (CC). Given an n-fold extension F with initial object F_n and initial arrows $f_i\colon F_n \to F_{n\setminus\{i\}}$, we write $k_i\colon K_i = \mathrm{Ker}(f_i) \to F_n$ for all $i \in n$. We say that F is H-**central** when

$$\Big[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\setminus I} K_i\Big]^{(\mathrm{Huq})} = 0 \qquad (\mathbf{B})$$

for all $I \subseteq n$. Here the commutators are either Huq or Higgins commutators (Subsection 1.8); this explains the "H" (see Lemma 2.5). The category $\mathcal{A}$ satisfies the **Commutator Condition (CC)** when H-centrality is equivalent to centrality with respect to $\mathsf{Ab}(\mathcal{A})$ in the Galois-theory sense (Subsection 1.4).

The condition (CC) falls apart in one version for each degree of extension n: the category $\mathcal{A}$ satisfies **(CCn)** when an n-fold extension in it is H-central if and only if it is central. The principal result in this work is to show that (CC2) implies (CC) (Theorem 2.8).

1.6. One-fold extensions and (CC1). Recall that an extension $f\colon X \to Z$ in the category of groups is central (with respect to Ab) when $[\mathrm{Ker}(f), X] = 0$. This result was adapted to a semi-abelian context in [25, 8]: the one-fold central extensions (in the sense of Galois theory) may be characterised through the Smith commutator of equivalence relations as those extensions $f\colon X \to Z$ such that $[\mathrm{R}[f], \nabla_X]^{\mathrm{S}} = \Delta_X$, where $\mathrm{R}[f]$ denotes the kernel pair of f. This means that $\mathrm{R}[f]$ is a central equivalence relation (Subsection 1.1). A characterisation closer to the group case appears in [27] where the condition is reformulated in terms of the Huq commutator of normal subobjects so that it becomes $[\mathrm{Ker}(f), X]^{\mathrm{Huq}} = 0$. Hence f is central if and only if it is H-central, so that (CC1) is true in any semi-abelian category.

1.7. Double central extensions and (CC2). One level up, the double central extensions of groups vs. abelian groups were first characterised in [34]. A double extension (of Z) is a pushout square of regular epimorphisms

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "d"'] & C \arrow[d, "g"] \\ D \arrow[r, "f"'] & Z. \end{tikzcd} \tag{$\mathbf{C}$}$$

Let us write $K = \mathrm{Ker}(c)$, $L = \mathrm{Ker}(d)$ for the kernels of c and d and $R = \mathrm{R}[c]$, $S = \mathrm{R}[d]$ for the respective kernel pairs. Then $(\mathbf{C})$ is central when

$$[K, L] = 0 = [K \wedge L, X].$$

General versions of this characterisation were given in [26] for Mal'tsev varieties, then in [45] for semi-abelian categories and finally in [21] for exact Mal'tsev categories: the double extension $(\mathbf{C})$ is central if and only if

$$[R, S]^{\mathrm{S}} = \Delta_X = [R \wedge S, \nabla_X]^{\mathrm{S}}.$$

This means that the span (X, d, c) is a special kind of pregroupoid in the slice category $\mathcal{A}/Z$ (see [38] for the definition of a pregroupoid).

The problem we are now confronted with is that the correspondence between the Huq commutator of normal monomorphisms and the Smith commutator of equivalence relations which exists in level one is no longer there when we go up in degree. However, we know that always $[K \wedge L, X]^{\mathrm{Huq}} = 0$ if and only if $[R \wedge S, \nabla_X]^{\mathrm{S}} = \Delta_X$ by (CC1). Furthermore, $[R, S]^{\mathrm{S}} = \Delta_X$ implies $[K, L]^{\mathrm{Huq}} = 0$, so when $(\mathbf{C})$ is central it is also H-central. On the other hand, the Smith is Huq Condition says that $[K, L]^{\mathrm{Huq}} = 0$ implies $[R, S]^{\mathrm{S}} = \Delta_X$, so that the two concepts of centrality are equivalent—and hence (CC2) holds under (SH).

1.8. The Higgins commutator. Central extensions, relative to $\mathsf{Ab}(\mathcal{A})$, may also be characterised in terms of the Higgins commutator [29, 39], which in turn may be obtained through a co-smash product [12] or a cross-effect [16, 2, 28] of the identity functor on $\mathcal{A}$.

Given two objects K and L of $\mathcal{A}$, the **co-smash product** [12] of K and L

$$K \otimes L = \operatorname{Ker}\left[\left\langle \begin{smallmatrix} 1_K & 0 \\ 0 & 1_L \end{smallmatrix} \right\rangle : K + L \to K \times L\right]$$

behaves as a kind of "formal commutator" of K and L. (See [29] and [39]; this object is also written $K \diamond L$, or $(K|L)$ when it is interpreted as the **second cross-effect** of the identity functor $1_{\mathcal{A}}$ evaluated in K, L.) If now $k\colon K \to X$ and $l\colon L \to X$ are subobjects of an object X, their **Higgins commutator** $[K, L] \leqslant X$ is a subobject of X given by the image of the induced composite morphism

$$\begin{tikzcd} K \otimes L \arrow[r, tail, "\iota_{K,L}"] \arrow[dr, two heads] & K + L \arrow[r, "\left\langle \begin{smallmatrix} k \\ l \end{smallmatrix} \right\rangle"] & X. \\ & {[K, L]} \arrow[ur, tail] & \end{tikzcd}$$

When K and L are normal subobjects of X and $K \vee L = X$, the Higgins commutator $[K, L]$ is normal in X so that it coincides with the Huq commutator (Subsection 1.1). In particular, we always have $[K, X]^{\mathrm{Huq}} = [K, X]$. In general the Huq commutator is the normal closure of the Higgins commutator. So, $[K, L] \leqslant [K, L]^{\mathrm{Huq}}$ and $[K, L] = 0$ if and only if $[K, L]^{\mathrm{Huq}} = 0$. The Higgins commutator may also be used to measure normality of subobjects. In fact, a result in [39] states that $K \lhd X$ if and only if $[K, X] \leqslant K$, and is further refined in [29] as follows: the normal closure of K in X may be computed as the join $K \vee [K, X]$. In any case, an extension in $\mathcal{A}$ such as

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, tail, "k"] & X \arrow[r, two heads, "f"] & Z \arrow[r] & 0 \end{tikzcd}$$

is central if and only if $[K, X]^{\mathrm{Huq}} = [K, X] = 0$.

1.9. The ternary commutator. The Higgins commutator generally does not preserve joins, but the defect may be measured precisely—it is a ternary commutator which can be computed by means of a ternary co-smash product or a cross-effect of order three. Let us extend the definition above: given a third subobject $m\colon M \to X$ of the object X, the **ternary Higgins commutator**

$[K, L, M] \leqslant X$ is the image of the composite

$$\begin{tikzcd} K \otimes L \otimes M \arrow[r, tail, "\iota_{K,L,M}"] \arrow[dr, two heads] & K + L + M \arrow[r, "\left\langle \begin{smallmatrix} k \\ l \\ m \end{smallmatrix} \right\rangle"] & X, \\ & {[K, L, M]} \arrow[ur, tail] & \end{tikzcd}$$

where $\iota_{K,L,M}$ is the kernel of

$$K + L + M \xrightarrow{\left\langle \begin{smallmatrix} i_K & i_K & 0 \\ i_L & 0 & i_L \\ 0 & i_M & i_M \end{smallmatrix} \right\rangle} (K + L) \times (K + M) \times (L + M);$$

i_k, i_L and i_M denote the injection morphisms. The object $K \otimes L \otimes M$ is the **third cross-effect** of the identity functor $1_{\mathcal{A}}$ or **ternary co-smash product** evaluated in K, L and M.

Proposition 1.10. *If K, L, $M \leqslant X$ then*

$$[K, L \vee M] = [K, L] \vee [K, M] \vee [K, L, M].$$

Proof: Via the result in [29] or [30]. ■

1.11. (SH) and (CC2) via the ternary commutator. It is precisely the availability of this join decomposition which makes the Higgins commutator useful in what follows. This, and the fact that (SH) may be expressed in terms of ternary commutators. By the main result in [30], two normal subobjects K, $L \lhd X$ have Smith-commuting denormalisations when $[K, L] = 0 = [K, L, X]$. Hence the Smith is Huq Condition is equivalent to saying that $[K, L] = 0$ (they Huq- or Higgins-commute) implies $[K, L, X] = 0$ (what is missing for them to also Smith-commute).

What we shall be studying here (the Commutator Condition, at first for $n = 2$) is slightly weaker, because next to $[K, L] = 0$ we shall also assume $[K \wedge L, X] = 0$ to obtain the same conclusion $[K, L, X] = 0$. This will give us "H-centrality" implies "centrality" (Theorem 2.8) as in our paper [46]. Thus, (SH) $\Rightarrow$ (CC2) $\Rightarrow$ (CC).

Many other things can be said about these ternary commutators; let us just mention that they are generally not decomposable into iterated binary ones, and refer to [30] for further information.

2. Main result

In this section we prove our main result, Theorem 2.8: (CC2), the Commutator Condition in degree $n = 2$, implies (CC) in all degrees. So (CC) does not explode—in the sense that it would give rise to a new mysterious condition in each dimension separately—but instead stays within bounds, as it is implied by the well-studied condition (SH).

2.1. Degree two. We use the same notation as in Subsection 1.7 for double extensions in a semi-abelian category.

Lemma 2.2. *Let* $(\mathbf{C})$ *be a double extension in a semi-abelian category. Then*

$$[K, L] \vee [K \wedge L, X]$$

is normal in X*, and* $[K, L] \vee [K \wedge L, X] = [K, L]^{\mathrm{Huq}} \vee [K \wedge L, X]^{\mathrm{Huq}}$.

Proof: This follows form the fact that $[K, L]^{\mathrm{Huq}} \vee [K \wedge L, X]$ is normal in X while $[K, L]^{\mathrm{Huq}} = [K, L] \vee [[K, L], X]$ and $[K, L] \leqslant K \wedge L$. The second statement is now obvious. ∎

When in $\mathcal{A}$ (SH) holds, this implies that the normalisation of

$$[R, S]^{\mathrm{S}} \vee [R \wedge S, \nabla_X]^{\mathrm{S}}$$

is $[K, L] \vee [K \wedge L, X]$. Hence the centralisation of $(\mathbf{C})$ is its quotient

$$\begin{tikzcd} \frac{X}{[K,L]\vee[K\wedge L,X]} \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & Z. \end{tikzcd}$$

Recall that a **double presentation** of an object Z is a double extension such as $(\mathbf{C})$ in which the objects X, D and C are (regular epi)-projective.

Theorem 2.3. *Let* $\mathcal{A}$ *be a semi-abelian category with enough projectives and such that* (SH) *holds. Let* Z *be an object in* $\mathcal{A}$ *and* $(\mathbf{C})$ *a double presentation of* Z*. Then*

$$\mathrm{H}_3(Z, \mathsf{ab}) = \frac{K \wedge L \wedge [X, X]}{[K, L] \vee [K \wedge L, X]}.$$

When, moreover, $\mathcal{A}$ *is monadic over* Set*, these homology objects are comonadic Barr–Beck homology objects* [1] *with respect to the canonical comonad on* $\mathcal{A}$*.*

Proof: This follows from the main result of [18]; see also [20]. ∎

2.4. Higher degrees. Our purpose is now to prove that the Commutator Condition for double extensions (CC2) implies the Commutator Condition for all n-fold extensions (CC). Consequently, n-fold extensions are central if and only if they are H-central. We shall assume that a Higgins-style characterisation exists for the $(n-1)$-fold central extensions and prove that such a characterisation is also valid for n-fold central extensions. More precisely, we shall prove that under (CC2), the condition (CC$(n-1)$) implies (CCn).

We begin with a higher-dimensional version of the result above for double extensions which allows us to use either Huq or Higgins commutators in the definition of H-centrality and in (CC). We use the notation from Subsection 1.5.

Lemma 2.5. *Let F be an n-fold extension in a semi-abelian category. Then*

$$\bigvee_{I\subseteq n}\Big[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\backslash I} K_i\Big] = \bigvee_{I\subseteq n}\Big[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\backslash I} K_i\Big]^{\mathrm{Huq}} \qquad (\mathbf{D})$$

(so the join is normal in X).

Proof: We give an argument by induction on n of which Lemma 2.2 is the base step. (So we start at $n = 2$. The case $n = 1$ is of course also valid, even well known; see Subsection 1.8.) Suppose indeed that the claimed equality holds for some number $n-1$, then it will also hold for n: we may cut up ($\mathbf{D}$) into the n distinct equalities

$$\bigvee_{I\subseteq n\backslash\{k\}}\Big[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\backslash(I\cup\{k\})} K_i\Big] = \bigvee_{I\subseteq n\backslash\{k\}}\Big[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\backslash(I\cup\{k\})} K_i\Big]^{\mathrm{Huq}},$$

one for each $k \in n$, each of which holds by the induction hypothesis and the extension property of F—see Subsection 1.3. ∎

Before we prove that (CC2) implies (CCn), so that we can go *up* in dimension, let us first explain how to go *down*.

Proposition 2.6. *For any $n \geqslant 1$, the condition* (CC$(n+1)$) *implies* (CCn).

Proof: An n-fold arrow F is an n-fold extension if and only if the $(n+1)$-fold arrow $F \to 0$ is an $(n+1)$-fold extension. It follows immediately from the definitions that F is central (resp. H-central) precisely when $F \to 0$ is central (resp. H-central). ∎

Lemma 2.7. *If $F\colon X \to Z$ is an n-fold* H*-central extension, then also any of the two projections*

$$\pi_1, \pi_2 \colon \mathrm{R}[F] \to X$$

in its kernel pair are H*-central.*

Proof: We prove that $G = \pi_1$ is H-central. Consider $I \subseteq n$ and write

$$\mathrm{ker} g_i \colon \mathrm{Ker}(g_i) \to G_n$$

for the kernel of $g_i \colon G_n \to G_{n\setminus\{i\}}$. Then g_n is the "top morphism" of the first projection $\pi_1 \colon \mathrm{R}[F] \to X$; similarly, write h_n for the top morphism of $H = \pi_2$. Now $\bigwedge_{i\in I} \mathrm{ker} g_i$ and $\bigwedge_{i\in n\setminus I} \mathrm{ker} g_i$ commute: to see this, we compose them with the morphisms g_n and h_n, which form a jointly monic pair. Composing with g_n makes one of the intersections—the one containing the kernel of g_n—trivial, so already $g_n \bigwedge_{i\in I} \mathrm{ker} g_i$ and $g_n \bigwedge_{i\in n\setminus I} \mathrm{ker} g_i$ commute. On the other hand, the composites $h_n \bigwedge_{i\in I} \mathrm{ker} g_i$ and $h_n \bigwedge_{i\in n\setminus I} \mathrm{ker} g_i$ factor through the intersections $\bigwedge_{i\in I} k_i$ and $\bigwedge_{i\in n\setminus I} k_i$, respectively. These two intersections commute because F is H-central. ■

Theorem 2.8. *Every semi-abelian category with* (CC2) *satisfies the Commutator Condition* (CC).

Proof: We give a proof by induction on n: we show that under (CC2), for all $n \geqslant 3$ the condition (CC($n-1$)) implies (CCn).

Let F be an n-fold H-central extension, i.e., $[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\setminus I} K_i] = 0$ for all $I \subseteq n$. To prove that F is central, we must show that either one of the projections in the kernel pair of F is an n-fold trivial extension. Consider, for $i \in n$, the commutative diagram

$$\begin{tikzcd}
K_n \wedge M_i \arrow[r, "\overline{k_i}"] \arrow[d, "\overline{m_i}"'] \arrow[dr, phantom, "\lrcorner", very near start] & M_i \arrow[r, shift left=2, "\pi_1^i"] \arrow[r, leftarrow, "e_i" description] \arrow[r, shift right=2, "\pi_2^i"'] \arrow[d, "m_i"] & K_i \arrow[d, "k_i"] & \\
K_n \arrow[r, "\widetilde{k_n}"'] & \mathrm{R}[f_n] \arrow[r, shift left=2, "\pi_1"] \arrow[r, leftarrow, "e" description] \arrow[r, shift right=2, "\pi_2"'] \arrow[d, "R(f_i)"'] & F_n \arrow[r, "f_n"] \arrow[d, "f_i"] & F_{n-1} \arrow[d] \\
& \mathrm{R}[f_{n-1}] \arrow[r, shift left=2, "p_1"] \arrow[r, leftarrow] \arrow[r, shift right=2, "p_2"'] & F_{n\setminus\{i\}} \arrow[r, "f_{n-1}"'] & F_{n\setminus\{i,n-1\}},
\end{tikzcd} \qquad (\mathbf{E})$$

where $\widetilde{k_n}$ is the kernel of π_1, so $\pi_2\widetilde{k_n} = k_n$, while π_1^i, π_2^i, e_i and $R(f_i)$ are the induced morphisms and $M_i = \mathrm{Ker}(R(f_i))$.

By the induction hypothesis ($\mathrm{CC}(n-1)$), the first projection π_1 of the kernel pair of F is a trivial n-fold extension when the naturality square

$$\begin{tikzcd}
\mathrm{R}[f_n] \arrow[r, "\pi_1"] \arrow[d] & F_n \arrow[d] \\
\dfrac{\mathrm{R}[f_n]}{\bigvee_{I\subseteq n-1}\Big[\bigwedge_{i\in I} M_i, \bigwedge_{i\in(n-1)\setminus I} M_i\Big]} \arrow[r] & \dfrac{F_n}{\bigvee_{I\subseteq n-1}\Big[\bigwedge_{i\in I} K_i, \bigwedge_{i\in(n-1)\setminus I} K_i\Big]}
\end{tikzcd}$$

is a pullback. This amounts to proving that

$$\bigvee_{I\subseteq n-1}\Big[\bigwedge_{i\in I} M_i, \bigwedge_{i\in(n-1)\setminus I} M_i\Big] \cong \bigvee_{I\subseteq n-1}\Big[\bigwedge_{i\in I} K_i, \bigwedge_{i\in(n-1)\setminus I} K_i\Big].$$

As subobjects of $\mathrm{R}[f_n]$, we have

$$e\Big(\bigvee_{I\subseteq n-1}\Big[\bigwedge_{i\in I} k_i(K_i), \bigwedge_{i\in(n-1)\setminus I} k_i(K_i)\Big]\Big) \tag{$\mathbf{F}$}$$

$$= \bigvee_{I\subseteq n-1}\Big[\bigwedge_{i\in I} ek_i(K_i), \bigwedge_{i\in(n-1)\setminus I} ek_i(K_i)\Big] \tag{$\mathbf{G}$}$$

$$\leqslant \bigvee_{I\subseteq n-1}\Big[\bigwedge_{i\in I} m_i(M_i), \bigwedge_{i\in(n-1)\setminus I} m_i(M_i)\Big]. \tag{$\mathbf{H}$}$$

To prove the other inclusion, we shall decompose the subobject ($\mathbf{H}$) as a join using Proposition 1.10.

From $M_i = \overline{k_i}(K_n \wedge M_i) \vee e_i(K_i)$, we get

$$\begin{aligned}\bigwedge_{i\in I} m_i(M_i) = \bigwedge_{i\in I} m_i\overline{k_i}(K_n \wedge M_i) \vee \bigwedge_{i\in I} m_ie_i(K_i)&\\ = \bigwedge_{i\in I} \widetilde{k_n}\overline{m_i}(K_n \wedge M_i) \vee \bigwedge_{i\in I} ek_i(K_i)&\end{aligned}$$

for all $\varnothing \neq I \subseteq n-1$ and

$$\bigwedge_{i\in\varnothing} m_i(M_i) = \mathrm{R}[f_n] = \widetilde{k_n}(K_n) \vee e(F_n).$$

For $\varnothing \neq I \subseteq n-1$ we have

$$\Big[\bigwedge_{i\in I} m_i(M_i), \bigwedge_{i\in(n-1)\setminus I} m_i(M_i)\Big]$$
$$= \Big[\bigwedge_{i\in I} \widetilde{k_n}\overline{m_i}(K_n \wedge M_i) \vee \bigwedge_{i\in I} ek_i(K_i), \bigwedge_{i\in(n-1)\setminus I} m_i(M_i)\Big],$$

which decomposes to the join

$$\Big[\bigwedge_{i\in I} \widetilde{k_n}\overline{m_i}(K_n \wedge M_i), \bigwedge_{i\in(n-1)\setminus I} m_i(M_i)\Big] \vee \Big[\bigwedge_{i\in I} ek_i(K_i), \bigwedge_{i\in(n-1)\setminus I} m_i(M_i)\Big]$$
$$\vee \Big[\bigwedge_{i\in I} \widetilde{k_n}\overline{m_i}(K_n \wedge M_i), \bigwedge_{i\in I} ek_i(K_i), \bigwedge_{i\in(n-1)\setminus I} m_i(M_i)\Big]. \quad (\mathbf{I})$$

The first term of (**I**) vanishes by Lemma 2.7 and the assumption that F is H-central. In fact, the intersection $\bigwedge_{i\in I} \widetilde{k_n}\overline{m_i}(K_n \wedge M_i)$ may be written as $\widetilde{k_n}(K_n) \wedge \bigwedge_{i\in I} m_i(M_i)$, i.e., an intersection of kernels of the initial arrows of the first projection of $\mathrm{R}[F]$. Consequently, the commutator

$$\Big[\bigwedge_{i\in n-1} \widetilde{k_n}\overline{m_i}(K_n \wedge M_i), \mathrm{R}[f_n]\Big]$$

vanishes as it one of the commutators which express the H-centrality of the first projection of the kernel pair $\mathrm{R}[F]$. So by (CC2) also the last term in (**I**) is trivial, because it is smaller than

$$\Big[\bigwedge_{i\in I} \widetilde{k_n}\overline{m_i}(K_n \wedge M_i), \mathrm{R}[f_n], \bigwedge_{i\in(n-1)\setminus I} m_i(M_i)\Big] = 0$$

as explained in Subsection 1.11.

We now further decompose the second term of (**I**)

$$\Big[\bigwedge_{i\in I} ek_i(K_i), \bigwedge_{i\in(n-1)\setminus I} \widetilde{k_n}\overline{m_i}(K_n \wedge M_i) \vee \bigwedge_{i\in(n-1)\setminus I} ek_i(K_i)\Big]$$

into the join

$$\Big[\bigwedge_{i\in I} ek_i(K_i), \bigwedge_{i\in(n-1)\setminus I} \widetilde{k_n}\overline{m_i}(K_n \wedge M_i)\Big] \vee \Big[\bigwedge_{i\in I} ek_i(K_i), \bigwedge_{i\in(n-1)\setminus I} ek_i(K_i)\Big]$$
$$\vee \Big[\bigwedge_{i\in I} ek_i(K_i), \bigwedge_{i\in(n-1)\setminus I} \widetilde{k_n}\overline{m_i}(K_n \wedge M_i), \bigwedge_{i\in(n-1)\setminus I} ek_i(K_i)\Big]. \quad (\mathbf{J})$$

The first term of (**J**) vanishes, as the even larger subobjects

$$\bigwedge_{i\in I} m_i(M_i) \qquad \text{and} \qquad \bigwedge_{i\in(n-1)\setminus I} \widetilde{k_n}\overline{m_i}(K_n \wedge M_i)$$

commute, again by Lemma 2.7 and the assumption that F is H-central. By (CC2) also the last term in (**J**) is trivial, because it is smaller than

$$\Big[\bigwedge_{i\in I} m_i(M_i), \bigwedge_{i\in(n-1)\setminus I} \widetilde{k_n}\overline{m_i}(K_n \wedge M_i), \mathrm{R}[f_n]\Big] = 0.$$

So all commutators determined by $\varnothing \neq I \subseteq n-1$ in the join (**H**) are also in the join (**G**). As $I = \varnothing$ and $I = n-1$ give rise to the same commutator, this finally tells us that the join (**H**) is smaller than the join (**G**)—which finishes the proof that when F is H-central, then it is central.

The other implication was *almost* proved in [46]; the only difference between the result there and the present claim is that there, H-centrality was characterised in terms of Huq commutators, rather than Higgins commutators as in (**B**). But the two concepts are equivalent by Lemma 2.5. ■

Corollary 2.9. *Every semi-abelian category with* (SH) *satisfies the Commutator Condition* (CC). ■

Corollary 2.10. *Every semi-abelian category with* (CCn) *for some $n \geqslant 2$ satisfies the Commutator Condition* (CC). ■

This immediately gives us explicit versions of Hopf formulae obtained in [18, 20]. Recall that an n-fold extension of an object Z is an **n-fold presentation** of Z when all its objects, but its terminal object Z, are projective.

Theorem 2.11. *Let $\mathcal{A}$ be a semi-abelian category with enough projectives such that* (SH) *holds. Let Z be an object in $\mathcal{A}$ and F an n-fold presentation of Z. Then*

$$\mathrm{H}_{n+1}(Z, \mathsf{ab}) = \frac{[F_n, F_n] \wedge \bigwedge_{i\in n} K_i}{\bigvee_{I\subseteq n}\Big[\bigwedge_{i\in I} K_i, \bigwedge_{i\in n\setminus I} K_i\Big]}.$$

When, moreover, $\mathcal{A}$ is monadic over Set, *these homology objects are comonadic Barr–Beck homology objects with respect to the canonical comonad on $\mathcal{A}$.* ■

3. A counterexample

We prove that not every semi-abelian category needs to satisfy the Commutator Condition (CC): for instance, the category of loops and loop homomorphisms Loop doesn't. This is a refinement of the result from [30] saying that the category Loop does not satisfy (SH). Incidentally, our counterexample also works in the category of *commutative* loops CLoop, so it is a new example of a semi-abelian category where (SH) is not valid.

Let us recall a few basic notions. A **loop** is a quasigroup with a neutral element: an algebraic structure $(X, \cdot, \backslash, /, 1)$ that satisfies $x \cdot 1 = x = 1 \cdot x$ and

$$y = x \cdot (x\backslash y) \qquad\qquad y = x\backslash(x \cdot y)$$
$$x = (x/y) \cdot y \qquad\qquad x = (x \cdot y)/y.$$

We also write xy for the product $x \cdot y$. An associative loop is the same thing as a group. A **commutative** loop has $xy = yx$ for all x, $y \in X$—which doesn't yet imply that X is *abelian* in Loop: X carries an internal abelian group structure precisely when it is an abelian *group*, when it is commutative and associative. The defect in being associative is measured by means of the **associator elements**

$$[\![x, y, z]\!] = (xy \cdot z)/(x \cdot yz).$$

The associator elements are in the ternary commutator $[X, X, X]$ of X since they are expressions in three variables which vanish as soon as one of the variables is equal to 1.

We take X to be the non-associative commutative loop of which the multiplication table is Table 1. (Any Latin square determines a quasigroup, and a loop is a quasigroup with unit. It is commutative as the multiplication table is symmetric.) We take I to be its normal subloop $\{1, -1, i, -i\}$, as indicated in the multiplication table of X, and H the normal subloop $\{1, -1, h, -h\}$ of X. The normal subloop

$$Y = \{1, -1, i, -i, g, -g, h, -h\} = H \vee I$$

of X is actually an abelian group (isomorphic to the cube C_2^3 of the cyclic group of order two C_2), so that I and H commute in Y, hence in X. Furthermore, $A = H \wedge I = \{1, -1\}$ is central in X, as the multiplication on X restricts to a loop homomorphism $\cdot\colon A \times X \to X$. Hence we have the the equalities

$$[H, I] = 0 = [H \wedge I, X].$$

1	-1	i	$-i$	g	$-g$	h	$-h$	j	$-j$	k	$-k$	l	$-l$	m	$-m$
-1	1	$-i$	i	$-g$	g	$-h$	h	$-j$	j	$-k$	k	$-l$	l	$-m$	m
i	$-i$	1	-1	h	$-h$	g	$-g$	k	$-k$	j	$-j$	$-m$	m	l	$-l$
$-i$	i	-1	1	$-h$	h	$-g$	g	$-k$	k	$-j$	j	m	$-m$	$-l$	l
g	$-g$	h	$-h$	1	-1	i	$-i$	l	$-l$	m	$-m$	j	$-j$	k	$-k$
$-g$	g	$-h$	h	-1	1	$-i$	i	$-l$	l	$-m$	m	$-j$	j	$-k$	k
h	$-h$	g	$-g$	i	$-i$	1	-1	m	$-m$	l	$-l$	k	$-k$	j	$-j$
$-h$	h	$-g$	g	$-i$	i	-1	1	$-m$	m	$-l$	l	$-k$	k	$-j$	j
j	$-j$	k	$-k$	l	$-l$	m	$-m$	1	-1	i	$-i$	g	$-g$	h	$-h$
$-j$	j	$-k$	k	$-l$	l	$-m$	m	-1	1	$-i$	i	$-g$	g	$-h$	h
k	$-k$	j	$-j$	m	$-m$	l	$-l$	i	$-i$	1	-1	h	$-h$	g	$-g$
$-k$	k	$-j$	j	$-m$	m	$-l$	l	$-i$	i	-1	1	$-h$	h	$-g$	g
l	$-l$	$-m$	m	j	$-j$	k	$-k$	g	$-g$	h	$-h$	1	-1	i	$-i$
$-l$	l	m	$-m$	$-j$	j	$-k$	k	$-g$	g	$-h$	h	-1	1	$-i$	i
m	$-m$	l	$-l$	k	$-k$	j	$-j$	h	$-h$	g	$-g$	i	$-i$	1	-1
$-m$	m	$-l$	l	$-k$	k	$-j$	j	$-h$	h	$-g$	g	$-i$	i	-1	1

TABLE 1. The loop X with its normal subloops I and Y

On the other hand, the commutator $[H, I, X]$ is non-trivial, as $hi \cdot l = gl = j$ while $h \cdot il = h(-m) = -j$, so that

$$1 \neq (hi \cdot l)/(h \cdot il) = [\![h, i, l]\!] \in [H, I, X].$$

This violates the Commutator Condition for $n = 2$ (Subsection 1.11), since H and $I \lhd X$ determine a double extension (of $C_2 \cong X/Y$) which is H-central but not central.

A direct proof without ternary commutators goes as follows. Let R and S be the respective denormalisations of H and I. Then $(x, y) \in R$ (resp. $\in S$) when $xH = yH$ (resp. $xI = yI$). The Smith commutator $[R, S]^{\mathrm{S}}$ is the kernel pair of t in the colimit diagram

$$\begin{tikzcd} & R \arrow[dl, "\langle 1_R, \Delta_S r_2\rangle"'] \arrow[d, dotted] \arrow[dr, "r_1"] & \\ R \times_X S \arrow[r, dotted] & T & X \arrow[l, dotted, "t"'] \\ & S \arrow[ul, "\langle \Delta_R s_1, 1_S\rangle"] \arrow[u, dotted] \arrow[ur, "s_2"'] & \end{tikzcd}$$

(Subsection 1.1). We claim that t maps $[\![h,i,l]\!] \in X$ to 1, so that the couple $([\![h,i,l]\!],1)$ is a non-trivial element of $[R,S]^{\mathrm{S}}$. This violates the characterisation of double central extensions recalled in Subsection 1.7.

The above colimit may be computed as the pushout

$$\begin{tikzcd} R+S \arrow[r, "\left\langle\begin{smallmatrix} r_1 \\ s_2 \end{smallmatrix}\right\rangle"] \arrow[d, "\left\langle\begin{smallmatrix} 1_R & \Delta_S r_2 \\ \Delta_R s_1 & 1_S \end{smallmatrix}\right\rangle"'] & X \arrow[d, dotted, "t"] \\ R\times_X S \arrow[r, dotted] & T. \end{tikzcd}$$

Certainly the formal associator

$$[\![(h,1),(1,i),(l,l)]\!]$$

in $R+S$, where (l,l) is considered as belonging to R, is mapped to $[\![h,i,l]\!]$ in X. On the other hand, the arrow $\left\langle\begin{smallmatrix} 1_R & \Delta_S r_2 \\ \Delta_R s_1 & 1_S \end{smallmatrix}\right\rangle$ sends this associator to the element

$$[\![(h,1,1),(1,1,i),(l,l,l)]\!]$$

of the pullback $R\times_X S$. This element is equal to

$$([\![h,1,l]\!],[\![1,1,l]\!],[\![1,i,l]\!]) = (1,1,1),$$

because any associator containing 1 vanishes, so that indeed

$$(1,1) \neq ([\![h,i,l]\!],1) \in [R,S]^{\mathrm{S}}.$$

4. Further remarks

4.1. Modules. The Smith is Huq Condition implies that every action of an object on an abelian object A is a module (i.e., an abelian group in the slice category $\mathcal{A}/Z$): given any split epimorphism $f\colon X\to Z$, the equality $[\mathrm{R}[f],\mathrm{R}[f]]^{\mathrm{S}} = \Delta_X$ follows from

$$[\mathrm{Ker}(f),\mathrm{Ker}(f)]^{\mathrm{Huq}} = [A,A]^{\mathrm{Huq}} = 0.$$

All known counterexamples against (SH), in digroups [3, 7] or in loops [30], were examples of an action of an abelian object which is not a module, so where $[\mathrm{R}[f],\mathrm{R}[f]]^{\mathrm{S}}$ is bigger than Δ_X. Under (CC) the situation is different: considering f as a double extension

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Z \arrow[d, equal] \\ Z \arrow[r, equal] & Z, \end{tikzcd}$$

in order to make use of (CC2) we would have to assume the stronger condition

$$[A \wedge A, X]^{\mathrm{Huq}} = [A, X]^{\mathrm{Huq}} = 0;$$

by (CC1) this already implies the stronger $[\mathrm{R}[f], \nabla_X]^{\mathrm{S}} = \Delta_X$, which defeats the purpose.

4.2. Relative commutators. Many of the examples obtained in [20] through explicit calculations now become instances of Theorem 2.11, as do several other examples considered in the literature: groups vs. abelian groups, rings vs. zero rings, and Lie algebras vs. vector spaces, for instance. Nevertheless, there is still a whole class of examples missing, namely all those where the homology is not *absolute*, i.e., the functor which is being derived is not the abelianisation functor. Higher Hopf formulae exist e.g. for precrossed modules vs. crossed modules, groups vs. groups of a certain nilpotency or solvability class [20], loops vs. groups (in low dimensions) [22], compact groups vs. profinite groups [19] and Leibniz n-algebras vs. Lie n-algebras [13]. We hope to extend the results of the present paper to the relative case so that also these examples may become instances of the general theory. This problem is closely related to the results of [17, 22, 23], as it depends on a suitable notion of *relative commutator*. In the article [19] a solution is given for reflectors which are *protoadditive*.

4.3. Equivalence of (CC) and (SH). Another question we did not answer now is whether or not (CC) implies (SH). The problem is that already against (SH) alone the counterexamples are exotic, and now we would have to find a category which does not have (SH) but does satisfy (CC).

4.4. Exact Mal'tsev categories. Under (CC), higher central extensions in a semi-abelian category may be characterised in terms of binary Huq commutators. So under (SH), this characterisation may be reformulated using binary Smith commutators as follows.

Corollary 4.5. *Given an n-fold extension F in a semi-abelian category with* (SH)*,*

$$\bigvee_{I\subseteq n}\Big[\bigwedge_{i\in I}\mathrm{R}[f_i], \bigwedge_{i\in n\setminus I}\mathrm{R}[f_i]\Big]^{\mathrm{S}} = \Delta_{F_n}$$

if and only if F is central. ∎

We know, however, that when $n = 2$ this characterisation of double central extensions is valid in all exact Mal'tsev categories: in [21], the proof given in the article [45] in a semi-abelian context was replaced by a much more efficient one which avoids the use of the Huq commutator and doesn't need that the category is pointed nor that it is protomodular but works in the exact Mal'tsev context. This naturally leads to the following conjecture:

Conjecture 4.6. *The above characterisation of n-fold central extensions is also valid in exact Mal'tsev categories.*

The difficulty here may be better understood when observing the difference in underlying geometry between the vanishing of the Smith commutators that occur in Corollary 4.5 on the one hand, and the characterisation of higher centrality given in [46]—which is also geometrical in nature, and makes sense in the exact Mal'tsev context—on the other. One could argue that this latter characterisation of higher centrality leads to a "higher-order Smith commutator". This would be just one *n-ary* Smith commutator involving higher-order diamonds, instead of a join of several *binary* Smith commutators, each of which only gives rise to a fragment of the geometry of those higher-order diamonds. The question now essentially becomes whether the characterisation of *double* central extensions in terms of *binary* Smith commutators is a coincidence typical for degree two or not.

4.7. Higgins instead of Smith. Even when a semi-abelian category does not have the property (CC), the double central extensions in it may still be characterised in terms of Higgins commutators. The only problem is that *binary* commutators will not suffice, but rather a *ternary* commutator is needed: the result in [30] says that ($\mathbf{C}$) is central when the join $[K \wedge L, X] \vee [K, L] \vee [K, L, X]$ vanishes. An unpublished result by Tomas Everaert, on which the proof of Theorem 2.8 was based, gives the higher-dimensional analogue. It says that an n-fold extension F in a semi-abelian category is central if and only if the join of higher-order Higgins commutators [29]

$$\bigvee_{I_0 \cup \cdots \cup I_k = I \subseteq n} \Big[\bigwedge_{i \in I_0} \mathrm{Ker}(f_i), \ldots, \bigwedge_{i \in I_k} \mathrm{Ker}(f_i)\Big]$$

vanishes. The size of the commutators stays bounded, and the join finite, as a commutator in which an entry is repeated is smaller than the commutator with the repetition removed.

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