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PROPER MAPS FOR LAX ALGEBRAS AND THE KURATOWSKI-MRÓWKA THEOREM

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Abstract: The characterization of stably closed maps of topological spaces as the closed maps with compact fibres and the role of the Kuratowski-Mrówka' Theorem in this characterization are being explored in the general context of $(\mathbb{T}, V)$-algebras, for a quantale V and a **Set**-monad $\mathbb{T}$ with a lax extension to V-relations. The general results are being applied in standard (topological and metric) and non-standard (labeled graphs) contexts.

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1. Introduction

Bourbaki [2] emphasized the importance of *proper* maps of topological spaces, defined as the *stably closed* continuous maps. Point-set topologists prefer to introduce them as the closed continuous maps with compact fibres and to call them *perfect* ([8]), give or take Hausdorff separation conditions which, however, we will disregard in this paper. The statement that perfect maps are proper generalizes Kuratowski's Theorem which asserts that $X \to 1$ is proper when X is compact. Mrówka [16] showed that compactness of X is not only sufficient but also necessary for propriety of $X \to 1$, which then gives that proper maps are perfect.

Extrapolating from the Manes-Barr presentation (see [1]) of topological spaces as the relational algebras of the ultrafilter monad (induced by the underlying **Set**-functor of compact Hausdorff spaces [15]), in this paper we consider the question of to which extent the equivalence of the notions of proper and perfect may be transferable to the context of $(\mathbb{T}, V)$-algebras, as considered with slight variations in [3, 6, 17, 9] and other papers, where the

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quantale V replaces the two-element chain (so that V-relations replace ordinary relations) and the **Set**-monad $\mathbb{T}$ replaces the ultrafilter monad. In order not to lose the V-categorical intuition [13], we prefer to call $(\mathbb{T}, V)$-algebras and their lax homomorphisms $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors, respectively. With no obvious candidate for a notion of closedness at hand in the general context, we define proper $(\mathbb{T}, V)$-functors as in [4, 11] equationally, as the strict homomorphisms amongst lax, and call an object X compact when $X \to 1$ is proper, with 1 denoting the terminal object. The terminal structure on a singleton set will generally be distinct from its discrete structure, which is being used when forming fibres. Keeping this distinction in mind, with the known and easily-established pullback stability of proper morphisms one obtains that their fibres are proper as maps, and then compact as objects whenever the terminal structure is discrete.

We prove two versions of the proper=perfect paradigm in the general context, using two distinct approaches to a notion of "closed morphism". The first one was already used in [11] in some key examples and relies on assigning to every $(\mathbb{T}, V)$-category structure on X a V-category structure on TX in a functorial manner, such that in the example $V = 2$ and $\mathbb{T}$ the ultrafilter monad, closedness of a continuous map $f : X \to Y$ is equivalently described as propriety of the monotone map $Tf : TX \to TY$. This leads us to the general characterization of proper $(\mathbb{T}, V)$-functors as those f with proper fibres for which Tf is proper (Theorem 3.2), as presented by the second author at CT2011. The second version uses a family of closure operators and works well when V is constructively completely distributive. It requires us, however, to mimic Mrówska's result in the general context, which as in the topological role model relies on the provision of suitable "test objects" (Theorem 5.2). With that at hand, proper $(\mathbb{T}, V)$-functors can be characterized entirely in terms of closure (Theorem 6.1) which, however, has in this general context features not apparent at the level of the role model **Top**, and we illustrate them by non-standard examples that leave the realm of categories considered in [11], like the categories of metric and of topological spaces and their natural hybrid, the category of approach spaces [14].

The authors are indebted to Dirk Hofmann who advised them about his proof of Lemma 7.1 in the cases that the quantale V is the two-element chain or the extended non-negative real half line. The proof given here is an easy adaptation of his argumentation to our more general context.

2. The setting

Throughout the paper V is a cartesian closed, unital, associative and commutative quantale. Hence, V is a frame endowed with an associative and commutative binary operation $\otimes$ which, like the binary meet $\wedge$, preserves arbitrary joins in each variable; in addition, we assume that the top element $\top$ serves as the $\otimes$-neutral element.

We consider a monad $\mathbb{T} = (T, m, e)$ of the category **Set** and, for simplicity, assume that T is *taut*, so that T preserves inverse images (i.e., pullbacks of monomorphisms along arbitrary maps). In particular then, T preserves monomorphisms, and for $i : A \hookrightarrow X$ and $\mathfrak{x} \in TX$ we will often write $\mathfrak{x} \in TA$ when $\mathfrak{x} \in Ti(TA)$.

Furthermore, we assume that $\mathbb{T}$ comes with a fixed *lax extension* $\hat{T}$ to the category V-**Rel** of V-relations, that is: to the category with objects sets and morphisms $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ given by functions $r : X \times Y \to V$ whose composite with $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ is defined by

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$$

for all $x \in X$, $z \in Z$. Note that V-**Rel** has an involution $r \mapsto r^\circ : Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$ with $r^\circ(y, x) = r(x, y)$, and that every map $f : X \to Y$ may be considered a V-relation $f_\circ : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ with $f_\circ(x, y) = \top$ when $f(x) = y$, and $f_\circ(x, y) = \bot$ (the bottom element) otherwise. Unless $|V| = 1$ there is no danger in identifying $f_\circ$ with f; its converse, $f^\circ : Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$, serves as the right adjoint to $f_\circ$ in the 2-category V-**Rel**, the 2-cells of which are given by pointwise order: $r \leq r'$ if and only if $r(x, y) \leq r'(x, y)$ for all $x \in X$, $y \in Y$.

We must clarify what we mean by *lax extension*: $\hat{T}$ assigns to every V-relation r the V-relation $\hat{T}r : TX \relbar\joinrel\mapstochar\joinrel\rightarrow TY$ subject to the axioms (A)-(F) below.

(A) $Tf \leq \hat{T}f$, $(Tf)^\circ \leq \hat{T}(f^\circ)$,
(B) $r \leq r' \;\Rightarrow\; \hat{T}r \leq \hat{T}r'$,
(C) $\hat{T}s \cdot \hat{T}r \leq \hat{T}(s \cdot r)$,
(D) $\hat{T}\hat{T}r \cdot m_X^\circ = m_Y^\circ \cdot \hat{T}r$,
(E) $r \cdot e_X^\circ \leq e_Y^\circ \cdot \hat{T}r$,

for all $r, r' : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ and $f : X \to Y$. (A)-(E) mean equivalently that $\hat{T} : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ is a lax functor, $m^\circ : \hat{T} \to \hat{T}\hat{T}$ a natural transformation, and $e^\circ : \hat{T} \to 1$ a lax natural transformation, extending T laxly (in the sense of (A)). They imply in particular the identities

$$\hat{T}(s\cdot f)=\hat{T}s\cdot Tf,\ \hat{T}(g^\circ\cdot r)=(Tg)^\circ\cdot\hat{T}r,\ \hat{T}1_X=\hat{T}(e_X^\circ)\cdot m_X^\circ$$

(with $g:Z\to Y$), see [17, 18]. We require in addition:

(F) $\hat{T}(h\cdot r)=Th\cdot\hat{T}r$

(with $h:Y\to Z$). We do *not* assume a priori that $\hat{T}$ is *flat*, i.e., that $\hat{T}1_X=1_{TX}$, which forces the inequalities (A) to become identities.

A $(\mathbb{T},V)$-*category* (X,a) is a set X with a V-relation $a:TX\mathrel{-\!\!\!\mapsto} X$ with $1_X\le a\cdot e_X$ and $a\cdot\hat{T}a\le a\cdot m_X$. A $(\mathbb{T},V)$-*functor* $f:(X,a)\to(Y,b)$ is a map $f:X\to Y$ with $f\cdot a\le b\cdot Tf$. This defines the (ordinary) category $(\mathbb{T},V)$-**Cat**. For $\mathbb{T}=\mathbb{I}$ the identity monad (identically extended to V-**Rel**), $(\mathbb{T},V)$-**Cat** is the category V-**Cat**, i.e. the category of (small) categories enriched over the monoidal-closed category V.

The forgetful functor

$$(\mathbb{T},V)\text{-}\mathbf{Cat}\longrightarrow\mathbf{Set},\qquad (X,a)\longmapsto X,\qquad f\longmapsto f$$

is topological, hence $(\mathbb{T},V)$-**Cat** is complete and cocomplete. In particular, $(1,\top)$, with $1=\{*\}$ and $\top(\mathfrak{w},*)=\top$ for every $\mathfrak{w}\in T1$, is the terminal object, and the structure d on the pullback of $f:(X,a)\to(Z,c)$ and $g:(Y,b)\to(Z,c)$

$$\begin{tikzcd} X\times_Z Y \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

is given by

$$d(\mathfrak{w},(x,y))=a(Tp(\mathfrak{w}),x)\wedge b(Tq(\mathfrak{w}),y),$$

for any $\mathfrak{w}\in T(X\times_Z Y)$, $(x,y)\in X\times_Z Y$. The left adjoint to the forgetful functor assigns to each set X the *discrete structure*

$$1_X^\sharp=e_X^\circ\cdot\hat{T}1_X.$$

The monad $\mathbb{T}$ may be extended to become a monad of V-**Cat** which we again denote by $\mathbb{T}=(T,m,e)$: for a V-category (X,a_0), let $T(X,a_0)=(TX,\hat{T}a_0)$. There is a comparison functor

$$K:(V\text{-}\mathbf{Cat})^{\mathbb{T}}\longrightarrow(\mathbb{T},V)\text{-}\mathbf{Cat}$$

which commutes with the underlying-set functors; it sends $(X, a_0 : X \relbar\joinrel\mapstochar\joinrel\rightarrow X, \alpha : TX \to X)$ to $(X, a_0 \cdot \alpha : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X)$ (see [19]). It is less trivial and requires the full extent of hypothesis (D) to show that K has a left adjoint, which sends a $(\mathbb{T}, V)$-category (X, a) to $(TX, \hat{a}, m_X)$ and a $(\mathbb{T}, V)$-functor f to Tf, where

$$\hat{a} := \hat{T}a \cdot m_X^\circ$$

(see [10]). We will make use of the composite of this left adjoint with the forgetful functor $(V\text{-}\mathbf{Cat})^{\mathbb{T}} \to V\text{-}\mathbf{Cat}$:

$$(\mathbb{T}, V)\text{-}\mathbf{Cat} \longrightarrow V\text{-}\mathbf{Cat}, \qquad (X, a) \longmapsto (TX, \hat{a}), \qquad f \longmapsto Tf.$$

Examples 2.1. 1. For $V = 2 = \{\text{false} \leq \text{true}\}$, with $\otimes = \&$, an $(\mathbb{I}, 2)$-category (X, a) is a set X equipped with a pre-order, that is a relation $\leq$ on X with

$$x \leq x, \ \ (x \leq y \,\&\, y \leq z) \ \Rightarrow \ x \leq z,$$

for all $x, y, z \in X$ (no anti-symmetry assumed), while $(\mathbb{I}, 2)$-functors are exactly monotone maps. We write **Ord** for $(\mathbb{I}, 2)$-**Cat**=2-**Cat**.

If $V = [0, \infty]$ is the real half-line, ordered by the relation $\geq$, and $\otimes = +$ (with $v + \infty = \infty$ for every $v \in [0, \infty]$), then an $(\mathbb{I}, [0, \infty])$-category (X, a) is a set X equipped with a (generalized) metric a, that is a map $a : X \times X \to [0, \infty]$ such that

$$0 \geq a(x, x), \ \ a(x, y) + a(y, z) \geq a(x, z),$$

for all $x, y, z \in X$, and $(\mathbb{I}, [0, \infty])$-functors are non-expansive maps ([13]). We write **Met** for $(\mathbb{I}, [0, \infty])$-**Cat** $= [0, \infty]$-**Cat**.

2. Let $V = 2$ and $\mathbb{P} = (P, m, e)$ be the power-set monad in **Set**, extended to **Rel** by

$$A(\hat{P}r)B \ \Leftrightarrow \ \forall x \in A \ \exists y \in B \ : \ x \, r \, y,$$

for $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $A \subseteq X$ and $B \subseteq Y$. (Note that $\hat{P}$ is a non-flat extension of P.) Then, as shown in [17], $(\mathbb{P}, 2)$-**Cat** is isomorphic to **Ord**. In particular, every ordered set $(X, \leq)$ defines a $(\mathbb{P}, 2)$-category $(X, \preceq)$ via

$$A \preceq y \ :\Leftrightarrow \ \forall x \in A \ (x \leq y),$$

and vice-versa.

For every $(X,a) \in (\mathbb{P},2)$-**Cat**, $\hat{a} : PX \to\!\!\!\!\!+\; PX$ is defined by

$$\begin{aligned} A\,\hat{a}\,B &\Leftrightarrow \exists \mathcal{A} \in PX \;:\; m_X(\mathcal{A}) = A \;\&\; \mathcal{A}\,(\hat{P}a)\,B \\ &\Leftrightarrow \exists \mathcal{A} \in PX \;:\; \bigcup \mathcal{A} = A \;\&\; \forall A' \in \mathcal{A}\; \exists y \in B \;:\; A' \preceq y \\ &\Leftrightarrow \forall x \in A\; \exists y \in B \;:\; x \leq y. \end{aligned}$$

3. Let $V = 2$ and $\mathbb{F} = (F, m, e)$ be the filter monad on **Set**, extended to **Rel** by putting

$$\mathfrak{x}\,(\hat{F}r)\,\mathfrak{y} \;:\Leftrightarrow\; \forall B \in \mathfrak{y}\;\; \exists A \in \mathfrak{x}\;\; \forall x \in A\;\; \exists y \in Y \;:\; x\,r\,y,$$

   for a relation $r : X \to\!\!\!\!\!+\; Y$, $\mathfrak{x} \in FX$, $\mathfrak{y} \in FY$. As shown in [17], $(\mathbb{F},2)$-**Cat** is isomorphic to **Top**.
4. When restricted to ultrafilters, $\hat{F}$ gives the lax extension $\hat{U}$ of the ultrafilter **Set**-monad $\mathbb{U} = (U, m, e)$ to **Rel** which may be described by:

$$\mathfrak{x}\,(\hat{U}r)\,\mathfrak{y} \;\Leftrightarrow\; \forall A \in \mathfrak{x},\; B \in \mathfrak{y}\; \exists x \in A,\; y \in B \;:\; x\,r\,y,$$

   for a relation $r : X \to\!\!\!\!\!+\; Y$ and $\mathfrak{x} \in UX$, $\mathfrak{y} \in UY$. As shown by Barr [1], the category $(\mathbb{U},2)$-**Cat** is isomorphic to the category **Top** of topological spaces and continuous maps (see [3, 6] for details).

   If (X,a) is an $(\mathbb{U},2)$-category, then the ordered set $(UX, \hat{a})$ has the following structure:

$$\begin{aligned} \mathfrak{x}\,\hat{a}\,\mathfrak{y} &\Leftrightarrow \forall A \subseteq X\; A \text{ closed } (A \in \mathfrak{y} \;\Rightarrow\; A \in \mathfrak{x}) \\ &\Leftrightarrow \forall A \subseteq X\; A \text{ open } (A \in \mathfrak{x} \;\Rightarrow\; A \in \mathfrak{y}), \end{aligned}$$

   for all $\mathfrak{x}, \mathfrak{y} \in UX$.

   In fact, for any V, $\mathbb{U}$ has a flat extension to V-**Rel** given by:

$$(\hat{U}r)(\mathfrak{x},\mathfrak{y}) := \bigwedge_{A \in \mathfrak{x},\, B \in \mathfrak{y}} \; \bigvee_{x \in A,\, y \in B} r(x,y),$$

   for a relation $r : X \to\!\!\!\!\!+\; Y$, $\mathfrak{x} \in UX$, $\mathfrak{y} \in UY$.

   When $V = [0,\infty]$ is the real half-line, it was shown in [3] that $(\mathbb{U},[0,\infty])$-**Cat** is isomorphic to the category **App** of approach spaces and non-expansive maps [14]. The structure $\hat{a}$, for a given approach space (X,a), will be studied in Section 7.
5. Consider now the free-monoid monad $\mathbb{L} = (L, m, e)$ on **Set**, (flatly) extended to **Rel** by putting

$$\langle x_1, \cdots, x_n\rangle\,(\hat{L}r)\,\langle y_1, \cdots, y_m\rangle \;\Leftrightarrow\; n = m \;\&\; x_i\,r\,y_i, \text{ for all } i = 1, \cdots, n,$$

for $r : X \to\!\!\!\!\!\!\!\!\;\;\;\;\;\;\mapsto Y$, $\langle x_1, \cdots, x_n\rangle \in LX$, $\langle y_1, \cdots, y_m\rangle \in LY$. Then an $(\mathbb{L}, 2)$-category (X, a) is a *multi-ordered set*, that is, the relation $a : LX {\to\!\!\!\!\!\!\!\!\;\;\;\;\;\;\mapsto} X$ is such that

$$\langle x\rangle\, a\, x,$$

$$\langle\langle x_1^1, \cdots, x_{n_1}^1\rangle, \cdots, \langle x_1^l, \cdots, x_{n_l}^l\rangle\rangle\, (\hat{L}a)\, \langle y_1, \cdots, y_m\rangle\, a\, z \;\Rightarrow\; \langle x_1^1, \cdots, x_{n_l}^l\rangle\, a\, z.$$

6. For a monoid (H, μ, η), we consider the **Set**-monad $\mathbb{H} = (H \times -, m, e)$, with $m_X = \mu \times 1_X$ and $e_X = \langle \eta, 1_X\rangle$. $\mathbb{H}$ has a flat extension to **Rel** given by

$$(\alpha, x)\, (\hat{H}r)\, (\beta, y) \;\Leftrightarrow\; \alpha = \beta \;\&\; x\, r\, y,$$

for any $r : X {\to\!\!\!\!\!\!\!\!\;\;\;\;\;\;\mapsto} Y$, $(\alpha, x) \in H \times X$ and $(\beta, y) \in H \times Y$. Writing $x \xrightarrow{\alpha} y$ instead of $(\alpha, x)\, a\, y$ for a relation $a : H \times X {\to\!\!\!\!\!\!\!\!\;\;\;\;\;\;\mapsto} X$, an $(\mathbb{H}, 2)$-category (X, a) can be seen as an *H-labeled graph* such that

$$x \xrightarrow{\eta} x\;, \quad x \xrightarrow{\alpha} y \xrightarrow{\beta} z \;\;\Rightarrow\;\; x \xrightarrow{\alpha\cdot\beta} z\;,$$

for all $x, y, z \in X$ and $\alpha, \beta \in H$. An $(\mathbb{H}, 2)$-functor $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ satisfying the condition:

$$x \xrightarrow{\alpha} y \;\;\Rightarrow\;\; f(x) \xrightarrow{\alpha} f(y)\;.$$

For each H-labeled graph (X, a), the order $\hat{a}$ induced on $H \times X$ by a is given by:

$$(\alpha, x)\, \hat{a}\, (\beta, y) \;\Leftrightarrow\; \exists \gamma \in H\; (\alpha = \beta \cdot \gamma \;\&\; x \xrightarrow{\gamma} y\;).$$

3. Proper $(\mathbb{T}, V)$-functors

A $(\mathbb{T}, V)$-functor $f : (X, a) \to (Y, b)$ is *proper* if $f \cdot a = b \cdot Tf$. In order to be able to talk about fibres of f, we should first clarify that very term. For each $y \in Y$, the assignment $* \mapsto y$ defines a $(\mathbb{T}, V)$-functor $y : (1, 1^\sharp) \to (Y, b)$, where $1^\sharp = e_1^\circ \cdot \hat{T}1_1$ is the discrete structure on $1 = \{*\}$; explicitly, for $\mathfrak{w} \in T1$,

$$1^\sharp(\mathfrak{w}, *) = \hat{T}1_1(\mathfrak{w}, e_1(*)).$$

By *fibre of f on y* we mean the pullback $(f^{-1}y, \tilde{a}) \to (1, 1^\sharp)$ of f along the $(\mathbb{T}, V)$-functor $y : (1, 1^\sharp) \to (Y, b)$. We note that $(f^{-1}y, \tilde{a}) \to (X, a)$ is a monomorphism, but in general not regular, i.e., $\tilde{a}$ does not need to be the restriction of $a : TX \times X \to V$ to $T(f^{-1}y) \times f^{-1}y$:

$$\begin{aligned} \tilde{a}(\mathfrak{x}, x) &= a(\mathfrak{x}, x) \wedge 1^\sharp(T!(\mathfrak{x}), *) \qquad\qquad \text{(where } ! : f^{-1}y \to 1) \\ &= a(\mathfrak{x}, x) \wedge \hat{T}1_X(T!(\mathfrak{x}), e_1(*)), \end{aligned}$$

for every $\mathfrak{x} \in T(f^{-1}y)$ and $x \in f^{-1}y$.

Proper $(\mathbb{T}, V)$-functors have proper fibres, since:

Proposition 3.1 (See [4])**.** *Proper maps are stable under pullback in* $(\mathbb{T}, V)$-**Cat**.

Proof: Consider the pullback diagram of Section 2, with f proper. Then

$$\begin{array}{rcll} b\cdot Tq & = & (b\wedge b)\cdot Tq & \\ & \leq & ((g^\circ\cdot c\cdot Tg)\wedge b)\cdot Tq & \\ & = & (g^\circ\cdot c\cdot Tg\cdot Tq)\wedge b\cdot Tq & \\ & = & (g^\circ\cdot c\cdot Tf\cdot Tp)\wedge b\cdot Tq & (f \text{ proper}) \\ & = & (q\cdot p^\circ\cdot a\cdot Tp)\wedge(b\cdot Tq) & \\ & = & q\cdot((p^\circ\cdot a\cdot Tp)\wedge(q^\circ\cdot b\cdot Tq)) & (V \text{ cartesian closed}) \\ & = & q\cdot d. & \end{array}$$

■

We can now prove a first characterization theorem.

Theorem 3.2. *A* $(\mathbb{T}, V)$*-functor* $f : (X, a) \to (Y, b)$ *is proper if, and only if, all of its fibres are proper, and the* V*-functor* $Tf : (TX, \hat{a}) \to (TY, \hat{b})$ *is proper.*

Proof: If f is proper, from $b \cdot Tf = f \cdot a$ one obtains

$$\begin{array}{rcl} \hat{b}\cdot Tf \;=\; \hat{T}b\cdot m_Y^\circ\cdot Tf & = & \hat{T}b\cdot\hat{T}\hat{T}f\cdot m_X^\circ \qquad \text{(D)} \\ & \leq & \hat{T}(b\cdot\hat{T}f)\cdot m_X^\circ \qquad \text{(C)} \\ & \leq & \hat{T}(b\cdot Tf)\cdot m_X^\circ \qquad (*) \\ & = & \hat{T}(f\cdot a)\cdot m_X^\circ \\ & = & Tf\cdot\hat{T}a\cdot m_X^\circ \;=\; Tf\cdot\hat{a}; \quad \text{(F)} \end{array}$$

here (*) comes about since

$$\begin{array}{rcl} b\cdot\hat{T}f & = & b\cdot\hat{T}1_X\cdot Tf \;=\; b\cdot\hat{T}(e_X^\circ)\cdot m_X^\circ\cdot Tf \;\leq\; b\cdot\hat{T}b\cdot m_X^\circ\cdot Tf \\ & \leq & b\cdot m_X\cdot m_X^\circ\cdot Tf \;\leq\; b\cdot Tf. \end{array}$$

Conversely, assume all fibres of f to be proper in $(\mathbb{T}, V)$-**Cat** and Tf to be proper in V-**Cat**. Since

$$b = b\cdot e_{TY}^\circ\cdot m_Y^\circ \leq e_Y^\circ\cdot\hat{T}b\cdot m_Y^\circ = e_Y^\circ\cdot\hat{b},$$

for all $\mathfrak{x} \in TX$, $y \in Y$ one obtains:

$$\begin{aligned} b \cdot Tf(\mathfrak{x}, y) &= b(Tf(\mathfrak{x}), y) \\ &\leq \hat{b}(Tf(\mathfrak{x}), e_Y(y)) \\ &= \bigvee_{\mathfrak{z} \in (Tf)^{-1}(e_Y(y))} \hat{a}(\mathfrak{x}, \mathfrak{z}) \qquad (Tf \text{ proper}) \\ &= \bigvee_{\mathfrak{z} \in (Tf)^{-1}(e_Y(y))} (\hat{T}a \cdot m_X^\circ)(\mathfrak{x}, \mathfrak{z}) \\ &= \bigvee_{\mathfrak{z} \in (Tf)^{-1}(e_Y(y))} \bigvee_{\mathfrak{X} \in m_X^{-1}\mathfrak{x}} \hat{T}a(\mathfrak{X}, \mathfrak{z}) \otimes \top \end{aligned}$$

Since tautness of T guarantees that the following diagram is a pullback,

$$\begin{tikzcd} T(f^{-1}y) \arrow[r, "T!"] \arrow[d] & T1 \arrow[d, "Ty"] \\ TX \arrow[r, "Tf"] & TY \end{tikzcd}$$

every $\mathfrak{z} \in (Tf)^{-1}(e_Y(y)) = (Tf)^{-1}(Ty(e_1(*)))$ satisfies $\mathfrak{z} \in T(f^{-1}y)$ and $T!(\mathfrak{z}) = e_1(*)$. Using propriety of $(f^{-1}y, \tilde{a}) \to (1, 1^\sharp)$ one gets:

$$\begin{aligned} \bigvee_{Tf(\mathfrak{z}) = e_Y(y)} \bigvee_{\mathfrak{X} \in m_X^{-1}\mathfrak{x}} \hat{T}a(\mathfrak{X}, \mathfrak{z}) \otimes \top &\leq \bigvee_{T!(\mathfrak{z}) = e_1(*)} \bigvee_{\mathfrak{X} \in m_X^{-1}\mathfrak{x}} \hat{T}a(\mathfrak{X}, \mathfrak{z}) \otimes \bigvee_{x \in f^{-1}y} \tilde{a}(\mathfrak{z}, x) \\ &\leq \bigvee_{T!(\mathfrak{z}) = e_1(*)} \bigvee_{\mathfrak{X} \in m_X^{-1}\mathfrak{x}} \bigvee_{x \in f^{-1}y} \hat{T}a(\mathfrak{X}, \mathfrak{z}) \otimes a(\mathfrak{z}, x) \\ &\leq \bigvee_{\mathfrak{X} \in m_X^{-1}\mathfrak{x}} \bigvee_{x \in f^{-1}y} a(m_X(\mathfrak{X}), x) \\ &\leq \bigvee_{x \in f^{-1}y} a(\mathfrak{x}, x) \\ &= (f \cdot a)(\mathfrak{x}, y). \end{aligned}$$

Hence, f is proper. ∎

Next we show that propriety of fibres trivializes whenever the lax natural transformation $e^\circ : \hat{T} \to 1$ is strict.

Proposition 3.3. *If $e^\circ : \hat{T} \to 1$ is a natural transformation, then any $(\mathbb{T}, V)$-functor has proper fibres.*

Proof: For a $(\mathbb{T},V)$-functor $f:(X,a)\to(Y,b)$ and $y\in Y$, we must show that the diagram

$$\begin{tikzcd} T(f^{-1}y) \arrow[r, "T!"] \arrow[d, "\tilde a"', "|" marking] & T1 \arrow[d, "1^\sharp", "|" marking] \\ f^{-1}y \arrow[r, "!"] & 1 \end{tikzcd}$$

commutes, and for that it suffices to consider $\mathfrak{x}\in T(f^{-1}y)$ with $1^\sharp(T!(\mathfrak{x}),*)=\hat{T}1(T!(\mathfrak{x}),e_1(*))>\perp$ and show $\tilde a(\mathfrak{x},*)=\top$. From the commutativity of the diagram

$$\begin{tikzcd} T(f^{-1}y) \arrow[r, "T!"] \arrow[d, "e^\circ"', "|" marking] & T1 \arrow[r, "\hat{T}1", "|" marking] \arrow[d, "e_1^\circ"', "|" marking] & T1 \arrow[d, "e_1^\circ", "|" marking] \\ f^{-1}y \arrow[r, "!"] & 1 \arrow[r, "1"] & 1 \end{tikzcd}$$

we first obtain

$$\perp< e_1^\circ\cdot\hat{T}1\cdot T!(\mathfrak{x},*)=e_1^\circ\cdot T!(\mathfrak{x},*)=!\cdot e^\circ(\mathfrak{x},*)=\bigvee_{x\in f^{-1}y}e^\circ(\mathfrak{x},x)=\top,$$

and then

$$!\cdot\tilde a(\mathfrak{x},*)\geq !\cdot e^\circ(\mathfrak{x},x)=\top.$$

■

Corollary 3.4. *If $e^\circ:\hat{T}\to 1$ is a natural transformation, then a $(\mathbb{T},V)$-functor $f:(X,a)\to(Y,b)$ is proper if, and only if, the V-functor Tf is proper.*

Remark 3.5. This Corollary shows that, in Examples 2.1.4 and 2.1.5, propriety of $(\mathbb{T},V)$-functors can be characterized at the V-categorical level. However our main example, the ultrafilter monad, shows that the hypothesis that e° be a natural transformation is essential for the validity of the Corollary.

The notion of proper morphism leads to a natural notion of compactness: a $(\mathbb{T},V)$-category (X,a) is *compact* whenever $!_X:(X,a)\to(1,\top)$ is proper. When $T1\cong 1$, so that the generator $(1,1^\sharp)$ coincides with the terminal object $(1,\top)$, (X,a) is compact if, and only if, the only fibre of $!_X:(X,a)\to(1,\top)$ is proper. In general we can prove:

Proposition 3.6. *If (X,a) is a compact $(\mathbb{T},V)$-category, then the fibre of the $(\mathbb{T},V)$-functor $!_X:(X,a)\to(1,\top)$ is proper. Furthermore, when the two* *structures* $1^\sharp$ *and* $\top$ *on* 1 *coincide (in particular, when* $T1 \cong 1$*), the converse is true.*

Proof: Let (X, a) be compact and $t : (X, \tilde{a}) \to (1, 1^\sharp)$ be the fibre of $!_X$ along $* \in 1$. Then, for any $\mathfrak{x} \in TX$, since V is a frame,

$$\begin{aligned} t \cdot \tilde{a}(\mathfrak{x}, x) &= \bigvee_{x \in X} \tilde{a}(\mathfrak{x}, x) = \bigvee_{x \in X} (a(\mathfrak{x}, x) \wedge 1^\sharp(T!(\mathfrak{x}), *)) \\ &= \top \wedge 1^\sharp(T!(\mathfrak{x}, *)) = 1^\sharp(Tt(\mathfrak{x}), *), \end{aligned}$$

so that t is proper. ■

Corollary 3.7. *If* $\top$ *is the discrete structure on* 1*, then the following conditions are equivalent, for a* $(\mathbb{T}, V)$*-functor* f*:*

(i) f *is proper;*
(ii) Tf *is proper and* f *has compact fibres.*

Corollary 3.8. *If* $\top$ *is the discrete structure on* 1 *and* e° *a natural transformation, then every* $(\mathbb{T}, V)$*-category is compact.*

We point out that, when the lax extension $\hat{T}$ is flat, $\top = 1^\sharp$ if and only if $T1 \cong 1$, since flatness of $\hat{T}$ gives $1^\sharp(\mathfrak{x}, *) = e_1^\circ(\mathfrak{x}, *) = \top$ only if $\mathfrak{x} = e_1(*)$.

We will be able to demonstrate easily that Corollary 3.7 generalizes the characterization of the proper maps in **Top** as the closed maps with compact fibres once we have interpreted the condition that "Tf be proper" to mean equivalently that "f be closed". To this end, the next section introduces a suitable notion of closedness.

4. Closed $(\mathbb{T}, V)$-functors

Recall that an ordered set X is *constructively completely distributive (ccd)* if there are adjunctions

$$\Downarrow \dashv \bigvee \dashv \downarrow : X \longrightarrow \mathrm{Down}X$$

where $\mathrm{Down}X$ is the lattice of down-closed sets in X, ordered by inclusion. Writing $x \ll a$ instead of $x \in \Downarrow a$, one then has

$$x \ll a \;\Leftrightarrow\; \forall A \subseteq X \; (a \le \bigvee A \;\Rightarrow\; \exists y \in A \;:\; x \le y),$$

and $a = \bigvee \{x \in X \,|\, x \ll a\}$.

Throughout the remainder of the paper, we assume V to be ccd. Fixing $v \in V$, for a $(\mathbb{T}, V)$-category (X, a) and $A \subseteq X$ we let

$$A^{(v)} := \{x \in X \mid \bigvee_{\mathfrak{x} \in TA} a(\mathfrak{x}, x) \geq v\}.$$

For a $(\mathbb{T}, V)$-functor $f : (X, a) \to (Y, b)$ one then has

$$\bigcap_{u \ll v} f(A^{(u)}) \subseteq f(A)^{(v)}.$$

Indeed, if $y \in f(A^{(u)})$ for every $u \ll v$ in V, so that we can write $y = f(x)$ for some $x \in A^{(u)}$, we obtain

$$u \leq \bigvee_{\mathfrak{x} \in TA} a(\mathfrak{x}, x) \leq \bigvee_{\mathfrak{x} \in TA} b(Tf(\mathfrak{x}), y)$$

and, with $Tf(TA) = T(f(A))$ (Choice granted), $v \leq \bigvee_{\mathfrak{y} \in T(f(A))} b(\mathfrak{y}, y)$. We call $f : (X, a) \to (Y, b)$ *closed* if

$$\bigcap_{u \ll v} f(A^{(u)}) = f(A)^{(v)}$$

for all $v \in V$, $A \subseteq X$.

Proposition 4.1. *Every proper $(\mathbb{T}, V)$-functor is closed, and the converse statement holds in V-**Cat** (i.e., when $\mathbb{T} = \mathbb{I}$).*

Proof: Let $f : (X, a) \to (Y, b)$ in $(\mathbb{T}, V)$-**Cat** be proper, and $y \in f(A)^{(v)}$ for $v \in V$, so that

$$v \leq \bigvee_{\mathfrak{y} \in T(f(A))} b(\mathfrak{y}, y) = \bigvee_{\mathfrak{x} \in TA} b(Tf(\mathfrak{x}), y) \leq \bigvee_{\mathfrak{x} \in TA} \bigvee_{x \in f^{-1}y} a(\mathfrak{x}, x).$$

For every $u \ll v$ one then obtains $\mathfrak{x} \in TA$, $x \in f^{-1}y$ with $u \leq a(\mathfrak{x}, x)$, and $y \in \bigcap_{u \ll v} f(A^{(u)})$ follows.

Let now $\mathbb{T} = \mathbb{I}$ and f be closed. For all $x \in X$, $y \in Y$, with $v := b(f(x), y)$ and $A := \{x\}$, from

$$y \in f(A)^{(v)} \subseteq \bigcap_{u \ll v} f(A^{(u)})$$

one obtains for every $u \ll v$ some $z \in f^{-1}y$ with $a(x,z) \geq u$. Consequently,

$$v = b(f(x), y) \leq \bigvee_{z \in f^{-1}y} a(x,z),$$

as desired. ■

Corollary 4.2. *For every $(\mathbb{T}, V)$-functor $f : (X,a) \to (Y,b)$, the V-functor $Tf : (TX, \hat{a}) \to (TY, \hat{b})$ is proper if and only if it is closed.*

5. The Kuratowski-Mrówka Theorem

In order to be able to characterize compactness of a $(\mathbb{T}, V)$-category (X,a) by the condition

(KM) *the projection $X \times Z \to Z$ along any $(\mathbb{T}, V)$-category (Z,c) is closed,*

one needs to provide suitable test objects (Z,c) that can be used in the sufficiency proof of the condition. Hence, using a particular instance of a construction given in [5], for every set X and $\mathfrak{x} \in TX$ we consider the set

$$Z := X \cup \{\omega\} \text{ (for some } \omega \notin X)$$

and the V-relation $c : TZ \to\!\!\!\!\!\!\mapsto Z$ with

$$c(\mathfrak{z}, z) = \begin{cases} \top & \text{if } \mathfrak{z} = e_Z(z) \text{ or } (\mathfrak{z} = \mathfrak{x} \text{ and } z = \omega), \\ \bot & \text{else,} \end{cases}$$

for all $\mathfrak{z} \in TZ$, $z \in Z$, assuming $TX \subseteq TZ$ (and $TTX \subseteq TTZ$) without loss of generality. In order to determine when c will provide Z with the structure of a $(\mathbb{T}, V)$-category, we highlight two convenient properties of the V-relation c:

1. With i denoting the inclusion map $X \hookrightarrow Z$, c satisfies $i^\circ \cdot c = e_X^\circ \cdot (Ti)^\circ$. Consequently, *when $\hat{T}$ is flat, $(Ti)^\circ \cdot \hat{T}c = (Te_X)^\circ \cdot (TTi)^\circ$, in particular*

$$\hat{T}c(\mathfrak{Z}, \mathfrak{z}) > \bot \;\Rightarrow\; \mathfrak{Z} = Te_X(\mathfrak{z}) \tag{1}$$

   for all $\mathfrak{Z} \in TTZ$, $\mathfrak{z} \in TX$.
2. The V-relation c *has finite fibres*, that is:

$$c^\circ(z) = \{\mathfrak{z} \in TZ \mid c(\mathfrak{z}, z) > \bot\}$$

is finite for all $z \in Z$. Consequently, *if the lax natural transformation* $e^\circ : \hat{T} \to 1$ *is finitely strict, so that*

$$\begin{tikzcd} TX \arrow[r, "\hat{T}r", "\shortmid"{marking}] \arrow[d, "e_X^\circ"', "\shortmid"{marking}] & TY \arrow[d, "e_Y^\circ", "\shortmid"{marking}] \\ X \arrow[r, "r", "\shortmid"{marking}] & Y \end{tikzcd}$$

commutes strictly whenever r *has finite fibres, then* $e_Z^\circ \cdot \hat{T}c = c \cdot e_{TZ}^\circ$*, in particular*

$$\hat{T}c(\mathfrak{Z}, e_Z(z)) > \bot \;\Rightarrow\; \exists \mathfrak{w} \in TZ \; (\mathfrak{Z} = e_{TZ}(\mathfrak{w}) \;\&\; c(\mathfrak{w}, z) = \top) \tag{2}$$

for all $\mathfrak{Z} \in TTZ$*,* $z \in Z$*.*

Proposition 5.1. *If* $\hat{T}$ *is flat and* e° *finitely strict, then* (Z, c) *is a* $(\mathbb{T}, V)$*-category.*

Proof: It suffices to show

$$\hat{T}c(\mathfrak{Z}, \mathfrak{z}) \otimes c(\mathfrak{z}, z) > \bot \;\Rightarrow\; c(m_Z(\mathfrak{Z}), z) = \top$$

for all $\mathfrak{Z} \in TTZ$, $\mathfrak{z} \in TZ$, $z \in Z$. The premiss implies $\hat{T}c(\mathfrak{Z}, \mathfrak{z}) > \bot$ and $c(\mathfrak{z}, z) = \top$. If $\mathfrak{z} \in TX$, one obtains $\mathfrak{Z} = Te_X(\mathfrak{z}) = Te_Z(\mathfrak{z})$ from (1) and therefore $c(m_Z(\mathfrak{Z}), z) = c(\mathfrak{z}, z) = \top$. If $\mathfrak{z} \notin TX$, since $c(\mathfrak{z}, z) = \top$, we must have $z = \omega$ and $\mathfrak{z} = e_Z(\omega)$, and (2) gives $\mathfrak{w} \in TZ$ with $\mathfrak{Z} = e_{TZ}(\mathfrak{w})$, and we may conclude again $c(m_Z(\mathfrak{Z}), z) = c(\mathfrak{w}, z) = \top$. ∎

Theorem 5.2. *Let* $\hat{T}$ *be flat and* $e^\circ : \hat{T} \to 1$ *be finitely strict. Then a* $(\mathbb{T}, V)$*-category* (X, a) *is compact if, and only if, (KM) holds.*

Proof: As a pullback of $X \to 1$, the second projection $q : X \times Z \to Z$ is proper for every $(\mathbb{T}, V)$-category (Z, c) when (X, a) is compact, and therefore closed. Conversely, let (X, a) be such that (KM) holds. We must now show

$$\bigvee_{x \in X} a(\mathfrak{x}, x) = \top,$$

for every $\mathfrak{x} \in TX$. For $Z = X \cup \{\omega\}$ and c as defined above, one considers the set $\Delta_X = \{(x, x) \mid x \in X\} \subseteq X \times Z$. Since $q(\Delta_X) = X \subseteq Z$, from $c(\mathfrak{x}, \omega) = \top$ with $\mathfrak{x} \in TX$ one obtains $\omega \in q(\Delta_X)^{(\top)}$, hence

$$\omega \in \bigcap_{u \ll \top} q(\Delta_X^{(u)})$$

by hypothesis. Consequently, for all $u \ll \top$ one can find $x \in X$ with $(x, \omega) \in \Delta_X^{(u)}$, that is (using the product structure of $X \times Z$):

$$\bigvee_{\mathfrak{w} \in T\Delta_X} a(Tp(\mathfrak{w}), x) \wedge c(Tq(\mathfrak{w}), \omega) \geq u,$$

with $p : X \times Z \to X$ the first projection. For any $\mathfrak{w} \in T\Delta_X$ one has $Tq(\mathfrak{w}) \in TX$, so that when (without loss of generality) $u > \bot$, we must have $c(Tq(\mathfrak{w}), \omega) = \top$ with $Tq(\mathfrak{w}) = \mathfrak{x}$, and then also $Tp(\mathfrak{w}) = \mathfrak{x}$. Hence, for all $\bot < u \ll \top$ we have found an $x \in X$ with $a(\mathfrak{x}, x) \geq u$, which implies

$$\bigvee_{x \in X} a(\mathfrak{x}, x) = \top,$$

as desired. ■

6. Characterization of propriety via closure

We now have all the ingredients that allow for a characterization of propriety of a $(\mathbb{T}, V)$-functor $f : (X, a) \to (Y, b)$ in terms of closure, making essential use of the V-functor $Tf : (TX, \hat{a}) \to (TY, \hat{b})$ again. V continues to be constructively completely distributive.

Theorem 6.1. *Let $T1 \cong 1$, $\hat{T}$ be flat and e° be finitely strict. Then the following conditions are equivalent for a $(\mathbb{T}, V)$-functor f:*

(i) *f is proper;*
(ii) *every pullback of f is closed, and Tf is closed;*
(iii) *all fibres of f are compact, and Tf is closed.*

Proof: (i) ⇒ (ii): From Theorem 3.2 and Propositions 3.1 and 4.1. (ii) ⇒ (iii): From Theorem 5.2. (iii) ⇒ (i): From Corollary 3.7. ■

Remark 6.2. (1) Without the hypothesis $T1 \cong 1$, stably-closed maps need not be proper (see 7.2), and proper maps may have non-compact fibres (see 7.6).
(2) In Theorem 6.1 we do not know whether the condition that Tf be closed may be removed from (ii) or be replaced in (iii) by the condition that f be closed.

7. Examples

7.1. V-categories. (See [11].) By Corollary 3.8 every V-category is compact, and by Corollary 4.2 closed V-functors are proper. In case $V = 2$, for a monotone map $f : (X, \leq) \to (Y, \leq)$,

$$\begin{aligned} f \text{ proper } &\Leftrightarrow \ \forall x \in X \quad \uparrow_Y f(x) \subseteq f(\uparrow_X x) \\ &\Leftrightarrow \ \forall A \subseteq X \quad \uparrow_Y f(A) \subseteq f(\uparrow_X A), \end{aligned}$$

with $\uparrow_X A = \{x' \in X \,|\, \exists x \in A \,:\, x \leq x'\}$.

When $V = [0, \infty]$, for a non-expansive map $f : (X, a) \to (Y, b)$,

$$\begin{aligned} f \text{ proper } &\Leftrightarrow \ \forall x \in X,\, y \in Y \ \ b(f(x), y) = \inf\{a(x, x') \,|\, x' \in X,\ f(x') = y\} \\ &\Leftrightarrow \ \forall A \subseteq X,\, y \in Y \ \ b(f(A), y) = \inf\{a(A, x') \,|\, x' \in X,\ f(x') = y\}, \end{aligned}$$

with $a(A, x') = \inf\limits_{x \in A} a(x, x')$.

7.2. Ordered sets as $(\mathbb{P}, 2)$**-categories.** (See [11].) Consider the lax extension $\hat{P}$ of the power-set monad introduced in Example 2.1.2. Then a monotone map $f : (X, \preceq) \to (Y, \preceq)$ is proper if, and only if, for all $A \subseteq X$,

$$\Uparrow_Y f(A) \subseteq f(\Uparrow_X A), \tag{3}$$

where $\Uparrow_X A = \{x \in X \,|\, A \preceq x\}$. Taking $A = \emptyset$ in (3) one sees immediately that proper maps are surjective, while putting $A = \{x\}$ shows that they are $(\mathbb{I}, 2)$-proper. Here closedness of f is equivalent to surjectivity since, for any $A \subseteq X$, $A^{(\top)} = X$. So, stably-closed $(\mathbb{P}, 2)$-functors need not be proper. Note, however, that neither of the hypotheses of Theorem 6.1 is satisfied here.

7.3. Topological spaces as $(\mathbb{F}, 2)$**-categories.** If $\hat{F}$ is the lax (non-flat) extension of F considered in Example 2.1.3, an $(\mathbb{F}, 2)$-functor is proper if, and only if, it is closed (in the ordinary topological sense) and every fibre has a largest element with respect to the underlying order of X (that is, $x \leq x'$ when $e_X(x) \to x'$): see [11]. In particular, proper $(\mathbb{F}, 2)$-functors must be surjective stably-closed maps.

7.4. Topological and approach spaces as $(\mathbb{U}, V)$**-categories.** For an $(\mathbb{U}, V)$-category (X, a) and $\mathfrak{x}, \mathfrak{y} \in UX$ one has, by definition,

$$\hat{a}(\mathfrak{x}, \mathfrak{y}) = \bigvee_{\mathfrak{X} \in m_X^{-1}\mathfrak{x}} \hat{U}a(\mathfrak{X}, \mathfrak{y}) = \bigvee_{\mathfrak{X} \in m_X^{-1}\mathfrak{x}} \ \bigwedge_{\mathcal{A} \in \mathfrak{X},\, B \in \mathfrak{y}} \ \bigvee_{\mathfrak{z} \in \mathcal{A},\, y \in B} a(\mathfrak{z}, y).$$

Using the hypothesis that V is ccd, we first show that $\hat{a}(\mathfrak{x}, \mathfrak{y})$ can be written more conveniently, provided that V also satisfies the property

$$v \leq w \vee z \ \Rightarrow \ v \leq w \text{ or } v \leq z. \tag{$*$}$$

Lemma 7.1. *Under hypothesis* (*), $\hat{a}(\mathfrak{x},\mathfrak{y}) = \bigvee\{u \in V \mid \forall A \in \mathfrak{x} \ : \ A^{(u)} \in \mathfrak{y}\}$.

Proof: For "$\leq$", consider any $\mathfrak{X} \in UUX$ with $m_X(\mathfrak{X}) = \mathfrak{x}$. It suffices to show that every $u \ll \bigwedge_{\mathcal{A}\in\mathfrak{X},\, B\in\mathfrak{y}} \bigvee_{\mathfrak{z}\in\mathcal{A},\, y\in B} a(\mathfrak{z}, y)$ has the property that $A^{(u)} \in \mathfrak{y}$ for all $A \in \mathfrak{x}$. But if for $A \in \mathfrak{x}$ we assume $A^{(u)} \notin \mathfrak{y}$, so that $B := X \setminus A^{(u)} \in \mathfrak{y}$, considering

$$\mathcal{A} := A^\sharp = \{\mathfrak{z} \in UX \ : \ A \in \mathfrak{z}\} \in \mathfrak{X} \text{ (since } A \in \mathfrak{x})$$

we would conclude

$$u \ll \bigvee_{\mathfrak{z}\in\mathcal{A},\, y\in B} a(\mathfrak{z}, y)$$

and therefore $A^{(u)} \cap B \neq \emptyset$, a contradiction.

For "$\geq$", consider $v \ll \bigvee\{u \in V \mid \forall A \in \mathfrak{x} \ : \ A^{(u)} \in \mathfrak{y}\}$ in V. For all $A \in \mathfrak{x}$, $B \in \mathfrak{y}$, the ultrafilter $\mathfrak{y}$ contains $A^{(v)} \cap B \neq \emptyset$, so that $v \leq \bigvee_{\mathfrak{z}\in A^\sharp} a(\mathfrak{z}, y)$ for some $y \in B$, and

$$v \leq \bigwedge_{B\in\mathfrak{y}} \bigvee_{\mathfrak{z}\in A^\sharp,\, y\in B} a(\mathfrak{z}, y)$$

follows for every $A \in \mathfrak{x}$. Now,

$$\mathfrak{F} = \{\mathcal{A} \subseteq UX \mid A^\sharp \subseteq \mathcal{A} \text{ for some } A \in \mathfrak{x}\}$$

is a filter on UX, and

$$\mathfrak{J} := \{\mathcal{B} \subseteq UX \mid v \not\leq \bigwedge_{B\in\mathfrak{y}} \bigvee_{\mathfrak{z}\in\mathcal{B},\, y\in B} a(\mathfrak{z}, y)\}$$

is an ideal on UX that is disjoint from $\mathfrak{F}$. (Closure of $\mathfrak{J}$ under binary union needs (*).) There is therefore an ultrafilter $\mathfrak{X} \supseteq \mathfrak{F}$ on UX with $\mathfrak{X} \cap \mathfrak{J} = \emptyset$. By definition of $\mathfrak{F}$ one has $\mathfrak{x} = m_X(\mathfrak{X})$, and by definition of $\mathfrak{J}$

$$v \leq \bigwedge_{\mathcal{A}\in\mathfrak{X},\, B\in\mathfrak{y}} \bigvee_{\mathfrak{z}\in\mathcal{A},\, y\in B} a(\mathfrak{z}, y) \leq \hat{a}(\mathfrak{x}, \mathfrak{y})$$

follows. ■

Proposition 7.2. *Under hypothesis* (*), *for an* $(\mathbb{U}, V)$*-functor* $f : (X, a) \to (Y, b)$ *one has:*

$$f \text{ closed } \Leftrightarrow Uf \text{ closed.}$$

Proof: As a V-functor, Uf is closed if, and only if, it is proper. We must show that propriety of Uf is equivalent to closedness of f. First let f be closed. For $\mathfrak{x} \in UX$, $\mathfrak{y} \in UY$, we must show $\hat{b}(Uf(\mathfrak{x}), \mathfrak{y}) \leq \bigvee_{\mathfrak{z} \in (Uf)^{-1}\mathfrak{y}} \hat{a}(\mathfrak{x}, \mathfrak{z})$, and for that, by Lemma 7.1, it suffices to show that, whenever $u \ll v$ in V with $B^{(v)} \in \mathfrak{y}$ for all $B \in Uf(\mathfrak{x})$, one has some $\mathfrak{z} \in UX$ with $Uf(\mathfrak{z}) = \mathfrak{y}$ and $A^{(u)} \in \mathfrak{z}$ for all $A \in \mathfrak{x}$. But since f is closed, for every $A \in \mathfrak{x}$ one has $f(A)^{(v)} \subseteq f(A^{(u)}) \in \mathfrak{y}$. Therefore, any ultrafilter $\mathfrak{z}$ on X containing the filterbase $\{A^{(u)} \,|\, A \in \mathfrak{x}\}$ disjoint from the ideal $\{C \subseteq X \,|\, f(C) \notin \mathfrak{y}\}$ will be as required.

Conversely, let Uf be proper and $y \in f(A)^{(v)}$ with $A \subseteq X$, $v \in V$. For every $u \ll v$ we must show $y \in f(A^{(u)})$. Since every ultrafilter $\mathfrak{y}$ on Y containing $f(A)$ is the image of an ultrafilter $\mathfrak{x}$ on X containing A, one has:

$$\begin{aligned}
u \ll v \;\leq\; & \bigvee_{\mathfrak{y} \in Uf(A)} b(\mathfrak{y}, y) \\
= & \bigvee_{\mathfrak{y} \in Uf(A)} \hat{b}(\mathfrak{y}, e_Y(y)) \\
= & \bigvee_{\mathfrak{x} \in UA} \hat{b}(Uf(\mathfrak{x}), e_Y(y)) \\
= & \bigvee_{\mathfrak{x} \in UA} \; \bigvee_{\mathfrak{x}' \in (Uf)^{-1}(e_Y(y))} \hat{a}(\mathfrak{x}, \mathfrak{x}') \\
= & \bigvee_{\mathfrak{x} \in UA} \; \bigvee_{\mathfrak{x}' \in (Uf)^{-1}(e_Y(y))} \bigvee \{w \in V \,|\, \forall B \in \mathfrak{x} \,:\, B^{(w)} \in \mathfrak{x}'\}.
\end{aligned}$$

Hence there exist $\mathfrak{x} \in UA$, $\mathfrak{x}' \in (Uf)^{-1}(e_Y(y))$ and $w \geq u$ such that $B^{(w)} \in \mathfrak{x}'$ whenever $B \in \mathfrak{x}$. In particular, $A^{(u)} \in \mathfrak{x}'$, and so $f(A^{(u)}) \in Uf(\mathfrak{x}') = e_Y(y)$, that is $y \in f(A^{(u)})$. ■

Since $U1 = 1$, $\hat{U}$ is flat and e° is finitely strict (although not strict in general), Theorem 6.1 gives:

Corollary 7.3. *Under hypothesis* (*)*, for an* $(\mathbb{U}, V)$*-functor* $f : (X, a) \to (Y, b)$ *the following conditions are equivalent:*

(i) f *is proper;*
(ii) f *is stably closed;*
(iii) f *is closed with compact fibres.*

In case $V = 2$ this Theorem recovers the classical results for **Top**, while in case $V = [0, \infty]$ it recovers the results obtained in [7].

7.5. Multi-ordered sets as $(\mathbb{L}, 2)$-categories. The extension $\hat{L}$ of the free-monoid monad given in Example 2.1.5 is flat, with $L1 \not\cong 1$, and e° a strict natural transformation. Hence, by Corollary 3.4, an $(\mathbb{L}, 2)$-functor $f : (X, a) \to (Y, b)$ is proper whenever Lf is closed. As in Corollary 7.2, closedness of f does not imply propriety. In fact, an $(\mathbb{L}, 2)$-functor $f : (X, a) \to (Y, b)$ is:

(1) proper if, and only if, whenever $\langle f(x_1), \cdots, f(x_n)\rangle \, b \, y$, there exists $x \in f^{-1}y$ such that $\langle x_1, \cdots, x_n\rangle \, a \, x$;
(2) closed if, and only if, whenever $\langle f(x_1), \cdots, f(x_n)\rangle \, b \, y$, there exist a sublist $\langle x'_1, \cdots, x'_m\rangle$ of $\langle f(x_1), \cdots, f(x_n)\rangle$ and $x \in f^{-1}y$ such that $\langle x'_1, \cdots, x'_m\rangle \, a \, x$.

7.6. Labeled graphs as $(\mathbb{H}, 2)$-categories. For the flat extension $\hat{H}$ of $\mathbb{H} = (H \times -, m, e)$ (H a monoid) of Example 2.1.6, we have:

(1) Since e° is a natural transformation, every $(\mathbb{H}, 2)$-functor $f : (X, a) \to (Y, b)$ has proper fibres; hence,

$$f \text{ proper} \Leftrightarrow Hf \text{ proper} \Leftrightarrow Hf \text{ closed}.$$

(2) Although propriety of fibres is trivial, compactness is not: for (X, a) an H-labeled graph,

$$(X, a) \text{ compact} \;\Leftrightarrow\; \forall \alpha \in H,\, x \in X \;\exists x' \in X \;\; x \xrightarrow{\alpha} x' .$$

(3) Closed $(\mathbb{H}, 2)$-functors need not be proper: for an $(\mathbb{H}, 2)$-functor $f : (X, a) \to (Y, b)$,

$$f \text{ proper} \Leftrightarrow \forall \alpha \in H,\, x \in X,\, y \in Y \;\; f(x) \xrightarrow{\alpha} y \;\; \Rightarrow \exists x' \in f^{-1}y : \;\; x \xrightarrow{\alpha} x',$$

$$f \text{ closed} \Leftrightarrow \forall \alpha \in H, x \in X, y \in Y \;\; f(x) \xrightarrow{\alpha} y \;\; \Rightarrow \exists x' \in f^{-1}y, \beta \in H : \;\; x \xrightarrow{\beta} x'.$$

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