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DUALS OF WEAKLY MAL'TSEV CATEGORIES

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ABSTRACT: We study the dual of weakly Mal'tsev categories and show that examples of such categories include: (a) any quasi-adhesive category (b) any extensive category with pullback-stable epimorphisms (c) any solid quasi-topos. These capture many familiar aspects of topological spaces.

KEYWORDS: extensive category, weakly Mal'tsev category, quasi-adhesive category, pullback-stable epimorphism, stable coproduct, split exact cospan, topological space, Van Kampen square, solid quasi-topos.
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1. Introduction

In this paper we continue the study of weakly Mal'tsev categories. After having studied internal categorical structures, namely internal categories and internal groupoids [25], the connection with the classical definition of Mal'tsev category [6, 7] via strong relations [13], and considered the particular example of distributive lattices [26], we now turn to the dual of a weakly Mal'tsev category in general.

Our motivating example is the dual of the category of topological spaces and continuous maps. Indeed, as Zurab Janelidze observed (during the 2008 CT conference in Calais), the dual of the category of topological spaces is weakly Mal'tsev. It is also well known that the dual of any topos is Mal'tsev (but even more is true, see for instance [2, 1, 14]). Since every Mal'tsev category is in particular weakly Mal'tsev, it is only natural to look for those properties on a topos which should be maintained in a category if its dual is expected to be, not necessarily Mal'tsev, but weakly Mal'tsev. The particular case of topological spaces should be considered as the leading example. This work is devoted to giving a satisfactory answer to that question.

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In the next section we give a quick survey on weakly Mal'tsev categories and introduce a general class of examples containing in particular any Mal'tsev variety of universal algebras, the category of distributive lattices and the category of commutative magmas with cancellation, and also any quasi-subvariety of those classes of algebras (Proposition 2.1).

In Section 3 we introduce the notion of split exact cospan (Definition 3.3), as one which is obtained by pushout of a split monomorphism along a split monomorphism, and give a characterization of the dual of a weakly Mal'tsev category (Corollary 3.5).

Section 4 contains the main result stating that the dual of a category with pullbacks and pushouts of split monomorphisms along split monomorphisms, in which a cospan is jointly epimorphic whenever it is obtained by pulling back a split exact cospan, is weakly Mal'tsev (Theorem 4.1).

The main result is then used in Section 5 to prove that the dual of any quasi-adhesive category [19] is weakly Mal'tsev. The proof is based on the notion of Van Kampen square [3] and uses the fact that any split exact cospan, indeed any pushout along a regular monomorphism in a quasi-adhesive category, is part of a Van Kampen square (Proposition 5.1).

Finally, in the last section we show that the dual of any category with pullback-stable epimorphisms and stable coproducts in the sense of [10] is weakly Mal'tsev. Examples include for instance the dual of any extensive category with pullback-stable epimorphisms or the dual of any solid quasi-topos [15]. Keeping in mind that the original motivation was the case of topological spaces, we give specific references showing that many familiar categories of spaces fit into the above setting, such as the dual of any lax algebra, or (T, V)-category in the sense of [9].

2. A quick survey on weakly Mal'tsev categories

The notion of weakly Mal'tsev category, introduced in [25], has proved to be a convenient and general setting for working with internal categorical structures: every reflexive graph, in a weakly Mal'tsev category, can have at most one multiplicative graph structure and every multiplicative graph is automatically an internal category. Moreover, contrary to the well known case of Mal'tsev categories [6, 7], not every internal category is an internal groupoid. For instance the linearly ordered set of natural numbers is an internal category in the category of commutative monoids with cancellation (a weakly Mal'tsev category) and it is obviously not an internal groupoid.

The category of distributive lattices is another important example of a weakly Mal'tsev category which is not Mal'tsev [26]. However, the category of modular lattices is not weakly Mal'tsev [26]. Thus, in this way it is possible to cover a wider range of examples while still keeping some of the useful properties desirable for internal categorical structures. As proved in [27], in the context of weakly Mal'tsev categories, groupoids and internal categories coincide if and only if every reflexive and transitive relation (i.e. a preorder) is an equivalence relation. Remarkably, when the category is regular the weak Mal'tsev property is not necessary and groupoids coincide with internal categories as soon as preorders coincide with equivalence relations [29].

Mal'tsev categories are characterized by the fact that every reflexive relation is an equivalence relation. From [13] we now know that the weak Mal'tsev property can be characterized by the fact that every strong relation is difunctional, or equivalently that every reflexive and strong relation is an equivalence relation.

Another characterization of a Mal'tsev category, due to Bourn [2], is that every pair of local product injections is jointly strongly epimorphic. By definition a weakly Mal'tsev category is one where local product injections are jointly epimorphic (further details can be found in [13] but will not be needed here). A general example of a class of categories with the weak Mal'tsev property, including in particular every Mal'tsev variety of universal algebras (such as groups, rings, Lie-algebras, etc.), the category of distributive lattices or the category of commutative magmas with cancellation, is presented next.

Proposition 2.1. *Let I be a fixed set of indices and consider the category whose objects are triples $(B, p, (q_i)_{i\in I})$ where B is a set, p and q_i, $i \in I$, are ternary operations on the set B, and the following conditions are satisfied*

$$p(x, y, y) = p(y, y, x)$$

$$q_i(x, y, y) = q_i(y, x, x), \quad \textit{for all } i \in I$$

and if for some $a \in B$,

$$p(x, a, a) = p(x', a, a)$$

and

$$q_i(x, a, a) = q_i(x', a, a), \quad \textit{for all } i \in I,$$

then $x = x'$. The morphisms are the expected ones.

Then the category just described is a weakly Mal'tsev category.

Proof: The proof is a small variation of a similar result, involving only the ternary operation $p(x,y,z)$ above, that can be found in the introduction of [26]. We only observe that, given any diagram of the form

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[dr, "\alpha"'] & B \arrow[l, shift left, "r"] \arrow[r, shift right, "s"'] \arrow[d, "\beta"] & C \arrow[l, shift right, "g"'] \arrow[dl, "\gamma"] \\ & D & \end{tikzcd}$$

with $fr = 1_B = gs$ and $\alpha r = \beta = \gamma s$, and two morphisms $\varphi, \varphi' \colon A \times_B C \longrightarrow D$, such that for every $a \in A$ and $c \in C$,

$$\begin{aligned} \varphi(a, sf(a)) &= \varphi'(a, sf(a)) = \alpha(a) \\ \varphi(rg(c), c) &= \varphi'(rg(c), c) = \gamma(c) \end{aligned}$$

then, for every $a \in A$ and $c \in C$ with $f(a) = b = g(c) \in B$, we have

$$\begin{aligned} p(\varphi(a,c), \beta(b), \beta(b)) = p(\varphi'(a,c), \beta(b), \beta(b)) &= p(\alpha(a), \beta(b), \gamma(c)) \\ &= p(\gamma(c), \beta(b), \alpha(a)) \end{aligned}$$

and

$$\begin{aligned} q_i(\varphi(a,c), \beta(b), \beta(b)) = q_i(\varphi'(a,c), \beta(b), \beta(b)) &= q_i(\alpha(a), \gamma(c), \gamma(c)) \\ &= q_i(\gamma(c), \alpha(a), \alpha(a)), \end{aligned}$$

from which we conclude that $\varphi(a,c) = \varphi'(a,c)$. ■

Every Mal'tsev variety with a Mal'tsev term $m(x,y,y) = m(y,y,x) = x$ is an instance of the case above: choose I to be the empty set and put $p = m$. The case of distributive lattices is another instance of the above: take $I = \{1,2\}$ and define $p(x,y,z) = y$, $q_1(x,y,z) = x \wedge (y \vee z)$ and $q_2(x,y,z) = x \vee (y \wedge z)$. The case of commutative magmas with cancellation may be captured by choosing again I as the empty set and defining $p(x,y,z) = y \cdot (x \cdot z) = y \cdot (z \cdot x)$.

3. Exact and split exact cospans – the dual of a weakly Mal'tsev category

Recall that a category $\mathcal{C}$ is said to be weakly Mal'tsev [25] if it has pullbacks of split epimorphisms along split epimorphisms and if every two morphisms into a pullback of split epimorphisms, which are induced by the given sections of the respective split epimorphisms, form a jointly epimorphic cospan.

The following is an immediate consequence of the definition of a weakly Mal'tsev category.

Proposition 3.1. *Let $\mathcal{C}$ be a category with pushouts of split monomorphisms along split monomorphisms. The dual of the category $\mathcal{C}$ is weakly Mal'tsev if and only if for every diagram of the form*

$$\begin{tikzcd} A \arrow[r, shift left, "r"'] & B \arrow[l, shift left, "f"'] \arrow[r, shift left, "g"] & C \arrow[l, shift left, "s"] \end{tikzcd}$$

with $rf = 1_B = sg$, the two morphisms $p_1 = [1, fs] \colon A +_B C \longrightarrow A$ and $p_2 = [gr, 1] \colon A +_B C \longrightarrow C$, canonically induced from the pushout diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, shift right, "\iota_C"'] \\ A \arrow[r, shift left, "\iota_A"] & A +_B C \arrow[l, shift left, "p_1"] \arrow[u, shift right, "p_2"'] \end{tikzcd}$$

by the conditions

$$p_1 \iota_A = 1_A, \quad p_1 \iota_C = fs \quad \text{and} \quad p_2 \iota_A = gr, \quad p_2 \iota_C = 1_C,$$

are jointly monomorphic.

We will say that a cospan is exact if it is the pushout of its pullback. More specifically:

Definition 3.2. A cospan

$$\begin{tikzcd} A \arrow[r, "i"] & Q & B \arrow[l, "j"'] \end{tikzcd}$$

is an *exact cospan* if the pullback of i and j exists and, moreover, the square

$$\begin{tikzcd} A \times_Q B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "j"] \\ A \arrow[r, "i"] & Q \end{tikzcd}$$

is a pushout square.

It is clear from Proposition 3.1 that we will only be interested in cospans arising as the pushout of split monomorphisms along split monomorphisms. The following definition of split cospan is used to precisely capture that idea, as observed in Proposition 3.4.

Definition 3.3. An exact cospan

$$A \xrightarrow{i} Q \xleftarrow{j} B$$

is said to be *split exact* if there exist two morphisms

$$\begin{tikzcd} A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd}$$

such that

$$j\alpha = i\beta\alpha \quad , \quad i\beta = j\alpha\beta$$
$$\pi_1 = \beta\pi_2 \quad , \quad \alpha\pi_1 = \pi_2.$$

In other words, in a category with pullbacks, a cospan (i, j) is split exact if the square

$$\begin{tikzcd} A \times_Q B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "j"] \\ A \arrow[r, "i"] & Q \end{tikzcd}$$

is a pushout square and there exists

$$\begin{tikzcd} A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \end{tikzcd}$$

making the diagram

$$\begin{tikzcd} A \times_Q B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "i\beta"] \arrow[dl, shift left, "\beta"] \\ A \arrow[r, "j\alpha"'] \arrow[ur, shift left, "\alpha"] & Q \end{tikzcd}$$

commutative.

From now on we assume that our setting is at least a category with pullbacks and pushouts of split monomorphisms along split monomorphisms.

Proposition 3.4. *A cospan is split exact if and only if it is obtained as a pushout of a split monomorphism along a split monomorphism.*

Proof: Let f and g be split monomorphisms with the same codomain, as displayed in the diagram

$$\begin{tikzcd} A \arrow[r, shift right, "r"'] & B \arrow[l, shift right, "f"'] \arrow[r, shift left, "g"] & C \arrow[l, shift left, "s"] \end{tikzcd}$$

with the respective retractions, that is,

$$rf = 1_B = sg. \tag{1}$$

We have to show that the canonical inclusions, (ι_A, ι_C), into the pushout of f and g,

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\iota_C"] \\ A \arrow[r, "\iota_A"'] & A +_B C \end{tikzcd},$$

gives rise to a split exact cospan.

First we observe that the square above is a pullback square. Indeed, given any $u\colon Z \longrightarrow A$ and $v\colon Z \longrightarrow C$ with $\iota_A u = \iota_C v$, there exists $w\colon Z \longrightarrow B$ such that $fw = u$ and $gw = v$, namely $w = sv = ru$, as we shall now see. The equation $sv = ru$ follows from (1), which gives

$$[1, fs]\colon A +_B C \longrightarrow A,$$

in the following way

$$sv = rfsv = r[1, fs]\iota_C v = r[1, fs]\iota_A u = ru;$$

in addition $fru = u$ is obtained as

$$u = [1, fs]\iota_A u = [1, fs]\iota_C v = fsv = fru,$$

and similarly for $gsv = v$:

$$v = [gr, 1]\iota_C v = [gr, 1]\iota_A u = gru = gsv.$$

The uniqueness of w is guaranteed by the fact that f (and g) is a split monomorphism. This shows that the square above is a pullback square; indeed, this also follows from the square being obtained as a pushout along split monomorphisms, which is a well-known result. This shows that the cospan (ι_A, ι_C) is exact. In order to prove that it is split exact we simply observe that the two morphisms

$$\begin{tikzcd} A \arrow[r, shift left, "gr"] & C \arrow[l, shift left, "fs"] \end{tikzcd}$$

render the following diagram commutative:

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\iota_A f s"] \arrow[dl, shift left, "fs"] \\
A \arrow[r, "\iota_C g r"'] \arrow[ur, shift left, "gr"] & A +_B C
\end{tikzcd}$$

Conversely, given a diagram of the form

$$\begin{tikzcd}
A \arrow[rr, shift left, "\alpha"] \arrow[dr, "i"'] & & B \arrow[ll, shift left, "\beta"] \arrow[dl, "j"] \\
& Q &
\end{tikzcd}$$

such that the pullback square

$$\begin{tikzcd}
A \times_Q B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "j"] \\
A \arrow[r, "i"] & Q
\end{tikzcd}$$

is also a pushout square, and the diagram

$$\begin{tikzcd}
A \times_Q B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "i\beta"] \arrow[dl, shift left, "\beta"] \\
A \arrow[r, "j\alpha"'] \arrow[ur, shift left, "\alpha"] & Q
\end{tikzcd}$$

commutes, we have to show that π_1 and π_2 are split monomorphisms. The identity $i\beta\alpha = j\alpha$ induces a morphism

$$\langle \beta\alpha, \alpha \rangle \colon A \longrightarrow A \times_Q B$$

satisfying

$$\pi_1 \langle \beta\alpha, \alpha \rangle = \beta\alpha \text{ and } \pi_2 \langle \beta\alpha, \alpha \rangle = \alpha$$

and this morphism is such that

$$\begin{aligned}
\langle \beta\alpha, \alpha \rangle \pi_1 &= \langle \beta\alpha\pi_1, \alpha\pi_1 \rangle \\
&= \langle \beta\pi_2, \pi_2 \rangle \\
&= \langle \pi_1, \pi_2 \rangle = 1_{A \times_Q B}
\end{aligned}$$

proving that π_1 is a split monomorphism. Similarly, the identity $j\alpha\beta = i\beta$ induces

$$\langle \beta, \alpha\beta \rangle \colon B \longrightarrow A \times_Q B$$

such that

$$\begin{aligned} \langle \beta, \alpha\beta \rangle \pi_2 &= \langle \beta\pi_2, \alpha\beta\pi_2 \rangle \\ &= \langle \pi_1, \alpha\pi_1 \rangle \\ &= \langle \pi_1, \pi_2 \rangle = 1_{A \times_Q B} \end{aligned}$$

proving that π_2 is a split monomorphism. ■

As a result we obtain a new characterization for the weak Mal'tsev property (stated in the dual form).

Corollary 3.5. *Let $\mathcal{C}$ be a category with finite limits and pushouts of split monomorphisms along split monomorphisms. The following conditions are equivalent:*

(a) *$\mathcal{C}^{op}$ is weakly Mal'tsev.*

(b) *for every split exact copan, with specified α and β,*

$$\begin{tikzcd} A \arrow[rr, shift left, "\alpha"] \arrow[dr, "i"'] & & C \arrow[ll, shift left, "\beta"] \arrow[dl, "j"] \\ & Q & \end{tikzcd} \tag{2}$$

the induced morphism

$$\langle [1, \beta], [\alpha, 1] \rangle \colon Q \longrightarrow A \times C$$

is a monomorphism.

Proof: Using Proposition 3.1 and Proposition 3.4 we simply observe that given a split exact cospan as in (2), the two morphisms $p_1 = [1, \beta]$ and $p_2 = [\alpha, 1]$, induced from the pushout square presenting (i, j) as an exact cospan via the following commutative diagrams

$$\begin{tikzcd} A \times_Q C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "j"] \arrow[ddr, bend left, "\beta"] & \\ A \arrow[r, "i"] \arrow[drr, equal] & Q \arrow[dr, dashed, "p_1"] & \\ & & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \times_Q C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "j"] \arrow[ddr, equal] & \\ A \arrow[r, "i"] \arrow[drr, bend right, "\alpha"'] & Q \arrow[dr, dashed, "p_2"] & \\ & & C \end{tikzcd}$$

are well defined. ■

4. The main result

In this section we show that the dual of a category with pullbacks and pushouts of split monomorphisms along split monomorphisms, in which pullbacks of split exact cospans are jointly epimorphic, is weakly Mal'tsev.

Theorem 4.1. *Let $\mathcal{C}$ be a category with pushouts of split monomorphisms along split monomorphisms, pullbacks, and such that for every commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] & E \arrow[l] \arrow[d] \\ B \arrow[r] & D & F \arrow[l] \end{tikzcd} \tag{3}$$

where both squares are pullback squares, if the bottom cospan is split exact then the top one is jointly epimorphic. Then $\mathcal{C}^{op}$ is a weakly Mal'tsev category.

Proof: Consider a diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, shift left, leftarrow, "f"] \arrow[r, shift right, "r"'] & B \arrow[r, shift left, "g"] \arrow[r, shift right, leftarrow, "s"'] & C \end{tikzcd}$$

with $rf = 1_B = sg$, and take the pushout of the split monomorphism f along the split monomorphism g in order to obtain a *square of split monomorphisms*

$$\begin{tikzcd} B \arrow[r, shift right, "g"'] \arrow[r, shift left, leftarrow, "s"] \arrow[d, shift left, "f"] \arrow[d, shift right, leftarrow, "r"'] & C \arrow[d, shift right, "\iota_C"'] \arrow[d, shift left, leftarrow, "p_2"] \\ A \arrow[r, shift left, "\iota_A"] \arrow[r, shift right, leftarrow, "p_1"'] & A +_B C \end{tikzcd}$$

with $p_1 = [1, fs]$ and $p_2 = [gr, 1]$. It follows from Proposition 3.4 that the cospan

$$A \xrightarrow{\iota_A} Q \xleftarrow{\iota_C} C\ ,$$

with $Q = A +_B C$, is a split exact cospan.

Now, assume the existence of two morphisms

$$u, v \colon D \longrightarrow Q$$

with $p_1 u = p_1 v$ and $p_2 u = p_2 v$.

Our task is to prove $u = v$.

First consider the commutative diagram

$$\begin{tikzcd} D_A \arrow[r, "i"] \arrow[d, "u_A"'] & D \arrow[d, "u"] & D_C \arrow[l, "j"'] \arrow[d, "u_C"] \\ A \arrow[r, "\iota_A"] & Q & C \arrow[l, "\iota_C"'] \end{tikzcd}$$

obtained by pulling back u along ι_A and along ι_C. This means that the cospan (i, j) is obtained by pullback from a split exact cospan, and so, by assumption on $\mathcal{C}$, it is a jointly epimorphic cospan. As a consequence, the morphism u is completely determined by ui and uj and in fact we have

$$ui = \iota_A u_A \quad , \quad uj = \iota_C u_C;$$

similarly $v\colon D \longrightarrow Q$ is determined by vi and vj. Repeating the process, we obtain the two diagrams below

$$\begin{tikzcd} D_1 \arrow[r, "k_1"] \arrow[d, "v_1"'] & D_A \arrow[d, "vi"] & D_2 \arrow[l, "k_2"'] \arrow[d, "v_2"] \\ A \arrow[r, "\iota_A"] & Q & C \arrow[l, "\iota_C"'] \end{tikzcd} \qquad \begin{tikzcd} D_3 \arrow[r, "k_3"] \arrow[d, "v_3"'] & D_C \arrow[d, "vj"] & D_4 \arrow[l, "k_4"'] \arrow[d, "v_4"] \\ A \arrow[r, "\iota_A"] & Q & C \arrow[l, "\iota_C"'] \end{tikzcd}$$

by taking the pullbacks of vi and vj along ι_A and along ι_C, so that the four squares above are pullback squares.

Again, by assumption on $\mathcal{C}$, the cospans (k_1, k_2) and (k_3, k_4) are jointly epimorphic. This means that the four-tuple (ik_1, ik_2, jk_3, jk_4) is jointly epimorphic and therefore any morphism $h\colon D \longrightarrow Z$, with domain D, is uniquely determined by

$$h = (hik_1, hik_2, hjk_3, hjk_4).$$

Hence, v is completely determined by

$$v = (vi, vj) = ((\iota_A v_1, \iota_C v_2), (\iota_A v_3, \iota_C v_4)),$$

and if writing

$$u_1 = u_A k_1 \quad , \quad u_2 = u_A k_2 \quad , \quad u_3 = u_C k_3 \quad , \quad u_4 = u_C k_4$$

we have that u is also of the form

$$u = ((\iota_A u_1, \iota_A u_2), (\iota_C u_3, \iota_C u_4)).$$

The morphisms determining u and v may be displayed in the following diagram.

$$\begin{tikzcd}
D_1 \arrow[drr, shift left, "u_1"] \arrow[drr, shift right, "v_1"'] & & D_2 \arrow[d, "u_2"] \arrow[drr, "v_2"] & & D_3 \arrow[dll, "v_3"'] \arrow[d, "u_3"] & & D_4 \arrow[dll, shift right, "u_4"'] \arrow[dll, shift left, "v_4"] \\
& & A & & C & &
\end{tikzcd}$$

We conclude the proof by observing that $p_1u = p_1v$ and $p_2u = p_2v$ implies

$$v_1 = u_1, \quad \iota_C v_2 = \iota_A u_2, \quad \iota_A v_3 = \iota_C u_3, \quad u_4 = v_4.$$

Indeed,

$$p_1u = [1, fs]u = ((u_1, u_2), (fsu_3, fsu_4))$$

and

$$p_1v = [1, fs]v = ((v_1, fsv_2), (v_3, fsv_4)),$$

so that

$$\begin{aligned}
u_1 &= v_1 \\
u_2 &= fsv_2 \\
fsu_3 &= v_3 \\
fsu_4 &= fsv_4;
\end{aligned}$$

similarly, from $p_2u = p_2v$ or $[gr, 1]u = [gr, 1]v$, we obtain

$$\begin{aligned}
gru_1 &= grv_1 \\
gru_2 &= v_2 \\
u_3 &= grv_3 \\
u_4 &= v_4.
\end{aligned}$$

We already have $u_1 = v_1$ and $u_4 = v_4$, but from above we also deduce

$$\begin{aligned}
\iota_A u_2 &= \iota_A fsv_2 = \iota_C gsv_2 = \iota_C g1_B sv_2 = \iota_C grfsv_2 \\
&= \iota_C gru_2 = \iota_C v_2
\end{aligned}$$

and

$$\begin{aligned}
\iota_C u_3 &= \iota_C grv_3 = \iota_A frv_3 = \iota_A f1_B rv_3 = \iota_A fsgrv_3 \\
&= \iota_A fsu_3 = \iota_A v_3.
\end{aligned}$$

This shows that $u = v$ and completes the proof. ∎

5. Adhesive and quasi-adhesive categories

Adhesive categories were introduced in [18]. The category of directed graphs is an example of an adhesive category. The main feature of an adhesive category is the fact that pushouts along monomorphisms are well behaved with respect to pullbacks. As many interesting examples, mainly from computer science, were not covered by the setting of adhesive categories, the wider notion of quasi-adhesive category was introduced in [19], see also [16]. In a quasi-adhesive category not all monomorphisms are require to induce well behaved pushouts; this is only demanded from strong monomorphisms.

A category is quasi-adhesive when it has pullbacks, pushouts along regular monomorphisms and such pushouts are Van Kampen squares [3]. As is clear from the proof of the following result, if restricting the class of strong monomorphisms to the class of split monomorphisms in the definition of quasi-adhesive category, the result is still valid and hence the dual of such categories are weakly Mal'tsev.

Proposition 5.1. *The dual of any quasi-adhesive category is weakly Mal'tsev.*

Proof: Consider a commutative diagram of the form

$$\begin{tikzcd} E \arrow[r, "k"] \arrow[d, "f"'] & D \arrow[d, "g"'] & F \arrow[l, "l"'] \arrow[d, "h"] \\ A \arrow[r, "i"] & Q & B \arrow[l, "j"'] \end{tikzcd} \tag{4}$$

in which both squares are pullback squares and the cospan (i,j) is split exact. We will show that the cospan (k,l) is exact and hence in particular jointly epimorphic. The proof is then completed using Theorem 4.1.

The square

$$\begin{tikzcd} A \times_Q B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "j"] \\ A \arrow[r, "i"'] & Q \end{tikzcd}$$

is a Van Kampen square. Indeed, it is a pushout (since (i,j) is exact) and π_1 is a strong monomorphism (in fact it is a split monomorphism because (i,j) is split exact — Proposition 3.4). Now, diagram (4) can be completed into a commutative cube

$$\begin{tikzcd}
E \times_D F \arrow[rr, "\pi_2'"] \arrow[dr, "\pi_1'"'] \arrow[dd, dashed] & & F \arrow[dr, "l"] \arrow[dd, "h" near end] & \\
& E \arrow[rr, "k" near start, crossing over] & & D \arrow[dd, "g"] \\
A \times_Q B \arrow[rr, "\pi_2" near start] \arrow[dr, "\pi_1"'] & & B \arrow[dr, "j"] & \\
& A \arrow[rr, "i"'] \arrow[from=uu, "f" near start, crossing over] & & Q
\end{tikzcd}$$

in which the top face is obtained by pulling back k and l, and the dashed arrow is given by

$$\langle f\pi_1', h\pi_2'\rangle \colon E \times_D F \longrightarrow A \times_Q B.$$

In order to show that the top face is a pushout (and hence (k,l) is an exact cospan) it suffices to prove that the left and rear squares are pullback squares (observe that the bottom face is a Van Kampen square).

In order to see that the square

$$\begin{tikzcd}
E \times_D F \arrow[r, "\pi_2'"] \arrow[d, "\langle f\pi_1', h\pi_2'\rangle"'] & F \arrow[d, "h"] \\
A \times_Q B \arrow[r, "\pi_2"] & B
\end{tikzcd}$$

is a pullback square, consider any two morphisms $x \colon Z \longrightarrow A \times_Q B$ and $y \colon Z \longrightarrow F$ with $\pi_2 x = hy$. It is not difficult to find a morphism

$$w \colon Z \longrightarrow E \times_D F$$

such that

$$\begin{aligned}
\pi_2' w &= y \\
\langle f\pi_1', h\pi_2'\rangle w &= x.
\end{aligned}$$

In fact, since $i\pi_1 x = gly$ and $k\langle \pi_1 x, ly\rangle = ly$ we have

$$w = \langle \langle \pi_1 x, ly\rangle, y\rangle,$$

with

$$\langle f\pi_1', h\pi_2'\rangle \langle \langle \pi_1 x, ly\rangle, y\rangle = \langle \pi_1 x, hy\rangle = \langle \pi_1 x, \pi_2 x\rangle = x.$$

It remains to prove that w is uniquely determined. Suppose there exists $w' = \langle u, y\rangle \colon Z \longrightarrow E \times_D F$ with $\langle f\pi_1', h\pi_2'\rangle\langle u, y\rangle = x$, that is

$$\langle fu, hy\rangle = \langle \pi_1 x, \pi_2 x\rangle$$

and consequently $fu = \pi_1 x$, hence $u = \langle fu, ku\rangle = \langle \pi_1 x, ly\rangle$. This shows uniqueness.

Similarly we prove that the left face in the cube is a pullback. ■

6. Stable coproducts, quasi-toposes and extensive categories

In a category with pullbacks and finite coproducts we say that coproducts are stable [10] if, given any commutative diagram

$$\begin{tikzcd} D \arrow[r, "k"] \arrow[d, "f"'] & F \arrow[d, "h"'] & E \arrow[l, "l"'] \arrow[d, "g"] \\ A \arrow[r, "i"] & C & B \arrow[l, "j"'] \end{tikzcd}$$

in which both squares are pullback squares, the square

$$\begin{tikzcd} D+E \arrow[r, "{[k,l]}"] \arrow[d, "f+g"'] & F \arrow[d, "h"'] \\ A+B \arrow[r, "{[i,j]}"] & C \end{tikzcd}$$

also is a pullback square.

Proposition 6.1. *Let $\mathcal{C}$ be a category with pullbacks, finite coproducts and pushouts of split monomorphisms. If in addition $\mathcal{C}$ has*

(a) *pullback-stable epimorphisms, and*
(b) *stable coproducts,*

then $\mathcal{C}^{\mathrm{op}}$ is weakly Mal'tsev.

Proof: Given a commutative diagram of the form

$$\begin{tikzcd} D \arrow[r, "k"] \arrow[d, "f"'] & F \arrow[d, "h"'] & E \arrow[l, "l"'] \arrow[d, "g"] \\ A \arrow[r, "i"] & C & B \arrow[l, "j"'] \end{tikzcd}$$

in which both squares are pullbacks and the bottom row is an exact cospan, the induced morphism $[i,j]\colon A+B \longrightarrow C$ is a regular epimorphism. Also, since epimorphisms are stable under pullback and the square

$$\begin{tikzcd} D+E \arrow[r, "{[k,l]}"] \arrow[d, "f+g"'] & F \arrow[d, "h"] \\ A+B \arrow[r, "{[i,j]}"] & C \end{tikzcd}$$

is a pullback, the induced morphism $[k,l]\colon D+E \longrightarrow F$ is an epimorphism. The result in Theorem 4.1 concludes the proof. ■

Recall from [5] that a category is extensive if and only if it has disjoint and universal finite coproducts. Coproducts are universal when the pullback of a coproduct diagram is also a coproduct diagram. Coproducts are said to be disjoint when coproduct inclusions are monomorphisms and the pullback of any coproduct diagram is the initial object.

It is not difficult to see that the duals of such categories, in which epimorphisms are persistent (in [17] an epimorphism f is called *persistent* if every pullback of f exists and is an epimorphism), are weakly Mal'tsev.

Corollary 6.2. *Let $\mathcal{C}$ be an extensive category with pullbacks, pushouts of split monomorphisms along split monomorphisms, and such that the pullback of an epimorphism along any morphism is still an epimorphism. Then $\mathcal{C}^{\mathrm{op}}$ is a weakly Mal'tsev category.*

Proof: An extensive category with pullbacks always has stable coproducts, see for instance [21] Proposition 1.2. ■

In particular the category Top of topological spaces and continuous maps is extensive and epimorphisms (i.e. surjections) are pullback stable (see for instance [8] and [17]).

Many other familiar categories of spaces share these properties, for instance (T,V)-categories [9], which include approach spaces [23], preordered sets and metric spaces [22], probabilistic metric spaces [28, 12], or closure spaces [30]. Indeed, any (T,V)-category, or lax (T,V)-algebra, is extensive ([24], Corollary 8) and has pullback stable epimorphisms [9].

In [20] it is proved that any topos is adhesive. Combining the results from that paper, in particular the theorem by Brown and Janelidze on Van Kampen squares [3], we also see that the dual of any extensive and locally cartesian closed category is weakly Mal'tsev. Indeed, in a locally cartesian closed category, a morphism is effective for descent if and only if it is a regular epimorphism ([20], Lemma 12). In an extensive category any split exact cospan gives rise to a Van Kampen square. This is just a particular case of Theorem 23 as stated in [20]; see also [3], where the morphisms are not arbitrary monomorphisms but are split monomorphisms.

In [4] it is proved that the category of Kelly spaces is a regular category, it is also extensive because *coproducts coincide with topological sums*, hence its dual is weakly Mal'tsev.

Finally, it is worth noting that every solid quasi-topos [15] is extensive and has pullback stable epimorphisms, hence its dual is weakly Mal'tsev.

Corollary 6.3. *The dual of a solid quasi-topos (one with disjoint coproducts) is a weakly Mal'tsev category.*

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