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A SEMIDEFINITE APPROACH TO THE K_i COVER PROBLEM

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Abstract: We apply theta body relaxations to the K_i cover problem and use this to show polynomial time solvability for certain classes of graphs. In particular, we show that the facets corresponding to K_i-p-holes can be optimized over in polynomial time, answering an open question of Conforti et al [1]. For the triangle free problem on K_n, we show that the theta body relaxations do not converge by $n/4$ steps; we also prove an integrality gap of 2 for the second theta body and all G.

1. Introduction

A common way to model a combinatorial optimization problem is as the optimization of a function over the set $S \subseteq \{0,1\}^n$ of characteristic vectors of the objects in question. When the objective function is linear, we may replace S by its convex hull $\mathrm{conv}(S)$. The problem can be solved efficiently if we can find a small description of this polytope. Since for NP hard problems we cannot expect this, we look instead for approximations to $\mathrm{conv}(S)$. One possibility is to use semidefinite approximations, as introduced by Lovász [9] with the construction of the *theta body* of the stable set polytope of a graph. Another famous example is the approximation algorithm for the max cut problem due to Goemans and Williamson [3]. In this paper we will use the semidefinite relaxations introduced by Gouveia, Parrilo and Thomas [5] to analyze the K_i *cover problem.*

Recall that K_i denotes the complete graph, or clique, on i vertices. Given a graph G, let $\mathbf{K}_j(G)$ be the collection of cliques in G of size j (usually, the graph is clear from context, and we write $\mathbf{K}_j$). A collection $C \subset \mathbf{K}_{i-1}$ is said to be a K_i-cover if for each $K \in \mathbf{K}_i$, there is some $H \in C$ with $H \subset K$. In this case we say that H covers K. The K_i cover problem is, given a graph G and a set of weights on $\mathbf{K}_{i-1}$, to compute the minimum weight K_i cover. The case $i = 2$ is more commonly known as the vertex cover problem, in

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which we seek a collection of vertices such that each edge in G contains at least one vertex from the collection. However, note that the usage of "cover" is reversed here: the vertex cover problem is the K_2 cover problem, not the K_1 cover problem.

A closely related problem, and the setting in which we will prove our results, is the K_i *free problem*. As before, we are given a graph and a collection of weights on $\mathbf{K}_{i-1}$. But now we seek the maximum weight collection $C \subseteq \mathbf{K}_{i-1}$ such that C is K_i-free. That is, for each $K \in \mathbf{K}_i$, there is some $H \in \mathbf{K}_{i-1}$, with $H \subset K$ and $H \notin C$. Again, the case $i = 2$ of this problem is well-known as the stable set problem: we seek a maximum weight *stable set* C, where C is stable if no two of its vertices are connected by an edge.

The vertex cover and stable set problems are related in the following sense: let $G = (V, E)$ be a graph. Then a subset C of vertices is a vertex cover if and only if $V \setminus C$ is a stable set. The same is true for the K_i cover and K_i free problems: a subset $C \subset \mathbf{K}_{i-1}$ is a K_i-cover if and only if $\mathbf{K}_{i-1} \setminus C$ is K_i-free. Therefore, for a given set of weights on $\mathbf{K}_{i-1}$, optimal solutions to the two problems are complementary, and so solving one solves the other.

In this paper, we consider the polytope associated with the K_i free problem. Let $P_i(G) = \mathrm{conv}(\{\chi_S : S \subset \mathbf{K}_{i-1}(G) \text{ and } S \text{ is } K_i\text{-free}\})$, the convex hull of the incidence vectors of the K_i free sets. Note that $P_i(G) \subseteq [0,1]^{\mathbf{K}_{i-1}(G)}$.

As the K_i free problem is NP-complete (see [1]), we cannot expect a small description of $P_i(G)$ for general graphs G. However, for certain classes of facets of $P_i(G)$, we can solve the separation problem in polynomial time. Conforti, Corneil, and Mahjoub [1] worked this out for several families of facets. We answer an open question from their paper by solving the separation problem for the K_i-p-hole facets.

The structure of this paper is: in section 2, we outline the main algebraic machinery, *theta bodies*, a semidefinite relaxation hierarchy. In section 3 we use theta bodies to give a separation algorithm for the K_i-p-hole facets. Finally, in section 4 we focus on the triangle free problem. We use a result of Krivelevich to show an integrality gap of 2 for the second theta body. On the other hand, we show that in the case of $G = K_n$, the theta body relaxations cannot converge in less than $n/4$ steps.

2. Theta bodies

Theta bodies are semidefinite approximations to the convex hull of an algebraic variety. For background, see [2] and [5]. Here we state the necessary results for this paper without proofs.

Let $V \subseteq \mathbb{R}^n$ be a finite point set. One description of the convex hull of V is as the intersection of all affine half spaces containing V:

$$\operatorname{conv}(V) = \{x \in \mathbb{R}^n : f(x) \geq 0 \text{ for all linear } f \text{ such that } f|_V \geq 0\}.$$

Since it is computationally intractable to find whether $f|_V \geq 0$, we relax this condition. Let I be the vanishing ideal of V, i.e., the set of all polynomials vanishing on V. Recall that $f \equiv g \mod I$ means $f - g \in I$, and implies that f and g agree on V. A function f is said to be a sum of squares of degree at most k mod I, or *k-sos mod I*, if there exist functions g_j, $j = 1, \ldots, m$ with degree at most k, such that $f \equiv \sum_{j=1}^m g_j^2 \mod I$. If f is k-sos mod I for any k, it is clear that $f|_V \geq 0$ since g_j^2 is visibly nonnegative on V. Therefore, we make the following definition of $\mathrm{TH}_k(I)$, the k-th theta body of I:

$$\mathrm{TH}_k(I) = \{x \in \mathbb{R}^n : f(x) \geq 0 \text{ for all linear } f \equiv k\text{-sos mod } I\}.$$

The reason why the theta bodies $\mathrm{TH}_k(I)$ provide a computationally tractable relaxation of $\operatorname{conv}(V)$ is that the membership problem for $\mathrm{TH}_k(I)$ can be expressed as a semidefinite program, using *moment matrices* that are reduced mod I.

For what follows, we will restrict ourselves to a special class of varieties, and suppose that our variety $V \subseteq \{0,1\}^n$ and is *down-closed*; i.e., if $x \leq y$ componentwise, and $y \in V$, then $x \in V$. Additionally, we will always assume that V contains the canonical basis of $\mathbb{R}^n$, $\{e_1, \cdots, e_n\}$, as otherwise we could restrict ourselves to a subspace. All combinatorial optimization problems of avoiding certain finite list of configurations, such as stable set, K_i free, etc., have down-closed varieties. The restriction to this class is not necessary, but makes the theta body exposition simpler. In particular, the ideal of a down-closed variety has the following simple description.

Lemma 2.1. *Let V be a down-closed subset of $\{0,1\}^n$. Then its vanishing ideal is given by*

$$I = \langle x_j^2 - x_j : j = 1, \ldots, n; x^S : S \notin V\rangle,$$

and a basis for $\mathbb{R}[V] = \mathbb{R}[x]/I$ is given by $B = \{x^S : S \in V\}$, where $x^S := \prod_{i \in S} x_i$ is a shorthand used throughout the paper.

Another important fact about $\mathrm{TH}_k(I)$ in this setting (when I is real radical) is that a linear inequality $f(x) \geq 0$ is valid on $\mathrm{TH}_k(I)$ if and only if f is actually k-sos modulo I. In section 3, we will prove that certain facet-defining inequalities of $P_i(G)$ are also valid on its theta relaxations $\mathrm{TH}_k(I)$ by presenting a sum of squares representation modulo the ideal. For now, we observe that by considering degrees, we can get a bound on which theta bodies are trivial; that is, equal to the hypercube $[0,1]^n$.

Lemma 2.2. *Let $V \subseteq \{0,1\}^n$ be down-closed, and suppose that all elements $x \notin V$ have $\sum_j x_j \geq k$. Let I be its vanishing ideal. Then for $l < k/2$, $\mathrm{TH}_l(I) = [0,1]^n$.*

Proof: Let f be linear with $f \equiv \sum_j g_j^2 \mod I$ with each g_j of degree at most l. Then $f - \sum_j g_j^2 =: F \in I$, and F has degree at most $2l$. But the basis from Lemma 2.1 is a Groebner basis, and the only elements with degree $2l$ or less are $x_j^2 - x_j$, so $F \in I' := \langle x_j^2 - x_j ; j = 1, \ldots, n\rangle$. Thus $\mathrm{TH}_l(I) \supseteq \mathrm{TH}_l(I') = [0,1]^n$. ■

Let V_k be the subset of V whose elements have at most k entries equal to one. For convenience, we will often identify the elements of V, characteristic vectors χ_S for $S \subseteq \{1, \ldots, n\}$, with their supports, via $S \leftrightarrow \chi_S$. Given $y \in \mathbb{R}^{V_{2k}}$ we denote the *reduced moment matrix* of y with respect to I to be the matrix $M_{V_k}(y) \in \mathbb{R}^{V_k \times V_k}$ defined by

$$[M_{V_k}(y)]_{X,Y} = \begin{cases} y_{X \cup Y} & \text{if } X \cup Y \in V, \\ 0 & \text{otherwise.} \end{cases}$$

With these matrices we can finally give a semidefinite description of $\mathrm{TH}_k(I)$.

Proposition 2.3. *With I and V as before, $\mathrm{TH}_k(I)$ is the projection onto the coordinates $(y_{e_1}, \cdots, y_{e_n})$ of the set*

$$\{y \in \mathbb{R}^{V_{2k}} \ : \ M_{V_k}(y) \succeq 0 \text{ and } y_0 = 1\}.$$

In particular, optimizing to arbitrary fixed precision over $\mathrm{TH}_k(I)$ can be done polynomially in n for fixed k.

Now we can consider the specific case of the K_i-free problem. Here the variety $V \subseteq \mathbb{R}^{\mathbf{K}_{i-1}(G)}$ is the set of characteristic vectors of K_i-free subsets of $\mathbf{K}_{i-1}(G)$, V_k is the subset of V of elements of size at most k, and I is the vanishing ideal of V, described by Lemma 2.1. Since the K_is in G are the minimal elements not in V, by Lemma 2.1 we can write the ideal I as follows.

$$I = \langle x_j^2 - x_j : j \in \mathbf{K}_{i-1}(G); \prod_{j \subseteq K} x_j : K \in \mathbf{K}_i(G) \rangle.$$

For example, let G be a triangle, with edges A, B, C, and consider the triangle free problem on G. Then the ideal is

$$I = \langle x_A^2 - x_A, x_B^2 - x_B, x_C^2 - x_C, x_A x_B x_C \rangle,$$

and the variety V is as follows.

$$V = \{\emptyset, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}\} \equiv \{0, 1, 2, 3, 4, 5, 6\}.$$

Note that here, we again use our identification of sets with their characteristic vectors. To avoid writing, e.g., $y_{\{A,C\}}$ or even $y_{\chi_{\{A,C\}}}$, we label the elements of V by numbers as above. Then the moment matrix $M_{V_2}(y)$ is as follows:

$$M_{V_2}(y) = \begin{bmatrix} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ y_1 & y_1 & y_4 & y_5 & y_4 & y_5 & 0 \\ y_2 & y_4 & y_2 & y_6 & y_4 & 0 & y_6 \\ y_3 & y_5 & y_6 & y_3 & 0 & y_5 & y_6 \\ y_4 & y_4 & y_4 & 0 & y_4 & 0 & 0 \\ y_5 & y_5 & 0 & y_5 & 0 & y_5 & 0 \\ y_6 & 0 & y_6 & y_6 & 0 & 0 & y_6 \end{bmatrix}$$

Projecting the set $\{y : y_0 = 1, M_{V_2}(y) \succeq 0\}$ onto (y_1, y_2, y_3) gives $\mathrm{TH}_2(I)$ for this graph.

3. Polynomial-time algorithm

In this section, we will give a polynomial-time separation algorithm for a class of facets of $P_i(G)$, thus answering an open question in Conforti, Corneil and Mahjoub [1]. The facets we consider are called the *K_i-p-hole facets.* A graph H is a K_i-p-hole if H contains p copies of K_i as subgraphs, $G_1, \ldots, G_p$, and G_j and G_l share a common K_{i-1} if and only if $j - l = \pm 1 \mod p$; see Figure 1. Theorem 3.5 in [1] establishes that for $i \geq 3$ and odd p, the inequality $\sum_{\mathbf{K}_{i-1}(H)} x_j \leq (\frac{p-1}{2})(2i-3) + i - 2$ defines a facet of $P_i(G)$ for each induced K_i-p-hole H of G. We will show that the facets corresponding to induced K_i-p-holes are valid on $\mathrm{TH}_{\lceil i/2 \rceil}(I)$, and therefore that there is a polynomial-time separation algorithm for them. Note that in this section, the ideal I always refers to the K_i free problem, and the associated graph G will be clear from context.

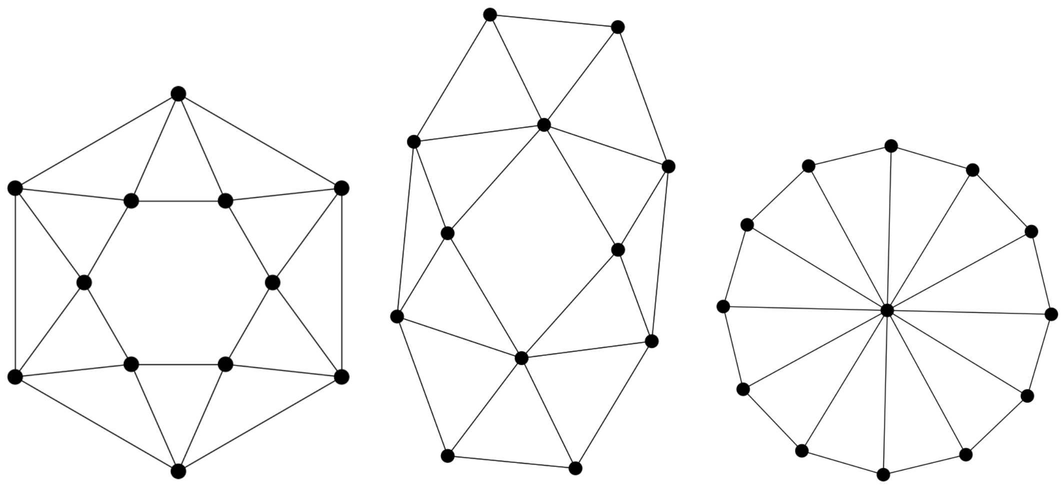

FIGURE 1. Three non-isomorphic K_3-12-holes.

The first lemma is an auxiliary result that a class of functions are sums of squares. For an ideal I, a function f is said to be *idempotent* mod I if $f^2 \equiv f$ mod I. Since an idempotent is visibly a square, we can use it as a summand in our sum of squares. In practice, idempotents end up being very useful in sums of squares.

Lemma 3.1. *Suppose* $A \subseteq B \subseteq \mathbf{K}_{i-1}(K_i)$. *Denote the variables in* $\mathbf{K}_{i-1}(K_i)$ *by* $\{x_k : 1 \leq k \leq i\}$. *Then* $f(x) = |B \setminus A| - x^A + x^B - \sum_{k \in B \setminus A} x_k$ *is* $|B|$*-sos mod* I.

Proof: Let $A = A_1 \subset A_2 \ldots \subset A_m = B$ be a maximal chain, where $A_k \cup \{x_k\} = A_{k+1}$, for $k = 1, \ldots, m-1$. Check that $g_k(x) = 1 - x_k - x^{A_k} + x^{A_{k+1}}$ is idempotent mod I. Adding them up we get that $f(x) = \sum_{k=1}^{m-1} g_k(x)$. Since each summand has degree at most $|B|$ the assertion holds. ■

The stable set polytope STAB(G) has a fractional relaxation FRAC(G), given by imposing nonnegativities $x_i \geq 0$, and inequalities $x_i + x_j \leq 1$ for each edge (i, j) of G. Similarly, we can define a fractional K_i free polytope FRAC$_i(G)$ by imposing nonnegativities, and the inequalities $\sum_{k \in \mathbf{K}_{i-1}(H)} x_k \leq i - 1$ for each $H \in \mathbf{K}_i(G)$. The following corollary shows that these inequalities are $\lceil i/2 \rceil$-sos, and therefore that the relaxation $\mathrm{TH}_{\lceil i/2 \rceil}(I) \subseteq \mathrm{FRAC}_i(G)$. This is parallel to the result that the Lovász theta body lies inside FRAC(G).

Corollary 3.2. *The inequality* $\sum_{k\in\mathbf{K}_{i-1}(H)} x_k \leq i-1$ *is valid on* $\mathrm{TH}_{\lceil i/2\rceil}(I)$ *for every* $H \in \mathbf{K}_i(G)$*.*

Proof: Let J be a subset of $\mathbf{K}_{i-1}(H)$ of size $\lceil i/2\rceil$. Applying Lemma 3.1 with $A=\emptyset$ and $B=J$ we see that

$$f(x) = |J| - 1 + x^J - \sum_{l\in J} x_l$$

is $|J|$-sos. Similarly

$$g(x) = |J^c| - 1 + x^{J^c} - \sum_{l\in J^c} x_l$$

is $|J^c|$-sos. Finally observe that $h(x) = 1 - x^J - x^{J^c}$ is idempotent. Since these polynomials are all $\lceil i/2\rceil$-sos, it remains to observe that their sum,

$$f(x)+g(x)+h(x) = i-1-\sum_{k\in\mathbf{K}_{i-1}(H)} x_k,$$

is also $\lceil i/2\rceil$-sos. ■

Now we are ready to prove that the K_i-p-hole inequalities are valid on $\mathrm{TH}_{\lceil i/2\rceil}(I)$. Recall that if H is a K_i-p-hole, we write $G_1,\ldots,G_p$ for the K_is in H, with adjacent K_i sharing a common K_{i-1}. If G has an induced K_i-p-hole H, then the inequality

$$k(2i-3)+i-2-\sum_{i\in H} x_i \geq 0$$

defines a facet of $P_i(G)$ for $i\geq 3$; see [1].

Lemma 3.3. *The* K_i*-*p*-hole inequalities are* $\lceil i/2\rceil$*-sos for* p *odd.*

Proof: Let $p=2k+1$. For each $l=1,\ldots,2k+1$, there is exactly one K_{i-1} common to G_l and G_{l-1} (taking indices mod $2k+1$). Denote this variable by x_l. Now fix l. Let the variables $\{y_k\}$ correspond to the K_{i-1} contained in only G_l. Then the variables corresponding to $\mathbf{K}_{i-1}(G_l)$ are $\{x_l, x_{l+1}, y_1,\ldots,y_{i-2}\}$. We will show that $p_l(x,y) = i-2-\sum y_k - x_l x_{l+1}$ is $\lceil i/2\rceil$-sos.

Let $J_1 = \{1,\ldots,\lceil i/2\rceil-2\}$ and $J_2 = \{\lceil i/2\rceil-1,\ldots,i-2\}$. Applying Lemma 3.1, we see that the following two functions are $\lceil i/2\rceil$-sos. First apply the lemma with $A=\{x_l,x_{l+1}\}$ and $B=\{y_j : j\in J_1\}\cup\{x_l,x_{l+1}\}$:

$$f(x,y) = |J_1| - x_l x_{l+1} + x_l x_{l+1} y^{J_1} - \sum_{j\in J_1} y_j.$$

Second, take $A = \emptyset$ and $B = J_2$:

$$g(x, y) = |J_2| - 1 + y^{J_2} - \sum_{j \in J_2} y_j.$$

Finally, observe that the following is idempotent:

$$h(x, y) = 1 - x_l x_{l+1} y^{J_1} - y^{J_2}.$$

Adding these up we get that $p_l(x, y) = f(x, y) + g(x, y) + h(x, y)$ is $\lceil i/2 \rceil$-sos. Now with $p(x, y) = \sum_{l=1}^{2k+1} p_l(x, y)$, we have that p is $\lceil i/2 \rceil$-sos:

$$p(x, y) = (2k + 1)(i - 2) - \sum_{l=1}^{2k+1} \sum_{y_k \subseteq G_l} y_k - \sum_{l=1}^{2k+1} x_l x_{l+1},$$

where the sum $\sum y_k$ is over all K_{i-1} contained in a unique K_i. It remains to show that $k - \sum x_l + \sum x_l x_{l+1}$ is $\lceil i/2 \rceil$-sos. Observe that this is attained by adding the following two quantities, each of which is a sum of idempotents.

$$\sum_{l=1}^{k} (1 - x_{2l-1} - x_{2l} - x_{2l+1} + x_{2l-1}x_{2l} + x_{2l-1}x_{2l+1} + x_{2l}x_{2l+1})$$

$$\sum_{l=2}^{k} (x_{2l-1} - x_{2l-1}x_1 - x_{2l-1}x_{2l+1} + x_{2l+1}x_1)$$

■

In section 3.3 of Conforti, Corneil, and Mahjoub [1], a polynomial-time separation oracle is given for the class of facets corresponding to odd wheels of order $i - 2$. These form a subclass of the K_i-odd hole inequalities, which at the time were not known to have such a separation oracle. Using Lemma 3.3, we can construct such an oracle.

Theorem 3.4. *The separation problem for the K_i-odd hole facets of $P_i(G)$ can be solved in polynomial time in the number of vertices of G, for fixed i.*

Proof: Let G have n vertices. By Lemma 3.3, the K_i-p-hole facets are valid on $\mathrm{TH}_{\lceil i/2 \rceil}(I)$. By Lemma 2.3,we can optimize over $\mathrm{TH}_{\lceil i/2 \rceil}(I)$ in time polynomial in the number of variables in $\mathbf{K}_{i-1}(G)$, at most $\binom{n}{i}$. But this is still polynomial in n. ■

4. Related Problems

Here we apply two results appearing in the literature to the triangle free problem.

4.1. Cuts, and a lower bound on theta convergence. In this section we use a result of Laurent on the max cut problem to give a negative result for the approximability of $P_3(K_n)$ by theta bodies. The max cut problem is the problem of finding a cut of maximum cardinality in a given graph. The theta body approach can be used also in this case, as in [4], providing us a hierarchy of approximation. We will compare these two theta bodies to prove a lower bound on the k such that $\mathrm{TH}_k(I) = P_3(K_n)$.

Let G be a graph with edge set E. A *cut* in G arises from a partition of the nodes of G into two sets S_1 and S_2, whereupon the associated cut is the set of edges from S_1 to S_2. Define C_G and $V_G \subseteq \{0,1\}^E$ to be the collections of characteristic vectors of cuts and triangle-free subgraphs, respectively. Then take their convex hulls, to get the associated polytopes $\mathrm{CUT}(G)$ and, as before, $P_3(G)$. Note that since a cut is by definition bipartite, it is also triangle-free. Therefore, we have $C_G \subseteq V_G$ and $\mathrm{CUT}(G) \subseteq P_3(G)$.

Lemma 4.1. *Let $X \subseteq Y$ be two real varieties, with ideals $I(X)$ and $I(Y)$. Then for any k, $\mathrm{TH}_k(I(X)) \subseteq \mathrm{TH}_k(I(Y))$.*

Proof: If $X \subseteq Y$, then the reverse inclusion holds for their ideals: $I(Y) \subseteq I(X)$. Any function which is k-sos mod $I(Y)$ is then also k-sos mod $I(X)$. The result follows from the definition of $\mathrm{TH}_k(I)$. ■

Consider the complete graph K_n, for odd n. The inequality

$$\sum_{e \in E} x_e \leq \frac{n^2-1}{4}$$

defines a facet of both $P_3(K_n)$ and $\mathrm{CUT}(K_n)$; see [8]. The results in [8] imply that for $k < \frac{n}{4}$, this inequality is not valid on $\mathrm{TH}_k(I(C_{K_n}))$. By Lemma 4.1, it is also not valid on $\mathrm{TH}_k(I(V_{K_n}))$. We have proved:

Theorem 4.2. *For $k < \frac{n}{4}$, $P_3(K_n) \subsetneq \mathrm{TH}_k(I(V_{K_n}))$.*

This implies that the theta body hierarchy fails to yield a polynomial time separation algorithm for the K_n inequalities, as the size of the reduced moment matrices associated with the $n/4$-th theta body is exponential in n. It is still an open question for which $k \geq \lceil n/4 \rceil$, in either the cut or triangle free case, $\mathrm{TH}_k(I) = P(G)$.

4.2. Tuva's conjecture, and an integrality gap. Let G be a graph. A *triangle packing* is a collection of triangles in G, no two of which share an edge. A *triangle cover* is a collection of edges, containing at least one edge from every triangle in G. Let $\tau(G)$ be the minimum-size triangle cover in G (in the language of the introduction, the K_3 cover problem with unit weights). Let $v(G)$ be the maximum-size triangle packing in G. It is an easy exercise to check that $v(G) \leq \tau(G) \leq 3v(G)$. However, Tuva conjectured in [10] that the stronger inequality $\tau(G) \leq 2v(G)$ holds for all graphs G. The problem is currently open; see [6] for more information.

Let E and T be the sets of edges and triangles in G. Krivelevich [7] defined the fractional relaxations of $\tau(G)$ and $v(G)$:

$$\tau^*(G) = \min \left\{ \sum_{e \in E} x_e : x \in [0,1]^E \text{ and for all triangles } \Delta, \sum_{e \in \Delta} x_e \geq 1 \right\}$$

$$v^*(G) = \max \left\{ \sum_{\Delta \in T} y_\Delta : y \in [0,1]^T \text{ and for all edges } e, \sum_{e \in \Delta} y_\Delta \leq 1 \right\}$$

Note that by LP strong duality, $\tau^*(G) = v^*(G)$.

Krivelevich proved that $\tau(G) \leq 2\tau^*(G)$, and that $v^*(G) \leq 2v(G)$. Due to the duality $\tau^*(G) = v^*(G)$, these are equivalent to the fractional Tuva conjecture: $\tau(G) \leq 2v^*(G)$ and $\tau^*(G) \leq 2v(G)$.

Let I be the ideal of the triangle cover problem. Define the following semidefinite relaxation:

$$\tau^\dagger(G) = \min \left\{ \sum_{e \in E} x_e : x \in \mathrm{TH}_2(I) \right\}.$$

Recall that S is a triangle cover if and only if $E \setminus S$ is triangle free. This implies that $x \in \mathrm{TH}_k$ for the triangle free problem if and only if $1 - x \in \mathrm{TH}_k$ for the triangle cover problem. Then by Corollary 3.2, $\tau^\dagger(G) \geq \tau^*(G)$.

We have proved the following integrality gap:

Theorem 4.3. *For any graph G, $\tau^\dagger(G) \geq \frac{\tau(G)}{2}$.*

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