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SPHERE ROLLING ON SPHERE - ALTERNATIVE APPROACH TO KINEMATICS AND CONSTRUCTIVE PROOF OF CONTROLLABILITY

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Abstract: We present an alternative approach to derive the kinematic equations for a system consisting of an Euclidean sphere rolling over another Euclidean sphere, subject to nonholonomic constraints of non-slip and non-twist, based on properties of rolling maps. This approach is suitable for the rolling of more general manifolds embedded in Euclidean space. It is well known that the sphere rolling on sphere system is controllable, except when the two spheres have equal radii. We also present a constructive proof of the controllability property, by showing how the forbidden motions can be performed by rolling without slip and twist. This is also illustrated for 2-dimensional spheres.

Keywords: Rolling maps, spheres, kinematics, non-twist, non-slip, geodesic rolls, tumbles, controllability.

1. Introduction

The most classical of all nonholonomic systems is the rolling sphere, rolling without slip or twist on its tangent plane at a point. Another interesting example of a system subject to nonholonomic constraints is that of a sphere rolling over another sphere of the same dimension. Rolling motions of manifolds embedded in Euclidean space $\mathbb{R}^n$ can be described by curves in the Lie group SE_n of orientation preserving isometries of the ambient space, as explained in Sharpe [12]. We take the definition in [12] and consequent properties of rolling maps to derive the kinematic equations for the rolling spheres. We also show how the forbidden motions, twists and slips, can be produced using rolling without slip/twist. This is a constructive proof of the complete controllability of the system, when the spheres have unequal radius.

The organization of the paper is as follows. The formal definition of rolling and properties of rolling maps appear in Section 2. The particular case of a sphere rolling on another sphere and the derivation of the corresponding kinematics are presented in Section 4. Finally, in Section 5 we include a constructive proof of controllability.

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In the final stage of preparation of this paper, we learned of the PhD thesis of Frenkel [2], which addresses the problem of controlling a 2-sphere rolling on another 2-sphere by means of motion along a minimal number of geodesic arcs. This is a generalization of the analogous problem for a 2-sphere rolling on a plane, posed by Kendall in the 1950's and solved by Hammersley [4], who has shown that three arcs are sufficient and necessary to steer any state to any other state. Frenkel's result is that four moves are sufficient for the former case. It is open whether three moves are sufficient in that situation. As progress towards solving these types of problems, the results for the 2-dimensional case in the second part of our paper lost originality, though our approach is different, and we believe more suitable towards obtaining explicit controllability of other rolling systems, a goal we are pursuing.

2. Rolling maps

We refer to Sharpe [12] and Lee [11] for details concerning differential and Riemannian geometry.

Let M and N be two smooth manifolds, with the same dimension, both isometrically embedded in the Euclidean space $\mathbb{R}^n$. Rolling maps describe how M rolls upon N, without slip or twist, along a curve α on M. Rolling is a rigid motion in the embedding space, subject to holonomic and nonholonomic constraints. A rolling motion is then described by the action of the isometry group on $\mathbb{R}^n$, preserving orientations. This is the special Euclidean group $\mathrm{SE}_n = \mathrm{SO}_n \ltimes \mathbb{R}^n = \{X = (R, s),\ R \in \mathrm{SO}_n,\ s \in \mathbb{R}^n\}$, with group operations

$$(R_1, s_1) \circ (R_2, s_2) = (R_1 R_2, R_1 s_2 + s_1),$$

$$(R, s)^{-1} = (R^{-1}, -R^{-1} s),$$

and the action in $\mathbb{R}^n$ is defined as

$$\begin{array}{ccc} \mathrm{SE}_n \ltimes \mathbb{R}^n & \rightarrow & \mathbb{R}^n \\ (X, p) & \mapsto & X(p) = R\,p + s. \end{array}$$

We adopt the definition of a rolling map given in Sharpe [12] and write some of the constraints in terms of R and s.

Definition 2.1. A rolling map of M upon N, without slip or twist, along a (piecewise) smooth curve $\alpha : [0, t_1] \rightarrow M$ is a mapping

$$\begin{array}{cccc} X: & [0, t_1] & \rightarrow & \mathrm{SE}_n = \mathrm{SO}_n \ltimes \mathbb{R}^n \\ & t & \mapsto & X(t) = (R(t), s(t)) \end{array} \qquad (2.1)$$

satisfying the following conditions, for all but finitely many $t \in [0, t_1]$:

- Rolling conditions
  - $X(t)(\alpha(t)) := \overline{\alpha}(t) \in N$.
  - $T_{X(t)(\alpha(t))}(X(t)(M)) = T_{\overline{\alpha}(t)}N$.
- No-slip condition
$$\dot{\overline{\alpha}}(t) = R(t)(\dot{\alpha}(t))$$
- No-twist conditions:
  - Tangential part: $\dot{R}(t)R^{\top}(t)\,(T_{\overline{\alpha}(t)}N) \subset (T_{\overline{\alpha}(t)}N)^{\perp}$.
  - Normal part: $\dot{R}(t)R^{\top}(t)\,(T_{\overline{\alpha}(t)}N)^{\perp} \subset T_{\overline{\alpha}(t)}N$.

The curve α on M is called the *rolling curve* and $\overline{\alpha}$ is called the *development* of α on N. The rolling conditions in the definition above are *holonomic constraints*, they correspond to admissible configurations of the two manifolds, while the non-slip and non-twist conditions are *nonholonomic constraints*. The second normal part of the no-twist conditions is always satisfied for manifolds of co-dimension 1. For the most classical of all rolling motions: the 2-sphere rolling on the tangent space at the south pole, the admissible configurations are all positions of the sphere in which it is tangent to the plane, while the nonholonomic constraints forbid any pure translation and any rotation around an axis orthogonal to the plane.

Remark 2.1. It has been proven in Sharpe [12] that for each piecewise smooth curve α on M there exists a unique rolling map having α as its rolling curve. In the situation when $M \equiv N$, the rolling map reduces to the identity map and the development curve coincides with the rolling curve. Also, if the rolling curve α belongs to the intersection of the two manifolds, then the corresponding rolling map reduces to the identity ($X(t) = (I, 0)$ satisfies all the conditions trivially) and $\alpha \equiv \overline{\alpha}$.

In what follows, if X is defined as in (2.1), $X(t)_*$ stands for the tangent map of $X(t)$ and X^{-1} stands for the mapping

$$\begin{array}{rccl} X^{-1}: & [0, t_1] & \to & \mathrm{SE}_n = \mathrm{SO}_n \ltimes \mathbb{R}^n \\ & t & \mapsto & X^{-1}(t) = (R^{-1}(t), -R^{-1}(t)s(t)) \end{array},$$

2.1. Properties of rolling motions. The following properties can easily be proven using the definition 2.1 and are of particular importance for our proposes. The first two have been derived in Sharpe [12]. Assume that three manifolds M_1, M_2 and M_3, embedded in Euclidean space, are tangent to each other at a point $p \in M_1 \cap M_2 \cap M_3$ and that $t \mapsto \alpha_1(t)$ is a curve in M_1 satisfying $\alpha_1(0) = p$.

(1) **Rolling motions are transitive**
Suppose that M_1 rolls on M_2 with rolling map X_1, rolling curve α_1, and development curve α_2. Also suppose that M_2 rolls on M_3 with rolling map X_2, rolling curve α_2, and development curve α_3. Then M_1 rolls on M_3 with rolling map $X_2 \circ X_1$, rolling curve α_1, and development curve α_3.

(2) **Rolling motions are symmetric**
Suppose that M_1 rolls on M_2 with rolling map X_1, rolling curve α_1, and development curve α_2. Then M_2 rolls on M_1 with rolling map ${X_1}^{-1}$, rolling curve α_2, and development curve α_1.

(3) **Rolling under a change of coordinates**
If M_1 rolls on M_2 with rolling map X_1, rolling curve α_1, and development curve $\overline{\alpha}_1$ and $X_c \in \mathrm{SE}_n$ is a fixed isometry, then $X_c(M_1)$ rolls on $X_c(M_2)$ with rolling map $X_c \circ X_1 \circ {X_c}^{-1}$, rolling curve $X_c(\alpha)$ and development curve $X_c(\overline{\alpha})$.

3. Kinematic equations of rolling

In this section we derived the kinematic equations for the motion of a smooth manifold rolling on the affine tangent space at a point. At first glance this may seem to be very restrictive. However, due to the definition of rolling and consequent properties, the results for this particular situation are the key to study more general rolling problems, as will be illustrated later for a sphere rolling on another sphere.

Assume that M is rolling on the affine tangent space at a point, i.e. $N = T^{\mathrm{aff}}_{p_0} M$, where $p_0 = \alpha(0) = \overline{\alpha}(0)$. The kinematic equations describe the translational and the rotational velocities of the rolling motion, starting from rest $(R(0), s(0)) = (I, 0)$ and so, they have the form

$$\begin{cases} \dot{s}(t) = u(t) \\ \dot{R}(t) = A(t)\, R(t) \end{cases},$$

for some vector valued function u taking values in $\mathbb{R}^n$ and A taking values in $\mathfrak{so}_n$ (the Lie algebra of SO_n, consisting of the skewsymmetric matrices). Conditions on these functions are determined from the holonomic and nonholonomic constraints.

When SO_n leaves M invariant, the rolling curve is always of the form $\alpha(t) = R(t)^\top p_0$, for some $R(t) \in \mathrm{SO}_n$. Under this assumption, the first rolling condition implies that $s(t) = \overline{\alpha}(t) - p_0 \in T_{p_0}M$ and, consequently, the no-slip condition becomes

$$\dot{s}(t) = -A(t)\, p_0.$$

On the other hand, the structure of $A(t) = \dot{R}(t)R^\top(t) \in \mathfrak{so}_n$ is determined from the no-twist conditions

$$\begin{aligned} A(t) \;\; T_{\overline{\alpha}(t)}N \qquad &\subset \; \left(T_{\overline{\alpha}(t)}N\right)^{\perp}, \\ A(t) \;\; \left(T_{\overline{\alpha}(t)}N\right)^{\perp} &\subset \; T_{\overline{\alpha}(t)}N. \end{aligned}$$

Consequently, for an appropriate choice of coordinates, the matrix function A has the following structure

$$A(t) = \left[\begin{array}{c|c} 0 & A_1(t) \\ \hline -A_1^\top(t) & 0 \end{array}\right], \tag{3.1}$$

where $A_1(t) \in \mathbb{R}^{m\times(n-m)}$. We can now write the kinematic equations for rolling the manifold M upon $N = T^{\mathrm{aff}}_{p_0}M$:

$$\left\{ \begin{array}{rcl} \dot{s}(t) & = & -A(t)\, p_0 \\ \dot{R}(t) & = & A(t)\, R(t) \end{array} \right., \tag{3.2}$$

where $A(t)$ has the structure (3.1).

Remark 3.1. When M is the $(n-1)$-sphere S centered at the origin, with radius ρ, and p_0 is its south or north pole, then

$$A_1(t) = \left[\begin{array}{c} u_1(t) \\ \vdots \\ u_{n-1}(t) \end{array}\right], \qquad A(t) = \textstyle\sum_{i=1}^{n-1} u_i(t) A_{i,n},$$

and the equations (3.2) for rolling S on its affine tangent space at p_0 reduce to the well know (see, for instance, [7]) kinematic equations

$$\left\{ \begin{array}{l} \dot{s}(t) = \varepsilon\rho u(t) \\ \dot{R}(t) = \left(\sum_{i=1}^{n-1} u_i(t) A_{i,n}\right) R(t) \end{array} \right.,$$

where $A_{i,j} = e_i e_j^\top - e_j e_i^\top$ are elementary skewsymmetric matrices, $\varepsilon = 1$ if p_0 is the south pole and $\varepsilon = -1$ if p_0 is the north pole. In this case, the rolling condition $X(t)(\alpha(t)) = \overline{\alpha}(t)$, where $X = (R, s)$, reduces to

$$R(t)\alpha(t) = p_0. \tag{3.3}$$

Remark 3.2. If the sphere is not centered at the origin, the kinematic equations can be easily derived from the above, using a convenient change of coordinates. For instance, consider the following case which will be useful later. Let M be a sphere of radius ρ, centered at the point $(0,\cdots,0,a)^\top$. We can obtain the rolling map for the rolling motion of M on its affine tangent space at the north pole $p_0 = (0,\cdots,0,a+\rho)^\top$ from the rolling map $X = (R,s)$ of the sphere S in previous remark and the isometry $X_\tau = (I,\tau) \in \mathrm{SE}_n$, where $\tau = (0,\cdots,0,a)^\top$. In this case, X_τ is a pure translation and the translation vector τ sends S to $M = S+\tau$. In this situation,

$$X_\tau \circ X \circ X_\tau^{-1} = (R, -R\tau + s + \tau)$$

is the rolling map for the rolling motion of M upon its affine tangent space at the point p_0, with rolling curve $\alpha+\tau$ and development $\overline{\alpha}+\tau$.

4. A sphere rolling on another sphere

The most classical of all nonholonomic problems is that of a sphere rolling on its tangent plane at a point. Other rolling spheres problems have been studied (see, for instance, Montgomery [10], Jurdjevic [7] and more recently Jurdjevic and Zimmerman [8], Bloch and Rojo [1]). The kinematic equations for rolling a sphere on another sphere are known, but we present here an alternative approach which uses the transitive and symmetric properties of rolling and the kinematic equations of a sphere rolling on its affine tangent space. This approach is simple and may be used with great success for other manifolds. We first analyze the case of a sphere rolling over the outside of another sphere (no restrictions on the size of their radius is necessary). We then make the obvious changes and comment on the situation when a sphere rolls over the inside of a sphere with larger radius. We will also make connections with results in the existing literature on the subject.

4.1. A sphere rolling over the outside of another sphere. In this section we consider two spheres of the same dimension m, embedded in the Euclidean space $\mathbb{R}^n$: S_1 with radius ρ_1 and S_2 with radius ρ_2. Suppose that the sphere S_2 is centered at the origin and is stationary. Assume that S_1 is centered at the point $c = (0,\cdots,0,-(\rho_1+\rho_2))^\top$, so that at time $t=0$ it is tangent to S_2 at the south pole of S_2, $p_0 = (0,\cdots,0,-\rho_2)^\top$. Assume now that S_1 starts rolling over S_2, along a piecewise smooth curve α, satisfying $\alpha(0) = p_0$.

Our objective is to derive the kinematic equations for the rolling motion of S_1 on the outside of the stationary sphere S_2. This will be accomplished by using the kinematic equations derived in Section 3, for rolling a manifold on the affine tangent space at a point, together with the symmetric and transitive properties and remarks contained in Section 2.

Let N denote the affine tangent space to S_2 at p_0, which also coincides with the affine tangent space to S_1 at the same point. We know how to roll the spheres S_1 and S_2 on N. Consequently, we know how to roll S_1 on N and N on S_2. Thus, by transitivity, we can achieve our goal.

• **Rolling S_1 over $N = T^{\mathrm{aff}}_{p_0} S_1$:**

For a sphere with radius ρ_1 centered at the origin and rolling on the affine tangent space at the north pole q_0, the kinematic equations are

$$\begin{cases} \dot{s} &= -A(t)\, q_0 \\ \dot{R} &= A(t)\, R \end{cases}, \tag{4.1}$$

where $A(t) = \sum_{i=1}^{n-1} u_i(t) A_{i,n}$, for some scalar functions $u_1, \cdots, u_{n-1}$. Also, from (3.3), the rolling curve α satisfies

$$R(t)\alpha(t) = q_0. \tag{4.2}$$

So, according to Remark 3.2, the rolling map for S_1 over N is defined by $X_1 = (R_1, s_1) = (R, -R\tau + s + \tau)$, where τ is the translation vector $(0, \cdots, 0, -(\rho_1 + \rho_2))^\top$ with kinematic equations

$$\begin{cases} \dot{s}_1 &= -A_1(t)\,(q_0 + R_1\tau) \\ \dot{R}_1 &= A_1(t)\, R_1 \end{cases}, \tag{4.3}$$

where $A_1 \equiv A$, having rolling curve $\alpha_1 = \alpha + \tau$ and development curve $\overline{\alpha_1} = \overline{\alpha} + \tau$. It follows from (4.2) that

$$R\alpha + \tau = p_0. \tag{4.4}$$

• **Rolling S_2 over $N = T^{\mathrm{aff}}_{p_0} S_2$:**

The sphere S_2 is centered at the origin and has radius ρ_2. So, $(X_2 = (R_2, s_2)$ is the rolling map for rolling S_2 over the affine tangent space at the south pole

p_0, and the corresponding kinematic equations are given by:

$$\left\{ \begin{array}{rcl} \dot{s}_2 & = & -A_2(t)\, p_0 \\ \dot{R}_2 & = & A_2(t)\, R_2 \end{array} \right. , \tag{4.5}$$

with $A_2(t) = \sum_{i=1}^{n-1} v_i(t) A_{i,n}$, for some scalar functions $v_1, \cdots, v_m$. Moreover, the rolling curve α_2 satisfies

$$R_2 \alpha_2 = p_0. \tag{4.6}$$

For our purpose, we assume that the development curve $\overline{\alpha_2}$ coincides with $\overline{\alpha_1}$. According to the symmetric property of rolling in Section 2, N rolls upon S_2 with rolling map $X_2 = (R_2, -{R_2}^\top s_2)$, rolling curve $\overline{\alpha_1} \in N$ and development curve $\overline{\alpha_2} \in S_2$.

• **Rolling S_1 over S_2:**

Applying now the transitive property of rolling in Section 2, with $M_1 = S_1$, $M_2 = N$ and $M_3 = S_2$, we conclude the following: S_1 rolls upon S_2 with rolling map $X_3 = {X_2}^{-1} \circ X_1 = ({R_2}^\top R_1, {R_2}^\top (s_1 - s_2))$, rolling curve $\alpha_1 \in S_1$ and development curve $\alpha_2 \in S_2$, so that

$$X_3(\alpha_1) = \alpha_2. \tag{4.7}$$

We now show that, under the assumption

$$\overline{\alpha_2} = \overline{\alpha_1}, \tag{4.8}$$

the matrices A_1 and A_2 in (4.3) and (4.5) respectively, are related through

$$A_2 = -\frac{\rho_1}{\rho_2} A_1. \tag{4.9}$$

This is a consequence of the following simple calculations, where the conditions (4.4) and (4.6) are used.

$$
\begin{aligned}
& X_3(\alpha_1) = \alpha_2 \\
\Leftrightarrow\ & {R_2}^\top R_1 \alpha_1 + {R_2}^\top (s_1 - s_2) = \alpha_2 \\
\Leftrightarrow\ & {R_2}^\top R_1 \alpha_1 + {R_2}^\top (s_1 - s_2) = {R_2}^\top p_0 \\
\Leftrightarrow\ & R_1 \alpha_1 + s_1 - s_2 = p_0 \\
\Leftrightarrow\ & R_1 \alpha + R_1 \tau + s_1 - s_2 = p_0 \\
\Leftrightarrow\ & p_0 - \tau + R_1 \tau + s_1 - s_2 = p_0 \\
\Leftrightarrow\ & s_1 - s_2 = \tau - R_1 \tau.
\end{aligned}
$$

Consequently,

$$\dot{s}_2 - \dot{s}_1 = \dot{R}_1 \tau = A_1 R_1 \tau. \tag{4.10}$$

On the other hand, using the kinematic equations (4.3) and (4.5), we have

$$\dot{s}_2 - \dot{s}_1 = -A_2 p_0 + A_1 q_0 + A_1 R_1 \tau, \tag{4.11}$$

and by comparison of (4.10) and (4.11), it follows that

$$A_1 q_0 = A_2 p_0. \tag{4.12}$$

Finally, the relationship $A_2 = \frac{\rho_1}{\rho_2} A_1$ follows from here, taking into account the particular structure of the matrices A_1 and A_2 and the fact that $p_0 = (0 \cdots, 0, -\rho_2)^\top$ and $q_0 = (0 \cdots, 0, \rho_1)^\top$. In conclusion, we may state the following.

Theorem 4.1. Suppose that S_1 starts rolling over S_2 without slip or twist along a curve α_1 satisfying $\alpha_1(0) = p_0$. Then, the corresponding rolling map is given by

$$X_3 = ({R_2}^\top R_1, {R_2}^\top (s_1 - s_2)),$$

where s_1, s_2, R_1 and R_2 are the solutions of the following differential equations

$$\left\{\begin{array}{lcl}\dot{s}_1 & = & -U(t)(p_0-\tau+R_1\tau)\\ \dot{s}_2 & = & -U(t)q_0\\ \dot{R}_1 & = & +U(t)R_1\\ \dot{R}_2 & = & -\frac{\rho_1}{\rho_2}U(t)R_2\end{array}\right. , \tag{4.13}$$

where

$$U(t)=\left[\begin{array}{c|c}0 & u(t)\\ \hline -u^\top(t) & 0\end{array}\right], \tag{4.14}$$

for some vector function u depending on the rolling curve α_1. Moreover, along the rolling motion, the point of contact p_0 traces out the curve $\alpha_2 = {R_2}^\top p_0$ on S_2.

It is straight forward to conclude from the above relations that $u(t) = -\frac{1}{\rho_1}\dot{\overline{\alpha_1}}$. Clearly u is a constant function if and only if $\overline{\alpha_1}$ is a geodesic on N. It is well known (Sharpe [12]) that the development of a geodesic curve is a geodesic. So, the case when u is constant corresponds to the situation when the rolling curve α_1 is a geodesic on S_1 and, consequently, its development α_2 is also a geodesic on S_2.

With appropriate changes in notation, the equations (4.13) are in accordance with Proposition 2.3 in [8].

4.2. A sphere rolling on the inside of another sphere. Now consider two spheres of the same dimension $n-1$, embedded in the Euclidean space $\mathbb{R}^n$, S_1 with radius ρ_1 and S_2 with radius $\rho_2 > \rho_1$. Suppose that the sphere S_2 is centered at the origin and is stationary. Assume that S_1 is centered at the point $c = (0,\cdots,0,-(\rho_2-\rho_1))^\top$, so that at time $t=0$ it is tangent to S_2 at the south pole of S_2, $p_0 = (0,\cdots,0,-\rho_2)^\top$, (which also coincides with the south pole of S_1). Assume now that S_1 starts rolling inside S_2, along a piecewise smooth curve α, satisfying $\alpha(0) = p_0$.

This situation is similar to the previous. It is enough to replace q_0 by $(0,\cdots,0,-\rho_1)^\top$ and τ by $(0,\cdots,0,-(\rho_2-\rho_1))^\top$.

5. Constructive proof of controllability

Consider again a sphere rolling on the outside of another sphere, as in Section 4.1. As before, the initial point of contact is p_0.

When $n=3$, the spheres are two-dimensional and when S_1 rolls over S_2 without twisting or slipping, the no-twist condition prevents rotations of S_1 about the axis $\overline{0p_0}$ (*twists at* p_0), while the no-slip condition forbids slipping motions which may be thought of as transport of S_1 over a geodesic or as rotations of S_1 about an axis through the center of S_2 (the origin) and perpendicular to $\overline{0p_0}$ (these we term *slips from* p_0).

It is well known that the system is controllable when the spheres have unequal radii. We prove that result by constructing motions that achieve the effects of twists and slips by means of rolling, in the spirit of [9]. This is first done for the case $n = 3$, in Section 5.2. In Section 5.3, we define the higher-dimensional analogues of twists and slips, show how to synthesize them using rolling motions and finally establish that those constructions suffice for controllabilty.

An Euclidean transformation $(R, s) \in \mathrm{SE}_n$ is *constructible* if we can exhibit a rolling motion $X(t)$ such that $X(T) = (R, s)$ at some $T > 0$. Our goal is to show that all twists and all slips are constructible. We will do so using only piecewise constant control functions.

It will also be convenient to re-parametrize system (4.13) so that t is arclength of the rolling and development curves. For ease of notation, we will henceforth assume $\rho_2 = 1$. There is also no loss in assuming $\rho_1 < 1$, and we put $\gamma = \rho_1$. Then

$$\begin{cases} \frac{d}{dt}(s_1 - s_2) & = & -\frac{1}{\gamma}U(t)R_1\tau \\ \frac{dR_1}{dt} & = & +\frac{1}{\gamma}U(t)R_1 \\ \frac{dR_2}{dt} & = & -U(t)R_2 \end{cases} . \tag{5.1}$$

5.1. Piecewise constant controls. As remarked at the end of Section 4.1, the control function $U(t)$ is piecewise constant iff both the rolling and the development are piecewise geodesics of their respective spheres. We now describe such rolling motions by integrating (5.1) from $(R_1, s_1) = (R_2, s_2) = (I, 0)$ at $t = 0$.

Let $\alpha_2 : [0, T] \mapsto S_2$ be a piecewise geodesic development curve starting at p_0. Given a partition $t_0 = 0 < t_1 < \cdots < t_N = T$ of $[0, T]$, let $I_i = (t_{i-1}, t_i)$, $\tau_i = t_i - t_{i-1}$. Choose a partition so that,

$$\alpha_2(t) = e^{(t-t_{i-1})V_i} e^{\tau_{i-1}V_{i-1}} \cdots e^{\tau_1 V_1} p_0, \ \forall t \in I_i, \ 1 \le i \le N. \tag{5.2}$$

Suppose $\bar{t} \in I_i$ for the remainder of the Subsection. From $\alpha_2 = R_2^\top p_0$,

$$R_2(\bar{t}) = \left(e^{\tau_{i-1}V_{i-1}} \cdots e^{\tau_1 V_1}\right)^{-1} e^{-\left(\bar{t} - t_{i-1}\right)V_i},$$

whence, from $\dot{R}_2 = -UR_2$, the functions $U(t)$ and $V(t)$ are related by

$$U(\bar{t}) = U_i = \left(e^{\tau_{i-1}V_{i-1}} \cdots e^{\tau_1 V_1}\right)^{-1} V_i \left(e^{\tau_{i-1}V_{i-1}} \cdots e^{\tau_1 V_1}\right). \tag{5.3}$$

We note that, for each $k = 1, \ldots, N$,

$$e^{\tau_k V_k} \cdots e^{\tau_1 V_1} = e^{\tau_1 U_1} \cdots e^{\tau_k U_k}. \tag{5.4}$$

We make explicit the induction step in checking the previous formula

$$\begin{aligned} & e^{\tau_{k+1}V_{k+1}} \left(e^{\tau_k V_k} \cdots e^{\tau_1 V_1}\right) \\ &= \left(\left(e^{\tau_k V_k} \cdots e^{\tau_1 V_1}\right) e^{\tau_{k+1}U_{k+1}} \left(e^{\tau_k V_k} \cdots e^{\tau_1 V_1}\right)^{-1}\right) \left(e^{\tau_k V_k} \cdots e^{\tau_1 V_1}\right) \\ &= \left(e^{\tau_1 U_1} \cdots e^{\tau_k U_k}\right) e^{\tau_{k+1}U_{k+1}}. \end{aligned}$$

Combining (5.3) followed by (5.4), we get

$$\begin{aligned} V_i &= \left(e^{\tau_{i-1}V_{i-1}} \cdots e^{\tau_1 V_1}\right) U_i \left(e^{\tau_{i-1}V_{i-1}} \cdots e^{\tau_1 V_1}\right)^{-1}. \\ &= \left(e^{\tau_1 U_1} \cdots e^{\tau_{i-1}U_{i-1}}\right) U_i \left(e^{\tau_1 U_1} \cdots e^{\tau_{i-1}U_{i-1}}\right)^{-1}. \end{aligned}$$

From $\dot{R}_1 = \frac{1}{\gamma} U R_1$ and the fact that $U(t)$ is piecewise constant,

$$R_1(\bar{t}) = e^{\frac{1}{\gamma}(\bar{t} - t_{i-1})U_i} e^{\frac{\tau_{i-1}}{\gamma}U_{i-1}} \cdots e^{\frac{\tau_1}{\gamma}U_1}.$$

Finally, from $\frac{d}{dt}(s_1 - s_2) = -\frac{1}{\gamma} U R_1 \tau = -\frac{d}{dt}(R_1)\tau$ we have

$$(s_1 - s_2)(t) = (I - R_1(t))\tau.$$

Remark 1. $U_i p_0$ is the pullback of $V_i \alpha_2$ along α_2.

5.2. The case n=3, main results. Choose coordinates so that the initial point of contact of the spheres is $p_0 = (0,0,1)$. Then $q_0 = (0,0,-\rho_1)$, and $\tau = p_0 - q_0 = (0,0,1+\rho_1)$.

Letting, as in Section 3, $A_{ij} = e_i e_j^\top - e_j e_i^\top$, we now define the 3×3 matrices

$$A_y := A_{13},\ A_x := A_{23},\ A_z := A_{12}$$

and also

$$A(\theta) := A_y \cos\theta + A_x \sin\theta. \tag{5.5}$$

Note that $\frac{d}{dt}\Big|_{t=0} \left(e^{tA(\theta)} p_0\right) = (\cos\theta, \sin\theta, 0)$, so that geometrically the rotation matrix $e^{tA(\theta)}$ moves the north pole p_0 in the direction with angle θ in the tangent space to S_2 at p_0, identified with the the xy-plane. With the above choices, a twist at p_0 is $\left(e^{\alpha A_z}, 0\right)$ and a slip from p_0 is $\left(e^{tA(\theta)}, 0\right)$.

If a constant control

$$U = \left[\begin{array}{cc|c} 0 & 0 & u_1 \\ 0 & 0 & u_2 \\ \hline -u_1 & -u_2 & 0 \end{array}\right]$$

has norm one ($u_1^2 + u_2^2 = 1$), then $U = A(\theta)$ for some θ. Moreover, $e^{2\pi U} = I$, as can be seen from (5.18) in the Auxiliary results.

Remark 5.1. Let α_2 be a piecewise constant development as in (5.2), with $U_i = A(\theta_i)$ for each i. Then α_2 takes a left turn (as seen from the outside of S_2) with internal angle $\Delta\theta$ at $\alpha_2(t_i)$ iff

$$\theta_{i+1} = \pi + \theta_i - \Delta\theta. \tag{5.6}$$

The analogous formula for a right turn at the same point is $\theta_{i+1} = \pi + \theta_i + \Delta\theta$.

A *half-tumble* is a rolling motion of S_1 over S_2 corresponding to a geodesic development with length $\pi\gamma$.

The development of a sequence of two half-tumbles is

$$\alpha_2(t) = \begin{cases} e^{tV_1}p_0, & t \in (0, \gamma\pi) \\ e^{(t-\pi\gamma)V_2}e^{\pi\gamma V_1}p_0, & t \in (\pi\gamma, 2\gamma\pi) \end{cases}.$$

Letting $U_i = A(\theta_i)$, $i = 1, 2$, relation (5.3) reads $U_1 = A(\theta_1) = V_1$

$$U_2 = A(\theta_2) = e^{-\pi\gamma V_1}V_2e^{\pi\gamma V_1}.$$

Noting that $\tau_1 = \tau_2 = \pi\gamma$, the resulting rolling motions satisfy

$$\begin{aligned} R_2(2\pi\gamma) &= \left(e^{\pi\gamma V_2}e^{\pi\gamma V_1}\right)^{-1} \\ &= \left(e^{\pi\gamma A(\theta_1)}e^{\pi\gamma A(\theta_2)}\right)^{-1} \\ &= e^{-\pi\gamma A(\theta_2)}e^{-\pi\gamma A(\theta_1)}, \\ R_1(2\pi\gamma) &= e^{\pi A(\theta_2)}e^{\pi A(\theta_1)}, \end{aligned}$$

and so

$$\begin{aligned} R\left(2\pi\gamma\right) &= e^{\pi\gamma A(\theta_1)} e^{\pi\gamma A(\theta_2)} e^{\pi A(\theta_2)} e^{\pi A(\theta_1)}, && (5.7) \\ \left(s_1 - s_2\right)\left(2\pi\gamma\right) &= \left(I - R_1\left(2\pi\gamma\right)\right)\tau, \\ &= \left(I - e^{\pi A(\theta_2)} e^{\pi A(\theta_1)}\right)\tau = 0, \\ s\left(2\pi\gamma\right) &= R_2^T\left(s_1 - s_2\right) = 0. && (5.8) \end{aligned}$$

A *tumble* is a rolling motion of S_1 over S_2 corresponding to a geodesic development of length $2\pi\gamma$. This is a sequence of two half-tumbles with $U = A\left(\theta_1\right) = A\left(\theta_2\right)$. From the preceding discussion, $R_1\left(2\pi\gamma\right) = I$, $R_2\left(2\pi\gamma\right) = e^{-2\pi\gamma A(\theta)}$, so that, at $t = 2\pi\gamma$,

$$(R, s) = \left(e^{2\pi\gamma U}, 0\right).$$

A concatenation of n tumbles from p_0 has development (5.2) with $\tau_i = 2\pi\gamma$ for each i and thus satisfies $\left(R\left(2\pi\gamma n\right), s\left(2\pi\gamma n\right)\right) = (R, 0)$ where

$$R = \left(e^{2\pi\gamma V_n} \cdots e^{2\pi\gamma V_1}, 0\right) = \left(e^{2\pi\gamma U_1} \cdots e^{2\pi\gamma U_n}, 0\right). \tag{5.9}$$

Proposition 5.1. When $0 < \gamma < \frac{1}{4}$ or $\frac{3}{4} < \gamma < 1$, any twist is constructible and can be realized with finitely many sequences of four tumbles.

Proof: We first prove the Proposition for $0 < \gamma < \frac{1}{4}$. Given $\alpha \in (0, \pi/2)$, consider a spherical quadrangle $[p_0, A, B, C]$ on S_2 with interior angle 2α at p_0 and at the opposite vertex B and each of whose four arcs has length $T = 2\pi\gamma$. Note that $0 < T < \pi/2$. For definiteness, the vertices are labeled counter-clockwise, as seen from the outside of S_2 (See Figure 5.1).

We compute the angle β at A and C. The spherical triangle $[p_0, A, B]$ has interior angles α, β, and again α at those vertices. Let W be the arc-angle of p_0B. By the law of sines of spherical trigonometry,

$$\frac{\sin\alpha}{\sin T} = \frac{\sin\beta}{\sin W} \tag{5.10}$$

and by the law of cosines,

$$\cos W = \cos^2 T + \sin^2 T \cos\beta. \tag{5.11}$$

For convenience, put $\Gamma = \cos(T)$, so that $0 < \Gamma < 1$. Then $\cos W = \Gamma^2 + \left(1 - \Gamma^2\right)\cos\beta$. The previous relations imply

$$\cos\beta = \frac{\Gamma^2 \sin^2\alpha - \cos^2\alpha}{\Gamma^2 \sin^2\alpha + \cos^2\alpha}, \tag{5.12}$$

which uniquely defines $\beta \in [0, \pi]$. For later use, we rewrite these relations in terms of $\eta = 2\alpha$. Relation (5.12) becomes

$$\cos\beta = \frac{\Gamma^2\left(\frac{1-\cos\eta}{2}\right) - \left(\frac{1+\cos\eta}{2}\right)}{\Gamma^2\left(\frac{1-\cos\eta}{2}\right) + \left(\frac{1+\cos\eta}{2}\right)} = \frac{\left(\Gamma^2 - 1\right) - \left(\Gamma^2 + 1\right)\cos\eta}{\left(\Gamma^2 + 1\right) - \left(\Gamma^2 - 1\right)\cos\eta}. \tag{5.13}$$

and, because $\beta, \eta \in (0, \pi)$,

$$\sin\beta = \frac{2\Gamma\sin\eta}{\left(\Gamma^2 + 1\right) - \left(\Gamma^2 - 1\right)\cos\eta}. \tag{5.14}$$

Note that, for fixed Γ, $\beta(\alpha, \Gamma)$ decreases from π to 0 as α increases from 0 to $\pi/2$. This is easily seen by considering

$$f(x) = \frac{\Gamma^2 - \left(\Gamma^2 + 1\right)x}{\Gamma^2 - \left(\Gamma^2 - 1\right)x},$$

so that $\cos\beta = f\left(\cos^2\alpha\right)$, and noting that $f(1) = -1$, $f(0) = 1$ and that $\frac{d}{d\alpha}\left(f\left(\cos^2\alpha\right)\right) < 0$.

We consider the rolling motion with $[p_{0,}A, B, C, p_0]$ as its development. We describe this development in the form (5.2) for suitable $A(\theta_i)$ by repeatedly applying (5.6). Let $\theta_1 = \alpha + \beta$. At vertex A, the internal angle must be β and thus $\theta_2 = \pi + (\alpha + \beta) - \beta = \pi + \alpha$. Likewise, the internal angle at B must be 2α, so $\theta_3 = \pi + (\pi + \alpha) - 2\alpha = -\alpha$ and the internal angle at C is β and so $\theta_4 = \pi + (-\alpha) - \beta = \pi - \alpha - \beta$.

Recalling (5.9), at $t = 4T$ the rolling motion is $(R(4T), s(4T)) = (R, 0)$, where

$$\begin{aligned} R =& e^{TA(\theta_1)} e^{TA(\theta_2)} e^{TA(\theta_2)} e^{TA(\theta_4)} \\ =& e^{TA(\alpha+\beta)} e^{TA(\pi+\alpha)} e^{TA(-\alpha)} e^{TA(\pi-\alpha-\beta)} \end{aligned}$$

Using (5.20),

$$R = e^{TA(\alpha+\beta)} e^{-TA(\alpha)} e^{TA(-\alpha)} e^{-TA(-\alpha-\beta)}.$$

We show in Lemma 5.5 that this last expression equals $e^{-(4\alpha+2\beta)A_z}$, a twist at p_0, as required.

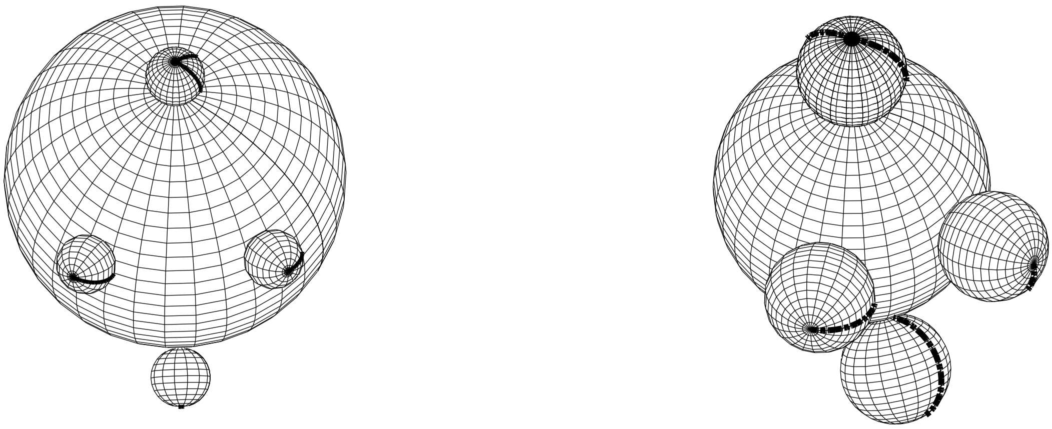

FIGURE 5.1. Two instances of maneuvers described in Propositions 5.1 and 5.2, for $0 < \gamma < \frac{1}{4}$, and then for $\frac{1}{4} < \gamma < \frac{1}{2}$. In either case, the rolling is counter-clockwise from the north pole.

Lemma 5.9 ensures that, for $\Gamma \neq 1$, the range I of $4\alpha+2\beta$ is never a singleton, as α takes values in $[0, \pi/2]$. This establishes that finitely many tumbles suffice to achieve any twist at p_0. (The number of tumbles required may be computed from I, which is made explicit in that Lemma.) We also remark that, as $\gamma \nearrow \frac{1}{4}$, I does narrow to a singleton.

If $\frac{3}{4} < \gamma < 1$, we define a similar quadrangle as before, but now with $T = 2\pi(1-\gamma)$. Consider a rolling motion (R, s) that traverses the same sequence of points

$$(p_0, A, B, C, p_0)$$

but now moving from p_0 to A along the *larger* portion (of length $\tilde{T} = 2\pi - T$) of the great circle containing those points, and so on. Since $e^{\tilde{T}A(\theta+\pi)} = e^{-TA(\theta+\pi)} = e^{TA(\theta)}$, we have again

$$R\left(4\tilde{T}\right) = e^{-(4\alpha+2\beta)A_z}.$$

■

Proposition 5.2. When $\frac{1}{4} < \gamma < \frac{1}{2}$ or $\frac{1}{2} < \gamma < \frac{3}{4}$, any twist is constructible and can be realized with finitely many sequences of four half-tumbles.

Proof: First we consider the case $\frac{1}{4} < \gamma < \frac{1}{2}$. Let $\tilde{\gamma} = \frac{\gamma}{2}$ and define the same quadrangle as in the previous Proposition, now with four arcs of length $T = 2\pi\tilde{\gamma} = \pi\gamma$. The two left arcs are a development with $\theta_1 = -\alpha$, $\theta_2 = \pi - \alpha + \beta$.

Using (5.8) and (5.7), the rolling motion from p_0 along the two left arcs results at $t = 2T$ in $(R_{12}, 0)$, with

$$R_{12} = e^{TA(\theta_1)} e^{(\pi+T)A(\theta_2)} e^{\pi A(\theta_1)}$$
$$= e^{TA(-\alpha)} e^{(\pi+T)A(\pi-\alpha+\beta)} e^{\pi A(-\alpha)}.$$

Analogously, the two right arcs are a development with $\theta_3 = \alpha$, $\theta_4 = \pi + \alpha - \beta$ and at $t = 2T$, the rolling motion from p_0 is $(R_{34}, 0)$

$$R_{34} = e^{TA(\theta_3)} e^{(\pi+T)A(\theta_4)} e^{\pi A(\theta_3)}$$
$$= e^{TA(\alpha)} e^{(\pi+T)A(\pi+\alpha-\beta)} e^{\pi A(\alpha)}.$$

Therefore, the rolling motion around the closed curve is, at $t = 4T$,

$$(R_{1234}, s) = \left(R_{34}^{-1}, 0\right)(R_{12}, 0).$$

Using Lemma 5.8,

$$R_{1234} = \left(e^{\pi A(\alpha)} e^{-(\pi+T)A(\pi+\alpha+\beta)} e^{-TA(\alpha)}\right)\left(e^{TA(-\alpha)} e^{(\pi+T)A(\pi-\alpha-\beta)} e^{\pi A(-\alpha)}\right)$$
$$= e^{(-4\alpha+2\beta)A_z}.$$

The case $\frac{1}{2} < \gamma < \frac{3}{4}$ may be handled in a way similar to the case $\frac{3}{4} < \gamma < 1$ at the end of proof of the previous Proposition.

■

Miming twists in the remaining cases is trivial. When $\gamma = \frac{1}{4}$ or $\gamma = \frac{3}{4}$, it is enough to consider a spherical triangle with a segment along the equator of S_2. When $\gamma = \frac{1}{2}$, it is enough to consider a spherical lune with the poles as endpoints. Thus, we have:

Corollary 5.1. Any twist is constructible with finitely many half-tumbles.

Proposition 5.3. Any slip $e^{WA(\theta)}$ from $p_0 = (0, 0, 1)$ with $0 < W < 2\pi\gamma$ is constructible as a twist followed by two half-tumbles or two tumbles.

Proof: We may take $\theta = 0$.

First we consider $0 < \gamma < \frac{1}{2}$. We exhibit a slip as a twist followed by a sequence of two tumbles. Consider a spherical triangle as in Figure 5.2 with vertices p_0, A, B, so that the arcs $[p_0, A]$ and $[A, B]$ have length $T = 2\pi\gamma$ and the arc $[B, p_0]$ has length W. (In this construction, one may take any

$0 < W < 4\pi\gamma$.) Let α be the common angle at p_0 and B and let β the angle at A. Again, from the laws of sines and cosines (5.10) and (5.11), but now solving for α, β, and using $F = \sin T > 0$, we get

$$\cos\beta = \frac{\Gamma^2 - \cos\xi}{\Gamma^2 - 1}$$

and

$$\cos\alpha = \pm\frac{\Gamma}{\sqrt{1-\Gamma^2}}\sqrt{\frac{1-\cos W}{1+\cos W}}.$$

Again recalling Remark 5.1, $[p_0, A, B]$ is a piecewise geodesic development arc with $\theta_1 = -\alpha$, $\theta_2 = \pi - \alpha - \beta$. The Proposition follows from equation (5.9) and then Lemma 5.6:

$$R = e^{TA(-\alpha)}e^{TA(\pi-(\alpha+\beta))} = e^{WA_y}e^{(\pi-(2\alpha+\beta))A_z}.$$

In the case $\frac{1}{2} < \gamma < 1$, we may proceed again by rolling along the same sequence points, but using the larger portion of the great circles, as in the previous Propositions.

If $\gamma = \frac{1}{2}$ we show that a slip $e^{WA(0)}$ with $0 < W < \pi$ is achieved as a twist e^{-WA_z} followed by two half-tumbles. Consider the same spherical triangle as before, but now with two sides of equal length $T = \pi\gamma = \pi/2$. Then $\Gamma = \cos T = 0$, $\cos\beta = \cos W$, β=W, $\cos\alpha$=0, $\alpha = \frac{\pi}{2}$. The motion after two half-tumbles is, according to (5.7), $(R, 0)$ with

$$R = e^{TA(\theta_1)}e^{(T+\pi)A(\theta_2)}e^{\pi A(\theta_1)}.$$

Choosing $\theta_1 = -\alpha$, $\theta_2 = \pi - \alpha - \beta = \pi - \alpha - W$, Lemma 5.8 provides

$$R = e^{\beta A_y}e^{(\pi-2\alpha+\beta)A_z} = e^{WA_y}e^{WA_z},$$

so that $e^{WA_y} = Re^{-WA_z}$, as required.

■

As a consequence of the preceding results, we have:

Corollary 5.2. Any slip is constructible.

5.3. The general case $n \geq 3$. As in Section 4.1, let S_1 and S_2 be a $(n-1)$-spheres in $\mathbb{R}^n$, of raddi $0 < \rho_1 = \gamma < 1$ and $\rho_2 = 1$, centered at c and the origin, respectively, and having $p_0 \in S_1 \cap S_2$ as the unique point of contact, so that $c = (1+\gamma)\,p_0$.

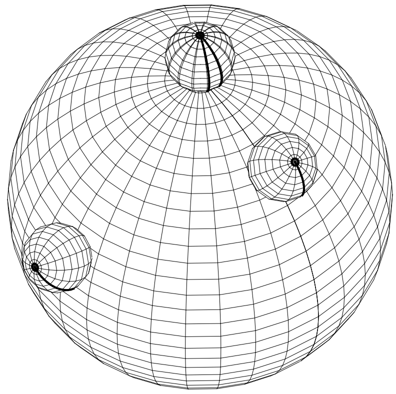

FIGURE 5.2. Achieving a slip.

Let $L = \mathrm{span}\{p_0\}$. A *twist at* p_0 is $(\exp(M), 0) \in \mathrm{SE}_n$, where $M \in \mathfrak{so}_n$ and $Mp_0 = 0$. A *slip from* p_0 is $(\exp(N), 0) \in \mathrm{SE}_n$ with $N \in \mathfrak{so}_n$, $N(L) \subset L^{\perp}$, and $N\left(L^{\top}\right) \subset L$.

In an orthogonal basis so that $p_0 = (0, \ldots, 0, 1) \in \mathbb{R}^n$, the requirements are simply that

$$M = \left[\begin{array}{c|c} \tilde{M} & 0 \\ \hline 0 & 0 \end{array}\right], \quad N = \left[\begin{array}{c|c} 0 & b \\ \hline -b^{\top} & 0 \end{array}\right], \tag{5.15}$$

with $\tilde{M}_{(n-1)\times(n-1)}$ skew-symmetric. (Having fixed a basis, we identify $\mathfrak{so}_n$ with a matrix subspace.)

We now show that these general twists and slips are constructible, in the sense defined in the introduction to this Section.

Proposition 5.4. If $\overline{X} = (\exp M, 0)$ is a twist at p_0, then there is a rolling motion $X : [0, T] \to \mathrm{SE}_n$ such that $X(T) = \overline{X}$.

Proof: A *Givens rotation* is a matrix of the form $\exp(tA_{ij})$, where the A_{ij} are elementary skew-symmetric matrices, as defined in Section 3. Let $\tilde{M}$ be the block of M indicated in (5.15). It is well known that there is a constructive procedure to write $\exp \tilde{M}$ as a finite product of Givens rotations [3]. In order to obtain each twist $\left(e^{tA_{ij}}, 0\right)$, it is enough to perform the maneuvers of the previous Section using only the control input entries u_i and u_j.

■

Proposition 5.5. If $\overline{X} = (\exp(N), 0)$ is a slip from p_0, then there is a rolling motion $X : [0, T] \to \mathrm{SE}_n$ such that $X(T) = \overline{X}$.

Proof: We use the same basis as in the preceding proof, and let b be defined by the second equality in (5.15). We claim that there are $n-1$ twists $(\exp(M_i), 0)$ at p_0, $2 \leq i \leq n$, for which, putting

$$K_n = \exp(M_n) \cdots \exp(M_2), \tag{5.16}$$

and $p = (0, \ldots, 0, t) \in \mathbb{R}^{n-1}$, one can write

$$N = \left[\begin{array}{c|c} 0 & b \\ \hline -b^\top & 0 \end{array}\right] = K_n \left[\begin{array}{c|c} 0 & p \\ \hline -p^\top & 0 \end{array}\right] K_n^{-1}.$$

We check this claim by induction on the dimension $n \geq 2$. The base case

$$N = \left[\begin{array}{c|c} 0 & b \\ \hline -b & 0 \end{array}\right] = \exp(M_2) \left[\begin{array}{c|c} 0 & p \\ \hline -p^\top & 0 \end{array}\right] \exp(-M_2)$$

holds with $M_2 = 0$ and $t = b$. For the step, suppose the dimension is $n+1$. Given $b \in \mathbb{R}^n$, choose a twist $(\exp M_{n+1}, 0)$,

$$M_{n+1} = \left[\begin{array}{c|c} \tilde{M}_{n+1} & 0 \\ \hline 0 & 0 \end{array}\right],$$

such that

$$\exp\left(\tilde{M}_{n+1}\right) b = c = [0, c_2, \ldots, c_n]^\top = [0|\tilde{c}]^\top.$$

By the induction hypothesis, there is a finite product K_n as in (5.16) for which

$$\left[\begin{array}{c|c} 0 & c \\ \hline -c^\top & 0 \end{array}\right] = S_n \left[\begin{array}{c|c} 0 & p \\ \hline -p^\top & 0 \end{array}\right] S_n^{-1}$$

and now we check

$$\begin{aligned} &\exp(M_{n+1}) K_n \left[\begin{array}{c|c} 0 & p \\ \hline -p^\top & 0 \end{array}\right] K_n^{-1} \exp(-M_{n+1}) \\ &= \left[\begin{array}{c|c} \exp \tilde{M}_{n+1} & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} 0 & c \\ \hline -c^\top & 0 \end{array}\right] \left[\begin{array}{c|c} \exp\left(-\tilde{M}_{n+1}\right) & 0 \\ \hline 0 & 1 \end{array}\right] \\ &= \left[\begin{array}{c|c} 0 & b \\ \hline -c^\top & 0 \end{array}\right] \left[\begin{array}{c|c} \exp\left(-\tilde{M}_{n+1}\right) & 0 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} 0 & b \\ \hline -b^\top & 0 \end{array}\right], \end{aligned}$$

as required.

The slip

$$\left(\exp\left[\begin{array}{c|c} 0 & p \\ \hline -p^\top & 0 \end{array}\right], 0\right)$$

corresponds to the slip $\left(e^{tA_x}, 0\right)$ with respect to the last three variables and is therefore constructible, as shown in Section 5.2. From Proposition 5.4, all of the twists $(\exp(M_i), 0)$ are constructible and therefore so is the slip $(\exp N, 0)$. ■

Now we recover the well-known result that the system consisting of S_1 rolling outside S_2 is fully controllable whenever the radii of those spheres are not equal. If M_1 and M_2 are m-dimensional submanifolds of Euclidean space $\mathbb{R}^n$, then, following Sharpe [12], we may take for the space of configurations as M_1 rolls over M_2 the space

$$\Omega(M_1, M_2) = \{(x_1, A, x_2) \subset M_1 \times \mathrm{SO}_n \times M_2 : AT_{x_1}M_1 = T_{x_2}M_2\}.$$

Remark 5.2. Alternatively, the space of configurations can be defined by identifying points of $\{(R_1, s_1, R_2, s_2) \in \mathrm{SO}_n \times \mathbb{R}^n \times \mathrm{SO}_n \times \mathbb{R}^n\}$ in an obvious way.

Let $I = [0, T]$ and $\omega_0 = (x_1, A, x_2) \in \Omega(M_1, M_2)$. A piecewise smooth rolling curve $\alpha_1 : I \to M_1$ with $\alpha_1(0) = x_1$ defines a rolling motion $X = (R, s) : I \to \mathrm{SE}_n$ with $X(0) = (A, x_1 - x_2)$ and thus also a development curve $\alpha_2 : I \to M_2$ by $\alpha_2(t) = R(t)\alpha_1(t) + s(t)$. In this way, each such α_1 lifts to a path $\omega : I \to \Omega(M_1, M_2)$ by $\omega(t) = (\alpha_1(t), R(t), \alpha_2(t))$ and we say that $\omega(T)$ *is reachable from* $\omega(0) = \omega_0$ in $\Omega(M_1, M_2)$.

Remark 5.3. For each $\xi \in \mathrm{SE}_n$ there is an obvious bijection $\phi_\xi : \Omega(\xi M_1, M_2) \to \Omega(M_1, M_2)$, namely $\phi_\xi(x_1, A, x_2) = \left(\xi^{-1}x_1, A \circ \xi, x_2\right)$, and it is clear that ω_1 is reachable from ω_0 in $\Omega(\xi M_1, M_2)$ iff $\phi_\xi\omega_1$ is reachable from $\phi_\xi\omega_0$ in $\Omega(M_1, M_2)$.

When M_1 and M_2 are oriented hypersurfaces of the ambient space, $\Omega(M_1, M_2)$ is partitioned into two subsets, corresponding to M_1 rolling on one of the sides of M_2. Since

$$T_{x_1}S_1 = \{w \in \mathbb{R}^n : \langle w, x_1 - c\rangle = 0\} \text{ and } T_{x_2}S_2 = \{w \in \mathbb{R}^n : \langle w, x_2\rangle = 0\},$$

we conclude that $\Omega(S_1, S_2) = \Omega_{-1} \cup \Omega_1$, where

$$\Omega_\sigma = \{(q, A, p) \subset S_1 \times SO_n \times S_2 : A(q - c) = \sigma\gamma p\}.$$

The configurations in Ω_{-1} correspond to S_1 rolling on the outside of S_2, and we write $\Omega := \Omega_{-1}$. With these definitions, $\omega_0 = (p_0, I, p_0) \in \Omega$, as may easily be checked:

$$A(p_0 - c) = A(p_0 - (1 + \gamma)p_0) = A(-\gamma p_0) = -\gamma p_0.$$

Proposition 5.6. Given $\omega_0, \omega_1 \in \Omega(S_1, S_2)$, ω_1 is reachable from ω_0.

Proof: Since the rolling system is reversible, it is enough to show that $\omega_0 = (p_0, I, p_0)$ is reachable from any $\omega_1 = (p_1, A, p_2) \in \Omega(S_1, S_2)$. Remark 5.3 is implicitly invoked at each step. Let $p_0 = e^{T\overline{U}}$ for some $T > 0$, $\overline{U} \in \mathfrak{so}_n$. By rolling with constant input $\overline{U}$, we reach a state $(\hat{p_1}, *, p_0)$. By a slip at $\hat{p_1}$, a configuration $\left(p_0, \hat{A}, p_0\right)$ is reached with $\hat{A}(p_0 - c) = -\gamma p_0$, that is $\hat{A}(p_0) = p_0$. Finally, a twist at p_0 allows us to reach (p_0, I, p_0).

■

5.4. Appendix - Auxiliary results.

Remark 5.4. We suppose throughout this Section that α, β are the angles and T, W are the arclengths of the sides of an isosceles spherical triangle in the unit sphere of Euclidean 3-space and thus those quantities satisfy relations (5.10) and (5.11).

The first Lemma concerns spherical trigonometry.

Lemma 5.1. If α, β, W,T are related as above, then

$$\cos T \sin T (1 - \cos\beta) = \sin W \cos\alpha.$$

Proof: Using (5.10) and then (5.13) and (5.14), we reduce the goal to

$$\cos T \sin T (2 + 2\cos(2\alpha)) = \frac{\cos\alpha \sin T}{\sin\alpha} \cdot 2\cos T \sin(2\alpha),$$

which is readily verified.

■

In this Section, we use the notation $A(\theta)$ introduced earlier in (5.5) for rotations in 3-space. Some of the results below may be geometrically obvious, but their proofs are included for the sake of completeness.

Lemma 5.2. The general equality holds:

$$e^{tA(\theta)} = e^{-\sigma A_z} e^{tA(\theta-\sigma)} e^{\sigma A_z}, \tag{5.17}$$

and thus also the special cases $\sigma = \theta$

$$e^{tA(\theta)} = e^{-\theta A_z} e^{tA_y} e^{\theta A_z}, \tag{5.18}$$

and the further particularization $\sigma = \theta = \pi$

$$e^{\pi A_z} e^{-tA_y} = e^{tA_y} e^{\pi A_z}. \tag{5.19}$$

Proof: We prove (5.17). Fix σ and θ. Equality holds at $t=0$. The left side obeys $\dot{X} = A(\theta) X$. As for the right side,

$$\frac{d}{dt}\left(e^{-\sigma A_z} e^{tA(\theta-\sigma)} e^{\sigma A_z}\right) = \left(e^{-\sigma A_z} A(\theta-\sigma) e^{\sigma A_z}\right)\left(e^{-\sigma A_z} e^{tA(\theta-\sigma)} e^{\sigma A_z}\right),$$

so it remains only to verify $e^{-\sigma A_z} A(\theta-\sigma) e^{\sigma A_z} = A(\theta)$, that is

$$e^{-\sigma A_z} A(\theta-\sigma) = A(\theta) e^{-\sigma A_z},$$

which is

$$\begin{bmatrix} \cos\sigma & -\sin\sigma & 0 \\ \sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cos(\theta-\sigma) \\ 0 & 0 & \sin(\theta-\sigma) \\ -\cos(\theta-\sigma) & -\sin(\theta-\sigma) & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 0 & 0 & \cos\theta \\ 0 & 0 & \sin\theta \\ -\cos\theta & -\sin\theta & 1 \end{bmatrix} \begin{bmatrix} \cos\sigma & -\sin\sigma & 0 \\ \sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Both sides equal

$$\begin{bmatrix} 0 & 0 & \cos\theta \\ 0 & 0 & \sin\theta \\ -\cos(\theta-\sigma) & -\sin(\theta-\sigma) & 1 \end{bmatrix}$$

and the Lemma is proved.

■

From (5.19), we derive the geometrically clear fact

$$e^{tA(\pi+\theta)} = e^{-\theta A_z} e^{-\pi A_z} e^{tA_y} e^{\pi A_z} e^{\theta A_z} = e^{-tA(\theta)}, \tag{5.20}$$

Lemma 5.3. For every $n \geq 1$ and numbers θ_i, t_i, σ the following are equivalent:

$$e^{t_1 A(\theta_1)} \cdots e^{t_n A(\theta_n)} = e^{t_0 A_z},$$

and

$$e^{t_1 A(\theta_1+\sigma)} \cdots e^{t_n A(\theta_n+\sigma)} = e^{t_0 A_z}.$$

Proof: This is an easy consequence of (5.17). ■

The following is a core technical result.

Lemma 5.4. If α, β, T are related as in Remark 5.4, then

$$e^{2\alpha A_z} e^{TA_y} e^{\beta A_z} e^{-TA_y} e^{2\alpha A_z} e^{TA_y} e^{\beta A_z} e^{-TA_y} = I. \tag{5.21}$$

Proof: Putting $M = e^{2\alpha A_z} e^{TA_y}$ and $N = e^{\beta A_z} e^{-TA_y}$, this last equation becomes

$$(MN)^2 = I.$$

Since each of $M, N, P = MN$ is orthogonal, it is enough to show that $MN = (MN)^\top$. This we accomplish by straightforward, but tedious, computations. For conciseness, write $\eta = 2\alpha$. Then

$$M = \begin{bmatrix} \cos\eta\cos T & \sin\eta & \cos\eta\sin T \\ -\sin\eta\cos T & \cos\eta & -\sin\eta\sin T \\ -\sin T & 0 & \cos T \end{bmatrix},$$

and

$$N = \begin{bmatrix} \cos\beta\cos T & \sin\beta & -\cos\beta\sin T \\ -\sin\beta\cos T & \cos\beta & \sin\beta\sin T \\ \sin T & 0 & \cos T \end{bmatrix},$$

and $P = MN$ is symmetric iff the following three equations hold: $P_{12} = P_{21}$, $P_{13} = P_{31}$, $P_{23} = P_{32}$, which are, in full,

$$\begin{aligned} &\cos\eta\cos T\sin\beta + \sin\eta\cos\beta + 0 \\ &= -\sin\eta\cos\beta\cos^2 T - \cos\eta\sin\beta\cos T - \sin\eta\sin^2 T, \end{aligned} \tag{5.22}$$

and

$$\begin{aligned} &\cos\eta\cos T\cos\beta\sin T - \sin\eta\sin\beta\sin T - \cos\eta\sin T\cos T \\ &= \sin T\cos\beta\cos T + 0 - \cos T\sin T, \end{aligned} \tag{5.23}$$

and

$$\begin{aligned} &-\sin\eta\cos T\cos\beta\sin T - \cos\eta\sin\beta\sin T + \sin\eta\sin T\cos T \\ &= \sin T\sin\beta + 0 + 0. \end{aligned} \tag{5.24}$$

Now, recalling that $\Gamma = \cos T$ and relations (5.13) and (5.14), the first equation (5.22) becomes,

$$\cos\beta\sin\eta\left(1+\Gamma^2\right) + 4\Gamma\sin\beta\cos\eta + \left(1-\Gamma^2\right)\sin\eta = 0.$$

After diving through by $\sin\eta$, we obtain

$$\begin{aligned} &\left(\left(\Gamma^2-1\right) - \left(\Gamma^2+1\right)\cos\eta\right)\left(1+\Gamma^2\right) + 4\Gamma^2\cos\eta \\ &+ \left(1-\Gamma^2\right)\left(\left(\Gamma^2+1\right) - \left(\Gamma^2-1\right)\cos\eta\right) = 0 \end{aligned}$$

and thus

$$\left(-\left(\Gamma^2+1\right)^2+\left(\Gamma^2-1\right)^2\right)\cos\eta+4\Gamma^2\cos\eta=0.$$

Similarly, the second equation (5.23), divided by $\sin T$,

$$\Gamma\cos\beta\cos\eta=\sin\beta\sin\eta+\Gamma\cos\eta+\Gamma\cos\beta-\Gamma$$

and again, dividing by Γ,

$$\begin{aligned}&\left(\left(\Gamma^2-1\right)-\left(\Gamma^2+1\right)\cos\eta\right)(\cos\eta-1)\\ =\;&2\sin^2\eta+(\cos\eta-1)\left(\left(\Gamma^2+1\right)-\left(\Gamma^2-1\right)\cos\eta\right)\end{aligned}$$

and so

$$\begin{aligned}&\cos^2\eta\left(-\left(\Gamma^2+1\right)+\left(\Gamma^2-1\right)\right)-2\sin^2\eta\\ +\;&\cos\eta\left(\left(1+\Gamma^2\right)+\left(\Gamma^2-1\right)-\left(\Gamma^2+1\right)-\left(\Gamma^2-1\right)\right)\\ =\;&\left(\Gamma^2-1\right)-\left(\Gamma^2+1\right).\end{aligned}$$

Finally, for the third equation (5.24), dividing through by $\sin T$,

$$-\Gamma\sin\eta\cos\beta-\cos\eta\sin\beta+\Gamma\sin\eta=\sin\beta$$

and so

$$\begin{aligned}-\;&\Gamma\left(\left(\Gamma^2-1\right)-\left(\Gamma^2+1\right)\cos\eta\right)\sin\eta-2\Gamma\sin\eta\cos\eta\\ +\;&\Gamma\left(\left(\Gamma^2+1\right)-\left(\Gamma^2-1\right)\cos\eta\right)\sin\eta=2\Gamma\sin\eta.\end{aligned}$$

■

The next result is used to compute a four-tumble sequence and will be used as a stepping stone to obtain more general results.

Lemma 5.5. If α,β,T are related as in Remark 5.4, then

$$e^{TA(\alpha+\beta)}e^{-TA(\alpha)}e^{TA(-\alpha)}e^{-TA(-\alpha-\beta)}=e^{-(4\alpha+2\beta)A_z}. \tag{5.25}$$

Proof: Repeatedly use (5.18) to rewrite the left-hand side of (5.25) as

$$\begin{aligned}&e^{TA(\alpha+\beta)}e^{-TA(\alpha)}e^{TA(-\alpha)}e^{-TA(-\alpha-\beta)}\\ =\;&\left(e^{-(\alpha+\beta)A_z}e^{TA_y}e^{(\alpha+\beta)A_z}\right)\left(e^{-\alpha A_z}e^{-TA_y}e^{\alpha A_z}\right)\\ &\left(e^{\alpha A_z}e^{TA_y}e^{-\alpha A_z}\right)\left(e^{(\alpha+\beta)A_z}e^{-TA_y}e^{-(\alpha+\beta)A_z}\right)\\ =\;&e^{-(\alpha+\beta)A_z}\left(e^{TA_y}e^{\beta A_z}e^{-TA_y}e^{2\alpha A_z}e^{TA_y}e^{\beta A_z}e^{-TA_y}\right)e^{-(\alpha+\beta)A_z}\end{aligned}$$

thus reducing (5.25) to

$$e^{TA_y}e^{\beta A_z}e^{-TA_y}e^{2\alpha A_z}e^{TA_y}e^{\beta A_z}e^{-TA_y} = e^{-2\alpha A_z},$$

which is proved in Lemma 5.4.

■

The next result is used to compute a two-tumble sequence.

Lemma 5.6. If α, β, T, W are related as in Remark 5.4, then

$$e^{TA(\alpha)}e^{-TA(\alpha+\beta)} = e^{WA_y}e^{(\pi+(2\alpha+\beta))A_z}.$$

and

$$e^{TA(-\alpha)}e^{-TA(-(\alpha+\beta))} = e^{WA_y}e^{(\pi-(2\alpha+\beta))A_z}$$

Proof: We check the first equation. First note that $e^{-WA_y}e^{TA(\alpha)}e^{-TA(\alpha+\beta)}$ leaves invariant the line spanned by $(0,0,1)$. This is geometrically clear, and may be shown by direct computation. Expand

$$\begin{aligned}
&e^{-WA_y}e^{TA(\alpha)}e^{-TA(\alpha+\beta)}\\
=&e^{-WA_y}\left(e^{-\alpha A_z}e^{TA_y}e^{\alpha A_z}\right)\left(e^{-(\alpha+\beta)A_z}e^{-TA_y}e^{(\alpha+\beta)A_z}\right)\\
=&e^{-WA_y}e^{-\alpha A_z}e^{TA_y}e^{-\beta A_z}e^{-TA_y}e^{(\alpha+\beta)A_z},
\end{aligned}$$

and then check that $e^{TA_y}e^{-\beta A_z}e^{-TA_y}e^{(\alpha+\beta)A_z}$ and $e^{\alpha A_z}e^{WA_y}$ take $(0,0,1)^\perp$ to, respectively,

$$\begin{bmatrix} -\cos T\sin T\cos\beta+\sin T\cos T\\ -\sin\beta\sin T\\ \sin^2 T\cos\beta+\cos^2 T\end{bmatrix} \text{ and } \begin{bmatrix}\sin W\cos\alpha\\ -\sin\alpha\sin W\\ \cos W\end{bmatrix},$$

which are equal by Lemma 5.1 and the laws of sines (5.10) and cosines (5.11). Therefore,

$$e^{-WA_y}e^{TA(\alpha)}e^{-TA(\alpha+\beta)} = e^{-\chi A_z} \tag{5.26}$$

for some χ, and thus

$$e^{TA(\alpha+\beta)}e^{-TA(\alpha)}e^{WA_y} = e^{\chi A_z}. \tag{5.27}$$

By symmetry with respect to the $yz-$plane, equation (5.26) becomes

$$e^{-WA_y}e^{TA(-\alpha)}e^{-TA(-(\alpha+\beta))} = e^{\chi A_z}. \tag{5.28}$$

Combining equations (5.27) and (5.28) we get

$$\left(e^{TA(\alpha+\beta)}e^{-TA(\alpha)}\right)\left(e^{TA(-\alpha)}e^{-TA(-(\alpha+\beta))}\right)=e^{2\chi A_z}.$$

By Lemma 5.5, $2\chi=-(4\alpha+2\beta)$. When $\alpha=0$, $\beta=\pi$, $W=2T$, we must have $\chi=0$ and therefore, by continuity, $\chi=\pi-(2\alpha+\beta)$. The second claimed equation is obtained from the first by symmetry with respect to the $yz-$plane. ■

The following is used to achieve a slip under certain conditions.

Lemma 5.7. If $\beta=W$ and $\alpha=T=\frac{\pi}{2}$, then

$$e^{TA(-\alpha)}e^{(\pi+T)A(\pi-\alpha-\beta)}e^{\pi A(-\alpha)}=e^{WA_y}e^{(\pi-2\alpha+\beta)A_z}.$$

Remark 5.5. More generally, the formula in the previous Lemma holds whenever α, β, T, W are related as in Remark 5.4, as does the analogous

$$e^{TA(\alpha)}e^{(\pi+T)A(\pi+\alpha+\beta)}e^{\pi A(\alpha)}=e^{WA_y}e^{(\pi+2\alpha-\beta)A_z},$$

though we make no use of these more general results.

Proof: We are to show that

$$e^{\frac{\pi}{2}A\left(-\frac{\pi}{2}\right)}e^{\frac{3}{2}\pi A\left(\frac{\pi}{2}-\beta\right)}e^{\pi A\left(-\frac{\pi}{2}\right)}=e^{\beta A_y}e^{\beta A_z}.$$

The right-hand-side is

$$\begin{bmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{bmatrix}\begin{bmatrix}\cos\beta & \sin\beta & 0\\ -\sin\beta & \cos\beta & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\cos^2\beta & \sin\beta\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta & 0\\ -\sin\beta\cos\beta & -\sin^2\beta & \cos\beta\end{bmatrix}.$$

Relation (5.17) with $\sigma=-\beta$ results in

$$e^{-\frac{\pi}{2}A\left(\frac{\pi}{2}-\beta\right)}=e^{\beta A_z}e^{-\frac{\pi}{2}A\left(\frac{\pi}{2}\right)}e^{-\beta A_z}=e^{\beta A_z}e^{-\frac{\pi}{2}A_x}e^{-\beta A_z}$$

and, from (5.20),

$$e^{\frac{\pi}{2}A\left(-\frac{\pi}{2}\right)}=e^{-\frac{\pi}{2}A\left(\frac{\pi}{2}\right)}=e^{-\frac{\pi}{2}A_x}$$

and

$$e^{\pi A\left(-\frac{\pi}{2}\right)}=e^{-\pi A\left(\frac{\pi}{2}\right)}=e^{\pi A_x}.$$

Then the left-hand-side of the goal becomes

$$e^{-\frac{\pi}{2}A_x}\left(e^{\beta A_z}e^{-\frac{\pi}{2}A_x}e^{-\beta A_z}\right)e^{\pi A_x}.$$

We compute this matrix explicitly:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ 0 & 0 & -1 \\ -\sin\beta & \cos\beta & 0 \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ 0 & 0 & 1 \\ \sin\beta & -\cos\beta & 0 \end{bmatrix} = \begin{bmatrix} \cos^2\beta & \sin\beta\cos\beta & \sin\beta \\ -\sin\beta & \cos\beta & 0 \\ -\sin\beta\cos\beta & -\sin^2\beta & \cos\beta \end{bmatrix}.$$

■

The next result is used to compute a sequence of four half-tumbles.

Lemma 5.8. If α, β, T, W are related as in Remark 5.4, then

$$e^{\pi A(\alpha)} e^{-(\pi+T)A(\pi+\alpha+\beta)} e^{-TA(\alpha)} e^{TA(-\alpha)} e^{(\pi+T)A(\pi-\alpha-\beta)} e^{\pi A(-\alpha)} = e^{(-4\alpha+2\beta)A_z}.$$

Proof: Relation (5.20) allows us to rewrite the goal as

$$e^{\pi A(\alpha)} e^{(\pi+T)A(\alpha+\beta)} e^{-TA(\alpha)} e^{TA(-\alpha)} e^{-(\pi+T)A(-\alpha-\beta)} e^{\pi A(-\alpha)} = e^{(-4\alpha+2\beta)A_z}$$

and then Lemma 5.5 alllows the further reduction to

$$e^{\pi A(\alpha)} e^{\pi A(\alpha+\beta)} e^{-(4\alpha+2\beta)A_z} e^{-\pi A(-\alpha-\beta)} e^{\pi A(-\alpha)} = e^{(-4\alpha+2\beta)A_z}.$$

As in the proof of that Lemma, use repeatedly equality (5.18) to rewrite this last equation as

$$\left(e^{-\alpha A_z} e^{\pi A_y} e^{\alpha A_z}\right) \left(e^{-(\alpha+\beta)A_z} e^{\pi A_y} e^{(\alpha+\beta)A_z}\right) e^{-(4\alpha+2\beta)A_z}$$
$$\cdot \left(e^{(\alpha+\beta)A_z} e^{\pi A_y} e^{-(\alpha+\beta)A_z}\right) \left(e^{\alpha A_z} e^{\pi A_y} e^{-\alpha A_z}\right) = e^{(-4\alpha+2\beta)A_z},$$

which simplifies to

$$e^{\pi A_y} e^{-\beta A_z} e^{\pi A_y} e^{-2\alpha A_z} e^{\pi A_y} e^{-\beta A_z} = e^{(-2\alpha+2\beta)A_z}.$$

Finally, relation (5.19) easily shows this last equation to be true.

■

The final result gives bounds for the magnitude of certain twists.

Lemma 5.9. If α, β, T are related as in Remark 5.4, then for any fixed $T \in (0, \pi/2)$, as α takes values in $(0, \pi/2)$, the range of $2\alpha + \beta$ is the interval with endpoints $2\arccos\left(\frac{\cos T - 1}{\cos T + 1}\right)$ and π.

Proof: As in relations (5.13) and (5.14), let $\eta = 2\alpha$ and $\Gamma = \cos T$. We are to determine the range of

$$F(\eta) = \eta + \beta = \eta + \arccos\left(f(\cos\eta)\right),$$

with

$$f(x) = \frac{\left(\Gamma^2 - 1\right) - \left(\Gamma^2 + 1\right)x}{\left(\Gamma^2 + 1\right) - \left(\Gamma^2 - 1\right)x}.$$

As noted before, $F(0) = F(\pi) = \pi$. Now compute the interior extrema of F by straightforward differentiation,

$$\begin{aligned} F'(\eta) &= 1 - \frac{1}{\sqrt{1 - f^2(\cos x)}} f'(\cos\eta)(-\sin\eta) \\ &= 1 + f'(\cos\eta) \\ &= 1 - \frac{4\Gamma^2}{\left(\left(\Gamma^2+1\right) - \left(\Gamma^2-1\right)\cos\eta\right)^2} = 0 \end{aligned}$$

iff

$$\left(\Gamma^2 + 1\right) - \left(\Gamma^2 - 1\right)\cos\eta = 2\Gamma$$

and thus the unique interior extremum of F occurs at η^* such that $\cos\eta^* = \frac{\Gamma-1}{\Gamma+1}$ with image

$$\begin{aligned} F(\eta^*) &= \eta^* + \arccos\left(f\left(\frac{\Gamma-1}{\Gamma+1}\right)\right) = . \\ &= \eta^* + \arccos\left(\frac{\Gamma-1}{\Gamma+1}\right) = 2\arccos\left(\frac{\Gamma-1}{\Gamma+1}\right) \end{aligned}$$

■

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