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FURTHER REMARKS ON THE "SMITH IS HUQ" CONDITION

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Dedicated to George Janelidze on the occasion of his sixtieth birthday

Abstract: We prove that the *Smith is Huq* condition for finitely cocomplete homological categories is equivalent to the Bourn-style condition that the inverse image functors of the fibration of points reflect Huq-commutativity of pairs of normal subobjects. We give a version of this result for pointed weakly Mal'tsev categories. In a semi-abelian context, we give a decomposition formula for the normal weighted commutator of Gran, Janelidze and Ursini.

Keywords: semi-abelian, finitely cocomplete homological, weakly Mal'tsev category; Smith, Huq, weighted commutator; normalisation functor.
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This article has the following purposes: first of all, to suggest that the context of pointed weakly Mal'tsev categories is still strong enough to work with commutators, and more generally with the *admissibility diagrams*—which are diagrams such as (**A**) below—introduced in [18]. We show that in this context the *Smith is Huq* condition is still equivalent to the condition that a reflexive graph is multiplicative if and only if the kernels of its domain and codomain morphisms commute (Proposition 3). We also characterise *Smith is Huq* in terms of the fibration of points (Proposition 2) and show that in the richer context of semi-abelian categories, the *Smith is Huq* condition is equivalent to several other conditions, relating thus many branches of categorical algebra (Theorem 3). Finally, we explain that the concept of *weighted commutativity* of Gran, Janelidze and Ursini [9] is essentially equivalent to admissibility in the above sense and show that *Smith is Huq* amounts to independence of the chosen weight for cospans of proper morphisms (Example 6, Proposition 5,

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Theorem 1 and 3). We also decompose their normal weighted commutator in terms of binary and ternary Higgins commutators [11] (Theorem 4).

Weakly Mal'tsev categories. A category is called **weakly Mal'tsev** [19] if and only if it has pullbacks of split epimorphisms along split epimorphisms and the splittings $e_1 = \langle 1_A, s\circ f\rangle$ and $e_2 = \langle r\circ g, 1_C\rangle$ induced by the sections r and s, respectively, are jointly epimorphic.

$$\begin{tikzcd} A \times_B C \arrow[r, shift left, "\pi_2"] \arrow[d, shift right, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[l, shift left, "e_2"] \arrow[d, shift right, "g"'] \\ A \arrow[u, shift right, "e_1"'] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "r"] \arrow[u, shift right, "s"'] \end{tikzcd}$$

In such categories, for any diagram of the shape

$$\begin{tikzcd} & C \arrow[ld, shift left, "e_2"] \arrow[rd, shift left, "g"] \arrow[rrd, bend left, "\gamma"] & & \\ A \times_B C \arrow[ru, shift left, "\pi_2"] \arrow[rd, shift right, "\pi_1"'] & & B \arrow[lu, shift left, "s"] \arrow[ld, shift left, "r"] \arrow[r, "\beta"] & D \\ & A \arrow[lu, shift right, "e_1"'] \arrow[ru, shift left, "f"] \arrow[rru, bend right, "\alpha"'] & & \end{tikzcd} \qquad (\mathbf{A})$$

with $f\circ r = 1_B = g\circ s$ and $\alpha\circ r = \beta = \gamma\circ s$ there exists at most one morphism $\varphi\colon A \times_B C \to D$ from the pullback of f and g to the object D such that $\varphi\circ e_1 = \alpha$ and $\varphi\circ e_2 = \gamma$. This means that if the **points** (= split epimorphisms with chosen splitting) (f, r) and (g, s) are fixed, then the existence of φ is a property of the triple (α, β, γ). We shall say that (α, β, γ) is **admissible with respect to** (f, r, g, s) whenever such a φ exists [18].

Every Mal'tsev category is in particular weakly Mal'tsev. Nevertheless, there are many examples of weakly Mal'tsev categories which do not satisfy the **Mal'tsev axiom** saying that every reflexive relation is an equivalence relation [6]. The category of *commutative magmas with cancellation and middle four interchange associativity* is one such example:

Example 1. We consider the quasivariety of algebras $(X, +)$ with one binary operation $+\colon X \times X \to X$ which satisfies

$$\begin{cases} x + y = y + x, \\ (x + y) + (z + w) = (x + z) + (y + w), \\ x + y = x' + y \quad \Rightarrow \quad x = x' \end{cases}$$

for all w, x, x', y, $z \in X$. In this category a triple (α, β, γ) is admissible with respect to (f, r, g, s) if and only if, for every $a \in A$ and $c \in C$ with $f(a) = b = g(c)$, the equation

$$x + \beta(b) = \alpha(a) + \gamma(c)$$

has a solution. See [8] for further details on this example.

This illustrates how the notion of admissibility changes from context to context and in general may not be immediately related to “commuting elements”. However:

Example 2. A classical instance of admissibility occurs when two equivalence relations on a given base object in a Mal’tsev category Smith-commute [30, 26, 3]. This happens when in (**A**) we take $B = D$ and $\beta = 1$ and the pairs (f, α), (g, γ) are jointly monomorphic. Then $(\alpha, 1, \gamma)$ is admissible with respect to (f, r, g, s) if and only if the relations (f, r, α) and (g, s, γ) commute—more details will be given later.

Example 3. In a Mal’tsev variety $\mathcal{V}$, a triple (α, β, γ) is admissible with respect to (f, r, g, s) as in (**A**) if and only if the function $\varphi\colon A \times_B C \to D$ defined by $\varphi(a, c) = p(\alpha(a), \beta(b), \gamma(c))$ is an algebra homomorphism; see [14]. Here $a \in A$ and $c \in C$ with $f(a) = b = g(c)$ and p is a Mal’tsev term of the theory of $\mathcal{V}$.

New examples may be constructed using the following result.

Proposition 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two categories with pullbacks and $F\colon \mathcal{A} \to \mathcal{B}$ a pullback-preserving faithful functor. If $\mathcal{B}$ is weakly Mal’tsev then so is $\mathcal{A}$.*

Proof: Considering a diagram of the form (**A**) in $\mathcal{A}$ with $f \circ r = 1_B = g \circ s$ and $\alpha \circ r = \beta = \gamma \circ s$, we only have to prove that whenever a morphism $\varphi\colon A \times_B C \to D$ with $\varphi \circ e_1 = \alpha$ and $\varphi \circ e_2 = \gamma$ exists, it is necessarily unique. This follows immediately from the uniqueness of $F(\varphi)$ because $\mathcal{B}$ is weakly Mal’tsev and F is faithful. ■

So when $\mathcal{B}$ is Mal’tsev, $\mathcal{A}$ will be weakly Mal’tsev. However, as soon as the functor F is not just faithful but also full, it is conservative, so it reflects also the Mal’tsev property.

Example 4. One simple example is obtained as follows. We consider the category whose objects are pairs (A_0, A) where $A = (A, +, 0)$ is an abelian group and $A_0 \subseteq A$ is a subset of A containing the zero element. A morphism $f\colon (A_0, A) \to (B_0, B)$ is a group homomorphism $f\colon A \to B$ such that

$f(A_0) \subseteq B_0$. Here (α, β, γ) is admissible with respect to (f, r, g, s) if and only if

$$\alpha(x) - \beta(y) + \gamma(z) \in D_0 \subseteq D$$

whenever $x \in A_0 \subseteq A$ and $z \in C_0 \subseteq C$ are such that

$$f(x) = g(z) = y \in B_0 \subseteq B.$$

When the category is also pointed (and hence has kernels of split epimorphisms) it turns out that sometimes the admissibility of (α, β, γ) with respect to (f, r, g, s) reduces to the admissibility of $(\alpha \circ \ker(f), 0, \gamma \circ \ker(g))$ with respect to $(0, 0, 0, 0)$. This happens for instance in the category of groups—see Example 8—but is far from being the case in general, even for protomodular categories:

Example 5. In the variety of algebras [13] with two binary operations $x + y$, $x - y$, a constant 0 and satisfying $x + 0 = x = 0 + x$, $(x + y) - y = x$, $(x - y) + y = x$, a triple (α, β, γ) is admissible with respect to (f, r, g, s) if and only if

$$\begin{aligned} \Big[\big((\alpha(x) + \gamma(y)) + \beta(b)\big) + \big((\alpha(x') + \gamma(y')) + \beta(b')\big)\Big] - \beta(b + b') \\ = \Big[\big((\alpha(x) + \beta(b)) + (\alpha(x') + \beta(b'))\big) - \beta(b + b')\Big] \\ + \Big[\big((\gamma(y) + \beta(b)) + (\gamma(y') + \beta(b'))\big) - \beta(b + b')\Big], \end{aligned}$$

for all x, $x' \in \mathrm{Ker}(f)$, y, $y' \in \mathrm{Ker}(g)$ and b, $b' \in B$. Observe that for groups, this condition simplifies to $\alpha(x) + \gamma(y) = \gamma(y) + \alpha(x)$ for all $x \in \mathrm{Ker}(f)$ and $y \in \mathrm{Ker}(g)$. The example in the category of loops considered in [11] may be used to show that in general, this latter condition on commuting kernels does not suffice for the triple (α, β, γ) to be admissible.

Another well-known counterexample, which was discovered by Janelidze, is the category of digroups; see [1, 2].

Example 6. In the article [9] the authors consider a weighted commutator which, depending on the chosen weight, captures several classical commutators. In a finitely cocomplete homological category, a **weighted cospan** is a triple of coterminal arrows

$$\begin{tikzcd} & W \arrow[d, "w"] & \\ X \arrow[r, "x"'] & D & Y \arrow[l, "y"] \end{tikzcd} \qquad (\mathbf{B})$$

in which (x, y) plays the role of cospan and w is the weight. It is immediately clear from the definitions that the **weighted centrality** of x and y over w from [9] amounts to admissibility of $\big(\left\langle \begin{smallmatrix} w \\ x \end{smallmatrix} \right\rangle, w, \left\langle \begin{smallmatrix} w \\ y \end{smallmatrix} \right\rangle\big)$ with respect to $\big(\left\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \right\rangle, \iota_W, \left\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \right\rangle, \iota_W\big)$ as in the diagram

$$
\begin{tikzcd}
W+X \arrow[r, shift left, "\left\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \right\rangle"] \arrow[dr, "\left\langle \begin{smallmatrix} w \\ x \end{smallmatrix} \right\rangle"'] & W \arrow[l, shift left, "\iota_W"] \arrow[r, shift right, "\iota_W"'] \arrow[d, "w"] & W+Y \arrow[l, shift right, "\left\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \right\rangle"'] \arrow[dl, "\left\langle \begin{smallmatrix} w \\ y \end{smallmatrix} \right\rangle"] \\
& D. &
\end{tikzcd}
\tag{\textbf{C}}
$$

In fact also the converse holds: admissibility may be expressed as weighted centrality (Theorem 1).

Reflected admissibility. Let $\mathcal{A}$ be a pointed weakly Mal'tsev category with pullbacks. In this paper we study properties of the **normalisation functor**

$$\mathrm{N}\colon \mathsf{Adm}(\mathcal{A}) \to \mathsf{Cospan}(\mathcal{A}),$$

where $\mathsf{Adm}(\mathcal{A})$ is the category of **admissibility diagrams** in $\mathcal{A}$, which are diagrams of shape

$$
\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[dr, "\alpha"'] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] \arrow[d, "\beta"] & C \arrow[l, shift right, "g"'] \arrow[dl, "\gamma"] \\
& D &
\end{tikzcd}
\tag{\textbf{D}}
$$

with $f \circ r = 1_B = g \circ s$ and $\alpha \circ r = \beta = \gamma \circ s$. The functor N maps such a diagram to the cospan

$$X = \mathrm{Ker}(f) \xrightarrow{\alpha \circ \ker(f)} D \xleftarrow{\gamma \circ \ker(g)} \mathrm{Ker}(g) = Y \tag{\textbf{E}}$$

in $\mathcal{A}$. Note that by taking the pullback of f with g, any admissibility diagram such as (**D**) may be extended to (**A**) in which the square is a double split epimorphism. We say that the triple (α, β, γ) is **admissible with respect to** (f, r, g, s) if there is a (necessarily unique) morphism $\varphi\colon A \times_B C \to D$ such that $\varphi \circ e_1 = \alpha$ and $\varphi \circ e_2 = \gamma$. We say that α and γ **Huq-commute** if $(\alpha, 0, \gamma)$ is admissible with respect to $(0, 0, 0, 0)$. (That is to say, B is zero. This is clearly equivalent with the definition from [12, 1].) We say that the functor N

reflects admissibility if and only if (α, β, γ) is admissible whenever $\alpha \circ \ker(f)$ and $\gamma \circ \ker(g)$ Huq-commute.

Smith-commuting reflexive graphs. Consider a pair of reflexive graphs (R, S) on a common object D as on the left

$$
\begin{tikzcd}
R \arrow[r, shift left=2, "r_0"] \arrow[r, shift right=2, "r_1"'] & D \arrow[l, "\langle 1_D, 1_D\rangle" description] \arrow[r, "\langle 1_D, 1_D\rangle" description] & S, \arrow[l, shift right=2, "s_1"'] \arrow[l, shift left=2, "s_0"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R \arrow[dl, "\langle 1_R, \langle 1_D, 1_D\rangle \circ r_1\rangle"'] \arrow[dr, "r_0"] & \\
R \times_D S \arrow[rr, dotted, "\theta" description] & & D \\
 & S \arrow[ul, "\langle \langle 1_D, 1_D\rangle \circ s_0, 1_S\rangle"] \arrow[ur, "s_1"'] &
\end{tikzcd}
$$

and consider the induced pullback $R \times_D S$ of r_1 and s_0. The pair (R, S) **Smith-commutes** when there is a (necessarily unique) morphism θ such that the above diagram on the right is commutative. A span

$$
\begin{tikzcd}
 & D \arrow[dl, "d"'] \arrow[dr, "c"] & \\
D_0 & & D_0'
\end{tikzcd}
\tag{\textbf{F}}
$$

is a **pregroupoid** when the kernel pairs of c and d Smith-commute [26, 30].

Smith versus Huq. As explained in [9], taking $W = 0$ in Example 6 captures Huq (x and y are central over 0 if and only if they Huq-commute), and $w = 1_D$ captures Smith (the respective normalisations x and y of two equivalence relations R and S on D are central over 1_D if and only if R and S Smith-commute).

It is well known that Smith-commuting equivalence relations always have Huq-commuting normalisations [4]. However, the converse need not hold: counterexamples exist in the category of digroups [1, 2] and in the category of loops [11]. A pointed weakly Mal'tsev category with pullbacks satisfies the **Smith is Huq** condition **(SH)** if and only if two equivalence relations on a given object always commute as soon as their normalisations do.

The condition (SH) is fundamental in the study of internal categorical structures: it is shown in [23] that, for a semi-abelian category, this condition holds if and only if every star-multiplicative graph is an internal groupoid. As explained in [15] and in [11] this is important when characterising internal crossed modules; furthermore, the condition has immediate (co)homological consequences [29].

Any pointed strongly protomodular exact category satisfies (SH) [4] (in particular, so does any Moore category [28]) as well as any action accessible category [5, 7] (in particular, any category of interest [24, 25]). Well-known concrete examples are the categories of groups, Lie algebras, associative algebras, non-unitary rings, and (pre)crossed modules of groups.

Conditions in terms of the fibration of points. Given any object B in $\mathcal{A}$, the category $\mathsf{Pt}_B(\mathcal{A})$ of points over B in $\mathcal{A}$ is still weakly Mal'tsev by Proposition 1.

Proposition 2. *In a pointed weakly Mal'tsev category with pullbacks $\mathcal{A}$, the following conditions are equivalent:*

(2.1) *for every morphism $p\colon E \to B$ in $\mathcal{A}$, the pullback functor*

$$p^*\colon \mathsf{Pt}_B(\mathcal{A}) \to \mathsf{Pt}_E(\mathcal{A})$$

reflects Huq-commutativity;

(2.2) *for every object B of $\mathcal{A}$, the kernel functor* $\mathrm{Ker}\colon \mathsf{Pt}_B(\mathcal{A}) \to \mathcal{A}$ *reflects Huq-commutativity;*

(2.3) *the normalisation functor* N *reflects admissibility.*

Proof: Condition (2.2) is the special case of (2.1) where p is the unique morphism $!_B\colon 0 \to B$. To prove that (2.2) implies (2.3) it suffices to rewrite Diagram (**D**) in the shape

$$\begin{tikzcd}
A \arrow[r, "{\langle \alpha, f\rangle}"] \arrow[dr, shift left, "f"] & D \times B \arrow[d, shift right, "\pi_B"'] & C \arrow[l, "{\langle \gamma, g\rangle}"'] \arrow[dl, shift left, "g"] \\
 & B \arrow[ul, shift left, "r"] \arrow[u, shift right, "{\langle \beta, 1_B\rangle}"'] \arrow[ur, shift left, "s"] &
\end{tikzcd}$$

and consider it as a cospan $(\langle \alpha, f\rangle, \langle \gamma, g\rangle)$ in $\mathsf{Pt}_B(\mathcal{A})$. Then taking kernels here gives the same result as taking the normalisation of the original diagram (**D**). So condition (2.3) is an instance of the kernel functors reflecting Huq-commutativity in the case where the point which is the codomain of the cospan is a product.

To see that (2.3) implies (2.2), let

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[dr, shift right, "f"'] & D \arrow[d, shift right, "p"'] & C \arrow[l, "\gamma"'] \arrow[dl, shift left, "g"] \\
& B \arrow[ul, shift right, "r"'] \arrow[u, shift right, "\beta"'] \arrow[ur, shift left, "s"] &
\end{tikzcd}
$$

be a cospan in $\mathsf{Pt}_B(\mathcal{A})$, and suppose that $\alpha \circ \ker(f)$ and $\gamma \circ \ker(g)$ commute in $\mathrm{Ker}(p)$. Then they certainly commute in D, and Condition (2.3) gives us a morphism $\varphi\colon A \times_B C \to D$ in $\mathcal{A}$ such that $\varphi \circ e_1 = \alpha$ and $\varphi \circ e_2 = \gamma$. We only need to check that this φ is a morphism of points, of which the domain is $\overline{p} = f \circ \pi_A = g \circ \pi_B\colon A \times_B C \to B$ with section $\overline{\beta} = e_1 \circ r = e_2 \circ s\colon B \to A \times_B C$. Now

$$p \circ \varphi \circ e_1 = p \circ \alpha = f = f \circ \pi_A \circ e_1 = \overline{p} \circ e_1$$

and, similarly, $p \circ \varphi \circ e_2 = \overline{p} \circ e_2$, so that $\overline{p} = p \circ \varphi$ by the weak Mal'tsev property of $\mathcal{A}$. Furthermore, $\varphi \circ \overline{\beta} = \varphi \circ e_1 \circ r = \alpha \circ r = \beta$.

The following standard trick shows (2.2) $\Rightarrow$ (2.1). For any $p\colon E \to B$ we have the induced inverse image functors

$$\mathsf{Pt}_B(\mathcal{A}) \xrightarrow{p^*} \mathsf{Pt}_E(\mathcal{A}) \xrightarrow{!_E^*} \mathsf{Pt}_0(\mathcal{A}) \cong \mathcal{A}.$$

Clearly $!_E^* \circ p^* = !_B^* = \mathrm{Ker}$. By assumption, this functor reflects Huq-commutativity. But the kernel functor $!_E^*$ also *preserves* Huq-commutating pairs of morphisms, and these two properties together give us (2.1). ■

Further conditions. In terms of internal categorical structures we have the following basic conditions.

Proposition 3. *In a pointed weakly Mal'tsev category with pullbacks, the following conditions are equivalent:*

(3.1) *every two effective equivalence relations over the same base object Smith-commute as soon as their normalisations Huq-commute;*
(3.2) *every span is a pregroupoid provided the kernels of its domain and codomain morphisms Huq-commute;*
(3.3) *every reflexive graph with commuting kernels of the domain and codomain morphisms is an internal groupoid;*
(3.4) *every reflexive graph with commuting kernels of the domain and codomain morphisms is an internal category;*

(3.5) *every reflexive graph with commuting kernels of the domain and codomain morphisms is a multiplicative graph.*

Proof: To obtain (3.2) from (3.1) it suffices to take the kernel pairs of the domain and codomain morphisms. (3.3) is a particular case of (3.2) where the span has $D_0 = D_0'$, and d and c are split by the same morphism. (3.4) follows from (3.3) since every internal groupoid is an internal category. Finally (3.5) follows form (3.4) because every internal category is also a multiplicative graph. To obtain (3.1) from (3.5) we may repeat the proof of Theorem 1.6 in [23] as it is still valid in the present context. ■

Thus we see that in particular, in this context, the *Smith is Huq* property (3.1) implies that internal groupoids, internal categories and multiplicative graphs are the same (cf. [22]).

Proposition 4. *In a pointed weakly Mal'tsev category with pullbacks $\mathcal{A}$, each of the following conditions implies the next one:*

(4.1) *the conditions (2.1)–(2.3) of Proposition 2;*
(4.2) *the normalisation functor* N *reflects admissibility for those diagrams* (**D**) *where α and γ are regular epimorphisms;*
(4.3) *any two reflexive graphs over the same base object Smith-commute as soon as their normalisations Huq-commute;*
(4.4) *any two reflexive relations over the same base object Smith-commute as soon as their normalisations Huq-commute;*
(4.5) *any two equivalence relations over the same base object Smith-commute as soon as their normalisations Huq-commute;*
(4.6) *the conditions (3.1)–(3.5) of Proposition 3.*

Furthermore, (4.6) implies (4.5). When $\mathcal{A}$ is a semi-abelian category, (4.5) implies (4.2), so that conditions (4.2)–(4.6) are equivalent.

Proof: Condition (4.3) is the particular case of condition (4.2) with $B = D$ and $\beta = 1_B$. (4.4) is a particular case of (4.3) where the graphs are jointly monic spans. (4.5) is a particular case of (4.4) where the reflexive relations are also symmetric and transitive.

(4.6) $\Rightarrow$ (4.5) is the content of Remark 1.7 in [23], but in the semi-abelian case we can go up to (4.2) in one step as follows.

Given (α, β, γ) as in (**D**) we define a span (**F**) by $d = \mathrm{coeq}(\alpha, \beta \circ f) = \mathrm{coker}(\alpha \circ \ker(f))$ and $c = \mathrm{coeq}(\beta \circ g, \gamma) = \mathrm{coker}(\gamma \circ \ker(g))$. Since the direct image of a kernel along a regular epimorphism is a kernel, $\alpha \circ \ker(f) = \ker(d) \circ \alpha'$

and $\gamma \circ \ker(g) = \ker(c) \circ \gamma'$ where α' and γ' are regular epi. Suppose that $\alpha \circ \ker(f)$ and $\gamma \circ \ker(g)$ commute. Then we have $\phi\colon X \times Y \to D$, which factors as $\phi' \circ (\alpha' \times \gamma')$ where $\alpha' \times \gamma'$ is again regular epi. It follows that $\ker(d)$ and $\ker(c)$ Huq-commute. But this gives a pregroupoid structure on D by assumption, so that the triple (α, β, γ) is admissible. Indeed, the needed $\varphi\colon A \times_B C \to D$ is given by $\varphi(a, b, c) = \theta(\alpha(a), \beta(b), \gamma(c))$ where R and S are the respective kernel relations of d and c and $\theta\colon R \times_D S \to D$ is the pregroupoid structure on (d, c). ■

Even in the semi-abelian case, there is a gap between (4.1)—the conditions of Proposition 2—and the formally weaker (4.2)–(4.6): the following example in the category of Heyting semi-lattices shows that indeed, (4.1) is strictly stronger. On the other hand, in the category of groups, condition (4.1) holds: see Example 8.

Example 7. An exact Mal'tsev category is called **arithmetical** [27, 1] when every internal groupoid is an equivalence relation. In presence of finite colimits, then the Smith commutator $[R, S]^{\mathrm{S}}$ of two equivalence relations R and S on an object D is their intersection $R \wedge S$. It is also well known that if the category is, moreover, pointed, then the only abelian object is the zero object, and the Huq commutator $[X, Y]^{\mathrm{H}}$ of normal subobjects X, $Y \lhd D$ is the intersection $X \wedge Y$. Since the normalisation functor preserves intersections, it follows that any pointed arithmetical category with finite colimits has the *Smith is Huq* property.

Two examples of this situation which are relevant to us are the category HSLat of Heyting (meet) semi-lattices and DLat of distributive lattices. The latter is only weakly Mal'tsev [21], but it is easily seen that it fits the above picture, hence (trivially) satisfies *Smith is Huq*. On the other hand, the category HSLat is semi-abelian [16]—in fact it even is a Moore category [28]—and satisfies the conditions (4.2)–(4.6), but nevertheless does not satisfy (4.1). This tells us that the condition (4.1) is strictly stronger than the *Smith is Huq* property, while it is not implied by strong protomodularity.

A concrete counterexample is the diagram (**D**) in HSLat defined as follows: $A = D = \{0, \frac{1}{2}, 1\}$ and $B = \{0, 1\}$ with the natural order, and C is

$$\begin{tikzcd} & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ a \arrow[dr, dash] & & b; \arrow[dl, dash] \\ & 0 & \end{tikzcd}$$

the tables

A	0	$\frac{1}{2}$	1
f	0	1	1
α	0	$\frac{1}{2}$	1

B	0	1
r	0	1
s	0	1
β	0	1

C	0	a	b	1
g	0	0	1	1
γ	0	1	1	1

determine the morphisms between them. Let $X = \{\frac{1}{2}, 1\}$ and $Y = \{b, 1\}$ be the kernels of f and g, respectively. Then their direct images along α and γ become $\{\frac{1}{2}, 1\}$ and $\{1\}$, which Huq-commute in D. On the other hand, the triple (α, β, γ) is not admissible with respect to (f, r, g, s). Indeed, the pullback of f and g is given by the following diagram.

$$\begin{tikzcd} & (1,1) \arrow[dl, dash] \arrow[dr, dash] & & \\ (1,b) \arrow[dr, dash] & & (\frac{1}{2},1) \arrow[dl, dash] \arrow[dr, dash] & \\ & (\frac{1}{2},b) \arrow[dr, dash] & & (0,a) \arrow[dl, dash] \\ & & (0,0) & \end{tikzcd}$$

Hence if a function φ as in the definition of admissibility exists, then

$$\varphi(0, a) = \varphi(rg(a), a) = \gamma(a) = 1$$

and $\varphi(\frac{1}{2}, 1) = \varphi(\frac{1}{2}, sf(\frac{1}{2})) = \alpha(\frac{1}{2}) = \frac{1}{2}$; but this function cannot preserve the order.

Expressed in terms of the normalisation functor N the problem is the following: reflection of diagrams (**D**) where α and γ are regular epimorphisms need not imply that N reflects *all* diagrams (**D**). However, the conditions of Proposition 2 do hold in the category of groups:

Example 8. We prove that when $\mathcal{A}$ is the category Gp of groups, the normalisation functor does reflect admissibility for all diagrams (**D**). In what follows we use additive notation, also for non-abelian groups. Consider in Gp the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "k"] & A \arrow[l, shift left, dotted, "k'"] \arrow[r, shift left, "f"] \arrow[dr, "\alpha"'] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] \arrow[d, "\beta"] & C \arrow[l, shift right, "g"'] \arrow[r, shift right, dotted, "l'"'] \arrow[dl, "\gamma"] & Y \arrow[l, shift right, "l"'] \\ & & D & & \end{tikzcd}$$

in which $k = \ker(f)$, $l = \ker(g)$ and where k', l' are the unique functions (not homomorphisms) with the property that $kk' = 1_A - rf$ and $ll' = -sg + 1_C$ (so that $k'k = 1_X$ and $l'l = 1_Y$). Note that

$$a = kk'(a) + rf(a) \qquad \text{and} \qquad c = sg(c) + ll'(c)$$

for all $a \in A$, $c \in C$.

Assuming that αk and γl commute, we have to construct a suitable group homomorphism $\varphi\colon A \times_B C \to D$ to show that (α, β, γ) are admissible. We define

$$\varphi(a, c) = \alpha kk'(a) + \gamma(c)$$

and prove that $\varphi(a + a', c + c') = \varphi(a, c) + \varphi(a', c')$. Note that

$$\begin{aligned} k'(a + a') &= 1_A(a + a') - rf(a + a') \\ &= kk'(a) + rf(a) + kk'(a') + rf(a') - rf(a + a') \\ &= \underbrace{kk'(a)}_{\in X} + \underbrace{rf(a) + kk'(a') - rf(a)}_{\in X}. \end{aligned}$$

Now for all $x \in X$, $b \in B$, we have that

$$\alpha k(r(b) + k(x) - r(b)) = \beta(b) + \alpha k(x) - \beta(b).$$

Hence, on the one hand,

$$\begin{aligned} &\varphi(a + a', c + c') \\ &= \alpha kk'(a + a') + \gamma(c + c') \\ &= \alpha k(kk'(a) + rf(a) + kk'(a') - rf(a)) + \gamma(c) + \gamma(c') \\ &= \alpha kk'(a) + \alpha k(r(b) + kk'(a') - r(b))) + \gamma(sg(c) + ll'(c)) + \gamma(c') \\ &= \alpha kk'(a) + \beta(b) + \alpha kk'(a') - \beta(b) + \beta(b) + \gamma ll'(c) + \gamma(c') \\ &= \alpha kk'(a) + \beta(b) + \alpha kk'(a') + \gamma ll'(c) + \gamma(c'), \end{aligned}$$

where $f(a) = b = g(c)$, while on the other hand

$$\begin{aligned}\varphi(a', c') + \varphi(a', c') &= \alpha kk'(a) + \gamma(c) + \alpha kk'(a') + \gamma(c')\\ &= \alpha kk'(a) + \beta(b) + \gamma ll'(c) + \alpha kk'(a') + \gamma(c').\end{aligned}$$

Since, by assumption, $\gamma ll'(c) + \alpha kk'(a') = \alpha kk'(a') + \gamma ll'(c)$, these two expressions are equal to each other, and φ is a homomorphism.

Furthermore, in any finitely cocomplete homological category the conditions (2.1)–(2.3) are equivalent to the weighed commutator being independent of the chosen weight (Proposition 5).

Theorem 1. *In a finitely cocomplete homological category, consider a diagram* (**D**) *and its normalisation* (**E**)*. The triple* (α, β, γ) *is admissible with respect to* (f, r, g, s) *if and only if* $x = \alpha \circ \ker(f)$ *and* $y = \gamma \circ \ker(g)$ *are weighted central over* β*.*

Proof: It suffices to compare Diagram (**D**) with the induced diagram (**C**). In fact there is a regular epimorphism of admissibility diagrams from the latter to the former which keeps D fixed and makes

$$\begin{tikzcd}
B + X \arrow[r, shift left, "\left\langle \begin{smallmatrix} 1_B \\ 0 \end{smallmatrix} \right\rangle"] \arrow[d, "\left\langle \begin{smallmatrix} r \\ \ker(f) \end{smallmatrix} \right\rangle"'] & B \arrow[l, shift left, "\iota_B"] \arrow[r, shift right, "\iota_B"'] \arrow[d, equal] & B + Y \arrow[l, shift right, "\left\langle \begin{smallmatrix} 1_B \\ 0 \end{smallmatrix} \right\rangle"'] \arrow[d, "\left\langle \begin{smallmatrix} s \\ \ker(g) \end{smallmatrix} \right\rangle"] \\
A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] & C \arrow[l, shift right, "g"']
\end{tikzcd}$$

commute. This already proves the "only if" in our claim. For the "if" suppose that x and y are weighted central over β. For the induced arrow

$$\varphi\colon (B + X) \times_B (B + Y) \to D$$

to factor over the regular epimorphism

$$\left\langle \begin{smallmatrix} r \\ \ker(f) \end{smallmatrix} \right\rangle \times_B \left\langle \begin{smallmatrix} s \\ \ker(g) \end{smallmatrix} \right\rangle\colon (B + X) \times_B (B + Y) \to A \times_B C,$$

we only need that it vanishes on $\mathrm{Ker}\big(\big\langle {r \atop \ker(f)} \big\rangle\big) \times \mathrm{Ker}\big(\big\langle {s \atop \ker(g)} \big\rangle\big)$. Indeed,

$$\begin{aligned}
&\varphi\circ\big(\ker\big(\big\langle {r \atop \ker(f)} \big\rangle\big) \times \ker\big(\big\langle {s \atop \ker(g)} \big\rangle\big)\big)\circ\langle 1,0\rangle\\
&= \varphi\circ\big\langle 1_{W+X}, \iota_W\circ\big\langle {1_W \atop 0} \big\rangle\big\rangle\circ\ker\big(\big\langle {r \atop \ker(f)} \big\rangle\big)\\
&= \big\langle {\beta \atop x} \big\rangle\circ\ker\big(\big\langle {r \atop \ker(f)} \big\rangle\big)\\
&= \alpha\circ\big\langle {r \atop \ker(f)} \big\rangle\circ\ker\big(\big\langle {r \atop \ker(f)} \big\rangle\big)
\end{aligned}$$

is trivial, and similarly for $\mathrm{Ker}\big(\big\langle {s \atop \ker(g)} \big\rangle\big)$. ■

Thus we see that in this context, the theory of admissibility diagrams is essentially equivalent to weighted commutator theory. This gives a new conceptual interpretation for admissibility of diagrams such as (**D**)—as a way to encode weighted commutativity—while, on the other hand, we may analyse some of the concepts in [9] from a different point of view. First of all we obtain new conditions, equivalent to those of Proposition 2:

Proposition 5. *In a finitely cocomplete homological category, the following are equivalent:*

(5.1) *the conditions (2.1)–(2.3) of Proposition 2;*
(5.2) *if a cospan (x, y) Huq-commutes, then x and y are weighted central over every w which makes* (**B**) *a weighted cospan;*
(5.3) *weighted centrality is independent of the chosen weight.* ■

These "strong" conditions of Proposition 2 may be expressed in terms of binary and ternary Higgins commutators as in the following theorem. We recall the needed definitions from [10, 17, 11].

If $k\colon K \to X$ and $l\colon L \to X$ are subobjects of an object X in a finitely cocomplete homological category, then the **(Higgins) commutator** $[K, L] \leqslant X$ is the image of the induced morphism

$$K \otimes L \xrightarrow{\iota_{K,L}} K + L \xrightarrow{\left\langle {k \atop l} \right\rangle} X,$$

where

$$K \otimes L = \mathrm{Ker}\Big(\Big\langle \begin{smallmatrix} 1_K & 0 \\ 0 & 1_L \end{smallmatrix} \Big\rangle : K + L \to K \times L\Big).$$

When also $m\colon M \to X$ is a subobject of X, the **ternary commutator** $[K, L, M] \leqslant X$ is the image of the composite

$$K \otimes L \otimes M \xrightarrow{\iota_{K,L,M}} K + L + M \xrightarrow{\left\langle \begin{smallmatrix} k \\ l \\ m \end{smallmatrix} \right\rangle} X,$$

where $\iota_{K,L,M}$ is the kernel of the morphism

$$K + L + M \xrightarrow{\left\langle\begin{smallmatrix} i_K & i_K & 0 \\ i_L & 0 & i_L \\ 0 & i_M & i_M \end{smallmatrix}\right\rangle} (K + L) \times (K + M) \times (L + M).$$

Given any diagram (**D**), let $\overline{k}\colon \overline{K} \to D$ be the image of $\alpha\circ\ker(f)$, $\overline{l}\colon \overline{L} \to D$ the image of $\gamma\circ\ker(g)$ and $\overline{\beta}\colon \overline{B} \to D$ the image of β.

Theorem 2. *In any finitely cocomplete homological category, the following are equivalent:*

(1) *the conditions of Proposition 2;*
(2) $[\overline{K}, \overline{L}] = 0$ *implies* $[\overline{K}, \overline{L}, \overline{B}] = 0$ *for every diagram such as* (**D**).

Proof: The equivalence between (2.3) and (2) is a key result in [11]. ■

It is explained in [11] that the *Smith is Huq* condition is equivalent to the vanishing of the ternary commutator $[K, L, X]$ whenever K, $L \lhd X$ are commuting normal subobjects of an object X. A priori, condition (2) in Theorem 2 is stronger, as it considers the same implication but in a setting which is wider. Indeed, $\overline{K}$ and $\overline{L}$ need not be normal in D, and the diagram (**D**) need not be induced by their denormalisations. This makes us conclude that if we restrict all conditions in Proposition 2 to the case of proper morphisms—so that the images of these morphisms are normal monomorphisms—we should obtain an equivalence with the standard *Smith is Huq* condition. This is the content of Theorem 3. Let us first explain what we mean with “restricting to proper morphisms.”

Recall that a morphism is called **proper** when it factorises as a regular epimorphism followed by a **normal monomorphism**, that is, the normalisation of an equivalence relation—which in a homological category is the same thing as a direct image of a kernel along regular epimorphism [17]. We shall restrict the first two conditions in Proposition 2 and the conditions of Proposition 5 to Huq-commutativity of cospans of proper morphisms. In the case of (2.3), we only consider diagrams (**D**) for which the two arrows in the induced cospan (**E**) are proper.

Theorem 3. *In a finitely cocomplete homological category $\mathcal{A}$, consider the following conditions.*

(1) *the* Smith is Huq *condition;*

(2) *the conditions (2.1)–(2.3) of Proposition 2, restricted to pairs of proper morphisms;*
(3) *the conditions (3.1)–(3.5) of Proposition 3.*
(4) *the conditions (4.2)–(4.6) in Proposition 4;*
(5) *the conditions (5.1)–(5.3) of Proposition 5, restricted to pairs of proper morphisms.*

Always (1)–(3) *are equivalent to* (5)*, and as soon as* $\mathcal{A}$ *is semi-abelian, these conditions are also equivalent to* (4)*.*

Proof: The implication (1) $\Rightarrow$ (2) is again a consequence of the result in [11]. To see that (2) implies (3), consider a diagram (**D**) and its normalisation (**E**). Condition (2.3) says that (α, β, γ) is admissible with respect to (f, r, g, s) as soon as $\alpha \circ \ker(f)$ and $\gamma \circ \ker(g)$ are proper and Huq-commute. Condition (3.1) is just a special case where $B = D$ and $\beta = 1_B$. Note that indeed, the normalisation of an effective equivalence relation is a kernel. ∎

Remark 1. Note that in the semi-abelian case, condition (4.2) is immediately implied by the restriction of (2.3) to pairs of proper morphisms, as then the direct image of a kernel (= normal monomorphism) along a regular epimorphism is a kernel.

Remark 2. We did not investigate any further connections between the conditions of Proposition 2 and the kernel reflected admissibility property, called *condition (III)* in [20], but it is not difficult to observe that the latter implies the former.

Thus in a semi-abelian category, *Smith is Huq* amounts to independence of the chosen weight for weighted proper cospans. This is further refined by the following decomposition result, a consequence of Theorem 2.

Theorem 4. *Given a weighted cospan* (**B**) *in a semi-abelian category, weighted centrality of normal monomorphisms* x *and* y *over* w *is equivalent to the vanishing of the commutators* $[X, Y]$ *and* $[X, Y, \mathrm{Im}(w)]$*. Hence the* (W, w)*-weighted normal commutator*

$$N[(X, x), (Y, y)]_{(W,w)}$$

of [9] *is the normal closure of* $[X, Y] \vee [X, Y, \mathrm{Im}(w)]$*.*

Proof: By Theorem 2 it suffices to prove that x and y coincide with the images of $\left\langle \begin{smallmatrix} w \\ x \end{smallmatrix} \right\rangle \circ \ker\left(\left\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \right\rangle\right)$ and $\left\langle \begin{smallmatrix} w \\ y \end{smallmatrix} \right\rangle \circ \ker\left(\left\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \right\rangle\right)$, respectively, as in (**C**). To see this, we consider the diagram with short exact rows

$$\begin{tikzcd}
 & & X \arrow[dl, dotted, "\eta_X^W"'] \arrow[d, "\iota_X"] & & \\
0 \arrow[r] & W\flat X \arrow[r, tail, "\kappa_{B,X}"'] \arrow[d, dotted, "\xi"'] & W+X \arrow[r, two heads, "\left\langle\begin{smallmatrix}1_W\\0\end{smallmatrix}\right\rangle"] \arrow[d, "\left\langle\begin{smallmatrix}w\\x\end{smallmatrix}\right\rangle"] & W \arrow[r] \arrow[d, "d\circ w"] & 0 \\
0 \arrow[r] & X \arrow[r, tail, "x"'] & D \arrow[r, two heads, "d"'] & D_0 \arrow[r] & 0.
\end{tikzcd}$$

It is clear that $\left\langle\begin{smallmatrix}1_W\\0\end{smallmatrix}\right\rangle\circ\iota_X = 0$ induces the factorisation η_X^W of ι_X over the kernel $\kappa_{B,X}$ of $\left\langle\begin{smallmatrix}1_W\\0\end{smallmatrix}\right\rangle$. Similarly, since

$$d\circ\left\langle\begin{smallmatrix}w\\x\end{smallmatrix}\right\rangle\circ\kappa_{B,X} = d\circ w\circ\left\langle\begin{smallmatrix}1_W\\0\end{smallmatrix}\right\rangle\circ\kappa_{B,X}$$

is trivial we obtain the dotted factorisation ξ. Now

$$x\circ\xi\circ\eta_X^W = \left\langle\begin{smallmatrix}w\\x\end{smallmatrix}\right\rangle\circ\kappa_{B,X}\circ\eta_X^W = \left\langle\begin{smallmatrix}w\\x\end{smallmatrix}\right\rangle\circ\iota_X = x,$$

so $\xi\circ\eta_X^W = 1_X$ because x is a monomorphism. In particular, ξ is a regular epimorphism. It follows that x is the image of $\left\langle\begin{smallmatrix}w\\x\end{smallmatrix}\right\rangle\circ\kappa_{B,X}$. ■

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