Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 12–43

CURVATURE PROPERTIES OF 3-QUASI-SASAKIAN MANIFOLDS

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Abstract: We find some curvature properties of 3-quasi-Sasakian manifolds which are similar to some well-known identities holding in the Sasakian case. As an application, we prove that any 3-quasi-Sasakian manifold of constant horizontal sectional curvature is necessarily either 3-α-Sasakian or 3-cosymplectic.

Keywords: quasi-Sasakian, 3-quasi-Sasakian, 3-α-Sasakian, 3-Sasakian, 3-cosymplectic.
AMS Subject Classification (2010): 53C12 (Primary), 53C25, 57R30 (Secondary).

1. Introduction

An important topic in contact Riemannian geometry is the study of curvature properties of almost contact metric manifolds (see [1] for details). In some cases it is in fact possible to characterize a manifold in terms of it curvature tensor field. The typical example is given by Sasakian manifolds, which are characterized by the well-known condition

$$R_{XY}\xi = \eta(Y)X - \eta(X)Y. \tag{1.1}$$

A key role in this area is played by the interaction between the curvature and the structure tensors (ϕ, ξ, η) of an almost contact metric manifold. For instance, in any Sasakian manifold one has

$$\begin{aligned} R(X,Y,Z,W) = R(X,Y,\phi Z,\phi W) + g(X,Z)g(Y,W) + g(X,W)g(Y,Z) \\ + g(X,\phi Z)g(Y,\phi W) - g(X,\phi W)g(Y,\phi Z) \end{aligned} \tag{1.2}$$

and in any cosymplectic manifold

$$R(X,Y,Z,W) = R(X,Y,\phi Z,\phi W) \tag{1.3}$$

for any vector fields X, Y, Z, W. The relations (1.2) and (1.3) turn out to be useful for studying the ϕ-sectional curvature and the Ricci tensor and deriving other properties on the geometry of the manifold. A generalization of (1.2) and (1.3) was proposed by Janssens and Vanecke in [9]. They defined

Received November 14, 2012.

a *$C(\alpha)$-manifold* as a normal almost contact metric manifold whose curvature tensor satisfies the condition

$$R(X,Y,Z,W) = R(X,Y,\phi Z,\phi W) + \alpha\big(g(X,Z)g(Y,W) + g(X,W)g(Y,Z) + g(X,\phi Z)g(Y,\phi W) - g(X,\phi W)g(Y,\phi Z)\big),$$

for some $\alpha \in \mathbb{R}$. $C(\alpha)$-manifolds include Sasakian, cosymplectic and Kenmotsu manifolds. Another generalization, due to Blair, is given by the notion of *quasi-Sasakian structure* ([2]). By definition, a quasi-Sasakian manifold is a normal almost contact metric manifold whose fundamental 2-form $\Phi := g(\cdot,\phi\cdot)$ is closed. This class includes Sasakian and cosymplectic manifolds and can be viewed as an odd-dimensional counterpart of Kähler structures. Although quasi-Sasakian manifolds were studied by several different authors and are considered a well-established topic in contact Riemannian geometry, only little about their curvature properties is known. With this regard we mention the attempts of Olszak ([10]) and Rustanov ([12]). On the other hand, if a quasi-Sasakian manifold is endowed with two additional quasi-Sasakian structures defining a 3-quasi-Sasakian manifold then, as shown in [3] and [4], the quaternionic-like relations force the three structures to satisfy more restrictive geometric conditions.

Motivated by these considerations, in this paper we study the curvature properties of 3-quasi-Sasakian manifolds. We are able to find conditions similar to (1.1), (1.2), and (1.3) for the 3-quasi-Sasakian case. Moreover, we present one application of these properties by proving a formula relating the three ϕ-sectional curvatures of a 3-quasi-Sasakian manifold. We then obtain that a 3-quasi-Sasakian manifold has constant horizontal sectional curvature if and only if it is either 3-c-Sasakian or 3-cosymplectic. In the first case it is a space of constant curvature $c^2/4$ and in the latter case it is flat. The last result extends to the quasi-Sasakian setting a famous theorem of Konishi ([7]).

2. Preliminaries

A quasi-Sasakian manifold (M,ϕ,ξ,η,g) of dimension $2n+1$ is said to be of rank $2p$ (for some $p \leq n$) if $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$ on M, and to be of rank $2p+1$ if $\eta \wedge (d\eta)^p \neq 0$ and $(d\eta)^{p+1} = 0$ on M (cf. [2, 13]). It was proven in [2] that there are no quasi-Sasakian manifolds of (constant) even rank. Particular subclasses of quasi-Sasakian manifolds are *c-Sasakian manifolds* (usually called α-Sasakian), which have rank $2n+1$, and *cosymplectic manifolds* (rank 1) according to satisfy, in addition, $d\eta = c\Phi$ ($c \neq 0$) and $d\eta = 0$, respectively. For $c = 2$ we obtain the well-known Sasakian manifolds.

If on the same manifold M there are given three distinct almost contact structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3) satisfying the following relations, for any even permutation (α, β, γ) of $\{1, 2, 3\}$,

$$\begin{aligned} \phi_\gamma = \phi_\alpha\phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta\phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\ \xi_\gamma = \phi_\alpha\xi_\beta = -\phi_\beta\xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha, \end{aligned} \tag{2.1}$$

we say that $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha \in \{1, 2, 3\}$, is an almost contact 3-structure. Then the dimension of M is necessarily of the form $4n + 3$. This notion was introduced independently by Kuo ([8]) and Udriste ([14]). An almost 3-contact manifold M is said to be *hyper-normal* if each almost contact structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ is normal.

In [8] Kuo proved that given an almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$, there exists a Riemannian metric g compatible with each of the three almost contact structure and hence we can speak of *almost contact metric 3-structure*. It is well known that in any almost 3-contact metric manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal with respect to the compatible metric g. Moreover, by putting $\mathcal{H} = \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ we obtain a codimension 3 distribution on M and the tangent bundle splits as the orthogonal sum $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle$. The distributions $\mathcal{H}$ and $\mathcal{V}$ are called, respectively, *horizontal* and *Reeb distribution*.

A 3-*quasi-Sasakian structure* is an almost contact metric 3-structure such that each structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is quasi-Sasakian. Remarkable subclasses are 3-Sasakian and 3-cosymplectic manifolds. Another subclass of 3-quasi-Sasakian structures is given by almost contact metric 3-structures $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ such that each structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is c_α-Sasakian. It is proven in [6] that the non-zero constants c_1, c_2, c_3 must coincide. Therefore we speak of 3-*c-Sasakian manifolds*. Many results on 3-quasi-Sasakian manifolds were obtained in [3] and [4]. We collect some of them in the following theorem.

Theorem 2.1 ([3, 4]). *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold of dimension $4n + 3$. Then, for any even permutation (α, β, γ) of $\{1, 2, 3\}$, the Reeb vector fields satisfy*

$$[\xi_\alpha, \xi_\beta] = c\xi_\gamma, \tag{2.2}$$

for some $c \in \mathbb{R}$. Moreover, the 1-forms η_1, η_2, η_3 have the same rank, called the rank *of the 3-quasi-Sasakian manifold M. The rank of M is 1 if and only if M is 3-cosymplectic and it is an integer of the form $4l+3$, for some $l \leq n$, in the other cases. Furthermore, any 3-quasi-Sasakian manifold of rank $4n+3$ is necessarily 3-c-Sasakian.*

We point out that the constant c in (2.2) is zero if and only if the manifold is 3-cosymplectic. Moreover, for any 3-quasi-Sasakian manifold of rank $4l+3$ one can consider the distribution

$$\mathcal{E}^{4m} := \{X \in \mathcal{H} | i_X \eta_\alpha = 0, i_X d\eta_\alpha = 0 \text{ for any } \alpha = 1,2,3\} \quad (l+m=n)$$

and its orthogonal complement $\mathcal{E}^{4l+3} := (\mathcal{E}^{4m})^\perp$. A remarkable property of 3-quasi-Sasakian manifolds, which in general does not hold for a single quasi-Sasakian structure, is that both $\mathcal{E}^{4l+3}$ and $\mathcal{E}^{4m}$ are integrable and define Riemannian foliations with totally geodesic leaves. In particular it follows that $\nabla \mathcal{E}^{4l+3} \subset \mathcal{E}^{4l+3}$ and $\nabla \mathcal{E}^{4m} \subset \mathcal{E}^{4m}$.

All manifolds considered in the paper are assumed to be connected. The Spivak's conventions for the differential, the wedge product and the interior product are adopted.

3. Main results

We recall that in any 3-quasi-Sasakian manifold of rank $4l+3$ for each $\alpha \in \{1,2,3\}$ one defines two tensors ψ_α and θ_α by

$$\psi_\alpha := \begin{cases} \phi_\alpha, & \text{on } \mathcal{E}^{4l+3} \\ 0, & \text{on } \mathcal{E}^{4m} \end{cases} \qquad \theta_\alpha := \begin{cases} 0, & \text{on } \mathcal{E}^{4l+3} \\ \phi_\alpha, & \text{on } \mathcal{E}^{4m}. \end{cases}$$

Moreover we define $\Psi_\alpha(X,Y) := g(X, \psi_\alpha Y)$ and $\Theta_\alpha(X,Y) := g(X, \theta_\alpha Y)$ for all $X, Y \in \Gamma(TM)$. The tensors ψ_α and Ψ_α satisfy

$$d\eta_\alpha = c\Psi_\alpha, \qquad \nabla \xi_\alpha = -\frac{c}{2}\psi_\alpha \tag{3.1}$$

(cf. ([4, (4.8)] and [4, Theorem 4.3]). Since $\phi_\alpha = \psi_\alpha + \theta_\alpha$ one has that $\Phi_\alpha = \Psi_\alpha + \Theta_\alpha$. Consequently, due to (3.1), Ψ_α and Θ_α are closed 2-forms. We start with a few lemmas. The first is immediate.

Lemma 3.1. *In any 3-quasi-Sasakian manifold of rank $4l+3$ one has,*

$$g(\psi_\alpha^2 X, Y) = g(X, \psi_\alpha^2 Y), \tag{3.2}$$

$$\psi_\alpha^3 = -\psi_\alpha, \tag{3.3}$$

$$\nabla \eta_\alpha = \frac{c}{2}\Psi_\alpha. \tag{3.4}$$

Lemma 3.2. *In any 3-quasi-Sasakian manifold of rank* $4l+3$ *one has*

$$(\nabla_X \psi_\alpha)Y = \frac{c}{2}\left(\eta_\alpha(Y)\psi_\alpha^2 X - g(\psi_\alpha^2 X, Y)\xi_\alpha\right). \tag{3.5}$$

Proof: Let $X \in \Gamma(TM)$. According to the orthogonal decomposition $TM = \mathcal{E}^{4l+3} \oplus \mathcal{E}^{4m}$ we may distinguish the following two cases. (i) Assume $Y \in \Gamma(\mathcal{E}^{4l+3})$. Then, since $\nabla\mathcal{E}^{4l+3} \subset \mathcal{E}^{4l+3}$, we have $(\nabla_X\psi_\alpha)Y = \nabla_X(\psi_\alpha Y) - \psi_\alpha\nabla_X Y = \nabla_X(\phi_\alpha Y) - \phi_\alpha\nabla_X Y = (\nabla_X\phi_\alpha)Y$. The assertion then follows from [4, (4.9)]. (ii) If $Y \in \Gamma(\mathcal{E}^{4m})$, then, as $\nabla\mathcal{E}^{4m} \subset \mathcal{E}^{4m}$, one has $(\nabla_X\psi_\alpha)Y = \nabla_X(\psi_\alpha Y) - \psi_\alpha\nabla_X Y = 0$. On the other hand $\eta_\alpha(Y) = 0$ and $\psi_\alpha^2 X \in \mathcal{E}^{4m}$ and thus

$$\frac{c}{2}\left(\eta_\alpha(Y)\psi_\alpha^2 X - g(\psi_\alpha^2 X, Y)\xi_\alpha\right) = 0. \qquad \blacksquare$$

By using (3.5) and (3.3) we get straightforwardly the following formula for $\nabla\psi_\alpha^2$.

Lemma 3.3. *In any 3-quasi-Sasakian manifold of rank* $4l+3$ *one has*

$$(\nabla_X \psi_\alpha^2)Y = \frac{c}{2}\left(\Psi_\alpha(X, Y)\xi_\alpha - \eta_\alpha(Y)\psi_\alpha X\right). \tag{3.6}$$

Theorem 3.4. *In any 3-quasi-Sasakian manifold the following formula holds*

$$R_{XY}\xi_\alpha = \frac{c^2}{4}\left(\eta_\alpha(X)\psi_\alpha^2 Y - \eta_\alpha(Y)\psi_\alpha^2 X\right).$$

Proof: If the manifold is 3-cosymplectic, i.e. $c = 0$, the claim follows easily from the property that each ξ_α is parallel. Thus we can assume that M has rank $4l+3$. By using (3.1), (3.5), and (3.2), we have

$$\begin{aligned}
R_{XY}\xi_\alpha &= \frac{c}{2}\left(\nabla_Y(\psi_\alpha X) - \nabla_X(\psi_\alpha Y) + \psi_\alpha[X, Y]\right)\\
&= \frac{c}{2}\left((\nabla_Y\psi_\alpha)X - (\nabla_X\psi_\alpha)Y\right)\\
&= \frac{c^2}{4}\left(\eta_\alpha(X)\psi_\alpha^2 Y - g(\psi_\alpha^2 Y, X)\xi_\alpha - \eta_\alpha(Y)\psi_\alpha^2 X + g(\psi_\alpha^2 X, Y)\xi_\alpha\right)\\
&= \frac{c^2}{4}\left(\eta_\alpha(X)\psi_\alpha^2 Y - \eta_\alpha(Y)\psi_\alpha^2 X\right). \qquad \blacksquare
\end{aligned}$$

Theorem 3.5. *Let M be a 3-quasi-Sasakian manifold of rank $4l+3$. Then,*

$$\begin{aligned} R_{XY}\phi_\alpha Z - \phi_\alpha R_{XY} Z = \frac{c^2}{4}\big(& (\Psi_\alpha(Y,\psi_\alpha Z) - \eta_\alpha(Y)\eta_\alpha(Z))\,\psi_\alpha X - (\Psi_\alpha(X,\psi_\alpha Z) \\ & -\eta_\alpha(X)\eta_\alpha(Z))\,\psi_\alpha Y - \Psi_\alpha(Y,Z)\psi_\alpha^2 X + \Psi_\alpha(X,Z)\psi_\alpha^2 Y \\ & + (\eta_\alpha(X)\Psi_\alpha(Y,Z) - \eta_\alpha(Y)\Psi_\alpha(X,Z))\,\xi_\alpha\big). \end{aligned}$$

Proof: The claim follows from a long computation using (3.5), (3.6) and (3.3). ∎

Corollary 3.6. *In any 3-quasi-Sasakian manifold of rank $4l+3$ one has*

$$g(R_{XY}\phi_\alpha Z, W) + g(R_{XY}Z, \phi_\alpha W) = -P_\alpha(X,Y,Z,W),$$

where P_α is the tensor defined by

$$\begin{aligned} P_\alpha(X,Y,Z,W) = \frac{c^2}{4}\big(& \Psi_\alpha(Y,Z)\Psi_\alpha(X,\psi_\alpha W) - \Psi_\alpha(X,Z)\Psi_\alpha(Y,\psi_\alpha W) \\ & + \Psi_\alpha(Y,\psi_\alpha Z)\Psi_\alpha(X,W) - \Psi_\alpha(X,\psi_\alpha Z)\Psi_\alpha(Y,W) \\ & - \eta_\alpha(X)\eta_\alpha(W)\Psi_\alpha(Y,Z) - \eta_\alpha(Y)\eta_\alpha(Z)\Psi_\alpha(X,W) \\ & + \eta_\alpha(Y)\eta_\alpha(W)\Psi_\alpha(X,Z) + \eta_\alpha(X)\eta_\alpha(Z)\Psi_\alpha(Y,W)\big). \end{aligned}$$

Corollary 3.7. *In any 3-quasi-Sasakian manifold of rank $4l+3$ one has*

$$\begin{aligned} g(R_{\phi_\alpha X\phi_\alpha Y}\phi_\alpha Z, \phi_\alpha W) = \frac{c^2}{4}\big(& g(R_{XY}Z,W) + \Psi_\alpha(Z,X)\Psi_\alpha(W,\psi_\alpha\phi_\alpha Y) \\ & + \Psi_\alpha(Z,\psi_\alpha X)\Psi_\alpha(W,\phi_\alpha Y) \\ & + \Psi_\alpha(\phi_\alpha X, Z)\Psi_\alpha(\phi_\alpha Y, \psi_\alpha\phi_\alpha W) \\ & + \Psi_\alpha(\phi_\alpha X, \psi_\alpha Z)\Psi_\alpha(\phi_\alpha Y, \phi_\alpha W)\big), \end{aligned}$$

for any $X,Y,Z,W \in \Gamma(\mathcal{H})$.

Proof: By using Corollary 3.6 twice, one obtains

$$\begin{aligned} g(R_{\phi_\alpha X\phi_\alpha Y}\phi_\alpha Z, \phi_\alpha W) = & g(R_{XY}Z,W) - P_\alpha(Z,W,X,\phi_\alpha Y) \\ & - P_\alpha(\phi_\alpha X, \phi_\alpha Y, Z, \phi_\alpha W). \end{aligned}$$

Next, by using (3.2) and the property that ϕ_α and ψ_α commute, we get that

$$\begin{aligned}P_\alpha(Z,W,X,\phi_\alpha Y)+P_\alpha(\phi_\alpha X,\phi_\alpha Y,Z,\phi_\alpha W) = -\frac{c^2}{4}\big(&\Psi_\alpha(Z,X)\Psi_\alpha(W,\psi_\alpha\phi_\alpha Y)\\ &+\Psi_\alpha(Z,\psi_\alpha X)\Psi_\alpha(W,\psi_\alpha Y)\\ &+\Psi_\alpha(\phi_\alpha X,Z)\Psi_\alpha(\phi_\alpha Y,\psi_\alpha\phi_\alpha W)\\ &+\Psi_\alpha(\phi_\alpha X,\psi_\alpha Z)\Psi_\alpha(\phi_\alpha Y,\phi_\alpha W)\big).\end{aligned}$$

Thus the assertion follows. ■

We recall that on an almost contact metric manifold (M,ϕ,ξ,η,g) one defines a *ϕ-section* as the 2-plane spanned by X and ϕX, where X is a unit vector field orthogonal to ξ. Then the sectional curvature $H(X) := K(X,\phi X) = g(R_{X\phi X}\phi X, X)$ is called *ϕ-sectional curvature*. In a 3-quasi-Sasakian manifold M, we denote by H_α the ϕ_α-sectional curvature.

Theorem 3.8. *For any $X\in\Gamma(\mathcal{H})$ the ϕ_α-sectional curvatures of a 3-quasi-Sasakian manifold of rank $4l+3$ satisfy the following relation*

$$H_1(X)+H_2(X)+H_3(X)=\frac{3c^2}{4}g(X_{\mathcal{E}^{4l}},X_{\mathcal{E}^{4l}})^2, \tag{3.7}$$

where $X_{\mathcal{E}^{4l}}$ denotes the projection of X onto the distribution $\mathcal{E}^{4l}$. In particular,

$$H_1(X)+H_2(X)+H_3(X)=\begin{cases}\frac{3c^2}{4}, & \text{for any } X\in\Gamma(\mathcal{E}^{4l});\\ 0, & \text{for any } X\in\Gamma(\mathcal{E}^{4m}).\end{cases} \tag{3.8}$$

Proof: From Corollary 3.6 it follows that, for any $X,Y,Z,W\in\Gamma(\mathcal{H})$,

$$g(R_{XY}\phi_\alpha Z,\phi_\alpha W) = g(R_{XY}Z,W) \tag{3.9}$$
$$+\frac{c^2}{4}\big(\Psi_\alpha(Y,\psi_\alpha Z)g(\psi_\alpha X,\phi_\alpha W)$$
$$-\Psi_\alpha(X,\psi_\alpha Z)g(\psi_\alpha Y,\phi_\alpha W) \tag{3.10}$$
$$-\Psi_\alpha(Y,Z)g(\psi_\alpha^2 X,\phi_\alpha W) \tag{3.11}$$
$$+\Psi_\alpha(X,Z)g(\psi_\alpha^2 Y,\phi_\alpha W)\big).$$

In (3.11) we put $\alpha = 1$, $Z = X$ and $Y = W = \phi_3 X$, getting

$$\begin{aligned}
-g(R_{X\phi_3 X}\phi_1 X, \phi_2 X) = g(R_{X\phi_3 X}X, \phi_3 X) & \\
+ \frac{c^2}{4}\big(&-g(\phi_3 X, \psi_1^2 X)g(\psi_1 X, \phi_2 X) \\
&+ g(X, \psi_1^2 X)g(\psi_1\phi_3 X, \phi_2 X) \\
&+ g(\phi_3 X, \psi_1 X)g(\psi_1^2 X, \phi_2 X) \\
&- g(X, \psi_1 X)g(\psi_1^2\phi_3 X, \phi_2 X)\big).
\end{aligned}$$

By using the definition of the operators ψ_α and the property that $g(\phi_\alpha\cdot, \cdot) = -g(\cdot, \phi_\alpha\cdot)$, one proves that $g(\psi_1 X, \phi_2 X)$, $g(\phi_3 X, \psi_1 X)$, and $g(X, \psi_1 X)$ vanish. Hence the previous relation becomes

$$\begin{aligned}
-g(R_{X\phi_3 X}\phi_1 X, \phi_2 X) &= g(R_{X\phi_3 X}X, \phi_3 X) + \frac{c^2}{4}g(X, \psi_1^2 X)g(\psi_1\phi_3 X, \phi_2 X) \\
&= -H_3(X) + \frac{c^2}{4}g(X_{\mathcal{E}^{4l}}, X_{\mathcal{E}^{4l}})^2 \qquad (3.12)
\end{aligned}$$

since

$$\begin{aligned}
g(X, \psi_1^2 X)g(\psi_1\phi_3 X, \phi_2 X) &= -g(X, \phi_1^2 X_{\mathcal{E}^{4l}})g(\phi_2 X_{\mathcal{E}^{4l}}, \phi_2 X) \\
&= g(X_{\mathcal{E}^{4l}}, X)^2 = g(X_{\mathcal{E}^{4l}}, X_{\mathcal{E}^{4l}})^2.
\end{aligned}$$

Making cyclic permutations of $\{1, 2, 3\}$, one gets

$$-g(R_{X\phi_1 X}\phi_2 X, \phi_3 X) = -H_1(X) + \frac{c^2}{4}g(X_{\mathcal{E}^{4l}}, X_{\mathcal{E}^{4l}})^2 \qquad (3.13)$$

$$-g(R_{X\phi_2 X}\phi_3 X, \phi_1 X) = -H_2(X) + \frac{c^2}{4}g(X_{\mathcal{E}^{4l}}, X_{\mathcal{E}^{4l}})^2. \qquad (3.14)$$

Then by summing (3.12), (3.13), (3.14), the claim follows from the Bianchi identity. ■

The notion of *horizontal sectional curvature* ([7]) plays in the context of 3-structures the same role played by the ϕ-sectional curvature in contact metric geometry. Let $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be an almost contact 3-structure on M. Let X be a horizontal vector at a point x. Then one can consider the 4-dimensional subspace $\mathcal{H}_x(X)$ of $T_x M$ defined by $\mathcal{H}_x(X) = \langle X, \phi_1 X, \phi_2 X, \phi_3 X\rangle$. $\mathcal{H}_x(X)$ is called the *horizontal section* determined by X. If the sectional curvature for any two vectors belonging to $\mathcal{H}_x(X)$ is a constant $k(X)$ depending only upon the fixed horizontal vector X at x, then $k(X)$ is said to be the horizontal sectional curvature with respect to X at x. Now let X be an arbitrary horizontal vector field on M. If the horizontal section $\mathcal{H}_x(X)$ at any point x of M has a horizontal sectional curvature whose value $k(X)$ is independent of X, we say that the manifold M is of *constant horizontal sectional curvature* at x. It is known ([7]) that a 3-Sasakian manifold has constant horizontal sectional curvature if and only if it has constant curvature 1. We now consider the 3-quasi-Sasakian setting.

Theorem 3.9. *A 3-quasi-Sasakian manifold has constant horizontal sectional curvature if and only if it is either 3-c-Sasakian or 3-cosymplectic. In the first case it is a space of constant curvature $c^2/4$, in the latter it is flat.*

Proof: We distinguish the case when M is 3-cosymplectic and M is 3-quasi-Sasakian of rank $4l+3$. Let M be a 3-cosymplectic manifold of constant horizontal sectional curvature k and let x be a point of M. There exists a local Riemannian submersion π defined on an open neighborhood of x with base space a hyper-Kähler manifold (M', J'_α, g'). We recall the well-known O'Neill formula ([11]) relating the sectional curvatures of the total and base spaces

$$K(Y,Z) = K'(Y,Z) - 3\,\|A_Y Z\| = K'(Y,Z), \tag{3.15}$$

A denoting the O'Neill tensor, which in this case vanishes identically since the distribution $\mathcal{H}$ is integrable. As the value of k does not depend of the horizontal section $\mathcal{H}_x(X)$ at x, we can choose X to be a basic vector field. Since for any $\alpha, \beta \in \{1,2,3\}$, $\mathcal{L}_{\xi_\alpha}\phi_\beta = 0$, $\mathcal{H}_x(X)$ projects to a horizontal section $\mathcal{H}_{x'}(X')$ on $x' = \pi(x)$. Then, (3.15) implies that M' has constant horizontal sectional curvature k. It is well known that a hyper-Kähler manifold of constant horizontal sectional curvature is flat, hence by using (3.15) again we get that M is horizontally flat. On the other hand, for any $Z \in \Gamma(TM)$, we have $K(Z,\xi_\alpha) = 0$ (cf. [5, Lemma 2]). Thus M is flat. Let us now suppose that M is a 3-quasi-Sasakian manifold of rank $4l+3$ with constant horizontal sectional curvature k. By definition of horizontal sectional curvature, $k = k(X) = H_1(X) = H_2(X) = H_3(X)$. Suppose the rank of M is not maximal, that is $\mathcal{E}^{4l}$ does not coincide with $\mathcal{H}$. Then, from (3.8), we get that $k(X) = \frac{c^2}{4}$ for $X \in \Gamma(\mathcal{E}^{4l})$ and $k(X) = 0$ for $X \in \Gamma(\mathcal{E}^{4m})$. This is in contrast with the fact that the value of k does not depend of X. Thus M is necessarily of maximal rank and $k = \frac{c^2}{4}$. Hence, due to [4, Corollary 4.4], M is 3-c-Sasakian. Observe now that one can apply a homothety to the given structure, that is a change of the structure tensors of the following type

$$\bar{\phi}_\alpha := \phi_\alpha, \quad \bar{\xi}_\alpha := \frac{2}{c}\xi_\alpha, \quad \bar{\eta}_\alpha := \frac{c}{2}\eta_\alpha, \quad \bar{g} := \frac{c^2}{4}g, \tag{3.16}$$

Then it is easy to check that the resulting structure $(\bar{\phi}_\alpha, \bar{\xi}_\alpha, \bar{\eta}_\alpha, \bar{g})$ is 3-Sasakian and its horizontal sectional curvature is proportional to that of $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$. Therefore, due to [7], $(M, \bar{\phi}_\alpha, \bar{\xi}_\alpha, \bar{\eta}_\alpha, \bar{g})$ is a space of constant sectional curvature and therefore the same is true for $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$. Its sectional curvature is $k = \frac{c^2}{4}$. ∎

Acknowledgments. Research partially supported by CMUC and FCT (Portugal), through European program COMPETE/FEDER, and grants PTDC/MAT/099880/2008, MTM2009-13383 (A. D. N.), and SFRH/BPD/31788/2006 (I. Y.).

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