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VARIETAL TECHNIQUES FOR n-PERMUTABLE CATEGORIES

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Dedicated to George Janelidze on the occasion of his sixtieth birthday

Abstract: We prove that varietal techniques based on the existence of operations of a certain arity can be extended to n-permutable categories with binary coproducts using a method developed by D. Bourn and Z. Janelidze [2]. In particular, we give a categorical version of the characterisation theorems for n-permutable varieties due to J. Hagemann and A. Mitschke [7].

Keywords: Mal'tsev, Goursat, n-permutable category; binary coproduct; approximate (co-)operation.
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Introduction

A variety of universal algebras is called *n-permutable* when its congruence relations satisfy the *n-permutability condition*: for congruences R and S on an algebra X, the equality $(R,S)_n = (S,R)_n$ holds, where $(R,S)_n = RSRS\cdots$ denotes the composition of n alternating factors R and S. In a categorical context, this notion was first considered by A. Carboni, G. M. Kelly and M. C. Pedicchio in the article [3]. Here an *n-permutable category* is defined as a regular category [1] in which the (effective) equivalence relations satisfy the n-permutability condition.

For a variety of universal algebras $\mathbb{V}$, it was shown by A. I. Mal'tsev in [9] that the 2-permutability of congruences is equivalent to the fact that the theory of $\mathbb{V}$ admits a ternary operation p such that $p(x,y,y) = x$ and $p(x,x,y) = y$. Then $\mathbb{V}$ is called a *Mal'tsev variety* and p a *Mal'tsev operation*. Similarly, for the strictly weaker 3-permutability condition [11], the theory admits quaternary operations p and q such that $p(x,y,y,z) = x$, $p(x,x,y,y) = q(x,x,y,y)$ and $q(x,y,y,z) = z$. Such varieties are called *Goursat varieties* and p and q are

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Goursat operations. More generally, following the results of [5, 12, 13], J. Hagemann and A. Mitschke showed in [7] that n-permutability of congruences can be characterised by the existence of $(n+1)$-arity operations satisfying suitable equations (see Theorem 1.3) or, equivalently, by the existence of ternary operations (see Theorem 1.4).

The first aim of this work is to give a categorical version of such $(n+1)$-ary and ternary operations for n-permutable categories. We do this in the context of regular categories with binary coproducts via *approximate (co-)operations* with a certain *approximation* (Definition 4.1 and Figures 3 and 4). This method extends D. Bourn and Z. Janelidze's approach to Mal'tsev categories through *approximate Mal'tsev operations* [2], which allows us to lift varietal techniques to the categorical level and obtain general versions of the characterisation theorems for n-permutable varieties mentioned above (Theorems 4.2 and 4.3). This aspect of our work gives a good illustration of the strength and generality of D. Bourn and Z. Janelidze's technique.

The second aim of our paper—in fact, the problem which we originally set out to solve—is answering the following question. In the article [7], J. Hagemann and A. Mitschke also characterise n-permutable varieties in terms which are purely categorical, as follows:

Theorem. *For any equational class $\mathbb{V}$, the following statements are equivalent:*

(a) *the congruence relations of every algebra of $\mathbb{V}$ are n-permutable;*
(b) *for $A \in \mathbb{V}$, every reflexive subalgebra R of $A \times A$ satisfies $R^{\mathrm{op}} \leqslant R^{n-1}$;*
(c) *for $A \in \mathbb{V}$, every reflexive subalgebra R of $A \times A$ satisfies $R^{n} \leqslant R^{n-1}$.* ∎

These three conditions make sense in arbitrary regular categories; nevertheless, they are not mentioned in the article [3], which surprised us. Since for the proof the authors refer to the unpublished work [6] we tried to find one ourselves. We did indeed manage to find a proof valid in varieties of algebras (see [10] for part of it) but failed to come up with a categorical argument. Somehow this was to be expected, given that the equivalences are missing in [3]—but then, what is it that makes the difference here between varieties of algebras and regular categories?

The answer seems to be: *binary coproducts*. This is what allows us to mimic the varietal arguments in terms of $(n+1)$-arity operations, using approximate co-operations instead. Thus we finally obtain a version of the above characterisation theorem, valid in any regular category with binary coproducts (Theorem 4.5). This, in turn, implies that in an n-permutable category with binary coproducts, any reflexive and transitive relation is symmetric (Corollary 4.6).

We do, however, not have a counterexample to show that our results fail in the absence of binary coproducts. Since in the statement of the above theorem they do not appear, it remains an open question whether or not binary coproducts are necessary for this result to be valid in a categorical setting.

Structure of the text. In Section 1 we give the background concerning n-permutable varieties and categories needed for this work. We recall the main points of the approach to Mal'tsev categories via approximate Mal'tsev operations in Section 2. We then extend this approach to Goursat categories in Section 3. Finally, Section 4 gives the full approach to n-permutable categories via approximate (co-)operations.

1. n-Permutability

We recall the main definitions and properties known for n-permutable varieties from [7] and for n-permutable categories we follow [3].

1.1. Relations. A category $\mathbb{C}$ with finite limits is called a **regular** category [1] when every kernel pair has a quotient and regular epimorphisms are stable under pulling back. In a regular category any morphism $f\colon A \to B$ can be decomposed into $f = mp$, where p is a regular epimorphism and m is a monomorphism. Regular categories give a suitable context for composing relations.

A **relation** R from A to B is a subobject $\langle r_1, r_2\rangle\colon R \rightarrowtail A \times B$. The opposite relation, denoted R^{op}, is the relation from B to A determined by the subobject $\langle r_2, r_1\rangle\colon R \rightarrowtail B \times A$. Given another relation S from B to C, the composite relation of R and S is a relation, denoted SR, from A to C.

Given morphisms $a\colon X \to A$ and $b\colon X \to B$, we say that $\langle a, b\rangle$ **belongs** to R when there exists a morphism $\chi\colon X \to R$ such that $r_1\chi = a$ and $r_2\chi = b$; we write $\langle a, b\rangle \in R$. For any morphism $c\colon X \to C$, we have $\langle a, c\rangle \in SR$ if and only if there exists a regular epimorphism $\zeta\colon Z \twoheadrightarrow X$ and a morphism $x\colon Z \to B$ such that $\langle a\zeta, x\rangle \in R$ and $\langle x, c\zeta\rangle \in S$ (see Proposition 2.1 in [3]). This observation trivially extends to the composite of more than two relations. Moreover, when R' is another relation from A to B, then $R \leqslant R'$ if and only if any pair of morphisms $\langle a, b\rangle$ that belongs to R also belongs to R'.

A relation R from an object A to A is simply called a **relation on** A. We say that R is **reflexive** when $1_A \leqslant R$, **symmetric** when $R^{\mathrm{op}} = R$ and **transitive** when $RR = R$. As usual, a relation R on A is an **equivalence relation** when it is reflexive, symmetric and transitive. In particular, a kernel pair $\langle f_1, f_2\rangle\colon \mathrm{R}[f] \rightarrowtail A \times A$ of a morphism $f\colon A \to B$ is an **effective equivalence relation** (also called a **congruence**).

1.2. ***n*-Permutable varieties** [7]**.** A **Mal'tsev** (or **2-permutable**) variety of universal algebras is such that the composition of congruences is 2-permutable, i.e., $RS = SR$, for any pair of congruences R and S on the same object. **Goursat** (or **3-permutable**) varieties satisfy the strictly weaker 3-permutability condition: $RSR = SRS$. More generally, ***n*-permutable** varieties satisfy the n-permutability condition $(R, S)_n = (S, R)_n$, where $(R, S)_n = RSRS \cdots$ denotes the composite of n alternating factors R and S. We write $R^n = (R, R)_n$ for the n-th power of R.

It is well known that an n-permutable variety of universal algebras is characterised by the fact that its theory contains $n + 1$ operations of arity $n + 1$ or, equivalently, $n - 1$ ternary operations satisfying appropriate identities:

Theorem 1.3 (Theorem 1 of [7])**.** *For any equational class $\mathbb{V}$, the following statements are equivalent:*

(a) *the congruence relations of every algebra of $\mathbb{V}$ are n-permutable;*
(b) *there exist $(n+1)$-ary algebraic operations $v_0, \ldots, v_n$ of $\mathbb{V}$ for which the identities*

$$\begin{cases} v_0(x_0, \ldots, x_n) = x_0, \\ v_{i-1}(x_0, x_0, x_2, x_2, \ldots) = v_i(x_0, x_0, x_2, x_2, \ldots), & i \text{ even}, \\ v_{i-1}(x_0, x_1, x_1, x_3, x_3, \ldots) = v_i(x_0, x_1, x_1, x_3, x_3, \ldots), & i \text{ odd}, \\ v_n(x_0, \ldots, x_n) = x_n \end{cases}$$

hold. ■

Theorem 1.4 (Theorem 2 of [7]))**.** *For any equational class $\mathbb{V}$, the following statements are equivalent:*

(a) *the congruence relations of every algebra of $\mathbb{V}$ are n-permutable;*

(b) *there exist ternary algebraic operations* $w_1, \ldots, w_{n-1}$ *of* $\mathbb{V}$ *for which the identities*

$$\begin{cases} w_1(x,y,y) = x, \\ w_i(x,x,y) = w_{i+1}(x,y,y), \quad \text{for } i \in \{1,\ldots,n-2\}, \\ w_{n-1}(x,x,y) = y \end{cases}$$

hold. ■

In particular, a Mal'tsev variety has a **Mal'tsev ternary operation** p such that

$$\begin{cases} p(x,y,y) = x, \\ p(x,x,y) = y. \end{cases} \tag{A}$$

A Goursat variety can be characterised by the existence of two **Goursat quaternary operations**, p and q, satisfying the identities

$$\begin{cases} p(x,y,y,z) = x, \\ p(x,x,y,y) = q(x,x,y,y), \\ q(x,y,y,z) = z \end{cases} \tag{B}$$

or, equivalently, by the existence of two **Goursat ternary operations**, r and s, such that

$$\begin{cases} r(x,y,y) = x, \\ r(x,x,y) = s(x,y,y), \\ s(x,x,y) = y. \end{cases} \tag{C}$$

Remark 1.5. The equivalence between the Goursat quaternary and ternary operations is given by the identities

$$p(x,y,z,w) = r(x,y,z) \qquad \text{and} \qquad q(x,y,z,w) = s(y,z,w)$$

on the one hand,

$$r(x,y,z) = p(x,y,z,z) \qquad \text{and} \qquad s(x,y,z) = q(x,x,y,z)$$

on the other.

Remark 1.6. [7] More generally, the equivalence between $(n+1)$-ary and ternary operations for n-permutable varieties is given by the identities

$$\begin{cases} v_0(x_0,\ldots,x_n) = x_0, \\ v_i(x_0,\ldots,x_n) = w_i(x_{i-1},x_i,x_{i+1}), \quad \text{for } i \in \{1,\ldots,n-1\}, \\ v_n(x_0,\ldots,x_n) = x_n \end{cases}$$

and $w_i(x, y, z) = v_i(\underbrace{x, \ldots, x}_{i}, y, \underbrace{z, \ldots, z}_{n-i})$ for $i \in \{1, \ldots, n-1\}$.

J. Hagemann and A. Mitschke also claim alternative characterisations which involve certain conditions on reflexive relations:

Theorem 1.7 ([7], see also [10])**.** *For any equational class $\mathbb{V}$, the following statements are equivalent:*

(a) *the congruence relations of every algebra of $\mathbb{V}$ are n-permutable;*
(b) *for $A \in \mathbb{V}$, every reflexive subalgebra R of $A \times A$ satisfies $R^{\mathrm{op}} \leqslant R^{n-1}$;*
(c) *for $A \in \mathbb{V}$, every reflexive subalgebra R of $A \times A$ satisfies $R^n \leqslant R^{n-1}$.* ■

1.8. n-Permutable categories [3]**.** The notion of n-permutable variety has been made categorical. We say that a regular category is an **n-permutable category** when the composition of (effective) equivalence relations on a given object is n-permutable: for two (effective) equivalence relations R and S on the same object, we have $(R, S)_n = (S, R)_n$. In fact, it suffices to have one of the inequalities, say $(R, S)_n \leqslant (S, R)_n$. Equivalently, these categories can be characterised by the fact that, for every reflexive relation E, $(E, E^{\mathrm{op}})_{n-1}$ is an equivalence relation or, simply, $(E, E^{\mathrm{op}})_{n-1}$ is a transitive relation.

2. Mal'tsev categories with binary coproducts

We gather some basic ideas concerning approximate Mal'tsev operations introduced in [2] by D. Bourn and Z. Janelidze.

Given a category with binary products, we denote the powers of an object X by $X^2 = X \times X$, $X^3 = X \times X \times X$, ..., and the projections by π_1, π_2, π_3, etc. We denote the diagonal by $\langle 1_X, 1_X \rangle = \Delta_X \colon X \to X^2$. Dually, we write $2X = X + X$, $3X = X + X + X$, ..., ι_1, ι_2, ι_3, ..., for the coproduct inclusions and $\left\langle \begin{smallmatrix} 1_X \\ 1_X \end{smallmatrix} \right\rangle = \nabla_X \colon 2X \to X$ for the codiagonal.

Definition 2.1. Let $\mathbb{C}$ be a category with binary products. We say that a morphism $p \colon X^3 \to A$ in $\mathbb{C}$ is an **approximate Mal'tsev operation** (on X)

with approximation $\alpha\colon X \to A$ when the diagram

$$\begin{tikzcd} & X \arrow[d, dotted, "\alpha"] & \\ X^2 \arrow[ur, "\pi_1"] \arrow[dr, "1_X\times\Delta_X"'] & A & X^2 \arrow[ul, "\pi_2"'] \arrow[dl, "\Delta_X\times 1_X"] \\ & X^3 \arrow[u, dotted, "p"'] & \end{tikzcd} \tag{\textbf{D}}$$

commutes. We call p a **universal** approximate Mal'tsev operation with approximation α when A is the colimit of the outer solid square in diagram (**D**).

Note that the morphism α is uniquely determined by p since we have $\alpha = p\langle 1_X, 1_X, 1_X, 1_X\rangle$. Furthermore, p is an approximate Mal'tsev operation with approximation α if and only if, for any object W and any two morphisms x, $y\colon W \to X$, the identities

$$p\langle x, y, y\rangle = \alpha x \qquad \text{and} \qquad p\langle x, x, y\rangle = \alpha y$$

hold (cf. (**A**)).

When $\mathbb{C}$ is a category with binary products and finite colimits, then there always exists a universal approximate Mal'tsev operation given by the colimit of the outer diagram of (**D**).

If $A = X$ then p represents a "real" Mal'tsev operation on X: it equips X with an internal Mal'tsev structure. In the particular case where $\mathbb{C} = \mathsf{Set}$ this means that (X, p) is a Mal'tsev algebra.

Remark 2.2. Let $\mathbb{C}$ be the functor category $\mathbb{C} = \mathbb{D}^{\mathbb{E}}$, where $\mathbb{D}$ is a category with binary products.

1. A natural transformation $p\colon X^3 \Rightarrow A$ is an approximate Mal'tsev operation on X in $\mathbb{D}^{\mathbb{E}}$ with approximation $\alpha\colon X \Rightarrow A$ if and only if, for every object E in $\mathbb{E}$, the E-component $p_E\colon (X(E))^3 \to A(E)$ is an approximate Mal'tsev operation on $X(E)$ in the category $\mathbb{D}$ with approximation $\alpha_E\colon X(E) \to A(E)$, the E-component of α. Clearly, p is universal if and only if every E-component p_E is universal.
2. Let $\mathbb{D}$ be a category with binary colimits. Given a functor $X\colon \mathbb{E} \to \mathbb{D}$, then, for each E in $\mathbb{E}$, taking the universal approximate Mal'tsev operation $p_E\colon (X(E))^3 \to A(E)$ on $X(E)$ in $\mathbb{D}$ gives a unique universal approximate Mal'tsev operation $p\colon X^3 \Rightarrow A$ on X in $\mathbb{D}^{\mathbb{E}}$.

We now consider a category $\mathbb{X}$ with finite limits and binary coproducts and take $\mathbb{C} = \mathbb{X}^{\mathbb{X}^{\mathrm{op}}}$. We shall work in the dual category $\mathbb{C}^{\mathrm{op}} = \mathbb{X}^{\mathbb{X}}$. So, a (universal) approximate Mal'tsev operation $p\colon X^3 \to A$ with approximation $\alpha\colon X \to A$ in $\mathbb{C}$ is, in fact, a **(universal) approximate Mal'tsev co-operation** $p\colon A \to 3X$ with approximation $\alpha\colon A \to X$ in $\mathbb{C}^{\mathrm{op}}$. We consider the particular case when $X = 1_{\mathbb{X}}$.

In this setting, the universal approximate Mal'tsev co-operation always exists: p and α are natural transformations defined, for each object X in $\mathbb{X}$, by the limit of the outer solid square

$$\begin{tikzcd} & X \arrow[dl, "\iota_1"'] \arrow[dr, "\iota_2"] & \\ 2X & A(X) \arrow[u, dotted, "\alpha_X"'] \arrow[d, dotted, "p_X"] & 2X. \\ & 3X \arrow[ul, "1_X+\nabla_X"] \arrow[ur, "\nabla_X+1_X"'] & \end{tikzcd}$$

The existence of a (universal) approximate Mal'tsev co-operation whose approximation is "special", gives a characterisation for regular Mal'tsev categories with binary coproducts.

Theorem 2.3 (Theorem 4.2 of [2]). *Let $\mathbb{X}$ be a regular category with binary coproducts. The following statements are equivalent:*

(a) *the approximation $\alpha\colon A \Rightarrow 1_{\mathbb{X}}$ of the universal approximate Mal'tsev co-operation on $1_{\mathbb{X}}$ is a natural transformation, all of whose components are regular epimorphisms in $\mathbb{X}$;*
(b) *there exists an approximate Mal'tsev co-operation on $1_{\mathbb{X}}$ such that the approximation $\alpha\colon A \Rightarrow 1_{\mathbb{X}}$ is a natural transformation, all of whose components are regular epimorphisms in $\mathbb{X}$;*
(c) *$\mathbb{X}$ is a Mal'tsev category.* ∎

Following [8] we may consider a morphism $A(X) \to Y$ as an **imaginary morphism** $X \rightsquigarrow Y$ **from** X **to** Y. It is said to be **real** when it factors through $\alpha_X\colon A(X) \to X$. As soon as $\mathbb{X}$ is Mal'tsev such factorisations are uniquely determined, and furthermore any object X comes equipped with a **canonical imaginary Mal'tsev co-operation** $p_X\colon X \rightsquigarrow 3X$. This point of view also makes sense for the approximate co-operations considered in the following sections.

3. Goursat categories with binary coproducts

We adapt D. Bourn and Z. Janelidze's approach for Mal'tsev categories via approximate Mal'tsev operations recalled in Section 2 to the context of Goursat categories. In our proof of Theorem 3.2—a Goursat version of Theorem 2.3—we shall, however, use purely varietal techniques rather than a modification of the arguments of [2].

Definition 3.1. Let $\mathbb{C}$ be a category with binary products. We say that morphisms p, $q\colon X^4 \to A$ are **approximate Goursat operations** (on X) **with approximation** $\alpha\colon X \to A$ if the diagram

$$
\begin{tikzcd}
 & X \arrow[dd, dotted, "\alpha"] & \\
X^3 \arrow[ur, "\pi_1"] \arrow[dd, "1_X\times\Delta_X\times 1_X"'] & & X^3 \arrow[ul, "\pi_3"'] \arrow[dd, "1_X\times\Delta_X\times 1_X"] \\
 & A & \\
X^4 \arrow[ur, dotted, "p"] & & X^4 \arrow[ul, dotted, "q"'] \\
 & X^2 \arrow[ul, "\Delta_X\times\Delta_X"] \arrow[ur, "\Delta_X\times\Delta_X"'] &
\end{tikzcd}
\qquad (\mathbf{E})
$$

commutes. If A is the colimit of the outer solid diagram, then p and q are called **universal** approximate Goursat operations with approximation α.

The approximation α is uniquely determined by p (or by q) since we have $\alpha = p\langle 1_X, 1_X, 1_X, 1_X\rangle$ (or $\alpha = q\langle 1_X, 1_X, 1_X, 1_X\rangle$). Moreover, to say that p and q are approximate Goursat operations with approximation α is equivalent to having, for every object W and any three morphisms x, y, $z\colon W \to X$, the identities

$$p\langle x,y,y,z\rangle = \alpha x, \;\; p\langle x,x,y,y\rangle = q\langle x,x,y,y\rangle \text{ and } q\langle x,y,y,z\rangle = \alpha z$$

similarly to (**B**).

When $\mathbb{C}$ is a category with binary products and finite colimits, then there always exist universal approximate Goursat operations given by the colimit of the outer solid hexagon in (**E**).

If $A = X$ then p and q are "real" Goursat operations on X. For instance, in Set this means that (X, p, q) is a Goursat algebra.

The statements of Remark 2.2 obviously extend to the Goursat context. Again, we work in the dual category $\mathbb{C}^{\mathrm{op}} = \mathbb{X}^{\mathbb{X}}$, where $\mathbb{X}$ is a category with finite limits and binary coproducts. So, (universal) approximate Goursat operations p, $q\colon X^4 \to A$ with approximation $\alpha\colon X \to A$ in $\mathbb{C}$ are, in fact, **(universal) approximate Goursat co-operations** p, $q\colon A \to 4X$ with approximation $\alpha\colon A \to X$ in $\mathbb{C}^{\mathrm{op}}$. We consider the particular case when $X = 1_{\mathbb{X}}$. As before, universal approximate Goursat co-operations always exist: p, q and α are natural transformations defined, for each object X in $\mathbb{X}$, by the limit of the outer solid hexagon

$$
\begin{tikzcd}
 & & X \arrow[lld, "\iota_1"'] \arrow[rrd, "\iota_3"] & & \\
3X & & & & 3X \\
 & & A(X) \arrow[uu, dotted, "\alpha_X"'] \arrow[lld, dotted, "p_X"'] \arrow[rrd, dotted, "q_X"] & & \\
4X \arrow[uu, "1_X+\nabla_X+1_X"] \arrow[rrd, "\nabla_X+\nabla_X"'] & & & & 4X. \arrow[uu, "1_X+\nabla_X+1_X"'] \arrow[lld, "\nabla_X+\nabla_X"] \\
 & & 2X & &
\end{tikzcd}
\qquad (\mathbf{F})
$$

We now extend Theorem 2.3 to the context of Goursat categories. However, our proof is based on the one for varieties, where the role of the quaternary operations (**B**) is now played by the approximate Goursat co-operations. This gives a categorical version of Theorem 1.3 for $n = 3$.

Theorem 3.2. *Let $\mathbb{X}$ be a regular category with binary coproducts. The following statements are equivalent:*

(a) *the approximation $\alpha\colon A \Rightarrow 1_{\mathbb{X}}$ of the universal approximate Goursat co-operations on $1_{\mathbb{X}}$ is a natural transformation, all of whose components are regular epimorphisms in $\mathbb{X}$;*

(b) *there exist approximate Goursat co-operations on $1_{\mathbb{X}}$ such that the approximation $\alpha\colon A \Rightarrow 1_{\mathbb{X}}$ is a natural transformation, all of whose components are regular epimorphisms in $\mathbb{X}$;*

(c) *$\mathbb{X}$ is a Goursat category.*

Proof: (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c) Let R and S be equivalence relations on a given object Y. We must prove that $RSR \leqslant SRS$. Let a, $b\colon X \to Y$ be morphisms such that $\langle a, b\rangle \in$

RSR. Then there exist a regular epimorphism $\zeta\colon Z \twoheadrightarrow X$ and morphisms x, $y\colon Z \to Y$ such that $\langle a\zeta, x\rangle \in R$, $\langle x, y\rangle \in S$ and $\langle y, b\zeta\rangle \in R$ (see Subsection 1.1). Since S is an equivalence relation, we have $\langle a\zeta, a\zeta\rangle$, $\langle x, x\rangle$, $\langle x, y\rangle$, $\langle b\zeta, b\zeta\rangle \in S$. Thus

$$\left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ x \\ b\zeta \end{smallmatrix} \right\rangle, \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle \right\rangle \in S$$

implies

$$\begin{cases} \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ x \\ b\zeta \end{smallmatrix} \right\rangle p_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle p_Z \right\rangle = \left\langle a\zeta\alpha_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle p_Z \right\rangle \in S \\ \\ \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ x \\ b\zeta \end{smallmatrix} \right\rangle q_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle q_Z \right\rangle = \left\langle b\zeta\alpha_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle q_Z \right\rangle \in S. \end{cases}$$

Similarly, R is an equivalence relation so $\langle a\zeta, a\zeta\rangle$, $\langle x, a\zeta\rangle$, $\langle y, b\zeta\rangle$, $\langle b\zeta, b\zeta\rangle \in R$, which gives

$$\left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle, \left\langle \begin{smallmatrix} a\zeta \\ a\zeta \\ b\zeta \\ b\zeta \end{smallmatrix} \right\rangle \right\rangle \in R$$

and thus

$$\left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle p_Z, \left\langle \begin{smallmatrix} a\zeta \\ a\zeta \\ b\zeta \\ b\zeta \end{smallmatrix} \right\rangle p_Z \right\rangle \in R \qquad \text{and} \qquad \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle q_Z, \left\langle \begin{smallmatrix} a\zeta \\ a\zeta \\ b\zeta \\ b\zeta \end{smallmatrix} \right\rangle q_Z \right\rangle \in R.$$

We see that

$$\left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle p_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle q_Z \right\rangle \in R^{\mathrm{op}}R = R$$

since

$$\left\langle \begin{smallmatrix} a\zeta \\ a\zeta \\ b\zeta \\ b\zeta \end{smallmatrix} \right\rangle p_Z = \left\langle \begin{smallmatrix} a\zeta \\ b\zeta \end{smallmatrix} \right\rangle (\nabla_Z + \nabla_Z) p_Z = \left\langle \begin{smallmatrix} a\zeta \\ b\zeta \end{smallmatrix} \right\rangle (\nabla_Z + \nabla_Z) q_Z = \left\langle \begin{smallmatrix} a\zeta \\ a\zeta \\ b\zeta \\ b\zeta \end{smallmatrix} \right\rangle q_Z.$$

From

$$\left\langle a\zeta\alpha_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle p_Z \right\rangle \in S, \quad \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle p_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle q_Z \right\rangle \in R, \quad \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \\ b\zeta \end{smallmatrix} \right\rangle q_Z, b\zeta\alpha_Z \right\rangle \in S$$

we get $\langle a\zeta\alpha_Z, b\zeta\alpha_Z\rangle \in SRS$. Since ζ and α_Z are regular epimorphisms, via the argument from Subsection 1.1 we can conclude that $\langle a, b\rangle \in SRS$.

(c) $\Rightarrow$ (a) We must prove that α_X in the limit diagram (**F**) is a regular epimorphism for every object X in $\mathbb{X}$. Consider $R = \mathrm{R}[\nabla_X + \nabla_X]$ and $S = \mathrm{R}[1_X + \nabla_X + 1_X]$. For the coproduct inclusions ι_1, ι_2, ι_3, $\iota_4\colon X \to 4X$, we have $\langle \iota_1, \iota_2\rangle \in R$, $\langle \iota_2, \iota_3\rangle \in S$ and $\langle \iota_3, \iota_4\rangle \in R$, so that $\langle \iota_1, \iota_4\rangle \in RSR = SRS$. So there exists a regular epimorphism $\zeta \colon Z \twoheadrightarrow X$ together with morphisms x, $y \colon Z \to 4X$ such that $\langle \iota_1\zeta, x\rangle \in S$, $\langle x, y\rangle \in R$, $\langle y, \iota_4\zeta\rangle \in S$ (Subsection 1.1). It is easy to see that ζ, x and y give a cone for the outer diagram (**F**) and, consequently, there exists a unique morphism $\lambda \colon Z \to A(X)$ such that $\zeta = \alpha_X\lambda$, $x = p_X\lambda$ and $y = q_X\lambda$. We conclude that α_X is a regular epimorphism since ζ is. ■

As for varieties, we also have a correspondence between quaternary co-operations and ternary co-operations (see Subsection 1.2; in particular equations (**B**), (**C**) and Remark 1.5). Similarly, (universal) approximate ternary Goursat co-operations r, s with approximation β are natural transformations defined, for each object X in $\mathbb{X}$, by the (limit of the outer solid) commutative diagram

$$\begin{tikzcd}
 & & X \arrow[dll, "\iota_1"'] \arrow[drr, "\iota_2"] & & \\
2X & & & & 2X \\
 & & B(X) \arrow[uu, dotted, "\beta_X"'] \arrow[dll, dotted, "r_X"'] \arrow[drr, dotted, "s_X"] & & \\
3X \arrow[uu, "1_X+\nabla_X"] \arrow[drr, "\nabla_X+1_X"'] & & & & 3X. \arrow[uu, "\nabla_X+1_X"'] \arrow[dll, "1_X+\nabla_X"] \\
 & & 2X & &
\end{tikzcd} \tag{$\mathbf{G}$}$$

Theorem 1.4, for $n = 3$, translates to:

Theorem 3.3. *Let $\mathbb{X}$ be a regular category with binary coproducts. The following statements are equivalent:*

(a) *the approximation $\beta \colon B \Rightarrow 1_{\mathbb{X}}$ of the universal approximate ternary Goursat co-operations on $1_{\mathbb{X}}$ is a natural transformation, all of whose components are regular epimorphisms in $\mathbb{X}$;*

(b) *there exist approximate ternary Goursat co-operations on $1_{\mathbb{X}}$ such that the approximation $\beta \colon B \Rightarrow 1_{\mathbb{X}}$ is a natural transformation, all of whose components are regular epimorphisms in $\mathbb{X}$;*

(c) *$\mathbb{X}$ is a Goursat category.*

Proof: (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c) This proof is similar to the one given in Theorem 3.2. However, here we take

$$\langle a\zeta, a\zeta\rangle, \langle x, x\rangle, \langle x, y\rangle \in S \Rightarrow \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ x \end{smallmatrix} \right\rangle r_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \end{smallmatrix} \right\rangle r_Z \right\rangle = \left\langle a\zeta\alpha_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \end{smallmatrix} \right\rangle r_Z \right\rangle \in S$$

$$\langle x, x\rangle, \langle y, x\rangle, \langle b\zeta, b\zeta\rangle \in S \Rightarrow \left\langle \left\langle \begin{smallmatrix} x \\ y \\ b\zeta \end{smallmatrix} \right\rangle s_Z, \left\langle \begin{smallmatrix} x \\ x \\ b\zeta \end{smallmatrix} \right\rangle s_Z \right\rangle = \left\langle \left\langle \begin{smallmatrix} x \\ y \\ b\zeta \end{smallmatrix} \right\rangle s_Z, b\zeta\alpha_Z \right\rangle \in S.$$

and

$$\begin{aligned} \langle a\zeta, a\zeta\rangle, \langle x, a\zeta\rangle, \langle y, b\zeta\rangle \in R \quad &\Rightarrow \quad \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \end{smallmatrix} \right\rangle r_Z, \left\langle \begin{smallmatrix} a\zeta \\ a\zeta \\ b\zeta \end{smallmatrix} \right\rangle r_Z \right\rangle \in R \\ \langle a\zeta, x\rangle, \langle b\zeta, y\rangle, \langle b\zeta, b\zeta\rangle \in R \quad &\Rightarrow \quad \left\langle \left\langle \begin{smallmatrix} a\zeta \\ b\zeta \\ b\zeta \end{smallmatrix} \right\rangle s_Z, \left\langle \begin{smallmatrix} x \\ y \\ b\zeta \end{smallmatrix} \right\rangle s_Z \right\rangle \in R \\ &\Rightarrow \quad \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ y \end{smallmatrix} \right\rangle r_Z, \left\langle \begin{smallmatrix} x \\ y \\ b\zeta \end{smallmatrix} \right\rangle s_Z \right\rangle \in RR = R. \end{aligned}$$

(c) $\Rightarrow$ (a) We already know that (c) implies condition (a) from Theorem 3.2. So, we just need to show that condition (a) of Theorem 3.2 implies (a). We suppose that diagram (**F**) represents a limit where α_X is a regular epimorphism. For the limit of the outer diagram (**G**), we want to prove that β_X is a regular epimorphism. The morphisms

$$\left\langle \begin{smallmatrix} \iota_1 \\ \iota_2 \\ \iota_3 \\ \iota_3 \end{smallmatrix} \right\rangle p_X \colon A(X) \to 3X \quad \text{and} \quad \left\langle \begin{smallmatrix} \iota_1 \\ \iota_1 \\ \iota_2 \\ \iota_3 \end{smallmatrix} \right\rangle q_X \colon A(X) \to 3X$$

(see Remark 1.5) together with the regular epimorphism $\alpha_X \colon A(X) \twoheadrightarrow X$ give another cone of the outer diagram (**G**). So, there exists a unique morphism $\lambda\colon A(X) \to B(X)$ such that, in particular, $\alpha_X = \beta_X\lambda$. Hence β_X is a regular epimorphism. ∎

To finish this section, for $n = 3$ we generalise the first equivalence of Theorem 1.7 to a categorical context. The corresponding generalisation of the other equivalences in their full generality is given in Theorem 4.5.

Theorem 3.4. *Let $\mathbb{X}$ be a regular category with binary coproducts. The following statements are equivalent:*

(a) *$\mathbb{X}$ is a Goursat category;*
(b) *for every reflexive relation R we have $R^{\mathrm{op}} \leqslant RR$.*

Proof: (a) $\Rightarrow$ (b) Let R be a reflexive relation on Y and consider morphisms x, $y\colon X \to Y$ such that $\langle x, y\rangle \in R^{\mathrm{op}}$; hence $\langle y, x\rangle \in R$. Since $\mathbb{X}$ is a Goursat category, there exist approximate ternary Goursat co-operations r and s with approximation β defined, for each object X in $\mathbb{X}$, as in (**G**), where β_X is a regular epimorphism. Since R is a reflexive relation, we have $\langle x,x\rangle$, $\langle y,x\rangle$, $\langle y,y\rangle \in R$, so that also

$$\Big\langle\Big\langle \begin{smallmatrix} x \\ y \\ y \end{smallmatrix}\Big\rangle, \Big\langle \begin{smallmatrix} x \\ x \\ y \end{smallmatrix}\Big\rangle\Big\rangle \in R,$$

which gives

$$\begin{cases} \Big\langle\Big\langle \begin{smallmatrix} x \\ y \\ y \end{smallmatrix}\Big\rangle r_X, \Big\langle \begin{smallmatrix} x \\ x \\ y \end{smallmatrix}\Big\rangle r_X\Big\rangle = \big\langle x\beta_X, \big\langle \begin{smallmatrix} x \\ y \end{smallmatrix}\big\rangle(\nabla_X + 1_X)r_X\big\rangle \in R \\ \\ \Big\langle\Big\langle \begin{smallmatrix} x \\ y \\ y \end{smallmatrix}\Big\rangle s_X, \Big\langle \begin{smallmatrix} x \\ x \\ y \end{smallmatrix}\Big\rangle s_X\Big\rangle = \big\langle\big\langle \begin{smallmatrix} x \\ y \end{smallmatrix}\big\rangle(1_X + \nabla_X)s_X, y\beta_X\big\rangle \in R. \end{cases}$$

We can conclude that

$$\langle x\beta_X, y\beta_X\rangle = \langle x,y\rangle\beta_X \in RR \Rightarrow \langle x,y\rangle \in RR$$

because β_X is a regular epimorphism (Subsection 1.1).

(b) $\Rightarrow$ (a) For any object X in $\mathbb{X}$, consider the following reflexive graph and the reflexive relation R on $2X$ which arises from the (regular epi, mono) factorisation in

$$\begin{tikzcd} 3X \arrow[rr, shift left=2, "\nabla_X+1_X"] \arrow[rr, shift right=2, "1_X+\nabla_X"'] \arrow[dr, two heads, "\pi"'] & & 2X. \arrow[ll] \arrow[dl, shift right] \\ & R \arrow[ur, shift left, "r_1"] \arrow[ur, shift right, "r_2"'] & \end{tikzcd}$$

From $\langle \iota_1, \iota_2\rangle \in R$ we get $\langle \iota_2, \iota_1\rangle \in R^{\mathrm{op}} \leqslant RR$. So, there exists a regular epimorphism $\zeta\colon Z \twoheadrightarrow X$ and a morphism $x\colon Z \to 2X$ such that $\langle \iota_2\zeta, x\rangle$, $\langle x, \iota_1\zeta\rangle \in R$. Let i, $j\colon Z \to R$ be the morphisms such that $\langle r_1, r_2\rangle i = \langle \iota_2\zeta, x\rangle$ and $\langle r_1, r_2\rangle j = \langle x, \iota_1\zeta\rangle$. From the pullback

$$\begin{tikzcd} B(X) \arrow[r, "\langle s_X, r_X\rangle"] \arrow[d, two heads, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & 3X \times 3X \arrow[d, two heads, "\pi\times\pi"] \\ Z \arrow[r, "\langle i,j\rangle"'] & R \times R \end{tikzcd}$$

we get morphisms r_X, s_X and a regular epimorphism defined by $\beta_X = \zeta\pi'$ such that (**G**) commutes. Then $\mathbb{X}$ is a Goursat category by Theorem 3.3. ∎

$$\begin{tikzcd}
 & & & X \arrow[dd, dotted, "\alpha"] & & & \\
X^{k+1} \arrow[urrr, "\pi_1"] \arrow[dd, "1_X\times\Delta_X^{k-1}\times 1_X"'] & & & & & & X^{k+1} \arrow[ulll, "\pi_{k+1}"'] \arrow[dd, "1_X\times\Delta_X^{k-1}\times 1_X"] \\
 & & & A & & & \\
X^{2k=n+1} \arrow[urrr, dotted, "v_1"] & & & & & & X^{2k=n+1} \arrow[ulll, dotted, "v_{n-1}"'] \\
X^k \arrow[u, "\Delta_X^k"] \arrow[r, "\Delta_X^k"'] & X^{2k} \arrow[uurr, dotted, "v_2"] & & & & X^{2k} \arrow[uull, dotted, "v_{n-2}"'] & X^k \arrow[u, "\Delta_X^k"'] \arrow[l, "\Delta_X^k"] \\
 & X^{k+1} \arrow[u] \arrow[r] & X^{2k} \arrow[uuur, dotted, "v_3"] & & \cdots & &
\end{tikzcd}$$

FIGURE 1. Approximate operation, case $n = 2k - 1$ for $k \geqslant 2$

4. n-Permutable categories with binary coproducts

In this last section we fully generalise the approach to Mal'tsev and Goursat categories via approximate operations to the context of n-permutable categories. We extend the arguments used in Section 3 to an arbitrary n.

In the proofs of this section we shall repeatedly use the technique from Subsection 1.1 without further mention.

Definition 4.1. Let $\mathbb{C}$ be a category with binary products. We say that morphisms $v_1, \ldots, v_{n-1}\colon X^{n+1} \to A$ are **approximate operations** (on X) **with approximation** $\alpha\colon X \to A$ if either the diagram in Figure 1 commutes (when n is odd) or the diagram in Figure 2 (n even).

If A is the colimit of the outer solid diagram, then the $v_1, \ldots, v_{n-1}$ are called **universal** approximate operations with approximation α.

The approximation α is uniquely determined by any of its operations v_i since we have $\alpha = v_i\langle 1_X, 1_X, 1_X, 1_X\rangle$. To say that $v_1, \ldots, v_{n-1}$ are approximate operations with approximation α is equivalent to having, for every object W and any morphisms $x_0, \ldots, x_n\colon W \to X$, identities which correspond to those given in Theorem 1.3.

$$
\begin{tikzcd}
 & & & X \arrow[dd, dotted, "\alpha"] & & & \\
X^{k+1} \arrow[urrr, "\pi_1"] \arrow[dd, "1_X \times \Delta_X^k"'] & & & & & & X^{k+1} \arrow[ulll, "\pi_{k+1}"'] \arrow[dd, "\Delta_X^k \times 1_X"] \\
 & & & A & & & \\
X^{2k+1=n+1} \arrow[urrr, dotted, "v_1"] & & & & & & X^{2k+1=n+1} \arrow[ulll, dotted, "v_{n-1}"'] \\
X^{k+1} \arrow[u, "\Delta_X^k \times 1_X"] \arrow[r, "\Delta_X^k \times 1_X"'] & X^{2k+1} \arrow[uurr, dotted, "v_2"] & & & & X^{2k+1} \arrow[uull, dotted, "v_{n-2}"'] & X^{k+1} \arrow[u, "1_X \times \Delta_X^k"'] \arrow[l, "1_X \times \Delta_X^k"] \\
 & X^{k+1} \arrow[u] \arrow[r] & X^{2k+1} \arrow[uuur, dotted, "v_3"] & & \cdots & &
\end{tikzcd}
$$

FIGURE 2. Approximate operation, case $n = 2k$ for $k \geqslant 2$

When $\mathbb{C}$ is a category with binary products and finite colimits, then there always exist universal approximate operations given by the colimit of the outer diagram of Figure 1, when n is odd, or of Figure 2, when n is even.

If $A = X$ then the $v_1, \ldots, v_{n-1}$ are "real" operations on X which provide X with a structure of *internal n-permutable object*. For instance, in Set this means that $(X, v_1, \ldots, v_n)$ is an *n-permutable algebra*, a kind of generalised Mal'tsev algebra.

As before, the statements of Remark 2.2 also extend to the n-permutable context. Again, we work in the dual category $\mathbb{C}^{\mathrm{op}} = \mathbb{X}^{\mathbb{X}}$, where $\mathbb{X}$ is a category with finite limits and binary coproducts. So, (universal) approximate operations $v_1, \ldots, v_{n-1}\colon X^{n+1} \to A$ with approximation $\alpha\colon X \to A$ in $\mathbb{C}$ are, in fact, **(universal) approximate co-operations** $v_1, \ldots, v_{n-1}\colon A \to (n+1)X$ with approximation $\alpha\colon A \to X$ in $\mathbb{C}^{\mathrm{op}}$. We consider the particular case when $X = 1_{\mathbb{X}}$. Again, universal approximate co-operations always exist: $v_1, \ldots, v_{n-1}$ and α are natural transformations defined, for each object X in $\mathbb{X}$, by the limit of one of the outer solid diagrams in Figure 3 and Figure 4.

We now extend Theorem 2.3 and Theorem 3.2 to the context of n-permutable categories. Again, our proof is based on the one for varieties, where the $(n+1)$-ary operations from Theorem 1.3 are replaced by approximate co-operations. This leads to the following categorical version of Theorem 1.3.

$$
\begin{tikzcd}
 & & & X \arrow[dlll, "\iota_1"'] \arrow[drrr, "\iota_{k+1}"] & & & \\
(k+1)X & & & & & & (k+1)X \\
 & & & A(X) \arrow[uu, dotted, "\alpha_X"'] \arrow[dlll, dotted, "(v_1)_X"'] \arrow[ddll, dotted, "(v_2)_X"] \arrow[dddl, dotted, "(v_3)_X"] \arrow[ddrr, dotted, "(v_{n-2})_X"'] \arrow[drrr, dotted, "(v_{n-1})_X"] & & & \\
(n+1=2k)X \arrow[uu, "1_X+(k-1)\nabla_X+1_X"] \arrow[d, "k\nabla_X"'] & & & & & & (n+1=2k)X \arrow[uu, "1_X+(k-1)\nabla_X+1_X"'] \arrow[d, "k\nabla_X"] \\
kX & 2kX \arrow[l, "k\nabla_X"] \arrow[d] & & & & 2kX \arrow[r, "k\nabla_X"'] & kX \\
 & (k+1)X & 2kX \arrow[l] & & \cdots & &
\end{tikzcd}
$$

FIGURE 3. Approximate co-operation, case $n = 2k - 1$ for $k \geqslant 2$

$$
\begin{tikzcd}
 & & & X \arrow[dlll, "\iota_1"'] \arrow[drrr, "\iota_{k+1}"] & & & \\
(k+1)X & & & & & & (k+1)X \\
 & & & A(X) \arrow[uu, dotted, "\alpha_X"'] \arrow[dlll, dotted, "(v_1)_X"'] \arrow[ddll, dotted, "(v_2)_X"] \arrow[dddl, dotted, "(v_3)_X"] \arrow[ddrr, dotted, "(v_{n-2})_X"'] \arrow[drrr, dotted, "(v_{n-1})_X"] & & & \\
(n+1=2k+1)X \arrow[uu, "1_X+k\nabla_X"] \arrow[d, "k\nabla_X+1_X"'] & & & & & & (n+1=2k+1)X \arrow[uu, "k\nabla_X+1_X"'] \arrow[d, "1_X+k\nabla_X"] \\
(k+1)X & (2k+1)X \arrow[l, "k\nabla_X+1_X"] \arrow[d] & & & & (2k+1)X \arrow[r, "1_X+k\nabla_X"'] & (k+1)X \\
 & (k+1)X & (2k+1)X \arrow[l] & & \cdots & &
\end{tikzcd}
$$

FIGURE 4. Approximate co-operation, case $n = 2k$ for $k \geqslant 2$

Theorem 4.2. *Let $\mathbb{X}$ be a regular category with binary coproducts. The following statements are equivalent:*

(a) *the approximation $\alpha\colon A \Rightarrow 1_{\mathbb{X}}$ of the universal approximate co-operations on $1_{\mathbb{X}}$ is a natural transformation, all of whose components are regular epimorphisms in $\mathbb{X}$;*

(b) *there exist approximate co-operations on* $1_{\mathbb{X}}$ *such that the approximation* $\alpha\colon A \Rightarrow 1_{\mathbb{X}}$ *is a natural transformation, all of whose components are regular epimorphisms in* $\mathbb{X}$*;*

(c) $\mathbb{X}$ *is an* n*-permutable category.*

Proof: (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c) We follow the proof of the same implication in Theorem 3.2 and give just the main points. Let R and S be equivalence relations on an object Y. We must prove that $(R,S)_n \leqslant (S,R)_n$. Let a, $b\colon X \to Y$ be morphisms such that $\langle a,b\rangle \in (R,S)_n$.

We first consider the case that n is odd. Then there exists a regular epimorphism $\zeta\colon Z \twoheadrightarrow X$ together with morphisms $x_1, \ldots, x_{n-1}\colon Z \to Y$ such that

$$\langle a\zeta, x_1\rangle \in R,\ \langle x_1,x_2\rangle \in S,\ \ldots,\ \langle x_{n-2},x_{n-1}\rangle \in S \text{ and } \langle x_{n-1}, b\zeta\rangle \in R.$$

Since S is an equivalence relation, we have $\langle a\zeta, a\zeta\rangle$, $\langle x_1,x_1\rangle$, $\langle x_1,x_2\rangle$, $\ldots$, $\langle x_{n-2},x_{n-2}\rangle$, $\langle x_{n-2},x_{n-1}\rangle$, $\langle b\zeta,b\zeta\rangle \in S$. This gives

$$\left\langle \left\langle \begin{smallmatrix} a\zeta \\ x_{\mathrm{odd}} \\ x_{\mathrm{odd}} \\ b\zeta \end{smallmatrix} \right\rangle, \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle \right\rangle \in S,$$

where we write

$$\left\langle \begin{smallmatrix} a\zeta \\ x_{\mathrm{odd}} \\ x_{\mathrm{odd}} \\ b\zeta \end{smallmatrix} \right\rangle = \langle a\zeta, x_1, x_1, x_3, x_3, \ldots, x_{n-2}, x_{n-2}, b\zeta\rangle^{\mathrm{T}}$$

and

$$\left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle = \langle a\zeta, x_1, x_2 \ldots, x_{n-2}, x_{n-1}, b\zeta\rangle^{\mathrm{T}}$$

to simplify notation. By precomposing with each of the approximate co-operations, we get

$$\left\langle a\zeta\alpha_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle (v_1)_Z \right\rangle \in S, \qquad \left\langle b\zeta\alpha_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle (v_{n-1})_Z \right\rangle \in S$$

and

$$\begin{cases} \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle (v_2)_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle (v_3)_Z \right\rangle \in SS^{\mathrm{op}} = S \\ \quad\vdots \\ \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle (v_{n-3})_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle (v_{n-2})_Z \right\rangle \in SS^{\mathrm{op}} = S \end{cases}$$

since $(1_Z + (k-1)\nabla_Z + 1_Z)\,(v_j)_Z = (1_Z + (k-1)\nabla_Z + 1_Z)\,(v_{j+1})_Z$ for j even in $\{2,\dots,n-3\}$. Similarly, R is an equivalence relation and we have

$$\begin{aligned}&\langle a\zeta, a\zeta\rangle,\ \langle x_1, a\zeta\rangle,\ \langle x_2, x_3\rangle,\ \langle x_3, x_3\rangle,\ \dots,\\ &\qquad\langle x_{n-3}, x_{n-2}\rangle,\ \langle x_{n-2}, x_{n-2}\rangle,\ \langle x_{n-1}, b\zeta\rangle,\ \langle b\zeta, b\zeta\rangle \in R.\end{aligned}$$

This gives

$$\left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle, \left\langle \begin{smallmatrix} a\zeta \\ a\zeta \\ x_{\mathrm{odd}} \\ x_{\mathrm{odd}} \\ b\zeta \\ b\zeta \end{smallmatrix} \right\rangle \right\rangle \in R,$$

where we write

$$\left\langle \begin{smallmatrix} a\zeta \\ a\zeta \\ x_{\mathrm{odd}} \\ x_{\mathrm{odd}} \\ b\zeta \\ b\zeta \end{smallmatrix} \right\rangle = \langle a\zeta, a\zeta, x_3, x_3, \dots, x_{n-2}, x_{n-2}, b\zeta, b\zeta\rangle^{\mathrm{T}}$$

to simplify notation. By precomposing with each of the approximate co-operations, we get

$$\begin{cases} \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle (v_1)_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle (v_2)_Z \right\rangle \in R^{\mathrm{op}}R = R \\ \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle (v_3)_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle (v_4)_Z \right\rangle \in R^{\mathrm{op}}R = R \\ \quad\vdots \\ \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle (v_{n-2})_Z, \left\langle \begin{smallmatrix} a\zeta \\ x \\ b\zeta \end{smallmatrix} \right\rangle (v_{n-1})_Z \right\rangle \in R^{\mathrm{op}}R = R \end{cases}$$

since $(k\nabla_Z)\,(v_j)_Z = (k\nabla_Z)\,(v_{j+1})_Z$, for j odd, $j \in \{1,\dots,n-2\}$. We can now conclude that $\langle a\zeta\alpha_Z, b\zeta\alpha_Z\rangle \in (S,R)_n$, which implies that $\langle a, b\rangle \in (S,R)_n$, since ζ and α_Z are regular epimorphisms.

For n even the proof is similar. Now we have $\langle a\zeta, x_1\rangle \in S$, $\langle x_1, x_2\rangle \in R$, $\dots$, $\langle x_{n-2}, x_{n-1}\rangle \in S$ and $\langle x_{n-1}, b\zeta\rangle \in R$ and should consider

$$\begin{aligned}&\langle a\zeta, a\zeta\rangle, \langle x_1, x_1\rangle,\ \langle x_1, x_2\rangle,\ \dots,\\ &\qquad\langle x_{n-3}, x_{n-3}\rangle,\ \langle x_{n-3}, x_{n-2}\rangle,\ \langle x_{n-1}, x_{n-1}\rangle,\ \langle x_{n-1}, b\zeta\rangle \in R\end{aligned}$$

and

$$\begin{aligned}&\langle a\zeta, a\zeta\rangle, \langle x_1, a\zeta\rangle, \langle x_2, x_3\rangle,\ \langle x_3, x_3\rangle,\ \dots,\\ &\qquad\langle x_{n-3}, x_{n-3}\rangle,\ \langle x_{n-2}, x_{n-1}\rangle,\ \langle x_{n-1}, x_{n-1}\rangle,\ \langle b\zeta, b\zeta\rangle \in S.\end{aligned}$$

(c) $\Rightarrow$ (a) We must prove that α_X in the limit diagram in Figure 3, if n is odd, or in Figure 4, if n is even, is a regular epimorphism for every object X

in $\mathbb{X}$. If n is odd, we consider $R = \mathrm{R}[k\nabla_X]$ and $S = \mathrm{R}[1_X + (k-1)\nabla_X + 1_X]$, and if n is even, we consider $R = \mathrm{R}[1_X + k\nabla_X]$ and $S = \mathrm{R}[k\nabla_X + 1_X]$. For the coproduct inclusions $\iota_1, \ldots, \iota_{n+1} \colon X \to (n+1)X$ we have

$$\langle \iota_1, \iota_{n+1} \rangle \in (R, S)_n = (S, R)_n.$$

So there exist a regular epimorphism $\zeta \colon Z \twoheadrightarrow X$ as well as morphisms $x_1, \ldots, x_{n-1} \colon Z \to (n+1)X$ such that

$$\begin{cases} \langle \iota_1\zeta, x_1 \rangle \in S,\ \langle x_1, x_2 \rangle \in R,\ \langle x_2, x_3 \rangle \in S,\ \ldots, & \\ \quad \langle x_{n-2}, x_{n-1} \rangle \in R,\ \langle x_{n-1}, \iota_{n+1}\zeta \rangle \in S, & n \text{ odd} \\ & \\ \langle \iota_1\zeta, x_1 \rangle \in R,\ \langle x_1, x_2 \rangle \in S,\ \langle x_2, x_3 \rangle \in R,\ \ldots, & \\ \quad \langle x_{n-2}, x_{n-1} \rangle \in R,\ \langle x_{n-1}, \iota_{n+1}\zeta \rangle \in S, & n \text{ even.} \end{cases}$$

It is easy to see that ζ, $x_1, \ldots, x_{n-1}$ give a cone for the outer diagram in Figure 3 or in Figure 4. Consequently, there exists a unique morphism $\lambda \colon Z \to A(X)$ such that, in particular, $\zeta = \alpha_X \lambda$. Now α_X is a regular epimorphism since ζ is. ■

As for n-permutable varieties, we also have a correspondence between approximate $(n+1)$-ary co-operations and approximate ternary co-operations (Remark 1.6). Similarly, (universal) approximate ternary co-operations $w_1, \ldots, w_{n-1}$ with approximation β are natural transformations defined, for each object X in $\mathbb{X}$, by the (limit of the outer solid) commutative diagram in Figure 5.

Now we obtain the claimed categorical version of Theorem 1.4:

Theorem 4.3. *Let $\mathbb{X}$ be a regular category with binary coproducts. The following statements are equivalent:*

(a) *the approximation $\beta \colon B \Rightarrow 1_{\mathbb{X}}$ of the universal approximate ternary co-operations on $1_{\mathbb{X}}$ is a natural transformation, all of whose components are regular epimorphisms in $\mathbb{X}$;*
(b) *there exist approximate ternary co-operations on $1_{\mathbb{X}}$ such that the approximation $\beta \colon B \Rightarrow 1_{\mathbb{X}}$ is a natural transformation, all of whose components are regular epimorphisms in $\mathbb{X}$;*
(c) *$\mathbb{X}$ is an n-permutable category.*

Proof: (a) $\Rightarrow$ (b) is obvious.

$$
\begin{tikzcd}
 & & & & X \arrow[dllll, "\iota_1"'] \arrow[drrrr, "\iota_2"] & & & & \\
2X & & & & & & & & 2X \\
 & & & & B(X) \arrow[uu, dotted, "\beta_X"'] \arrow[dllll, dotted, "(w_1)_X"'] \arrow[ddlll, dotted, "(w_2)_X"] \arrow[dddll, dotted, "(w_3)_X"] \arrow[ddrrr, dotted, "(w_{n-2})_X"'] \arrow[drrrr, dotted, "(w_{n-1})_X"] & & & & \\
3X \arrow[uu, "1_X+\nabla_X"] \arrow[d, "\nabla_X+1_X"'] & & & & & & & & 3X \arrow[uu, "\nabla_X+1_X"'] \arrow[d, "1_X+\nabla_X"] \\
2X & 3X \arrow[l, "1_X+\nabla_X"] \arrow[d] & & & & & & 3X \arrow[r, "\nabla_X+1_X"'] & 2X \\
 & 2X & 3X \arrow[l] & & & \cdots & & &
\end{tikzcd}
$$

FIGURE 5. Approximate ternary co-operations

(b) $\Rightarrow$ (c) This proof is similar to the same implication in Theorem 4.3, thus we give just the main points. If n is odd, then $\langle a\zeta, a\zeta\rangle$, $\langle x_1, x_1\rangle$, $\langle x_1, x_2\rangle$, $\ldots$, $\langle x_{n-2}, x_{n-2}\rangle$, $\langle x_{n-2}, x_{n-1}\rangle$, $\langle b\zeta, b\zeta\rangle \in S$. By precomposing each triple successively with the approximate ternary co-operations, this gives

$$
\begin{cases}
\left\langle a\zeta\alpha_Z, \left\langle \begin{smallmatrix} a\zeta \\ x_1 \\ x_2 \end{smallmatrix} \right\rangle (w_1)_Z \right\rangle \in S \\
\left\langle \left\langle \begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix} \right\rangle (w_2)_Z, \left\langle \begin{smallmatrix} x_2 \\ x_3 \\ x_4 \end{smallmatrix} \right\rangle (w_3)_Z \right\rangle \in SS^{\mathrm{op}} = S \\
\qquad \vdots \\
\left\langle \left\langle \begin{smallmatrix} x_{n-4} \\ x_{n-3} \\ x_{n-2} \end{smallmatrix} \right\rangle (w_{n-3})_Z, \left\langle \begin{smallmatrix} x_{n-3} \\ x_{n-2} \\ x_{n-1} \end{smallmatrix} \right\rangle (w_{n-2})_Z \right\rangle \in SS^{\mathrm{op}} = S \\
\left\langle b\zeta\beta_Z, \left\langle \begin{smallmatrix} x_{n-2} \\ x_{n-1} \\ b\zeta \end{smallmatrix} \right\rangle (w_{n-1})_Z \right\rangle \in S
\end{cases}
$$

since $(\nabla_Z + 1_Z)(w_j)_Z = (1_Z + \nabla_Z)(w_{j+1})_Z$, for j even, $j \in \{2, \ldots, n-3\}$. Similarly, as

$\langle a\zeta, a\zeta\rangle$, $\langle x_1, a\zeta\rangle$, $\langle x_2, x_3\rangle$, $\langle x_3, x_3\rangle, \ldots, \langle x_{n-2}, x_{n-2}\rangle$, $\langle x_{n-1}, b\zeta\rangle$ and $\langle b\zeta, b\zeta\rangle$

are all in R, we get

$$\begin{cases} \left\langle \left\langle \begin{smallmatrix} a\zeta \\ x_1 \\ x_2 \end{smallmatrix} \right\rangle (w_1)_Z, \left\langle \begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix} \right\rangle (w_2)_Z \right\rangle \in R^{\mathrm{op}}R = R \\ \quad \vdots \\ \left\langle \left\langle \begin{smallmatrix} x_{n-3} \\ x_{n-2} \\ x_{n-1} \end{smallmatrix} \right\rangle (w_{n-2})_Z, \left\langle \begin{smallmatrix} x_{n-2} \\ x_{n-1} \\ b\zeta \end{smallmatrix} \right\rangle (w_{n-1})_Z \right\rangle \in R^{\mathrm{op}}R = R. \end{cases}$$

If n is even, then we start from

$$\langle a\zeta, a\zeta\rangle,\ \langle x_1, x_1\rangle,\ \langle x_1, x_2\rangle,\ \ldots,\ \langle x_{n-3}, x_{n-2}\rangle,\ \langle x_{n-1}, x_{n-1}\rangle,\ \langle x_{n-1}, b\zeta\rangle \in R$$

and

$$\begin{aligned} &\langle a\zeta, a\zeta\rangle, \langle x_1, a\zeta\rangle,\ \langle x_2, x_3\rangle,\ \langle x_3, x_3\rangle,\ \ldots, \\ &\qquad\qquad \langle x_{n-2}, x_{n-1}\rangle,\ \langle x_{n-1}, x_{n-1}\rangle,\ \langle b\zeta, b\zeta\rangle \in S \end{aligned}$$

and proceed as above.

(c) $\Rightarrow$ (a) We already know that (c) implies condition (a) from Theorem 4.2. So, we just need to show that condition (a) of Theorem 4.2 implies (a). We suppose that Figure 3, for n odd, or Figure 4, for n even, represents a limit where α_X is a regular epimorphism. For the limit of the outer diagram in Figure 5, we want to prove that β_X is a regular epimorphism. The regular epimorphism $\alpha_X\colon A(X) \twoheadrightarrow X$ and the morphisms

$$(\nabla_{iX} + 1_X + \nabla_{n-iX})\,(v_i)_X\colon A(X) \to 3X, \quad i \in \{1, \ldots, n-1\},$$

where $(\nabla_i)_X = \langle 1_X, \ldots, 1_X\rangle^{\mathrm{T}}\colon iX \to X$ (see Remark 1.6), give another cone of the outer diagram (5); in particular, $(\nabla_1)_X = 1_X$ and $(\nabla_2)_X = \nabla_X$. This guarantees the existence of a unique morphism $\lambda\colon A(X) \to B(X)$ such that, in particular, $\alpha_X = \beta_X\lambda$. Hence β_X is a regular epimorphism. ■

We finish this work with the full generalisation of Theorem 1.7.

Lemma 4.4. *Let $\mathbb{X}$ be a regular category such that, for some natural number $n \geqslant 2$, we have $E^n \leqslant E^{n-1}$ for every reflexive relation E. Then $\mathbb{X}$ is $(2n-2)$-permutable.*

Proof: $(n = 2)$ By assumption, every reflexive relation E is transitive. Hence $\mathbb{X}$ is a Mal'tsev category, so it is 2-permutable—see Subsection 1.8.

$(n = 3)$ Let R and S be equivalence relations on a given object Y. Then $E = SR$ is a reflexive relation, since $1_Y \leqslant R$, S. By assumption we have $SRSRSR \leqslant SRSR$. But $RSRS \leqslant S(RSRS)R \leqslant SRSR$, which shows that the category $\mathbb{X}$ is 4-permutable.

The proof for $n = 3$ is easily extended to an arbitrary n. Now we have $(S,R)_{2n} \leqslant (S,R)_{2n-2}$ by assumption. But

$$(R,S)_{2n-2} \leqslant S(R,S)_{2n-2}R = (S,R)_{2n} \leqslant (S,R)_{2n-2},$$

which proves our claim. ∎

Theorem 4.5. *Let $\mathbb{X}$ be a regular category with binary coproducts. The following statements are equivalent:*

(a) *$\mathbb{X}$ is an n-permutable category;*
(b) *for every reflexive relation R, we have $R^{\mathrm{op}} \leqslant R^{n-1}$;*
(c) *for every reflexive relation R, we have $R^n \leqslant R^{n-1}$.*

Proof: (a) ⇒ (b) We extend the proof of the same implication from Theorem 3.4. Let R be a reflexive relation on Y and consider morphisms x, $y\colon X \to Y$ such that $\langle x, y\rangle \in R^{\mathrm{op}}$; hence $\langle y, x\rangle \in R$. Since $\mathbb{X}$ is an n-permutable category, there exist approximate ternary co-operations $w_1, \ldots, w_{n-1}$ with approximation β. These are defined, for each object X in $\mathbb{X}$, as in Figure 5, where β_X is a regular epimorphism. Since R is a reflexive relation, we have $\langle x, x\rangle$, $\langle y, x\rangle$, $\langle y, y\rangle \in R$, so that also

$$\left\langle \left\langle \begin{smallmatrix} x \\ y \\ y \end{smallmatrix} \right\rangle, \left\langle \begin{smallmatrix} x \\ x \\ y \end{smallmatrix} \right\rangle \right\rangle \in R.$$

Precomposing with each approximate ternary co-operation, we get

$$\begin{cases} \left\langle x\beta_X, \left\langle \begin{smallmatrix} x \\ x \\ y \end{smallmatrix} \right\rangle (w_1)_X \right\rangle \in R \\ \left\langle \left\langle \begin{smallmatrix} x \\ y \\ y \end{smallmatrix} \right\rangle (w_2)_X, \left\langle \begin{smallmatrix} x \\ x \\ y \end{smallmatrix} \right\rangle (w_2)_X \right\rangle \in R \\ \quad \vdots \\ \left\langle \left\langle \begin{smallmatrix} x \\ y \\ y \end{smallmatrix} \right\rangle (w_{n-2})_X, \left\langle \begin{smallmatrix} x \\ x \\ y \end{smallmatrix} \right\rangle (w_{n-2})_X \right\rangle \in R \\ \left\langle \left\langle \begin{smallmatrix} x \\ y \\ y \end{smallmatrix} \right\rangle (w_{n-1})_X, y\beta_X \right\rangle \in R. \end{cases}$$

From $(\nabla_X + 1_X)(w_j)_X = (1_X + \nabla_X)(w_{j+1})_X$, for $j \in \{1, \ldots, n-2\}$, we can conclude that

$$\langle x\beta_X, y\beta_X\rangle = \langle x, y\rangle\beta_X \in R^{n-1}.$$

So $\langle x, y\rangle \in R^{n-1}$, since β_X is a regular epimorphism.

(b) ⇒ (a) Again we extend the proof of the same implication from Theorem 3.4. For any object X in $\mathbb{X}$, consider the following reflexive graph and the reflexive relation R on $2X$ which results by taking the (regular epi, mono) factorisation in

$$\begin{tikzcd}
3X \arrow[rr, shift left=2, "\nabla_X+1_X"] \arrow[rr, shift right=2, "1_X+\nabla_X"'] \arrow[dr, two heads, "\pi"'] & & 2X. \arrow[ll] \arrow[dl] \\
& R \arrow[ur, shift left=2, "r_1"] \arrow[ur, shift right=2, "r_2"'] &
\end{tikzcd}$$

From $\langle \iota_1, \iota_2 \rangle \in R$ we get $\langle \iota_2, \iota_1 \rangle \in R^{\mathrm{op}} \leqslant R^{n-1}$. So, there exists a regular epimorphism $\zeta\colon Z \twoheadrightarrow X$ together with morphisms $x_1, \ldots, x_{n-2}\colon Z \to 2X$ such that

$$\langle \iota_2\zeta, x_1 \rangle,\ \langle x_1, x_2 \rangle, \ldots,\ \langle x_{n-3}, x_{n-2} \rangle,\ \langle x_{n-2}, \iota_1\zeta \rangle \in R.$$

Let $k_1, \ldots, k_{n-1}\colon Z \to R$ be the morphisms such that $\langle r_1, r_2 \rangle k_1 = \langle \iota_2\zeta, x_1 \rangle$, $\langle r_1, r_2 \rangle k_i = \langle x_{i-1}, x_i \rangle$, $i \in \{2, \ldots, n-2\}$, and $\langle r_1, r_2 \rangle k_{n-1} = \langle x_{n-2}, \iota_1\zeta \rangle$. From the pullback

$$\begin{tikzcd}
B(X) \arrow[r, "\langle (w_{n-1})_X, \ldots, (w_1)_X \rangle"] \arrow[d, two heads, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & (3X)^{n-1} \arrow[d, two heads, "\pi^{n-1}"] \\
Z \arrow[r, "\langle k_1, \ldots, k_{n-1} \rangle"'] & R^{n-1}
\end{tikzcd}$$

we get morphisms $(w_1)_X$, …, $(w_{n-1})_X$ and a regular epimorphism defined by $\beta_X = \zeta\pi'$ such that the diagram in Figure 5 commutes. Then $\mathbb{X}$ is an n-permutable category by Theorem 4.3.

(a) $\Rightarrow$ (c) Let R be a reflexive relation on Y and consider morphisms a, $b\colon X \to Y$ such that $\langle a, b \rangle \in R^n$. Then there exists a regular epimorphism $\zeta\colon Z \twoheadrightarrow X$ and there exist morphisms $x_1, \ldots, x_{n-1}\colon Z \to Y$ such that $\langle a\zeta, x_1 \rangle$, $\langle x_1, x_2 \rangle$, …, $\langle x_{n-2}, x_{n-1} \rangle$, $\langle x_{n-1}, b\zeta \rangle \in R$. Since $\mathbb{X}$ is an n-permutable category, there are approximate ternary co-operations w_1, …, w_{n-1} with approximation β defined, for each object X in $\mathbb{X}$, as in Figure 5, where β_X is a regular epimorphism. Since R is a reflexive relation, we have

$$\langle a\zeta, x_1\rangle, \langle x_1, x_1\rangle, \langle x_1, x_2\rangle \in R \Rightarrow \left\langle a\zeta\beta_X, \left\langle \begin{smallmatrix} x_1 \\ x_1 \\ x_2 \end{smallmatrix} \right\rangle (w_1)_X \right\rangle \in R,$$

$$\langle x_1, x_2\rangle, \langle x_2, x_2\rangle, \langle x_2, x_3\rangle \in R \Rightarrow \left\langle \left\langle \begin{smallmatrix} x_1 \\ x_2 \\ x_2 \end{smallmatrix} \right\rangle (w_2)_X, \left\langle \begin{smallmatrix} x_2 \\ x_2 \\ x_3 \end{smallmatrix} \right\rangle (w_2)_X \right\rangle \in R,$$

$$\vdots$$

$$\langle x_{n-3}, x_{n-2}\rangle, \langle x_{n-2}, x_{n-2}\rangle, \langle x_{n-2}, x_{n-1}\rangle \in R \Rightarrow \left\langle \left\langle \begin{smallmatrix} x_{n-3} \\ x_{n-2} \\ x_{n-2} \end{smallmatrix} \right\rangle (w_{n-2})_X, \left\langle \begin{smallmatrix} x_{n-2} \\ x_{n-2} \\ x_{n-1} \end{smallmatrix} \right\rangle (w_{n-2})_X \right\rangle \in R,$$

$$\langle x_{n-2}, x_{n-1}\rangle, \langle x_{n-1}, x_{n-1}\rangle, \langle x_{n-1}, b\zeta\rangle \in R \Rightarrow \left\langle \left\langle \begin{smallmatrix} x_{n-2} \\ x_{n-1} \\ x_{n-1} \end{smallmatrix} \right\rangle (w_{n-1})_X, b\zeta\beta_X \right\rangle \in R.$$

Since $(\nabla_X + 1_X)(w_j)_X = (1_X + \nabla_X)(w_{j+1})_X$ for $j \in \{1, \ldots, n-2\}$, we conclude

$$\langle a\zeta\beta_X, b\zeta\beta_X\rangle = \langle a, b\rangle\zeta\beta_X \in R^{n-1},$$

so $\langle a, b\rangle \in R^{n-1}$ since ζ and β_X are regular epimorphisms.

(c) $\Rightarrow$ (b) By Lemma 4.4, we know that $\mathbb{X}$ is $(2n-2)$-permutable. Let R be a reflexive relation. Using the equivalence (a) $\Leftrightarrow$ (b) for $(2n-2)$-permutability, we have $R^{\mathrm{op}} \leqslant R^{2n-3}$. Using our assumption $R^n \leqslant R^{n-1}$ several (in fact, $n-2$) times we obtain

$$R^{\mathrm{op}} \leqslant R^{2n-3} \leqslant R^{2n-2} \leqslant \cdots \leqslant R^n \leqslant R^{n-1}.$$

This finishes the proof. ■

Corollary 4.6. *In an n-permutable category with binary coproducts, any reflexive and transitive relation is symmetric.*

Proof: It suffices to combine $R^{\mathrm{op}} \leqslant R^{n-1}$ for R reflexive with $RR \leqslant R$ for R transitive to see that $R^{\mathrm{op}} \leqslant R$. ■

Remark 4.7. We do not now whether the converse (n-permutability follows from any reflexive and transitive relation being symmetric) holds in categories; for varieties this is true, see [4]. However, any regular category in which every reflexive and symmetric relation is transitive has the Goursat property. In fact, for any reflexive relation E, the relation EE^{op} is reflexive and symmetric, hence it is transitive by assumption. So the category is 3-permutable, and consequently, n-permutable, by 1.8.

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