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WHY CONTROLLABILITY OF ROLLING MAY FAIL: A FEW ILLUSTRATIVE EXAMPLES

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ABSTRACT: We derive a distribution for a rolling map of a smooth surface in $\mathbb{R}^n$ rolling, without slip and twist, on its affine tangent space at a point. Sufficient conditions for local controllability are related to Gaussian curvature of the surface. Examples in $\mathbb{R}^3$ illustrate where these conditions may fail. We also derive the kinematic equations for an ellipsoid rolling on its affine tangent space at a point.

1. Introduction

It is well known that in Euclidean space a sphere rolling on its tangent affine space is globally controllable, *cf.* [20]. A constructive proof of this fact can be found in [6]. Recent studies consider rolling in Riemannian [7] and in pseudo-Riemannian spaces. Notably, the case of rolling in a space endowed with Lorentzian metric has been considered in [13]. It is shown there that rolling a sphere in this space is controllable.

What about other surfaces than a sphere? What are the necessary conditions a surface must satisfy so the rolling, without slip and twist, on its affine tangent space (a hyperplane) is controllable? In this note we review two simple cases of 2-dimensional surfaces, the unit sphere and a cylinder, in $\mathbb{R}^3$ and point out why the conditions for local controllability may fail. These two examples are easy to study because the two manifolds both have constant, yet different, Gaussian curvatures. A calculation is guided by an intuition, and the intuition is confirmed, the cylinder is not locally controllable, because one direction of rolling is impossible to achieve. This is the case in general: local controllability fails at the points where Gaussian curvature vanishes.

The paper is organised as follows. The definition of a rolling map appears in Section 2. Here we also unravel some properties of rolling without slip or twist and revisit the classical, simple, and intuitive example of the unit sphere rolling on a hyperplane. To study rolling maps in more detail it may be necessary to review a few geometric operations that are relevant in this

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paper. This is done in Section 3 where the first and the second fundamental forms, the Weingarten map, and Gaussian curvature are described. A standard configuration space and a distribution on it, for the rolling map, are presented in Section 4. The equations that define these spaces are then used in Section 5 to compare rolling a sphere and a cylinder in $\mathbb{R}^3$. Implications to controllability are discussed in Section 6. In particular, we prove sufficient conditions for local controllability of rolling a surface on its affine tangent space in Section 7. In addition, the kinematic equations for rolling an ellipsoid are presented in Section 8. Future directions of our research are given in Section 9.

2. Rolling Maps

The operation of rolling a surface on another surface appeared in the literature as early as in 1919 (Paul Appell, "Traité de Mécanique Rationnelle"). Later on a modern treatment of rolling without slip and twist was investigated by Nomizu [15]. Geometric definition of a rolling map in Euclidean space is given in [17] and its definition in a more general Riemannian framework appeared in [7]. In the previous situations, the rolling manifolds are assumed to be embedded in a bigger manifold. A somewhat different approach, where the embedding is not assumed, is the intrinsic rolling of Riemannian manifolds studied in [4].

The rolling map serves as an example of motions in classical mechanics. Such motions have holonomic constraints imposed by rolling and also by additional "no-slip" and "no-twist" conditions impose the non-holonomic constraints. Another interesting approach is taken by Rojo and Bloch in [16], who make use of the isomorphism between the rolling map of the unit sphere and the precession of a $1/2$ spin in the presence of a time dependant magnetic field. Thus the results of quantum physics make their way to study rolling!

To start our discussion we give the general definition of a rolling map now. The definition is followed by its interpretation and properties. Throughout this paper it is assumed that all manifolds are connected and orientable.

2.1. Definition

Let $\mathsf{Isom}(\mathbf{M})$ denote the group of isometries on $\mathbf{M}$. It is known that $\mathsf{Isom}(\mathbf{M})$ is a Lie group, whose maximal dimension is equal to $m(m+1)/2$ whenever $\mathbf{M}$ is isometric to $\mathbb{R}^m$, m-sphere $\mathbf{S}^m$, real projective space $\mathbb{RP}(m)$

or a hyperbolic space $\mathbf{H}^m$, *cf.* [11]. In other words, the four above spaces are maximally symmetric.

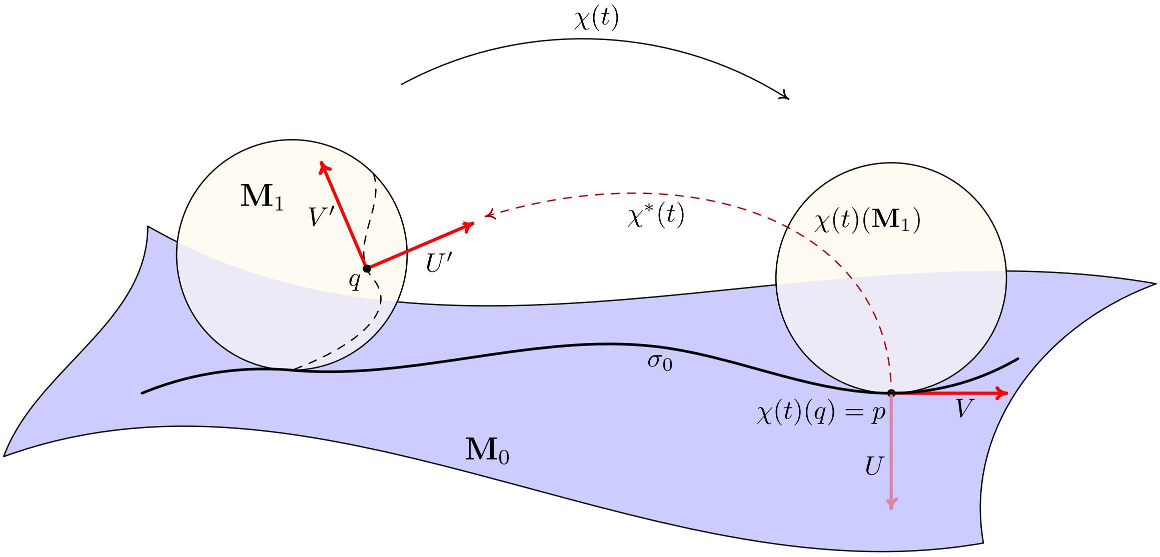

FIGURE 1. Manifold $\mathbf{M}_1$ is rolling on $\mathbf{M}_0$ along the developing curve σ_0 and the pull-back $\boldsymbol{\chi}^*$ of the rolling map

Let $I \subset \mathbb{R}$ denote a closed interval.

Definition 1 (rolling map). *Let* $\mathbf{M}_0$ *and* $\mathbf{M}_1$ *be two* n*-manifolds isometrically embedded in an* m*-dimensional Riemannian manifold* $\mathbf{M}$ *and* $\sigma_1 : I \to \mathbf{M}_1$ *a piecewise smooth curve in* $\mathbf{M}_1$*. A* rolling *of* $\mathbf{M}_1$ *on* $\mathbf{M}_0$ *along the curve* σ_1*, without slipping or twisting, is a map* $\boldsymbol{\chi} : I \to \mathsf{Isom}(\mathbf{M})$ *satisfying the following conditions:*

rolling: *for all* $t \in I$*:*

(a) $\boldsymbol{\chi}(t)(\sigma_1(t)) \in \mathbf{M}_0$*, and*

(b) $\mathbf{T}_{\boldsymbol{\chi}(t)(\sigma_1(t))}(\boldsymbol{\chi}(t)(M_1)) = \mathbf{T}_{\boldsymbol{\chi}(t)(\sigma_1(t))}\mathbf{M}_0$*;*

the curve $\sigma_0 : I \to \mathbf{M}_0$ *defined by* $\sigma_0(t) \stackrel{\text{def}}{=} \boldsymbol{\chi}(t)(\sigma_1(t))$ *is called the* development curve *of* σ_1*;*

no-slip: $\dot{\sigma}_0(t) = \boldsymbol{\chi}(t)_*(\dot{\sigma}_1(t))$*, for almost all* $t \in I$*, where* $\boldsymbol{\chi}_*$ *is the push-forward of* $\boldsymbol{\chi}$ *and* $\boldsymbol{\chi}(t)_* : \mathbf{TM} \to \mathbf{TM}$*;*

no-twist: *two complementary conditions, for almost all* $t \in I$*:*

tangential: $(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}(t)^{-1})_*(\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0) \subset \left(\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0\right)^{\perp}$*;*

normal: $(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}(t)^{-1})_*(\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0^{\perp}) \subset \mathbf{T}_{\sigma_0(t)}\mathbf{M}_0$*.*

Remark 2. *The above "rolling" conditions imply that at each point of contact, both manifolds, $\mathbf{M}_0$ and $\boldsymbol{\chi}(t)(\mathbf{M}_1)$, have the same tangent space. This is identified as a subspace of the tangent space of $\mathbf{M}$ at the specified point.* ❆

Remark 3. *The "no-slip" condition in Definition 1 is equivalent to*

$$\dot{\boldsymbol{\chi}}(t)(\sigma_1(t)) = (\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}^{-1}(t))(\sigma_0(t)) = 0. \tag{1}$$

That is, the infinitesimal transformation $\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}^{-1}(t) \colon \mathbf{M} \to \mathbf{TM}$ maps $\sigma_0(t)$ to the zero vector (the origin of the tangent space $\mathbf{T}_{\sigma_0(t)}\mathbf{M}$). ❆

Remark 4. *It is not apparent from the definition of rolling that operation $(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}(t)^{-1})_*$ is well defined. It turns out that $(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}(t)^{-1})_*$ is equal to the covariant derivative along curve $\boldsymbol{\chi}(t)(\sigma_1(t_0))$ in the ambient space $\mathbf{M}$, cf. [7]. Therefore, its image is confined to the tangent bundle $\mathbf{TM}$.* ❆

Remark 5. *If the "non-twist" conditions are satisfied, then in suitable coordinates in a neighbourhood of $p \in \sigma_0(I) \subset \mathbf{M}_0$ we may choose an orthonormal basis in $\mathbf{T}_p\mathbf{M} = \mathbf{T}_p\mathbf{M}_0 \oplus (\mathbf{T}_p\mathbf{M}_0)^{\perp}$ so that the operator $\left(\dot{\boldsymbol{\chi}} \circ \boldsymbol{\chi}^{-1}\right)_*$ has a matrix representation of the following form*

$$\left(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}(t)^{-1}\right)_*(p) = \left[\begin{array}{c|c} 0 & X_{n\times r} \\ \hline Y_{r\times n} & 0 \end{array}\right], \quad \textit{where} \quad n + r = m. \tag{2}$$

❆

In the Euclidean case, when the ambient space $\mathbf{M} = \mathbb{R}^m$, the group of orientation preserving isometries $\mathsf{Isom}(\mathbb{R}^m)$ is the Euclidean group of motions $\mathbb{SE}(m) = \mathbb{SO}(m) \ltimes \mathbb{R}^m$, the semi-direct product of the special orthogonal group $\mathbb{SO}(m)$ and $\mathbb{R}^m$. An element $\boldsymbol{\chi}(t)$ of $\mathbb{SE}(m)$ will be denoted by the pair $(R(t), s(t))$, where $R(t) \in \mathbb{SO}(m)$ is an orthogonal matrix describing rotations and $s(t) \in \mathbb{R}^m$ is a vector describing translations. $\mathbb{SE}(m)$ acts on $\mathbb{R}^m$ in the usual way

$$\begin{aligned} (R, s) \colon \mathbb{R}^m &\to \mathbb{R}^m, \\ p &\mapsto R\,p + s. \end{aligned} \tag{3}$$

The group operations on $\mathbb{SE}(m)$ are defined as follows

$$\begin{aligned} (R_1, s_1) \circ (R_2, s_2) &= (R_1 \cdot R_2, R_1 s_2 + s_1) \quad \text{and} \\ (R, s)^{-1} &= (R^{-1}, -R^{-1}\,s). \end{aligned}$$

Note that in Euclidean case the push-forward $\left(\dot{\boldsymbol{\chi}} \circ \boldsymbol{\chi}^{-1}\right)_*$ can be represented by a skew-symmetric matrix

$$(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}(t)^{-1})_* = \dot{\boldsymbol{\chi}}_*(t) \circ \boldsymbol{\chi}_*(t)^{-1} = \dot{R}(t)R^{\mathrm{T}}(t) \in \mathfrak{so}(m),$$

where $\mathfrak{so}(m)$ is the Lie algebra of $\mathbb{SO}(m)$. Because this operation plays an essential role in our approach we denote it by the matrix $\boldsymbol{A}(t) \stackrel{\text{def}}{=} \dot{R}(t)R^{\mathrm{T}}(t)$. Now equation (1) reads

$$\begin{aligned} 0 &= (\dot{R}, \dot{s}) \circ (R^{\mathrm{T}}, -R^{\mathrm{T}}\, s)(\sigma_0) = (\dot{R}R^{\mathrm{T}}, -\dot{R}R^{\mathrm{T}}\, s + \dot{s})(\sigma_0) \\ &= (\boldsymbol{A}, -\boldsymbol{A}\, s + \dot{s})(\sigma_0) = \boldsymbol{A}\, \sigma_0 - \boldsymbol{A}\, s + \dot{s} = \boldsymbol{A}\,(\sigma_0 - s) + \dot{s}. \end{aligned}$$

The above calculations show that the "no-slip" condition has an equivalent formulation as

$$\boldsymbol{A}(t)(\sigma_0(t) - s(t)) = -\dot{s}(t). \tag{4}$$

Also, the tangential and normal parts of the "no-twist" condition may be now rewritten, respectively, as

$$\boldsymbol{A}(t)(\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0) \subset \left(\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0\right)^{\perp}; \tag{5}$$

$$\boldsymbol{A}(t)(\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0)^{\perp} \subset \mathbf{T}_{\sigma_0(t)}\mathbf{M}_0. \tag{6}$$

Remark 6. *From Remark 5, it follows that if $\left(\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0\right)^{\perp}$ is one dimensional, then the normal part of the "no-twist" condition is always satisfied, cf.* [18]*. It also follows from (5) that the normal part of the no-twist condition always holds for* 1*-dimensional manifolds.* ❄

2.2. Interpretation

If $\mathbf{M} = \mathbb{R}^3$, we can give a simple interpretation of Definition 1 of the rolling map. Notice first, that the matrix $\boldsymbol{A} = (\dot{\boldsymbol{\chi}} \circ \boldsymbol{\chi}^{-1})_* = \dot{\boldsymbol{\chi}}_* \circ \boldsymbol{\chi}_*^{-1} = \dot{R}R^{\mathrm{T}}$ is the "spatial angular velocity" written in matrix form bellow, or the equivalent vector $\omega \in \mathbb{R}^3$ defined through a cross product, with the standard identification $\boldsymbol{A} \Leftrightarrow w$:

$$\boldsymbol{A} = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix}, \quad \text{where} \quad \boldsymbol{A}\, v = \omega \times v, \quad \text{for any} \quad v \in \mathbb{R}^3.$$

Hence, for a two dimensional compact surface the "no-slip" condition (4) becomes $\omega \times (\sigma_0 - s) = -\dot{s}$ and the "no-twist" condition (5) means that ω has no component normal to $\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0$, i.e., it is tangent to $\mathbf{M}_0$ at the point of contact.

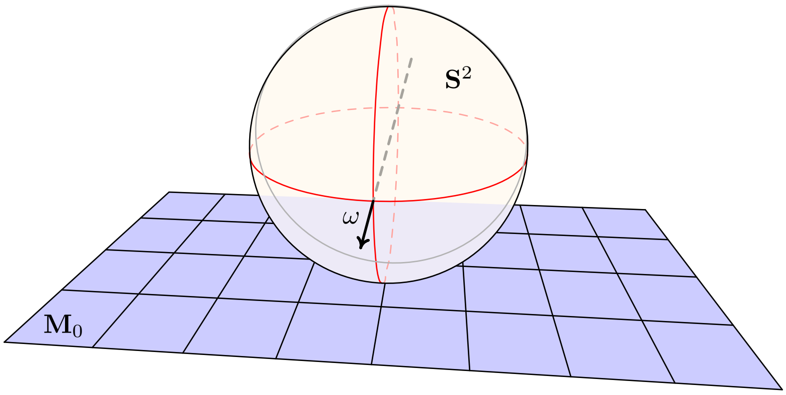

FIGURE 2. The unit sphere $\mathbf{S}^2$ rolling on its affine tangent plane in $\mathbb{R}^3$; the "no-twist" condition implies that spatial angular velocity ω is parallel to the plane $\mathbf{M}_0$

Example 7. *Consider the unit sphere $\mathbf{S}^n$ rolling on its affine tangent space at a point $p_0 \in \mathbf{S}^n$, cf [8]. Here, $\mathbf{M}_1 = \mathbf{S}^n \subset \mathbb{R}^{n+1}$ and $\mathbf{M}_0$ can be defined by*

$$\mathbf{M}_0 \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^{n+1} \ : \ x = p_0 + \Omega\, p_0 \quad \text{and} \quad \Omega \in \mathfrak{so}(n+1) \right\}.$$

Let $\boldsymbol{\chi} = (R^{\mathrm{T}}, s)$ be a rolling map satisfying $\boldsymbol{\chi}(0) = (I, 0)$ and σ_1 be the rolling curve starting at point p_0, i.e., $\sigma_1(0) = p_0$. It can be shown (cf. [8]) that the rolling map satisfies the following set of differential equations

$$\begin{cases} \dot{s}(t) &= u(t), \\ \dot{R}(t) &= R(t)\,(u(t)\,p_0^{\mathrm{T}} - p_0\,u^{\mathrm{T}}(t)), \end{cases} \tag{7}$$

with initial conditions $s(0) = 0$, $R(0) = I$, for some control function $t \mapsto u(t) \in \mathbb{R}^{n+1}$, that satisfies $\langle p_0, u(t) \rangle = 0$, for all $t \geq 0$. Moreover, the rolling and development curves are given by $t \mapsto \sigma_1(t) = R(t)\,p_0$ and $t \mapsto \sigma_0(t) = s(t) + p_0$, respectively. Equations (7) are the kinetic equations for the rolling sphere.

Before we have a closer look at the case of rolling a surface on another surface in Euclidean space we shall make a connection between rolling and the Gaussian curvature. In order to study geometry of rolling maps it is necessary to recall some fundamental operations on submanifolds.

3. Differential Geometric Operators

We shall give a brief account on covariant differentiation on a Riemannian submanifold, second fundamental form and *Weingarten map* [2].

Let $(\mathbf{M}, \overline{g})$ be a Riemannian structure with metric $\overline{g}$ and $(\mathbf{N}, g)$ be a Riemannian submanifold isometrically embedded in M, endowed with Riemannian metric g inherited from $\overline{g}$. That is, the metric g agrees with $\overline{g}$ on $\mathbf{TN}$.

Let $\gamma\colon I \to \mathbf{N}$ be a smooth curve and $\Lambda \in \mathcal{T}\mathbf{N}^{\perp}$ be a unit length vector field normal to $\mathbf{N}$ along γ. The covariant derivative can be decomposed into tangential and normal part* in the embedding manifold

$$\widetilde{\nabla}_X \Lambda = (\widetilde{\nabla}_X \Lambda)^{\perp} + (\widetilde{\nabla}_X \Lambda)^{\top}, \quad \text{for any} \quad X \in \mathcal{T}\mathbf{N}. \tag{8}$$

Then, if $|\Lambda|$ is constant and, consequently, $(\widetilde{\nabla}_{\dot\gamma}\Lambda)^{\perp} = 0$, one has

$$\frac{d\Lambda}{dt} = \widetilde{\nabla}_{\dot\gamma}\Lambda = (\widetilde{\nabla}_{\dot\gamma}\Lambda)^{\perp} + (\widetilde{\nabla}_{\dot\gamma}\Lambda)^{\top} = (\widetilde{\nabla}_{\dot\gamma}\Lambda)^{\top}. \tag{9}$$

Equation (9) gives rise to the operation $\Xi(\Lambda, X) \stackrel{\text{def}}{=} (\widetilde{\nabla}_X \Lambda)^{\perp}$ called the *Weingarten map*, where $\Xi\colon (\mathbf{TN})^{\perp} \times \mathcal{T}\mathbf{N} \to \mathcal{T}\mathbf{N}$ is an endomorphism inducing two linear transformations

$$\begin{aligned} \Xi^X \ &: \ (\mathbf{TN})^{\perp} \to \mathcal{T}\mathbf{N}, \quad \text{where} \quad \Xi^X(\Lambda) \stackrel{\text{def}}{=} \Xi(\Lambda, X), \quad \text{and} \\ \Xi_\Lambda \ &: \ \mathcal{T}\mathbf{N} \to \mathcal{T}\mathbf{N}, \quad \text{where} \quad \Xi_\Lambda(X) \stackrel{\text{def}}{=} \Xi(\Lambda, X). \end{aligned}$$

Moreover, it is known that $\Xi\colon (\mathbf{TN})^{\perp} \times \mathcal{T}\mathbf{N} \to \mathcal{T}\mathbf{N}$ is a tensor field, *cf.* [12], that has the following relationship with the second fundamental form

$$\langle \Xi_\Lambda(X), Y \rangle = -\langle \mathrm{II}(X, Y), \Lambda \rangle. \tag{10}$$

The relationship (10) between the second fundamental form and the Weingarten map can be illustrated by the commutative diagram in Figure 3. The two metric tensors $g_{\perp}\colon (\mathbf{TN})^{\perp} \times (\mathbf{TN})^{\perp} \to \mathbb{R}$ and $g_{\top}\colon \mathbf{TN} \times \mathbf{TN} \to \mathbb{R}$ are inherited from the metric $\widetilde{g}$ of the ambient space $\mathbf{M}$. The tangent metric tensor (field) $g = g_{\top}$ is sometimes called the *first fundamental form.*

Because the second fundamental form is symmetric, then by (10) the following symmetry of Ξ_Λ with respect to the metric holds

$$\langle \Xi_\Lambda(X), Y \rangle = \langle \Xi_\Lambda(Y), X \rangle.$$

This shows that Ξ_Λ is self-adjoin and so it is diagonalisable. Another important implication of the symmetries of the involved operators is that the second fundamental form II and the Weingarten map Ξ, although defined in

*Here we follow the notation and definition of the second fundamental according to modern books like [2, 14] and [3]. In other texts, notably in [12] and [15], the authors use a slightly different notation. Equation (8) is often written as

$$\nabla'_X \xi = -A_\xi(X) + D_X \xi, \quad \text{for any} \quad X \in \mathcal{T}\mathbf{N} \quad \text{and} \quad \xi \in (\mathbf{TN})^{\perp}.$$

where A (named there the *second fundamental form*) denotes the tangential part and D denotes the normal part of the covariant differentiation.

$$
\begin{tikzcd}
& \mathbf{TN} \times \mathbf{TN} \times (\mathbf{TN})^{\perp} \arrow[dl, bend right, "\cong"'] \arrow[dr, bend left, "\cong"] & \\
\mathbf{TN} \times (\mathbf{TN} \times (\mathbf{TN})^{\perp}) \arrow[dr, bend right, "g_{\top} \circ (\mathsf{id} \otimes \Xi)"] & & (\mathbf{TN} \times \mathbf{TN}) \times (\mathbf{TN})^{\perp} \arrow[dl, bend left, "-g_{\perp} \circ (\mathrm{II} \otimes \mathsf{id})"'] \\
& \mathbb{R} &
\end{tikzcd}
$$

FIGURE 3. Commutative diagram illustrating the relationship (10) between the second fundamental form and the Weingarten map

terms of the connection, which is a differential operator, both depend only on values of the vector fields at the point of evaluation.

3.1. Euclidean Hypersurfaces

In this section we confine ourselves to the Euclidean case, where $\mathbf{N}$ is an n-hypersurface of co-dimension one embedded in $\mathbf{M} = \mathbb{R}^{n+1}$. One considers the *scalar second fundamental form* h, the symmetric 2-tensor on $\mathbf{N}$, defined by

$$h(X,Y) = \langle \mathrm{II}(X,Y), \Lambda \rangle \quad \text{or equivalently} \quad \mathrm{II}(X,Y) = h(X,Y)\Lambda,$$

where Λ is a unit normal vector field on $\mathbf{N}$. Raising an index of the scalar second fundamental form yields the *shape operator* s, which is a tensor field

$$\langle X, sY \rangle \overset{\text{def}}{=} h(X,Y).$$

Since Λ is a unit normal vector, the *Weingarten equation* for Euclidean hypersurfaces assumes the following form, *cf.* [14]:

$$\widetilde{\nabla}_X \Lambda = -sX.$$

Because h is a self-adjoint bilinear form on $\mathbf{T}_p\mathbf{N}$, for any $p \in \mathbf{N}$, there are n real eigenvalues λ_i, and n pair-wise orthogonal eigenvectors $\mathbf{e}_i$, such that $s\mathbf{e}_i = \lambda_i \mathbf{e}_i$ or $h(X, \mathbf{e}_i) = \langle X, s\mathbf{e}_i \rangle = \lambda_i \langle X, \mathbf{e}_i \rangle$. In the standard basis $\{ \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \}$ the scalar second fundamental form is given by

$$h(X,Y) = \lambda_1 X^1 Y^1 + \lambda_2 X^2 Y^2 + \cdots + \lambda_n X^n Y^n = \sum_{i=1}^{n} \lambda_i X^i Y^i.$$

The eigenvalues λ_i are called the *principal curvatures* of $\mathbf{N}$ at p, and the corresponding eigenspaces are called the *principal directions*. The Gaussian curvature is defined as $K = \det(s) = \lambda_1 \lambda_2 \cdots \lambda_n$.

3.2. Connection to rolling

From (10), given unit length $\Lambda \in (\mathbf{TN})^{\perp}$, we get the following relations. By

$$\langle \Xi_\Lambda(X), Y \rangle = -\langle \mathrm{II}(X,Y), \Lambda \rangle = -h(X,Y) = -\langle sX, Y \rangle.$$

In this way we recover the Weingarten equation $\Xi_\Lambda(X) = -sX$.

Remark 8. *Suppose that the Gaussian curvature $K(p)$ is zero at $p \in \mathbf{N}$. From this, it follows that at least one eigenvalue λ_i is equal to zero. In other words, the principal directions do not span the whole tangent space $\mathbf{T}_p\mathbf{N}$ and both, the second fundamental form h and the Weingarten map Ξ_Λ fail to be surjective.*

The surjectivity of Ξ_Λ is a necessary condition for the rolling map according to the definition in Nomizu [15]. In view of Definition 1 the curvature of the manifolds involved does not enter in any of the conditions. However, as far as controllability is concerned, one needs the Weingarten map to be surjective. In Example 12, where we consider rolling a cylinder on the plane, this relationship is illustrated.

4. Configuration Space and Distribution

To investigate rolling hypersurfaces in Euclidean space we start with general results concerning configuration space, its tangent space and the distribution defined by rolling. Given a piecewise smooth curve σ_1 in $\mathbf{M}_1$, there exists a unique rolling map with rolling curve σ_1, *cf.* [17]. A somewhat similar result was obtained earlier by Nomizu [15] with the difference that he excluded points where the curvature of the two surfaces are the same. Nomizu considered such points as singular, where rolling is not defined. But Sharpe [17] defines rolling maps even when the rolling curve contains singular points. For instance, there is a rolling map, according to Sharpe's definition, that rolls the cylinder over a tangent plane along the line of contact.

To represent isometries of $\mathbf{M}$ consider the space $\mathcal{O}(\mathbf{M})$ of orthonormal frames on $\mathbf{M}$ and the group of automorphisms on $\mathcal{O}(\mathbf{M})$. Space $\mathcal{O}(\mathbf{M})$ is a principal bundle with group $\mathbb{O}(m)$. Since we assume that $\mathbf{M}$ is orientable then $\mathcal{O}(\mathbf{M})$ has two connected components, *cf.* [11, page 47].

Let $\mathcal{O}_p(\mathbf{M}_0)$ and $\mathcal{O}_q(\mathbf{M}_1)$ be the space of orthonormal frames on $\mathbf{M}_0$ at $p \in \mathbf{M}_0$ and on $\mathbf{M}_1$ at $q \in \mathbf{M}_1$, respectively. Choose an orthogonal frame $u_1 \in \mathcal{O}_q(\mathbf{M}_1)$. Define a *configuration space* $\mathscr{Z}$, i.e., the space of all positions of $\mathbf{M}_1$ tangent to $\mathbf{M}_0$, as follows:

$$\mathscr{Z} = \left\{ (p, R, q) \in \mathbf{M}_0 \times \mathbb{SO}(m) \times \mathbf{M}_1 \ : \ \widehat{R}(u_1) \in \mathcal{O}_p(\mathbf{M}_0) \right\}. \tag{11}$$

Here $\widehat{R}$ is the bundle automorphism of $\mathcal{O}(\mathbf{M})$ induced by R. If $\boldsymbol{\chi} = (R, s) \in \mathsf{Isom}(\mathbb{R}^m)$ is an isometry such that $\boldsymbol{\chi}(q) = p$ and $\boldsymbol{\chi}_*(\mathbf{T}_q\mathbf{M}_1) \subset \mathbf{T}_p\mathbf{M}_0$ then $(p, R, q) \in \mathscr{Z}$. By the uniqueness of isometries, there is a well defined map from the configuration space $\mathscr{Z}$ to $\mathsf{Isom}(\mathbb{R}^m)$ given by

$$\begin{array}{rcl} \psi\colon \mathscr{Z} & \to & \mathsf{Isom}(\mathbb{R}^m), \\ (p, R, q) & \mapsto & \boldsymbol{\chi} = (R, p - R\,q). \end{array} \tag{12}$$

We remark here that if $\boldsymbol{\chi}\colon I \to \mathsf{Isom}(\mathbb{R}^m)$ is a rolling map of $\mathbf{M}_1$ on $\mathbf{M}_0$, both embedded in $\mathbb{R}^m$, then there exists a unique induced curve $\gamma\colon I \to \mathscr{Z}$, such that $\boldsymbol{\chi} = \psi \circ \gamma$.

$$\begin{tikzcd} I \arrow[r, bend left, "\gamma"] \arrow[rr, bend right, "\boldsymbol{\chi}"'] & \mathscr{Z} \arrow[r, bend left, "\psi"] & \mathsf{Isom}(\mathbb{R}^m) \end{tikzcd}$$

Let II^0 and II^1 be the *second fundamental forms* on $\mathbf{M}_0$ and $\mathbf{M}_1$, and Ξ^0 and Ξ^1 be the Weingarten maps on $\mathbf{M}_0$ and $\mathbf{M}_1$, respectively, as they are defined in Section 3. First, we identify the tangent bundle of the configuration space and then we characterise the distribution of the rolling map. Both results can be found in [17].

Given a curve $(p, R, q)\colon I \to \mathscr{Z}$ let $(\dot{p}, \dot{R}, \dot{q})$ denote the corresponding velocity vector. The tangent space $\mathbf{T}_{(p,R,q)}\mathscr{Z}$ is given by the set of triples $(\dot{p}, \dot{R}, \dot{q})$ satisfying the following condition

$$\dot{R}\,R^{-1}V = \mathrm{II}^0(\dot{p}, V) - R\,\mathrm{II}^1(\dot{q}, R^{-1}V) \bmod \mathbf{T}_p\mathbf{M}_0, \quad \text{for all} \quad V \in \mathbf{T}_p\mathbf{M}_0.$$

Let $r = m - n$ be the co-dimension of the immersed manifolds $\mathbf{M}_0$ and $\mathbf{M}_1$. The tangent space is a vector space with dimension

$$\dim \mathbf{T}_{(p,R,q)}\mathscr{Z} = 2n + \frac{n\,(n-1)}{2} + \frac{r\,(r-1)}{2}.$$

In the case of hypersurfaces $r = 1$ and then $\dim \mathbf{T}_{(p,R,q)}\mathscr{Z} = n + \frac{n(n+1)}{2}$, the sum of $\dim \mathbf{M}_0$ and $\dim \mathsf{Isom}(\mathbb{R}^{n+1})$.

Lemma 9 (Sharpe [17])**.** *The mapping $\psi\circ\gamma\colon I\to \mathsf{Isom}(\mathbb{R}^m)$ is a rolling map if and only if γ is tangent to the n-dimensional distribution $\mathscr{D}$ on $\mathscr{L}$ given by the following set of differential equations*

(a) $\dot\sigma_0 = R\dot\sigma_1$;

(b) $\dot R\,R^{-1}V = \mathrm{II}^0(\dot\sigma_0, V) - R\,\mathrm{II}^1(R^{-1}\dot\sigma_0, R^{-1}V)$, *for all* $V\in \mathbf{T}_{\sigma_0(t)}\mathbf{M}_0$;

(c) $\dot R\,R^{-1}\Lambda = \Xi^0(\dot\sigma_0,\Lambda) - R\,\Xi^1(R^{-1}\dot\sigma_0, R^{-1}\Lambda)$, *for all* $\Lambda\in\big(\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0\big)^{\perp}$.

We shall illustrate consequences of Lemma 9 in the following Example 10, which is a continuation of Example 7.

Example 10. *Consider the unit sphere $\mathbf{S}^n$ rolling on its affine tangent space at a point $p_0\in\mathbf{S}^n$, as was introduced in Example 7. In this case $\mathbf{M}_0$ is a hyperplane whose curvature is equal to zero. Therefore, both the second fundamental form II^0 and the Weingarten map Ξ^0 are identically equal to zero. In the standard coordinates in $\mathbb{R}^{n+1}$, the n-sphere is defined by*

$$\mathbf{M}_1=\mathbf{S}^n=\left\{\, \left(x^1,x^2,\dots,x^{n+1}\right)\in\mathbb{R}^{n+1}\ :\ \sum_{i=1}^{n+1}\left(x^i\right)^2=1\,\right\}.$$

Take the manifold $\mathbf{M}_0$ to be its affine tangent space at the south pole, i.e.,

$$\mathbf{M}_0=\left\{\, \left(x^1,x^2,\dots,x^{n+1}\right)\in\mathbb{R}^{n+1}\ :\ x^{n+1}=-1\,\right\}.$$

Equations (b) and (c) of Lemma 9, defining the distribution $\mathscr{D}$ become, in this case,

$$\boldsymbol{A}\,V=-R\,\mathrm{II}^1(R^{-1}\dot\sigma_0,R^{-1}V)\quad \textit{and}\quad \boldsymbol{A}\,\Lambda=-R\,\Xi^1(R^{-1}\dot\sigma_0,R^{-1}\Lambda), \tag{13}$$

respectively. Note that both II^1 and Ξ^1 are tensors, therefore the right-hand sides of (13) are the push-forwards $\boldsymbol{\chi}_*\mathrm{II}^1$ and $\boldsymbol{\chi}_*\Xi^1$ (with negative sign) calculated at the point of contact $p=\sigma_0(t)$. These operations are well defined since, by the rolling condition (b) of Definition 1, the tangent spaces are identified at this point. Therefore, the normal spaces are also identified and we conclude, without calculations, that $\boldsymbol{A}\,V=-(\boldsymbol{\chi}_*\mathrm{II}^1)(\dot\sigma_0,V)$ lies in the normal space of $\mathbf{M}_0$. In matrix notation, equations (13) become

$$\left[\begin{array}{c|c}\widetilde{\boldsymbol{A}}_{n\times n} & X_{n\times 1}\\ \hline -X^{\mathrm{T}}_{1\times n} & 0\end{array}\right]\cdot\left[\begin{array}{c}\widetilde{V}_{n\times 1}\\ \hline 0\end{array}\right]=\left[\begin{array}{c}\widetilde{\boldsymbol{A}}_{n\times n}\cdot\widetilde{V}_{n\times 1}\\ \hline -X^{\mathrm{T}}_{1\times n}\cdot\widetilde{V}_{n\times 1}\end{array}\right]=\left[\begin{array}{c}\mathbf{0}_{n\times 1}\\ \hline *\end{array}\right]$$

and

$$\left[\begin{array}{c|c}\widetilde{\boldsymbol{A}}_{n\times n} & X_{n\times 1}\\ \hline -X^{\mathrm{T}}_{1\times n} & 0\end{array}\right]\cdot\left[\begin{array}{c}\mathbf{0}_{n\times 1}\\ \hline \widetilde{\Lambda}\end{array}\right]=\widetilde{\Lambda}\left[\begin{array}{c}X_{n\times 1}\\ \hline 0\end{array}\right]=\left[\begin{array}{c}*_{n\times 1}\\ \hline 0\end{array}\right].$$

As we pointed out in Remark 5, the latter of the above equalities, corresponding to the normal part of the "no-twist" condition, is always satisfied and the former equality implies that $\widetilde{\boldsymbol{A}}_{n\times n}=\mathbf{0}$. We shall explore the structure of the matrix $\boldsymbol{A}$ further in Section 6.1.

TABLE 1. Comparison of the scalar second fundamental form h and Weingarten map Ξ for the two surfaces $\mathbf{S}^2$ and $\mathbf{Cyl}$ in $\mathbb{R}^3$

	$\mathbf{S}^2$	$\mathbf{Cyl}$
$\mathbf{T}_x\mathbf{M}$	$\{ V \in \mathbb{R}^3 \ : \ \langle V, x \rangle = 0 \}$	$\{ V \in \mathbb{R}^3 \ : \ \langle V, \pi_{\mathbf{e}_2} x \rangle = 0 \}$
$(\mathbf{T}_x\mathbf{M})^\perp$	$\operatorname{span}(x)$	$\operatorname{span}(\pi_{\mathbf{e}_2} x)$
h	id_3	$\pi_{\mathbf{e}_2} \circ \mathsf{id}_3$
Ξ_Λ	$-\mathsf{id}_3$	$-\pi_{\mathbf{e}_2} \circ \mathsf{id}_3$
x_0, V, Λ	$(0,0,-1)$, $(v^1, v^2, 0)$, $(0,0,v^1)$	$(0,x^2,-1)$, $(v^1, v^2 x^2, 0)$, $(0,0,v^1)$

Here $\pi_{\mathbf{e}_2} \colon (x^1, x^2, x^3) \mapsto (x^1, 0, x^3)$ denotes the projection in the $\mathbf{e}_2$ direction.

5. Examples of Rolling in $\mathbb{R}^3$

Here we give a short account of two simple cases of rolling. This is to illustrate the relationship between curvature of rolling manifolds and the structure of the matrix $\boldsymbol{A}$.

5.1. A Comparison Between the 2-Sphere and a Cylinder in $\mathbb{R}^3$

As a simple illustration of the differences between rolling the sphere and a cylinder we show what constraints the two "non-twist" conditions (5) and (6) impose on the matrix $\boldsymbol{A}$ in these two cases.

We use the standard definition of the 2-sphere

$$\mathbf{S}^2 = \left\{ \left(x^1, x^2, x^3\right) \in \mathbb{R}^3 \ : \ \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 = 1 \right\}$$

and a cylinder

$$\mathbf{Cyl} = \left\{ \left(x^1, x^2, x^3\right) \in \mathbb{R}^3 \ : \ \left(x^1\right)^2 + \left(x^3\right)^2 = 1 \right\}.$$

In the sequel we make use of (13), and vectors and tensors summarised in Table 1.

Example 11 (The unit sphere). *The matrix $\boldsymbol{A} = \dot{R}R^{\mathrm{T}}$ is skew-symmetric, therefore at x_0 $\mathrm{II}(U,V) = \langle U, V \rangle\, x_0$ and, consequently,*

$$\begin{bmatrix} 0 & s_3 & s_1 \\ -s_3 & 0 & s_2 \\ -s_1 & -s_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} v^1 \\ v^2 \\ 0 \end{bmatrix} = \begin{bmatrix} s_3 v^2 \\ -s_3 v^1 \\ -s_1 v^1 - s_2 v^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix},$$

hence $s_3 = 0$. *For a normal vector* Λ *one has* $\Xi(\Lambda, V) = -V$ *and, consequently,*

$$\begin{bmatrix} 0 & 0 & s_1 \\ 0 & 0 & s_2 \\ -s_1 & -s_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ v^1 \end{bmatrix} = \begin{bmatrix} s_1 v^1 \\ s_2 v^1 \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}.$$

Example 12 (The cylinder)**.** *For any two tangent vectors* U, V *at* x_0 *one has* $\mathrm{II}(U, V) = \langle U, \pi_{\mathbf{e}_2} V \rangle\, x_0$. *So,*

$$\begin{bmatrix} 0 & s_3 & s_1 \\ -s_3 & 0 & s_2 \\ -s_1 & -s_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} v^1 \\ v^2 x^2 \\ 0 \end{bmatrix} = \begin{bmatrix} s_3 v^2 x^2 \\ -s_3 v^1 \\ -s_1 v^1 - s_2 v^2 x^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix}$$

hence $s_3 = 0$. *For a normal vector* Λ *one has* $\Xi(\Lambda, V) = -\pi_{\mathbf{e}_2} V$, *therefore*

$$\begin{bmatrix} 0 & 0 & s_1 \\ 0 & 0 & s_2 \\ -s_1 & -s_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ v^1 \end{bmatrix} = \begin{bmatrix} s_1 v^1 \\ s_2 v^1 \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix},$$

hence $s_2 = 0$.

The properties of the geometric tensors determine the structure of the matrix $\boldsymbol{A}$. The matrix $\boldsymbol{A}$ for $\mathbf{S}^2$ and $\mathbf{Cyl}$ has the two different structures

$$\boldsymbol{A}_{\mathbf{S}^2} = \begin{bmatrix} 0 & 0 & s_1 \\ 0 & 0 & s_2 \\ -s_1 & -s_2 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}_{\mathbf{Cyl}} = \begin{bmatrix} 0 & 0 & s_1 \\ 0 & 0 & 0 \\ -s_1 & 0 & 0 \end{bmatrix},$$

respectively. The two examples illustrate how singularity of the Weingarten map may put some constraints on the rolling of the cylinder.

6. Control Systems

Let $U \subset \mathbb{R}^m$ and $\mathbf{M}$ be a Riemannian manifold. Consider a control system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad \text{where} \quad x\colon I \to \mathbf{M} \quad \text{and} \quad u\colon I \to U.$$

If f is smooth in x and continuous in u, and control u is piecewise continuous, then the trajectory x is uniquely defined on I. We are concerned here with a more specific system, called *control-affine system*, which has the form

$$\dot{x}(t) = \mathscr{H}_0(x(t)) + \sum_{i=1}^{n} u_i(t)\, \mathscr{H}_i(x(t)),$$

where $\mathscr{H}_i$ are vector fields on $\mathbf{M}$ and the functions $u_i\colon I \to \mathbb{R}$ are controls. $\mathscr{H}_0$ is called the *drift* and a system without it is called *driftless*. We shall now reformulate the rolling equations as a control-affine system.

The formulation of the "no-slip" condition (4) leads to the following pair of kinematic equations on $R \in \mathbb{SO}(m)$ and $s \in \mathbb{R}^m$

$$\begin{cases} \dot{s} &= -\boldsymbol{A}\, R\, \sigma_1 \quad \text{and} \\ \dot{R} &= \boldsymbol{A}\, R. \end{cases} \tag{14}$$

Remark 13. *It is worth noticing that system (14) implies that $\dot{s}$ is orthogonal to $R\sigma_1$, because $\boldsymbol{A}$ is skew-symmetric. Therefore the distribution determined by (14) is restricted to a subspace $\mathcal{V} \subset \mathfrak{so}(m) \ltimes \mathbb{R}^m$, namely*

$$(\dot{R}, \dot{s}) \in \mathcal{V} \cong \mathfrak{so}(m) \ltimes \mathbb{R}^{m-1},$$

whose dimension is $\dim \mathcal{V} = m\,(m-1)\,/\,2 + m - 1 = (m+2)\,(m-1)\,/\,2$. In the case of co-dimension 1, when $m = n+1$, the $\dim \mathcal{V} = n\,(n+3)\,/\,2$ coincides with the dimension of $\dim \mathbf{T}_{(p,R,q)}\mathscr{X}$, the tangent space of the configuration space of the rolling map given in Section 4.

Remark 14. *The first equality in (14) may be also written in a more conventional but equivalent way, to give*

$$\dot{s} = \boldsymbol{A}\,(s - \sigma_0)\,.$$

However, from the viewpoint of Definition 1 of a rolling map, the curve σ_1 is assumed to be given. Therefore we always refer to (14).

We now write kinematic equations (14) as a control system, where the non-zero entries in matrix $\boldsymbol{A}$ are the controls.

6.1. The case of co-dimension one

Let us consider the case of a hypersurface rolling on a hyperplane. Here the embedding space is $\mathbb{R}^{n+1}$, the n-manifold $\mathbf{M}_1$ is a hypersurface, and $\mathbf{M}_0$ is the affine tangent space at a point $\mathbf{T}^{\mathrm{aff}}\mathbf{M}_0 \cong \mathbb{R}^n$, isomorphic to $\mathbb{R}^n$. In any neighbourhood $\mathcal{U} \subset \mathbf{M}$ of $x \in \mathcal{U}$ we denote a system of coordinates by $(\mathcal{U}, x)$. Local coordinates of a point $p \in \mathcal{U}$ denoted by $(p^1, p^2, \dots, p^{n+1})$ give rise to a basis of the tangent space $\mathbf{T}_p\mathbf{M}$ $\partial_i \stackrel{\text{def}}{=} \dfrac{\partial}{\partial x^i}$. Let $p = \sigma_0(t_0)$ be a point on the development curve at $t = t_0$. Choose system of coordinates $\{\,\mathcal{U}, p\,\}$ and the basis $\{\,\partial_i\,\}$ in such a way, that the first n coordinates correspond to the tangent space $\mathbf{T}_p\mathbf{M}_0$ and the last $n+1$-th to the normal space. More concisely

$$\operatorname{span}(\partial_1, \partial_2, \dots, \partial_n) = \mathbf{T}_p\mathbf{M}_0 \quad \text{and} \quad \operatorname{span}(\partial_{n+1}) = (\mathbf{T}_p\mathbf{M}_0)^{\perp}\,. \tag{15}$$

As pointed out in Remark 5, the "no-twist" conditions (5) and (6) impose the following structure on the skew-symmetric matrix $\boldsymbol{A} = \dot{R}R^{\mathrm{T}}$

$$\boldsymbol{A} = \left[\begin{array}{cccc|c} 0 & \dots & 0 & u_1 \\ 0 & \dots & 0 & u_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & u_n \\ \hline -u_1 & \dots & -u_n & 0 \end{array}\right] \quad \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \mathbf{T}_p\mathbf{M}_0 \\ (\mathbf{T}_p\mathbf{M}_0)^{\perp} \end{array} \tag{16}$$

This structure appears already in Example 10. Additional conditions follow from the equations (b)–(c) of Lemma 9 describing the distribution $\mathscr{D}$ of the rolling map. For $\mathbf{M}_0 = \mathbb{R}^n$ the second fundamental form II^0 and the Weingarten map Ξ^0 are both trivial and equal to zero. Then, the matrix $\boldsymbol{A}$ is governed by the two conditions, expressed with the push-forwards of the two tensors,

$$\begin{cases} \boldsymbol{A}\, V &= -(\boldsymbol{\chi}_*\mathrm{II}^1)(\dot{\sigma}_0, V), \qquad \text{for} \quad V \in \mathbf{T}_p\mathbf{M}_0 \quad \text{and} \\ \boldsymbol{A}\, \Lambda &= -(\boldsymbol{\chi}_*\Xi^1)(\dot{\sigma}_0, \Lambda), \qquad \text{for} \quad \Lambda \in (\mathbf{T}_p\mathbf{M}_0)^{\perp}, \end{cases}$$

where $\dot{\sigma}_0$, V and Λ are meant to belong to the tangent (normal) space of $\boldsymbol{\chi}(\mathbf{M}_1)$ at the point of contact, as noted in Remark 2. Examples of Section 5 illustrated that singularity of the Weingarten map may put some additional constraints on the entries of $\boldsymbol{A}$.

7. Controllability in Co-dimension One

To start our discussion on controllability of rolling it will be convenient to introduce the standard basis in the Lie algebra $\mathfrak{so}(n+1)$ in the following way

$$\boldsymbol{\Delta}_{i,j} = \partial_j \wedge \partial_i = \partial_i \otimes \partial_j - \partial_j \otimes \partial_i, \quad \text{for} \quad 1 \le i < j \le n+1, \tag{17}$$

where '$\otimes$' denotes the *outer* (Kronecker) product of two vectors. In these basis the skew-symmetric matrix $\boldsymbol{A} = \dot{R}R^{\mathrm{T}}$, given by (16), can be expressed as

$$\boldsymbol{A} = \sum_{i=1}^{n} u_i\, \boldsymbol{\Delta}_{i,n+1}.$$

The kinematic equations (14) in the basis (15) & (17) assume the following form

$$\begin{cases} \dot{R} &= \sum_{i=1}^{n} u_i\, \mathbf{\Delta}_{i,n+1}\, R \\ \dot{s}^i &= -u_i \sum_{j=1}^{n+1} R^{n+1}{}_j\, \sigma_1^j, \qquad \text{for} \quad 1 \le i \le n, \, . \\ \dot{s}^{n+1} &= \sum_{i=1}^{n} \sum_{j=1}^{n+1} u_i R^i{}_j\, \sigma_1^j \end{cases} \tag{18}$$

Equations (18), written as a control system, can further be written as

$$\begin{aligned} \dot{R}(t) &= \sum_{i=1}^{n} u_i(t)\, \mathscr{F}_i(R), \quad \text{where} \quad \mathscr{F}_i(R) &&= \mathbf{\Delta}_{i,n+1}\, R \quad \text{and} \\ \dot{s}(t) &= \sum_{i=1}^{n} u_i(t)\, \mathscr{G}_i(R), \quad \text{where} \quad \mathscr{G}_i(R) &&= \Big(\sum_{j=1}^{n+1} R^i{}_j\, \sigma_1^j\Big)\, \partial_{n+1} \\ & && \quad - \Big(\sum_{j=1}^{n+1} R^{n+1}{}_j\, \sigma_1^j\Big)\, \partial_i. \end{aligned} \tag{19}$$

Here, the functions $u_i\colon I \to \mathbb{R}$ are independent controls and $R^i{}_j$ denotes the component in i-th row and j-th column of matrix R. The pair of equations (19) defines a control system on $\mathbb{SO}(n+1) \times \mathbb{R}^{n+1}$ and can be written in condensed form as

$$(\dot{R}(t), \dot{s}(t)) = \sum_{i=1}^{n} u_i(t)\, (\mathscr{F}_i \oplus \mathscr{G}_i), \tag{20}$$

where the vector fields $\mathscr{F}_i$ and $\mathscr{G}_i$ are given in (19). Equation (20) also defines a distribution $\mathscr{D}$ over $\mathbb{SO}(n+1) \times \mathbb{R}^{n+1}$. Let $\mathscr{H}_i = \mathscr{F}_i \oplus \mathscr{G}_i$. At the identity $\mathsf{id}_{\mathbb{SO}(n+1)\times\mathbb{R}^{n+1}} = (\mathsf{id}_{\mathbb{SO}(n+1)}, \mathbf{0}_{\mathbb{R}^{n+1}})$, the vector field $\mathscr{H}_i$ is equal to

$$\mathscr{H}_i(\mathsf{id}) = (\mathbf{\Delta}_{i,n+1}) \oplus (\sigma_1^i\, \partial_{n+1} - \sigma_1^{n+1}\, \partial_i).$$

7.1. Bracket-generating property

In the sequel, we will show that the distribution defined by the kinematic equation (20) $\mathscr{D}$ is bracket-generating.

The Euclidean group $\mathbb{SE}(n+1)$ is a semi-direct product $\mathbb{SO}(n+1) \ltimes \mathbb{R}^{n+1}$. However, for the controllability of the kinematic equations we restrict to the situation when they evolve on the connected Lie group $\mathcal{G} = \mathbb{SO}(n+1) \times \mathbb{R}^{n+1}$, the Cartesian product of the multiplicative group $\mathbb{SO}(n+1)$ by the additive group R^{n+1}. Its Lie algebra $\mathbf{T}_{\mathsf{id}}\mathcal{G}$ is the direct sum $\mathfrak{g} = \mathfrak{so}(n+1) \oplus \mathbb{R}^{n+1}$, with Lie bracket defined by:

$$[A_1 \oplus f_1, A_2 \oplus f_2]_{\mathfrak{g}} = [A_1, A_2]_{\mathfrak{so}(n)} \oplus [f_1, f_2]_{R^{n+1}}. \tag{21}$$

(From now on, we write $[\ ,\ ]$ for any of these Lie brackets. This simplifies notations without compromising clarity.) The kinematic equation (20) defines the distribution $\mathscr{D}(\zeta) = \mathrm{span}(\{\, \mathscr{F}_i(\zeta) \oplus \mathscr{G}_i(\zeta)\,\}_{1 \le i \le n})$, for any $\zeta \in \mathcal{G}$. Although $\mathbf{T}_\zeta\mathcal{G} \neq \mathscr{D}(\zeta)$ we shall show that $\mathscr{D}$ is *bracket-generating*, i.e., $Lie\mathscr{D}(\zeta) = \mathbf{T}_\zeta\mathcal{G}$, for all $\zeta \in \mathcal{G}$. Note that by (14) the space spanned by $\{\, \mathscr{G}_i \,\}_{1 \le i \le n}$ is orthogonal to $R\sigma_1$, because the matrix $\boldsymbol{A}$ is skew-symmetric.

Lemma 15. *For $n > 1$, the distribution $\mathscr{D} = \mathscr{D}(\mathsf{id}) \subset \mathbf{T}_{\mathsf{id}}(\mathbb{SO}(n+1) \times \mathbb{R}^{n+1})$, given by (20), generates a subspace isometric to $\mathfrak{so}(n+1) \oplus \mathbb{R}^n$. More specifically, the following holds*

$$\mathscr{D} + [\mathscr{D}, \mathscr{D}] + [\mathscr{D}, [\mathscr{D}, \mathscr{D}]] = \mathfrak{so}(n+1) \times (\sigma_1)^\perp \cong \mathfrak{so}(n+1) \oplus \mathbb{R}^n.$$

Proof: We show first that

$$[\mathscr{D}, \mathscr{D}] = \mathrm{span}(\{\, \boldsymbol{\Delta}_{i,j} \oplus \mathbf{0} \,\}_{1 \le i < j \le n}).$$

This easily follows from (21) since

$$[\mathscr{H}_i, \mathscr{H}_j] = [\mathscr{F}_i \oplus \mathscr{G}_i, \mathscr{F}_j \oplus \mathscr{G}_j] = [\mathscr{F}_i, \mathscr{F}_j] \oplus [\mathscr{G}_i, \mathscr{G}_j]$$

and $[\mathscr{F}_i, \mathscr{F}_j] = [\boldsymbol{\Delta}_{i,n+1}, \boldsymbol{\Delta}_{j,n+1}] = \boldsymbol{\Delta}_{j,i}$ because, by (17), $[\boldsymbol{\Delta}_{i,n+1}, \boldsymbol{\Delta}_{j,n+1}] = [\partial_i \otimes \partial_{n+1} - \partial_{n+1} \otimes \partial_i, \partial_j \otimes \partial_{n+1} - \partial_{n+1} \otimes \partial_j] = -\partial_i \otimes \partial_j + \partial_j \otimes \partial_i = -\boldsymbol{\Delta}_{i,j}$. The vector fields $\{\, \partial_i \,\}$ are commutative therefore

$$[\mathscr{G}_i, \mathscr{G}_j] = [\sigma_1^i\, \partial_{n+1} - \sigma_1^{n+1}\, \partial_i, \sigma_1^j\, \partial_{n+1} - \sigma_1^{n+1}\, \partial_j] = \mathbf{0}.$$

Now, since

$$[\mathscr{H}_i, \boldsymbol{\Delta}_{i,j} \oplus \mathbf{0}] = [\boldsymbol{\Delta}_{i,n+1}, \boldsymbol{\Delta}_{i,j}] \oplus \mathbf{0} = \boldsymbol{\Delta}_{j,n+1} \oplus \mathbf{0},$$

it follows that

$$[\mathscr{D}, [\mathscr{D}, \mathscr{D}]] = \mathrm{span}(\{\, \boldsymbol{\Delta}_{i,n+1} \oplus \mathbf{0} \,\}_{1 \le i \le n}).$$

Therefore we may now write

$$\mathfrak{se}(n+1) \oplus \mathbf{0} = [\mathscr{D}, \mathscr{D}] + [\mathscr{D}, [\mathscr{D}, \mathscr{D}]].$$

Finally, because

$$\mathbf{\Delta}_{i,n+1} \oplus \mathscr{G}_i - \mathbf{\Delta}_{i,n+1} \oplus \mathbf{0} = \mathbf{0} \oplus (\sigma_1^i \, \partial_{n+1} - \sigma_1^{n+1} \, \partial_i)$$

and $\operatorname{span}(\{ \sigma_1^i \, \partial_{n+1} - \sigma_1^{n+1} \, \partial_i \}_{1 \le i \le n})$ is an n-dimensional subspace of $\mathbb{R}^{n+1}$ orthogonal to σ_1,

$$\mathscr{G}(\sigma_1) = \sum_{i=1}^{n} \mathscr{G}_i(\sigma_1) = \sum_{i=1}^{n} (\sigma_1^i \, \sigma_1^{n+1} - \sigma_1^{n+1} \, \sigma_1^i) = 0, \quad \text{where} \quad \sigma_1 = \sum_{i=1}^{n+1} \sigma_1^i \, x^i.$$

We conclude that

$$\mathscr{D} + [\mathscr{D}, \mathscr{D}] + [\mathscr{D}, [\mathscr{D}, \mathscr{D}]] \cong \mathfrak{so}(n+1) \oplus \mathbb{R}^n.$$

■

If one vector field $\mathscr{H}_i$ is missing, then the distribution $\mathscr{D}$ would not be bracket generating. In this sense, the distribution $\mathscr{D}$ is minimal.

Remark 16. *If one of the fields $\mathscr{H}_i$ is missing, then the $\mathbf{\Delta}_{i,j} \notin \mathrm{Lie}\mathscr{D}$, for all $1 \le i < j \le n+1$. Indeed, if $\mathscr{H}_i \notin \mathscr{D}$ then $\mathbf{\Delta}_{i,n+1} \notin \mathscr{D}$, because $\mathscr{F}_i \notin \mathscr{D}$. Further, since for any $1 \le a, b, c, d \le n+1$ the Lie bracket $[\mathbf{\Delta}_{a,b}, \mathbf{\Delta}_{c,d}]$ is equal to*

$$[\mathbf{\Delta}_{a,b}, \mathbf{\Delta}_{c,d}] = \delta_c^b \mathbf{\Delta}_{a,d} + \delta_d^a \mathbf{\Delta}_{b,c} - \delta_d^b \mathbf{\Delta}_{a,c} - \delta_c^a \mathbf{\Delta}_{b,d},$$

it is clear that $\mathbf{\Delta}_{i,j} \notin [\mathscr{D}, \mathscr{D}]$, where $1 \le j \le n+1$, for otherwise one of a, b, c or d would need to be equal to i. This is contrary to the assumption that $\mathbf{\Delta}_{i,j} \notin \mathscr{D}$. The property that $\mathbf{\Delta}_{i,j} \notin \mathrm{Lie}\mathscr{D}$, for all $1 \le i < j \le n+1$, is now easily seen by induction.

We are finally ready to present our main result. In the sequel, the term "free controls" is meant that functions $\{ u_i \}_{1 \le i \le n}$ are *not* linearly dependent. More precisely, let the n-tuple $u = (u_1, u_2, \ldots, u_n)$ denote a control function with values in $\mathbb{R}^n$, then the *affine hull* of the image $u(I)$ is equal to the whole vector space, i.e., $\operatorname{aff}(U) = \mathbb{R}^n$, where the *control set* U is the set of allowable values of $u(I)$, *cf.* [1].

Theorem 17. *If the last column entries u_1, u_2, $\ldots$, u_n of matrix $\boldsymbol{A}$ are free then the rolling is locally controllable.*

Proof: The result is an immediate consequence of Lemma 15. By Lemma 15 the distribution $\mathscr{D}$ associated to system (20) is bracket generating. Since the controls u_i are free, then by known results on controllability, the system is locally controllable *cf.* [1, pp. 374–376]. ■

Corollary 18. *Rolling of* $\mathbf{M}_1$ *upon its affine tangent space is locally controllable at* $p \in \mathbf{M}_1$ *if and only if the Gaussian curvature* $K(p)$ *at* p *is not equal to zero.*

Proof: Let $\{\, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n, \mathbf{e}_{n+1} \,\}$ be an orthonormal basis at $\mathbf{T}_p\mathbf{M}$ so that $\mathbf{e}_{n+1} \in (\mathbf{T}_p\mathbf{M}_0)^\perp$ and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the eigenvectors of the scalar second fundamental form h, *cf.* Section 3.1. The entries of the matrix $\boldsymbol{A}$ are then equal to $A^i{}_j = \langle \mathbf{e}_i, \boldsymbol{A}(\mathbf{e}_j) \rangle$. By equation (c) of Lemma 9,

$$u_i = A^i{}_{n+1} = \langle \mathbf{e}_i, \boldsymbol{A}(\mathbf{e}_{n+1}) \rangle = -\langle \mathbf{e}_i, \Xi_{\mathbf{e}_{n+1}} \dot{\sigma}_0 \rangle = h(\mathbf{e}_i, \dot{\sigma}_0) = \lambda_i \, \dot{\sigma}_0^i.$$

Hence u_i are free if and only if all $\lambda_i \neq 0$, i.e., $\prod_{i=1}^n \lambda_i = K(p) \neq 0$. The first part of result now follows from Theorem 17. If $K(p) = 0$ then at least one of the controls, say u_i is equal to zero. This amounts to $\mathscr{H}_i$ missing from $\mathscr{D}$. By Proposition 16 distribution $\mathscr{D}$ generates a Lie sub-algebra and so the system is not locally controllable because the system cannot be steered in some directions at the point of contact p. ∎

The two different situations are illustrated with our running examples of the unit sphere $\mathbf{S}^2$ and the cylinder **Cyl**, Example 11 & 12. In the case of $\mathbf{S}^2$ the $K(p) = 1$ and so the sphere is locally (of course it is known that it is globally) controllable on $\mathbb{SO}(3) \times \mathbb{R}^2$. However, because the Gaussian curvature K of **Cyl** is zero everywhere, rolling of the cylinder is not controllable. Our intuition is confirmed as the flatness of the cylinder induces the rolling distribution to generate a sub-algebra of $\mathbb{SO}(3) \times \mathbb{R}^2$. This simply means that the cylinder can roll on its affine tangent plane in only one direction given by the non-zero principal direction of the second fundamental form.

8. The Ellipsoid $\mathcal{E}^n$

We conclude our discussion on controllability by reviewing one more example of a hypersurface, an ellipsoid, rolling in Euclidean space. It will be convenient to define ellipsoid $\mathcal{E}^n$ embedded in $\mathbb{R}^{n+1}$ as follows, *cf.* [7]. For any symmetric positive definite matrix $\boldsymbol{B}$ define an inner product with respect to $\boldsymbol{B}$ by $\langle U, V \rangle_{\boldsymbol{B}} \stackrel{\text{def}}{=} \langle U, \boldsymbol{B}\, V \rangle$ and the norm $|U|_{\boldsymbol{B}} \stackrel{\text{def}}{=} \sqrt{\langle U, U \rangle_{\boldsymbol{B}}}$. Then

$$\mathcal{E}^n = \left\{\, p \in \mathbb{R}^{n+1} \;:\; |p|_{D^{-2}} = 1 \,\right\}, \quad \text{where} \quad D = \operatorname{diag}(d_1, d_2, \ldots, d_{n+1}) \succ 0. \tag{22}$$

Let $\gamma\colon (-\varepsilon, \varepsilon) \to \mathcal{E}^n$ be any differentiable curve in the ellipsoid such that $\gamma(0) = p$. Then $|\gamma|_{D^{-2}} = 1$. Differentiating this equality with respect to t

yields

$$0 = \frac{d}{dt}\,|\gamma|^2_{D^{-2}} = \frac{d}{dt}\langle \gamma, D^{-2}\gamma\rangle = \langle \dot\gamma, D^{-2}\gamma\rangle + \langle \gamma, D^{-2}\dot\gamma\rangle = 2\langle \dot\gamma, \gamma\rangle_{D^{-2}}.$$

Therefore the tangent space $\mathbf{T}_p\mathcal{E}^n$ is the subspace orthogonal to $D^{-2}p$ in $\mathbb{R}^{n+1}$. The unit normal vector $\Lambda \in (\mathbf{T}_p\mathcal{E}^n)^\perp$ is given by $\Lambda = D^{-2}p/\left|D^{-2}p\right|$. Hence the Weingarten map Ξ_Λ at $p \in \mathcal{E}^n$ is given by

$$\Xi_\Lambda(X) = -D^{-2}\left(\frac{X}{|D^{-2}p|} - \frac{p}{|D^{-2}p|^3}\langle D^{-2}X, D^{-2}p\rangle\right).$$

We can apply (10) to obtain the second fundamental form

$$\mathrm{II}(X,Y) = \frac{\langle D^{-1}X, D^{-1}Y\rangle}{|D^{-2}p|^2}\,D^{-2}p.$$

8.1. The Gaussian curvature of the ellipsoid

Since we didn't find in the literature any short method of deriving the formula (23) giving the Gaussian curvature of $\mathcal{E}^2$ at a point p, we present here an alternative proof, for the sake of completeness.

Proposition 19. *The Gaussian curvature $K(p)$ of $\mathcal{E}^2$ at a point p is given by*

$$K(p) = \frac{\det(D^{-2})}{|D^{-2}p|^4} > 0. \tag{23}$$

Proof: The Gaussian curvature of the ellipsoid will be derived without solving the Eigensystem. From the properties of the Weingarten map Ξ_Λ we know that

- $\Xi_\Lambda\colon \mathbf{T}_p\mathcal{E}^2 \to \mathbf{T}_p\mathcal{E}^2$, therefore its image lies in the tangent space, and
- Ξ_Λ is symmetric.

Let $X, Y \in \mathbf{T}_p\mathcal{E}^2$ be any two tangent vectors spanning the tangent space at $p \in \mathcal{E}^2$. Denote $q \overset{\text{def}}{=} D^{-2}p$ then $\langle X, q\rangle = 0 = \langle Y, q\rangle$, and $\langle p, q\rangle = 1$, where we identify points in $\mathbb{R}^3$ with vectors and the inner product is the standard 'dot' product in Euclidean space. On one hand we know that the cross product $X \times Y = \alpha\, q$, because the two vectors are orthogonal to q. On the other hand

$$\begin{aligned}\Xi_\Lambda(X) \times \Xi_\Lambda(Y) &= (a_{XX}X + a_{XY}Y) \times (a_{YX}X + a_{YY}Y)\\ &= a_{XX}a_{YY}X \times Y + a_{XY}a_{YX}Y \times X\end{aligned}$$

$$= \det(\Xi_\Lambda)\, X \times Y.$$

From the way Ξ_Λ is defined, the left hand side of the above equality becomes

$$\frac{1}{|q|^6}(-|q|^2\ D^{-2}X + \langle D^{-2}X, q\rangle\, q) \times (-|q|^2\ D^{-2}Y + \langle D^{-2}Y, q\rangle\, q).$$

Expanding the above product, gives three non vanishing terms and by taking the inner product with q only the following non zero term remains

$$\frac{1}{|q|^6}\langle(-|q|^2\ D^{-2}X) \times (-|q|^2\ D^{-2}Y), q\rangle = \frac{1}{|q|^2}\langle(D^{-2}X) \times (D^{-2}Y), q\rangle$$

$$= \frac{1}{|q|^2}\langle\det(D^{-2})\, D^2(X \times Y), q\rangle = \frac{\det(D^{-2})}{|q|^2}\langle X \times Y, p\rangle = \alpha\,\frac{\det(D^{-2})}{|q|^2},$$

where we have used the symmetry of D and the following property of the cross product in $\mathbb{R}^3$

$$(Mv_1) \times (Mv_2) = \det(M)\, M^{-\mathrm{T}}\,(v_1 \times v_2), \tag{24}$$

where M is an 3×3 non singular matrix, see Appendix A. Since $\langle X \times Y, q\rangle = \alpha\,|q|^2$, comparing the two sides of this equality yields the desired expression for the Gaussian curvature. ■

Remark 20. *By Corollary 18, we conclude that rolling of $\mathcal{E}^2$ is locally controllable.*

8.2. The kinematic equations for rolling the ellipsoid

According to Corollary 18, this rolling system is locally controllable. We will now establish kinematic equations for rolling the ellipsoid on its affine tangent space at a given point. As far as we know this has not been done before for $n > 2$. In the 2-dimensional case, what we derive here is an alternative to what appeared in [7].

To describe the rolling curve $\sigma_1 \colon I \to \mathcal{E}^n$ we use following construction. Let $R(\mathcal{E}^n)$, where $R \in \mathbb{SO}(n+1)$, be the image of the ellipsoid (22) by the rotation R. There exists a unique point $x \in R(\mathcal{E}^n)$ whose $n+1$-th coordinate $x^{n+1} = \min_{p \in R(\mathcal{E}^n)} p^{n+1}$. Then, $x + s$ is precisely the point of contact of $\mathcal{E}^n$ rolling on its affine tangent plane under the rolling $\boldsymbol{\chi} = (R, s)$. We assign the point x to σ_1. Because the above mapping $R(t) \mapsto \sigma_1(t)$ is not a bijection, σ_1 is not necessarily smooth even if $\boldsymbol{\chi}$ is. However, it is transitive by the following reasoning. Let $\pi \colon \mathcal{E}^n \to \mathbf{S}^n$ be a projection from the ellipsoid onto the sphere, i.e., $\pi\colon x \mapsto x/|x|$. Then σ_1 is an image of the composition of the following mappings acting on a point $p_0 \in \mathcal{E}^n$

$$\mathcal{E}^n \xrightarrow{D^{-1}} \mathbf{S}^n \xrightarrow{R^{-1}} \mathbf{S}^n \xrightarrow{D} \mathcal{E}^n \xrightarrow{\pi} \mathbf{S}^n \xrightarrow{D} \mathcal{E}^n$$

where $D \circ \pi \circ D\colon \mathbf{S}^n \to \mathcal{E}^n$ is a bijection whose inverse is given by $\pi \circ D^{-1} \circ D^{-1}\colon \mathcal{E}^n \to \mathbf{S}^n$. Since $\mathbb{SO}(n+1)$ acts transitively on $\mathbf{S}^n$ and D is bijective, the above composition is also transitive.

Proposition 21. *Let $p_0 = -d_{n+1}\,\mathbf{e}_{n+1}$ be the "south pole" of $\mathcal{E}^n$ and let $\sigma_1(t) = D^2\,R(t)^{\mathrm{T}}\,D^{-1}\,p_0/\big|D\,R(t)^{\mathrm{T}}\,D^{-1}\,p_0\big|$ be a curve in $\mathcal{E}^n$. Then the kinematic equations for rolling $\mathcal{E}^n$ on its affine tangent plane are given by*

$$\left\{\begin{array}{rl} \dot{s}(t) & = -\boldsymbol{A}(t)\,\dfrac{R(t)\,D^2\,R^{\mathrm{T}}(t)\,D^{-1}\,p_0}{|D\,R^{\mathrm{T}}(t)\,D^{-1}\,p_0|} \\ \dot{R}(t) & = \boldsymbol{A}(t)\,R(t) \end{array}\right. , \tag{25}$$

where the skew-symmetric matrix $\boldsymbol{A}$ has the structure (16).

Proof: We first show that $\boldsymbol{A}$ has the correct structure. This structure is determined by the constraints (b) and (c) in Lemma 9 for $\mathcal{E}^n$. Choose a $t_0 \in I$ and let $p = \sigma_0(t_0)$ be the point of contact. Then $q = \sigma_1(t_0)$ and $\boldsymbol{\chi}(t_0)(q) = p$. The tangent space $\mathbf{T}_p\mathbf{M}_0$ splits into $\mathbf{T}_p\mathbf{M} = \mathbf{T}_p\mathbf{M}_0 \oplus (\mathbf{T}_p\mathbf{M}_0)^{\perp}$. Choose the basis in $\mathbf{T}_p\mathbf{M}_0$ as in Remark 5, so that $\mathbf{e}_{n+1} \in (\mathbf{T}_p\mathbf{M}_0)^{\perp}$. We will now write explicitly

$$\boldsymbol{A}\,\Lambda = -(\boldsymbol{\chi}_*\Xi^1)(\dot{\sigma}_0, \Lambda),$$

where we take $\Lambda = \mathbf{e}_{n+1}$. From the rolling condition $R(t)^{-1}\dot{\sigma}_0(t) = \dot{\sigma}_1(t)$ it follows that $\boldsymbol{A}(\mathbf{e}_{n+1})|_p = -R\,\Xi^1_{\mathbf{e}_{n+1}}(\dot{\sigma}_1)$. We shall drop the parameter t whenever it will not lead to a confusion. The Weingarten map Ξ_Λ at $q \in \mathcal{E}^n$ is given by

$$\Xi^1_{\mathbf{e}_{n+1}}(\dot{\sigma}_1) = -D^{-2}\left(\frac{\dot{\sigma}_1}{|D^{-2}q|} - \frac{q}{|D^{-2}q|^3}\langle D^{-2}\dot{\sigma}_1, D^{-2}q\rangle\right),$$

where

$$\dot{\sigma}_1 = \frac{D^2\dot{R}^{\mathrm{T}}D^{-1}p_0}{|DR^{\mathrm{T}}D^{-1}p_0|} - D^2R^{\mathrm{T}}D^{-1}p_0\,\frac{\langle D\dot{R}^{\mathrm{T}}D^{-1}p_0, DR^{\mathrm{T}}D^{-1}p_0\rangle}{|DR^{\mathrm{T}}D^{-1}p_0|^3}.$$

By the expression of σ_1 there is $D^{-2}q = R^{\mathrm{T}} D^{-1} p_0 / \left| D R^{\mathrm{T}} D^{-1} p_0 \right|$ hence

$$\left| D^{-2} q \right| = \frac{\left| R^{\mathrm{T}} D^{-1} p_0 \right|}{\left| D R^{\mathrm{T}} D^{-1} p_0 \right|} = \frac{\left| D^{-1} p_0 \right|}{\left| D R^{\mathrm{T}} D^{-1} p_0 \right|}.$$

Now

$$\begin{aligned} \langle D^{-2} \dot{\sigma}_1, D^{-2} q \rangle \;&=\; \frac{\langle \dot{R}^{\mathrm{T}} D^{-1} p_0, R^{\mathrm{T}} D^{-1} p_0 \rangle}{\left| D R^{\mathrm{T}} D^{-1} p_0 \right|^2} \\ &\quad - \frac{\langle R^{\mathrm{T}} D^{-1} p_0, R^{\mathrm{T}} D^{-1} p_0 \rangle}{\left| D R^{\mathrm{T}} D^{-1} p_0 \right|^4} \langle D \dot{R}^{\mathrm{T}} D^{-1} p_0, D R^{\mathrm{T}} D^{-1} p_0 \rangle \\ &= \; - \frac{\left| D^{-1} p_0 \right|^2}{\left| D R^{\mathrm{T}} D^{-1} p_0 \right|^4} \langle D \dot{R}^{\mathrm{T}} D^{-1} p_0, D R^{\mathrm{T}} D^{-1} p_0 \rangle, \end{aligned}$$

because $R\dot{R}^{\mathrm{T}}$ is skew-symmetric. From the above equality it now follows that

$$\begin{aligned} D^{-2} q \, \frac{\langle D^{-2} \dot{\sigma}_1, D^{-2} q \rangle}{\left| D^{-2} q \right|^3} \;&=\; - \frac{R^{\mathrm{T}} D^{-1} p_0}{\left| D R^{\mathrm{T}} D^{-1} p_0 \right|} \frac{\left| D^{-1} p_0 \right|^2}{\left| D R^{\mathrm{T}} D^{-1} p_0 \right|^4} \\ &\qquad \langle D \dot{R}^{\mathrm{T}} D^{-1} p_0, D R^{\mathrm{T}} D^{-1} p_0 \rangle \frac{\left| D R^{\mathrm{T}} D^{-1} p_0 \right|^3}{\left| D^{-1} p_0 \right|^3} \\ &= \; - \frac{R^{\mathrm{T}} D^{-1} p_0}{\left| D^{-1} p_0 \right| \left| D R^{\mathrm{T}} D^{-1} p_0 \right|^2} \langle D \dot{R}^{\mathrm{T}} D^{-1} p_0, D R^{\mathrm{T}} D^{-1} p_0 \rangle. \end{aligned}$$

We will also need

$$\begin{aligned} -D^{-2} \frac{\dot{\sigma}_1}{\left| D^{-2} q \right|} &= - \frac{\left| D R^{\mathrm{T}} D^{-1} p_0 \right|}{\left| D^{-1} p_0 \right|} \\ &\left(\frac{\dot{R}^{\mathrm{T}} D^{-1} p_0}{\left| D R^{\mathrm{T}} D^{-1} p_0 \right|} - \frac{R^{\mathrm{T}} D^{-1} p_0}{\left| D R^{\mathrm{T}} D^{-1} p_0 \right|^3} \langle D \dot{R}^{\mathrm{T}} D^{-1} p_0, D R^{\mathrm{T}} D^{-1} p_0 \rangle \right). \end{aligned}$$

Adding the two equalities above, brings

$$\Xi^1_{\mathbf{e}_{n+1}}(\dot{\sigma}_1) = - \frac{\dot{R}^{\mathrm{T}} D^{-1} p_0}{\left| D^{-1} p_0 \right|} = - \dot{R}^{\mathrm{T}} D^{-1} p_0.$$

Finally, putting $D^{-1} p_0 = -\mathbf{e}_{n+1}$, the above calculations yield

$$\boldsymbol{A}(\mathbf{e}_{n+1})|_p = -R \, \Xi^1_{\mathbf{e}_{n+1}}(\dot{\sigma}_1) = R \dot{R}^{\mathrm{T}} D^{-1} p_0 = -R \dot{R}^{\mathrm{T}} \mathbf{e}_{n+1}, \tag{26}$$

what is in full agreement with the second kinematic equation $\dot{R} = \boldsymbol{A}R$. Equation (26) gives values of the last column of matrix $\boldsymbol{A}$. To find out the remaining part of $\boldsymbol{A}$ it remains to calculate condition (b) of Lemma 9

$$\boldsymbol{A}\,V = -(\boldsymbol{\chi}_*\mathrm{II}^1)(\dot{\sigma}_0, V),$$

where we take $V = \mathbf{e}_i$, for $1 \le i \le n$.

$$\begin{aligned}
\frac{\langle D^{-2}\dot{\sigma}_1, R^{-1}\mathbf{e}_i\rangle}{|D^{-2}q|^2}\,D^{-2}q &= \frac{\langle \dot{R}^{\mathrm{T}}D^{-1}p_0, R^{-1}\mathbf{e}_i\rangle\ \left|DR^{\mathrm{T}}D^{-1}p_0\right|^2}{|DR^{\mathrm{T}}D^{-1}p_0|\ |D^{-1}p_0|^2}\,\frac{R^{\mathrm{T}}D^{-1}p_0}{|DR^{\mathrm{T}}D^{-1}p_0|} \\
&\quad - \langle R^{\mathrm{T}}D^{-1}p_0, R^{-1}\mathbf{e}_i\rangle\,\frac{\langle D\dot{R}^{\mathrm{T}}D^{-1}p_0, DR^{\mathrm{T}}D^{-1}p_0\rangle\ \left|DR^{\mathrm{T}}D^{-1}p_0\right|^2}{|D^{-1}p_0|^2\ |DR^{\mathrm{T}}D^{-1}p_0|^3} R^{\mathrm{T}}D^{-1}p_0.
\end{aligned}$$

Because $\langle R^{\mathrm{T}}D^{-1}p_0, R^{-1}\mathbf{e}_i\rangle = \langle D^{-1}p_0, \mathbf{e}_i\rangle = -\langle \mathbf{e}_{n+1}, \mathbf{e}_i\rangle = 0$, the second term in the above equality vanishes, and its right hand side simplifies to

$$\begin{aligned}
\frac{\langle D^{-2}\dot{\sigma}_1, R^{-1}\mathbf{e}_i\rangle}{|D^{-2}q|^2}\,D^{-2}q &= -\frac{\langle \dot{R}^{\mathrm{T}}\mathbf{e}_{n+1}, R^{\mathrm{T}}\mathbf{e}_i\rangle}{|D^{-1}p_0|^2}\,R^{\mathrm{T}}D^{-1}p_0 \\
&= -\langle R\dot{R}^{\mathrm{T}}\mathbf{e}_{n+1}, \mathbf{e}_i\rangle\ R^{\mathrm{T}}D^{-1}p_0.
\end{aligned}$$

It now follows that $\boldsymbol{A}(\mathbf{e}_i) = -R\mathrm{II}^1(R^{-1}\dot{\sigma}_0, R^{-1}\mathbf{e}_i) = -\mathbf{e}_{n+1}\,\langle R\dot{R}^{\mathrm{T}}\mathbf{e}_{n+1}, \mathbf{e}_i\rangle$ and the entries of $\boldsymbol{A}$ are given by

$$A^j{}_i = -\langle \mathbf{e}_j, \mathbf{e}_{n+1}\rangle\ \langle R\dot{R}^{\mathrm{T}}\mathbf{e}_{n+1}, \mathbf{e}_i\rangle. \tag{27}$$

We conclude therefore that equalities (26) and (27) verify that the structure of $\boldsymbol{A}$ conforms with (16) and pose no further constraints on the n entries in the last column (or equivalently in the last row) of $\boldsymbol{A}$.

We are now ready to proceed with the proof that (25) are the kinematic equations. The expressions are somewhat by noting that $D^{-1}\,p_0 = -\mathbf{e}_{n+1}$. At first, we need to verify that $\sigma_1(t) \in \mathcal{E}^n$, for any $t \in I$. To do that it is enough to check that the $|\sigma_1|^2_{D^{-2}}$ is equal to 1. Indeed

$$|\sigma_1|^2_{D^{-2}} = \frac{\langle D^2\,R^{\mathrm{T}}\,\mathbf{e}_{n+1}, R^{\mathrm{T}}\,\mathbf{e}_{n+1}\rangle}{|D\,R^{\mathrm{T}}\,\mathbf{e}_{n+1}|^2} = \frac{\langle D\,R^{\mathrm{T}}\,\mathbf{e}_{n+1}, D\,R^{\mathrm{T}}\,\mathbf{e}_{n+1}\rangle}{|D\,R^{\mathrm{T}}\,\mathbf{e}_{n+1}|^2} = 1.$$

By transitivity arguments at the beginning of this section, it is clear that any curve on the ellipsoid can be written in this way.

Now we check if the conditions of the definition of rolling are satisfied. In fact we only need to check if the rolling conditions are satisfied. Since the "non-slip" and "no-twist" conditions have been already used in the structure of $\boldsymbol{A}$ we only need to check that rolling condition are satisfied. For this it is necessary that

$$R(\mathbf{T}_{\sigma_1}\mathcal{E}^n) = (p_0)^{\perp}, \tag{28}$$

where $(p_0)^{\perp}$ denotes the n-dimensional subspace orthogonal to p_0 in $\mathbb{R}^{n+1}$. Because $p_0 = -d_{n+1}\,\mathbf{e}_{n+1}$, in order to verify (28) it is enough to check for equality by pulling back $(p_0)^{\perp}$ onto $\mathbf{T}_{\sigma_1}\mathcal{E}^n$, i.e.,

$$\begin{aligned}\langle R^{\mathrm{T}}\mathbf{e}_i, D^{-2}\sigma_1\rangle &= \frac{\langle R^{\mathrm{T}}\,\mathbf{e}_i, D^{-2}\,D^2\,R^{\mathrm{T}}\,\mathbf{e}_{n+1}\rangle}{|D\,R^{\mathrm{T}}\,\mathbf{e}_{n+1}|}\\ &= \frac{\langle R^{\mathrm{T}}\,\mathbf{e}_i, R^{\mathrm{T}}\,\mathbf{e}_{n+1}\rangle}{|D\,R^{\mathrm{T}}\,\mathbf{e}_{n+1}|} = \frac{\langle \mathbf{e}_i, \mathbf{e}_{n+1}\rangle}{|D\,R^{\mathrm{T}}\,\mathbf{e}_{n+1}|} = 0,\end{aligned}$$

for any $1 \le i \le n$. Therefore, the tangent space $\mathbf{T}_{R\sigma_1}R\mathcal{E}^n$ is equal to $\operatorname{span}(\mathbf{e}_1,\dots,\mathbf{e}_n)$, which clearly verifies (28). ∎

We consider here a simple case of $\mathcal{E}^2$ embedded in $\mathbb{R}^3$. Since any curve on this ellipsoid is given by

$$\sigma_1 = -\frac{D^2\,R^{\mathrm{T}}\,\mathbf{e}_3}{|D\,R^{\mathrm{T}}\,\mathbf{e}_3|},$$

then, by Proposition 21, the first kinematic equation becomes

$$\dot{s}(t) = -\boldsymbol{A}\,R\,\sigma_1 = \boldsymbol{A}\,\frac{R\,D^2\,R^{\mathrm{T}}\,\mathbf{e}_3}{|D\,R^{\mathrm{T}}\,\mathbf{e}_3|}.$$

In the standard coordinates, this is given by

$$\begin{cases} \dot{s}^i &= u_i\,\left|D\,R^{\mathrm{T}}\,\mathbf{e}_3\right|, \qquad\qquad \text{for}\quad i=1,2\\ \dot{s}^3 &= -\displaystyle\sum_{i=1}^{2} u_i\,\frac{\langle D\,R^{\mathrm{T}}\,\mathbf{e}_i, D\,R^{\mathrm{T}}\,\mathbf{e}_3\rangle}{|D\,R^{\mathrm{T}}\,\mathbf{e}_3|}\end{cases}. \tag{29}$$

$$\dot{s}(t) = \sum_{i=1}^{2} u_i(t)\,\mathscr{G}_i, \quad \text{where} \quad \mathscr{G}_i = \left|D\,R^{\mathrm{T}}\,\mathbf{e}_3\right|\,\partial_i - \frac{\langle D\,R^{\mathrm{T}}\,\mathbf{e}_i, D\,R^{\mathrm{T}}\,\mathbf{e}_3\rangle}{|D\,R^{\mathrm{T}}\,\mathbf{e}_3|}\,\partial_3.$$

In this case the vector fields $\mathscr{G}_i = \mathscr{G}_i(R)$ depend on R only, and thus are time-invariant.

Remark 22. *In the case of the unit sphere $\mathbf{S}^2$, all the axes are the same length and the matrix D is equal to the identity. Then, the vector fields $\mathscr{G}_i$,*

$i = 1, 2,$ *become*

$$\mathscr{G}_i = \left|R^{\mathrm{T}}\,\mathbf{e}_3\right|\,\partial_i - \frac{\langle R^{\mathrm{T}}\,\mathbf{e}_i, R^{\mathrm{T}}\,\mathbf{e}_3\rangle}{\left|R^{\mathrm{T}}\,\mathbf{e}_3\right|}\,\partial_3 = |\mathbf{e}_3|\,\partial_i - \frac{\langle \mathbf{e}_i, \mathbf{e}_3\rangle}{|\mathbf{e}_3|}\,\partial_3 = \partial_i.$$

9. Future Work and Conclusion

In this paper we started the investigation on how the geometry of rolling manifolds poses restrictions on rolling maps. For some particular cases we have shown that singularities of the Gaussian curvature at the points of contact impose constraints on the directions of rolling. More precisely, rolling motions of Riemannian manifolds whose Gaussian curvature is zero at some points, are not locally controllable at these points. This behaviour agrees with our intuition in the case of a cylinder or a cone rolling on flat surfaces. In these cases the Gaussian curvature is zero everywhere. The results presented in this paper continue the study of controllability of rolling manifolds undertaken in [13, 6, 5]. Our aim, in the near future, is to find necessary and sufficient conditions for controllability of rolling motions in a more general Riemannian framework, as we setup in [7].

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Appendix A. Matrix Transformation of the Cross Product

For the standard cross product in $\mathbb{R}^3$ the identity (24) is well known. A more general cross product in $\mathbb{R}^n$ is defined in [19]. Here we derive an elegant generalisation of (24) in $\mathbb{R}^n$.

A.1. The general case

Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. The *pull-back* map F^* acts on tensors as follows

$$(F^*T)\,(v_1, v_2, \ldots, v_n) \stackrel{\text{def}}{=} T(Fv_1, Fv_2, \ldots, Fv_n).$$

We start with a *generalised cross* product, *cf.* [19]. Given $v_1, v_2, \ldots, v_{n-1} \in \mathbb{R}^n$, define a 1-form $\varphi \in \Lambda^1(\mathbb{R}^n)$ by

$$\varphi(w) = \det \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ w \end{bmatrix}, \quad \text{for any} \quad w \in \mathbb{R}^n.$$

There exists a unique $z \in \mathbb{R}^n$ (a dual) such that $\langle w, z\rangle = \varphi(w)$. This z denoted by $v_1 \times v_2 \times \cdots \times v_{n-1}$ is called the *cross product* of $v_1, v_2, \ldots, v_{n-1}$.

Since φ is a tensor, then the pull-back $F^*\varphi(w) = \varphi(Fw)$. Applying F^* to the determinant yields

$$F^* \det \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ w \end{bmatrix} = \det \begin{bmatrix} Fv_1 \\ Fv_2 \\ \vdots \\ Fv_{n-1} \\ Fw \end{bmatrix} = \det(F) \cdot \det \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ w \end{bmatrix} = \det(F) \cdot \varphi(w).$$

Putting these together, we can write

$$F^*\varphi(w) = \varphi(Fw) = \det(F) \cdot \varphi(w) = \det(F) \cdot \langle w, z\rangle,$$

yielding $\varphi(Fw) = \det(F) \cdot \langle F^{-1}Fw, z\rangle = \det(F) \cdot \langle Fw, F^{-\mathrm{T}}z\rangle$, for any vector w in $\mathbb{R}^n$.

A.2. The case of $\mathbb{R}^3$

When we restrict ourselves to R^3, we get the ordinary cross product $z = v_1 \times v_2$. Let M be a matrix associated with a linear transformation $F\colon \mathbb{R}^3 \to \mathbb{R}^3$. Then, it follows from the previous section that

$$(Mv_1) \times (Mv_2) = \det(M)\, M^{-\mathrm{T}}(v_1 \times v_2).$$

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