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SEMIDIRECT PRODUCTS AND SPLIT SHORT FIVE LEMMA IN NORMAL CATEGORIES

NELSON MARTINS-FERREIRA, ANDREA MONTOLI AND MANUELA SOBRAL

Abstract: In this paper we study a generalization of the notion of categorical semidirect product, as defined in [3], to a non-protomodular context of categories where internal actions are induced by points, like in any pointed variety. There we define semidirect products only for regular points, in the sense we explain below, provided the Split Short Five Lemma between such points holds, and we show that this is the case if the category is normal, as defined in [8]. Finally, we give an example of a category that is neither protomodular nor Mal'tsev where such generalized semidirect products exist.

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1. Introduction

The categorical definition of semidirect products was introduced by D. Bourn and G. Janelidze in [3], where they proved that, in the category of groups, this notion coincides with the classical one.

A characterization of pointed categories with categorical semidirect products was given in [10]. The existence of such products implies, in particular, that the category is protomodular. However there are many non-protomodular varieties where classical semidirect products exist and play an important role, like the category of monoids (and the same for the category of monoids with operations introduced in [11]).

Our main goal in [11] was to show that classical monoid actions/semidirect products correspond to a certain type of split extensions (the Schreier extensions), being distinct from internal actions/categorical semidirect products,

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in general.

Now we focus on the internal aspect in the context of pointed non-protomodular categories where every internal action is strict in the sense of [10]. In this case, if the category satisfies the Split Short Five Lemma for regular points, i.e. points such that the kernel and the section are jointly strongly epimorphic, then these points correspond to the internal actions via the generalized semidirect products. The equivalence between regular points and internal actions holds, in particular, in any normal variety and in any Barr-exact Mal'tsev normal category (in the sense of [8]).

The example of implication algebras shows that there are categories that are neither protomodular nor Mal'tsev where such generalized semidirect products exist.

2. Internal actions and categorical semidirect products

We start recalling the categorial definition of semidirect product introduced in [3]. For an object B of a category $\mathbb{C}$, we will denote by $Pt(B)$ the category of points (i.e. split epimorphisms) in $\mathbb{C}$ with codomain B.

Definition 2.1. *([3], Definition 3.2) A category $\mathbb{C}$ with split pullbacks is said to be a category with semidirect products if, for any arrow $p\colon E \to B$ in $\mathbb{C}$, the pullback functor $p^*\colon Pt(B) \to Pt(E)$ (has a left adjoint and) is monadic.*

In this case, denoting by T^p the monad defined by this adjunction, given a T^p-algebra (D,ξ) the semidirect product $(D,\xi) \rtimes (B,p)$ is the domain of the object in $Pt(B)$ corresponding to (D,ξ) via the canonical equivalence Φ:

$$\begin{tikzcd} & {[Pt(E)]^{T^p}} \arrow[d, shift right] \arrow[d, phantom, "\vdash"] \\ Pt(B) \arrow[ur, dashed, "\Phi"] \arrow[r, shift right, "p^*"'] \arrow[r, phantom, "\perp"] & Pt(E) \arrow[l, shift right, "p_!"'] \arrow[u, shift right] \end{tikzcd} \tag{1}$$

If $\mathbb{C}$ has split pullbacks, that is if we can define p^* for every morphism p, split pushouts of monomorphisms, so that the functors p^* have left adjoints $p_!$, and an initial object 0, then it is enough to consider the functors ${i_B}^*$ for the unique morphisms $i_B\colon 0 \to B$:

Proposition 2.2. *(*[12]*, Corollary* 3*) Let $\mathbb{C}$ be a category with finite limits, pushouts of split monomorphisms and initial object. Then the following statements are equivalent:*

(i) *all pullback functors i_B^* defined by the initial arrows are monadic;*
(ii) *for any morphism p in $\mathbb{C}$, the pullback functor p^* is monadic, i.e. $\mathbb{C}$ admits semidirect products.*

When the category $\mathbb{C}$ is pointed, the algebras for the monad (T^{i_B}, η, μ) are called *internal actions* in [1] and the endofunctor T^{i_B} is usually denoted by $B\flat(-)$.

We recall that η_X and μ_X are the unique morphisms such that $k_0\eta_X = \iota_X$ and $k_0\mu_X = [k_0, \iota_B]k_0'$, as displayed in the diagrams

$$\begin{tikzcd} B\flat X \arrow[r, "k_0"] & X+B \\ X \arrow[u, "\eta_X"] \arrow[ur, "\iota_X"'] \end{tikzcd} , \qquad \begin{tikzcd} B\flat(B\flat X) \arrow[r, "k_0'"] \arrow[d, "\mu_X"'] & (B\flat X)+B \arrow[d, "{[k_0,\iota_B]}"] \\ B\flat X \arrow[r, "k_0"'] & X+B, \end{tikzcd}$$

where k_0 and k_0' denote the kernels of $[0,1]\colon X+B \to B$ and of $[0,1]\colon (B\flat X)+B \to B$, respectively.

The algebras for this monad are pairs $(X, \xi\colon B\flat X \to X)$ satisfying the usual conditions:

$$\xi\eta_X = 1_X, \quad \text{and} \quad \xi\mu_X = \xi(1\flat\xi).$$

We denote by $Act(B)$ the category of algebras for the monad $B\flat(-)$, i.e. the category of internal actions, and by $\Phi_B\colon Pt(B) \to Act(B)$ the comparison functor of the adjunction $i_{B!} \dashv i_B^*$.

3. The comparison adjunction

Let $\mathbb{C}$ be a pointed, finitely complete and finitely cocomplete category. Then, in particular, the comparison functor Φ_B has a left adjoint L_B, for every object $B \in \mathbb{C}$. In this section we provide an explicit description of the corresponding comparison adjunction between the category of internal actions and the category of points.

Given a point (A, p, s) in $Pt(B)$, $\Phi_B(A, p, s)$ is a pair (X, ξ) where X is the kernel of p and ξ is the unique morphism induced by the universal property of the kernel, as in the following diagram:

$$\begin{tikzcd}
B\flat X \arrow[r, "k_0"] \arrow[d, "\xi"'] & X+B \arrow[r, shift left, "{[0,1]}"] \arrow[d, "{[k,s]}"'] & B \arrow[l, shift left, "\iota_B"] \arrow[d, equal] \\
X \arrow[r, "k"'] & A \arrow[r, shift left, "p"] & B. \arrow[l, shift left, "s"]
\end{tikzcd}$$

Given an internal action $(X, \xi) \in Act(B)$, consider the diagram

$$\begin{tikzcd}
B\flat X \arrow[r, "k_0"] \arrow[d, "\xi"'] & X+B \arrow[r, shift left, "{[0,1]}"] \arrow[d, "q"'] & B \arrow[l, shift left, "\iota_B"] \arrow[d, equal] \\
X \arrow[r, "q\iota_X"'] & Q \arrow[r, shift left, "p_\xi"] & B, \arrow[l, shift left, "s_\xi"]
\end{tikzcd}$$

where q is the coequalizer of k_0 and $\iota_X\xi$, $s_\xi = q\iota_B$ and p_ξ is defined by the universal property of q, since $[0,1]k_0 = 0 = [0,1]\iota_X\xi$. Hence $L_B(X,\xi) = (Q, p_\xi, s_\xi)$. We have that $p_\xi q\iota_X = 0$ but, in general, $q\iota_X$ is not the kernel of p_ξ. These are the object-functions of the two functors, being their definition on arrows straightforward.

The largest equivalence induced by the comparison adjunction $L_B \dashv \Phi_B$ is the adjoint equivalence

$$Fix(c) \underset{\Phi_B}{\overset{L_B}{\underset{\sim}{\leftrightarrows}}} Fix(u)$$

between the full subcategories $Fix(c)$ of $Pt(B)$ and $Fix(u)$ of $Act(B)$ whose objects are those for which the counit c and the unit u of the adjunction $L_B \dashv \Phi_B$ are isomorphisms, respectively.

Let $(A, p, s) \in Pt(B)$. Consider the diagram

$$\begin{tikzcd}
B\flat X \arrow[rr, "k_0"] \arrow[dr, "\xi'\cdot B\flat u_{(X,\xi)}"] \arrow[dd, "\xi"'] & & X+B \arrow[rr, shift left, "{[0,1]}"] \arrow[dr, "q"] \arrow[dd, "{[k,s]}"'] & & B \arrow[ll, shift left, "\iota_B"] \arrow[dr, equal] & \\
& X' \arrow[rr, "k'"] & & Q \arrow[rr, shift left, "p_\xi"] \arrow[dl, dotted, "c_{(A,p,s)}"] & & B \arrow[ll, shift left, "s_\xi"] \arrow[dl, equal] \\
X \arrow[ur, dotted, "u_{(X,\xi)}"] \arrow[rr, "k"'] & & A \arrow[rr, shift left, "p"] & & B, \arrow[ll, shift left, "s"] &
\end{tikzcd} \tag{2}$$

where $(X,\xi) = \Phi_B(A,p,s)$, $(Q,p_\xi,s_\xi) = L_B(X,\xi)$, and k and k' are the kernels of p and p_ξ, respectively. The two dotted morphisms are the component of the unit u and the counit c of the adjunction $L_B \dashv \Phi_B$: starting with $(X,\xi) \in Act(B)$, $u_{(X,\xi)}$ is the unique morphism such that $k'u_{(X,\xi)} = q\iota_X$, while, starting with $(A,p,s) \in Pt(B)$, $c_{(A,p,s)}$ is the unique morphism such that $c_{(A,p,s)}q = [k,s]$.

Therefore we have that $Fix(c)$ is the full subcategory of $Pt(B)$ whose objects are the points (A,p,s) such that the induced morphism $[k,s]$ from the coproduct $X+B$ is the coequalizer of k_0 and $\iota_X\xi$ and $Fix(u)$ is the full subcategory of $Act(B)$ whose objects are the internal actions (X,ξ) such that $q\iota_X$ is the kernel of p_ξ.

From now on, we will assume, in addition, that $\mathbb{C}$ is regular. We are going to analyze the categories $Fix(u)$ and $Fix(c)$. Let us start with the actions:

Proposition 3.1. *Let (X,ξ) be an internal action; the following conditions are equivalent:*

(i) *$(X,\xi) \in Fix(u)$, i.e. $u_{(X,\xi)}$ is an isomorphism;*
(ii) *$q\iota_X$ is a monomorphism;*
(iii) *the following square is a pullback:*

$$\begin{tikzcd} B\flat X \arrow[r, "k_0"] \arrow[d, "\xi"'] & X+B \arrow[d, "q"] \\ X \arrow[r, "q\iota_X"'] & Q. \end{tikzcd}$$

Proof: Consider the following diagram, where k' is the kernel of p_ξ:

$$\begin{tikzcd} B\flat X \arrow[r, "k_0"] \arrow[d, "\xi"'] & X+B \arrow[r, shift left, "{[0,1]}"] \arrow[d, "q"'] & B \arrow[l, shift left, "\iota_B"] \arrow[d, equal] \\ X \arrow[r, "q\iota_X"'] \arrow[d, "u_{(X,\xi)}"'] & Q \arrow[r, shift left, "p_\xi"] & B. \arrow[l, shift left, "s_\xi"] \\ X' \arrow[ur, "k'"'] & & \end{tikzcd}$$

Let us first observe that the square

$$\begin{tikzcd} B\flat X \arrow[r, "k_0"] \arrow[d, "u_{(X,\xi)}\xi"'] & X+B \arrow[d, "q"] \\ X' \arrow[r, "k'"'] & Q \end{tikzcd}$$

is a pullback. In fact, this is a particular case of the following known fact: in any commutative diagram

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "\alpha"'] & B \arrow[r, "f"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"] \\ A' \arrow[r, "k'"'] & B' \arrow[r, "f'"'] & C', \end{tikzcd}$$

if k is a kernel of f, k' is a kernel of f' and γ is a monomorphism, then the left-hand side square is a pullback.

Hence, since the category $\mathbb{C}$ is regular and q is a regular epimorphism, also $u_{(X,\xi)}\xi$ is, and so $u_{(X,\xi)}$ is always a regular epimorphism. Moreover, since $k'u_{(X,\xi)} = q\iota_X$ and k' is a monomorphism, we have that $u_{(X,\xi)}$ is a monomorphism (and hence an isomorphism) if and only if $q\iota_X$ is a monomorphism. This proves the equivalence between conditions (i) and (ii).

Let us now prove that (ii) implies (iii). Suppose that $q\iota_X$ is a monomorphism. If $f\colon C \to X$ and $g\colon C \to X+B$ are morphisms such that $qg = q\iota_X f$, then

$$[0,1]g = p_\xi qg = p_\xi q\iota_X f = 0,$$

and hence there exists a unique morphism $t\colon C \to B\flat X$ such that $k_0 t = g$. It remains to prove that $\xi t = f$, but this follows from the fact that

$$q\iota_X \xi t = qk_0 t = qg = q\iota_X f$$

and the fact that $q\iota_X$ is a monomorphism.

Finally, let us prove that (iii) implies (ii). Let $f_1, f_2 \colon C \to X$ be such that $q\iota_X f_1 = q\iota_X f_2$. Consider the following diagram:

$$
\begin{tikzcd}
P \arrow[r, "\bar{q}"] \arrow[d, dotted, shift right, "g_1"'] \arrow[d, dotted, shift left, "g_2"] \arrow[dd, bend right, "h"'] & C \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_2"] \\
B\flat X \arrow[r, "\xi"] \arrow[d, "k_0"] & X \arrow[d, "q\iota_X"] \\
X + B \arrow[r, "q"'] & Q,
\end{tikzcd}
$$

where the square below is a pullback and P is the pullback of q along $q\iota_X f_1 = q\iota_X f_2$. The two dotted arrows are induced by the universal property of the pullback, and we have

$$k_0 g_i = h, \quad \text{and} \quad \xi g_i = f_i \bar{q}, \quad i = 1, 2.$$

But k_0 is a monomorphism, so $g_1 = g_2$ and

$$f_1 \bar{q} = \xi g_1 = \xi g_2 = f_2 \bar{q}.$$

$\bar{q}$ is a regular epimorphism (because q is and the category is regular), hence $f_1 = f_2$ and $q\iota_X$ is a monomorphism. ∎

Internal actions satisfying Condition (iii) above were called *strict* in [10]. Under regularity of $\mathbb{C}$, they are exactly the objects of $Fix(u)$, as proved in Proposition 3.1, and so we denote this category by $StrAct(B)$. We point out that these are exactly what M. Hartl and B. Loiseau called internal actions in [5], in the context of homological categories.

The points for which the morphism $[k, s]$ is the coequalizer of k_0 and $\iota_X \xi$ were called *free split epimorphims* in [7]. Here we will denote $Fix(c)$ by $FPt(B)$ and call it the category of *free points.* By $RegPt(B)$ we denote the category of what we call *regular points* over B, i.e. points (A, p, s) such that $[k, s]$ is a regular epimorphism. It is clear that we have the inclusions

$$FPt(B) \subseteq RegPt(B) \subseteq Pt(B).$$

Both inclusions above are strict, in general. For example, in the category of monoids, if $\mathbb{N}$ is the monoid of natural numbers with the usual addition, the point

$$\mathbb{N} \xrightarrow{\langle 1,0\rangle} \mathbb{N}\times\mathbb{N} \underset{\langle 1,1\rangle}{\overset{\pi_2}{\rightleftarrows}} \mathbb{N};$$

is not regular, because $[\langle 1,0\rangle,\langle 1,1\rangle]\colon \mathbb{N}+\mathbb{N}\to\mathbb{N}\times\mathbb{N}$ is not a surjective homomorphism (hence a regular epimorphism): for instance, the element $(0,1)\in\mathbb{N}\times\mathbb{N}$ does not belong to its image. Moreover, as follows from Section 4 in [11], the point

$$\mathbb{N} \xrightarrow{\langle 1,0\rangle} A \underset{\langle 0,1\rangle}{\overset{\pi_2}{\rightleftarrows}} \mathbb{N},$$

where A is, as a set, the cartesian product of $\mathbb{N}$ with itself, and the monoid operation is defined by

$$(a_1,b_1)+(a_2,b_2)=(a_1+2^{b_1}a_2, b_1+b_2)$$

is regular but not free. In fact, the only free point over $\mathbb{N}$ with kernel $\mathbb{N}$ is the direct product $\mathbb{N}\times\mathbb{N}$.

It is well known that the comparison functor of an adjunction is fully faithful if and only if all the components of its counit are regular epimorphisms. In particular, for the adjunction

$$\begin{tikzcd} Pt(B) \arrow[r, shift right, "i_B^*"'] \arrow[r, phantom, "\perp"] & \mathbb{C} \arrow[l, shift right, "i_{B!}"'] \end{tikzcd},,$$

the components of the counit are $\varepsilon_{(A,p,s)}=[k,s]$ and so the comparison functor $\Phi\colon Pt(B)\to Act(B)$ is fully faithful, i.e. every point over B is free, if and only if $[k,s]$ is a regular epimorphism for every point (A,p,s). This means that $\mathbb{C}$ is protomodular [2].

Let us recall that a pointed category is protomodular if and only if the Split Short Five Lemma holds: for every morphism of points, i.e. for every commutative diagram of the form

$$\begin{tikzcd} X' \arrow[r, "k'"] \arrow[d, "g"'] & A' \arrow[r, shift left, "p'"] \arrow[d, "f"'] & B' \arrow[l, shift left, "s'"] \arrow[d, "h"] \\ X \arrow[r, "k"'] & A \arrow[r, shift left, "p"] & B, \arrow[l, shift left, "s"] \end{tikzcd} \tag{3}$$

where $p's' = 1$, $ps = 1$, k' is a kernel of p' and k is a kernel of p, if g and h are isomorphisms, then also f is. Hence we have the following:

Theorem 3.2. *(*[10]*, Theorem* 3.1*) A pointed, regular, finitely complete and finitely cocomplete category* $\mathbb{C}$ *has semidirect products if and only if the Split Short Five Lemma holds in* $\mathbb{C}$ *and every action is strict.*

4. The non-protomodular case

Now we are going to consider categories that are not protomodular, but where every action is strict. They obviously don't have semidirect products in the sense of [3], however we show that it is possible to obtain a sort of generalized semidirect products in this context.

Sufficient conditions for the internal actions in a category $\mathbb{C}$ to be strict were presented in [10]: this is true when $\mathbb{C}$ is a pointed variety of universal algebras and also when it is a Barr-exact, Mal'tsev ideal determined category. We recall from [6] the definition of ideal determined category:

Definition 4.1. *A pointed category* $\mathbb{C}$ *with finite limits and finite colimits is said to be ideal determined if the two following conditions hold:*

(A) *every morphism admits a pullback stable (normal epi, mono)-factorization, where a normal epimorphism is a cokernel of some morphism;*
(B) *for every commutative diagram*

$$\begin{tikzcd} F \arrow[r, "q"] \arrow[d, "w"'] & C \arrow[d, "v"] \\ E \arrow[r, "p"'] & B, \end{tikzcd}$$

where p *and* q *are normal epimorphisms,* v *and* w *are monomorphisms, if* w *is normal, then so is* v*.*

If $\mathbb{C}$ is regular, Condition (A) simply means that every regular epimorphism is normal. So, in our context, $\mathbb{C}$ satisfies Condition (A) if and only if it is *normal* in the sense of [8].

A pointed, regular, finitely complete and finitely cocomplete category $\mathbb{C}$ is normal if and only if every morphism with trivial kernel is a monomorphism and this is equivalent to the condition that every split epimorphism with trivial kernel is an isomorphism ([8], Propositions 3.9 and 3.12).

Let us also mention that S. Mantovani proved in [9] that a pointed Barr-exact Mal'tsev category is ideal determined provided that it is normal.

Lemma 4.2. *The Split Short Five Lemma holds for free points, i.e. if in the diagram (3) the two points involved are free and g and h are isomorphisms, then so is f.*

Proof: Without loss of generality, we can suppose that g and h are identities and consider the diagram

$$\begin{tikzcd}
B\flat X \arrow[rr, "k_0"] \arrow[dr, "\xi'"] \arrow[dd, "\xi"'] & & X+B \arrow[rr, shift left, "{[0,1]}"] \arrow[dr, "{[k',s']}"] \arrow[dd, "{[k,s]}"' near end] & & B \arrow[ll, shift left, "\iota_B"] \arrow[dr, equal] & \\
& X \arrow[rr, "k'"] & & A' \arrow[rr, shift left, "p'"] \arrow[dl, "f"'] & & B \arrow[ll, shift left, "s'"] \\
X \arrow[ur, equal] \arrow[rr, "k"] & & A \arrow[rr, shift left, "p"] & & B. \arrow[ll, shift left, "s"] \arrow[ur, equal] &
\end{tikzcd} \tag{4}$$

It is clear that $f[k',s'] = [k,s]$, then the triangle on the left commutes, and so $\xi = \xi'$. Since the points involved are free, both $[k,s]$ and $[k',s']$ are coequalizers of the pair $(k_0, \iota_X \xi)$, and this implies that f is an isomorphism. ∎

Theorem 4.3. *Let $\mathbb{C}$ be a pointed regular category with finite limits and finite colimits, such that every internal action is strict. The following conditions are equivalent:*

(i) *$RegPt(B) = FPt(B)$ for every $B \in \mathbb{C}$;*
(ii) *the Split Short Five Lemma holds for regular points.*

Proof: The implication (i) ⇒ (ii) follows immediately from Lemma 4.2. To prove the converse we consider the diagram

$$\begin{tikzcd}
B\flat X \arrow[rr, "k_0"] \arrow[dr, "\xi'\cdot B\flat u_{(X,\xi)}"] \arrow[dd, "\xi"'] & & X+B \arrow[rr, shift left, "{[0,1]}"] \arrow[dr, "q"] \arrow[dd, "{[k,s]}"' near end] & & B \arrow[ll, shift left, "\iota_B"] \arrow[dr, equal] & \\
& X' \arrow[rr, "k'"] & & Q \arrow[rr, shift left, "p_\xi"] \arrow[dl, dotted, "c_{(A,p,s)}"] & & B \arrow[ll, shift left, "s_\xi"] \\
X \arrow[ur, dotted, "u_{(X,\xi)}"'] \arrow[rr, "k"] & & A \arrow[rr, shift left, "p"] & & B, \arrow[ll, shift left, "s"] \arrow[ur, equal] &
\end{tikzcd} \tag{5}$$

where the point (A, p, s) is regular and q is the coequalizer of the pair $(k_0, \iota_x\xi)$. The morphism $u_{(X,\xi)}$ is an isomorphism by hypothesis, hence Condition (ii) implies that $c_{(A,p,s)}$ is an isomorphism, too. This means that $[k, s]$ is a coequalizer of the pair $(k_0, \iota_X\xi)$, and so the point (A, p, s) is free. ∎

Corollary 4.4. *If the equivalent conditions of Theorem 4.3 hold, the categories Act*(B) *and RegPt*(B) *are equivalent, for every object* $B \in \mathbb{C}$.

Proposition 4.5. *For a pointed, regular, finitely complete and finitely cocomplete category* $\mathbb{C}$*, the following conditions are equivalent:*

(i) $\mathbb{C}$ *is normal;*

(ii) *in the following commutative diagram*

$$\begin{tikzcd} X \arrow[r, "k'"] \arrow[d, equal] & A' \arrow[r, "p'"] \arrow[d, "f"'] & B \arrow[d, equal] \\ X \arrow[r, "k"'] & A \arrow[r, "p"'] & B, \end{tikzcd} \tag{6}$$

where k *is the kernel of* p *and* k' *is the kernel of* p'*, if* f *is a regular epimorphism, then it is an isomorphism;*

(iii) *in the following commutative diagram*

$$\begin{tikzcd} X \arrow[r, "k'"] \arrow[d, equal] & A' \arrow[r, shift left, "p'"] \arrow[d, "f"'] & B \arrow[l, shift left, "s'"] \arrow[d, equal] \\ X \arrow[r, "k"'] & A \arrow[r, shift left, "p"] & B, \arrow[l, shift left, "s"] \end{tikzcd}$$

where the two rows are points and the lower one is regular, f *is an isomorphism.*

Proof:

(i) ⇒ (ii) We have just to prove that f has trivial kernel $l : 0 \to A'$. Let $c \colon C \to A'$ be a morphism such that $fc = 0$. Hence $0 = pfc = p'c$, and since k' is the kernel of p', there exists a unique morphism $t \colon C \to X$ such that $c = k't$. Since

$$kt = fk't = fc = 0,$$

and k is a monomorphism, then $t = 0$ and so c factors (uniquely) through l. Hence $Ker(f)$ is trivial.

(ii) $\Rightarrow$ (iii) Given the diagram

$$\begin{tikzcd} X \arrow[r, "k'"] \arrow[d, equal] & A' \arrow[r, shift left, "p'"] \arrow[d, "f"'] & B \arrow[l, shift left, "s'"] \arrow[d, equal] \\ X \arrow[r, "k"'] & A \arrow[r, shift left, "p"] & B, \arrow[l, shift left, "s"] \end{tikzcd}$$

its commutativity implies that $f[k', s'] = [k, s]$, and since $[k, s]$ is a regular epimorphism, also f is. Hence the conclusion follows from (ii).

(iii) $\Rightarrow$ (i) It is enough to prove that every split epimorphism with trivial kernel is an isomorphism ([8], Propositions 3.9 and 3.12). Hence, given a split epimorphism f with trivial kernel, and section s, we consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & A \arrow[r, shift left, "f"] \arrow[d, "f"'] & B \arrow[l, shift left, "s"] \arrow[d, equal] \\ 0 \arrow[r] & B \arrow[r, shift left, "1_B"] & B. \arrow[l, shift left, "1_B"] \end{tikzcd}$$

Since the lower point is clearly a regular point, condition (iii) implies that f is an isomorphism.

■

Corollary 4.6. *If the category $\mathbb{C}$ is normal, then the Split Short Five Lemma holds for regular points.*

Proof: Condition (iii) in the previous Proposition obviously implies the Split Short Five Lemma for regular points. ■

Corollary 4.6 implies that the equivalent conditions of Theorem 4.3 hold when $\mathbb{C}$ is a normal variety and the same when $\mathbb{C}$ is a Barr-exact Mal'tsev normal category (which is then ideal determined, as already observed). So, in these categories, internal actions are equivalent to regular points. This can be considered a generalized semidirect product, in the sense that not every point corresponds to an action, but only the regular ones. This generalized semidirect product, although weaker, exists in a much wider context than the one considered in [3]. A concrete example is the following.

Example 4.7. *An implication algebra is a set X with a binary operation satisfying the following axioms:*

(1) $(xy)x = x$;
(2) $(xy)y = (yx)x$;
(3) $x(yz) = y(xz)$,

for every $x, y, z \in X$.

As observed in [4], these axioms imply that $xx = yy$ for every $x, y \in X$. Hence $1 := xx$ is an equationally defined constant satisfying $1x = x$ for every $x \in X$. Hence the category of implication algebras is pointed and, as proved in [4], it is a normal variety (actually it is an ideal determined category). But, as follows from a counterexample in [13], it is not a Mal'tsev category (because equivalence relations are not permutable), and hence it is not protomodular. Moreover, in [6], the authors used this example to prove that there are even ideal determined Mal'tsev varieties (hence Barr-exact ideal determined Mal'tsev categories) which are not protomodular.

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Nelson Martins-Ferreira
Departamento de Matemática, Escola Superior de Tecnologia e Gestão, Centro para o Desenvolvimento Rápido e Sustentado do Produto, Instituto Politécnico de Leiria, Leiria, Portugal
E-mail address: martins.ferreira@ipleiria.pt

Andrea Montoli
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: montoli@mat.uc.pt

Manuela Sobral
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: sobral@mat.uc.pt