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(SUB)FIT BIFRAMES AND NON-SYMMETRIC NEARNESS

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Abstract: The non-symmetric (quasi-)nearness and its generalized admissibility is studied both in its biframe and paircovers aspect and in the perspective of entourages. The necessary and sufficient condition for a biframe to carry such an enrichment is shown to be a biframe variant of subfitness (resp. fitness, in the hereditary case).

Keywords: Frame, locale, sublocale lattice, biframe, fitness and subfitness, paircover, entourage, nearness, quasi-nearness.

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Introduction

The standardly used concept of nearness in the pointfree context ([3, 1]) is that of a system of covers $\mathcal{N}$ of a frame L, admissible in the sense that each $a \in L$ is the join of all the x uniformly below it. It is expedient for many purposes, but sometimes it does call for modifications.

First, it makes sense in the regular frames only. No wonder, in this form of admissibility it is in fact the extension of Herrlich's *regular* nearness ([10, 11]) which in spaces needs regular carrier as well. The general space nearness can be defined on much more general spaces, and can be extended to the pointfree context so that it is definable on all subfit frames.

Second, one is sometimes interested in the non-symmetric variant which (even in the regular case) cannot be dealt simply with covers that make everything naturally symmetric.

In this paper we discuss the nearness extended in both the mentioned directions: it is generalized in the sense the cover nearness was generalized in [12], and it allows for non-symmetry as well. For the latter we exploit the

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Weil (entourage) approach ([17, 19]): unlike covers, the "neighbourhoods of the diagonal" do not create any a priori symmetry. But we do use the so called paircover approach as well ([9, 5]), and also the technique of biframes similarly as it has been used in more special context in [6, 7, 8] and other papers. In fact it turns out that the entourages naturally induce a biframe structure on a frame, too, so that the biframe context and techniques come quite organically, after all.

In the biframe discussion of (generalized) nearness one encounters inherent concepts of biframe fitness and subfitness analogous with the homonymous frame notions in the same way as the biframe regularity extending the frame one. It may be of interest that although one gets the subfitness as a necessary and sufficient condition of the existence of nearness in the quite general context again (and fitness as the hereditary variant), it is not quite a smooth extension (while the fitness is): one gets in fact a weaker and a stronger variant (the stronger one being the actual necessary and sufficient condition).

1. Preliminaries

1.1. We will use the standard terminology and notation for posets. In lattices the meet will be denoted as a rule by $a \wedge b$, $a_1 \wedge \cdots \wedge a_n$ etc., the meet (infimum) of a subset A in a complete lattice will be denoted by $\bigwedge A$; similarly we use $a \vee b$, $a_1 \vee \cdots \vee a_n$ etc. and $\bigvee A$ for joins (suprema).

The bottom (the smallest element) of a poset will be as a rule denoted by 0 and the top (the largest element) by 1.

Recall that *a Heyting algebra* is a lattice with an extra binary operation $x \to y$ on L satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c \tag{Heyt}$$

(there is hardly any danger that the "$\to$" be confused with the arrow sign for a mapping as in $f\colon A \to B$). From (Heyt) one can immediately infer rules like $(\bigvee a_i) \to b = \bigwedge(a_i \to b)$, that $a \to b = 1$ iff $a \leq b$, and further $a \leq b \to a$, or $(a \wedge b) \to c = a \to (b \to c)$, to be used without further mentioning.

1.2. Recall that a *frame* is a complete lattice L satisfying the distribution law

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\} \tag{frm}$$

for all subsets $A \subseteq L$ and elements $b \in L$.

Thus, the mapping $x \mapsto x \wedge a$ preserves suprema and has a right Galois adjoint $y \mapsto a \to y$ which makes a frame a Heyting algebra.

A *frame homomorphism* $h\colon L \to M$ preserves all joins and all finite meets. The resulting category is denoted by

$$\mathbf{Frm},$$

and its dual, the category of *locales* is denoted by

$$\mathbf{Loc}$$

and can be viewed as category of generalized spaces (the relations of frames and spaces is naturally contravariant).

The morphisms of **Loc** will be represented by the *localic maps* $f\colon L \to M$ defined as the right Galois adjoints of the frame homomorphisms $h\colon M \to L$ (that is, maps $L \to M$ such that $h(x) \le y$ iff $x \le f(y)$).

For more about frames and locales see, e.g., [23] or [14].

1.3. Subspaces of locales (viewed as generalized spaces) are represented by *sublocales*. A sublocale S of a frame (locale) L is a subset $S \subseteq L$ such that

(S1) for every $M \subseteq S$, $\bigwedge M \in S$ (thus in particular, the top 1 is in S), and
(S2) for every $s \in S$ and every $x \in L$, $x \to s$ is in S.

Sublocales are precisely such subsets for which the embedding map $j\colon S \overset{\subseteq}{\to} L$ is a (one-one) localic map; thus, the embedding of sublocales are precisely the right adjoints of the onto frame homomorphisms (which are often used to represent generalized subspaces). The frame homomorphism associated with the embedding $j\colon S \to L$, called *nucleus*, will be denoted by

$$\nu_S\colon L \to S.$$

Sublocales of L ordered by inclusion constitute a co-frame $\mathcal{S}\ell(L)$ (a complete lattice in which one has the distribution rule dual to (frm) from 1.2) with the meets coinciding with intersections, the suprema given by

$$\bigvee S_i = \{\bigwedge M \mid M \subseteq \bigcup S_i\},$$

the zero $\mathsf{O} = \{1\}$ and the top $\mathsf{1} = L$. The correspondence $S \mapsto \nu_S$ is a dual isomorphism between $\mathcal{S}\ell(L)$ and the lattice (frame) of nuclei, in which the meet is computed as $(\bigwedge \nu_i)(x) = \bigwedge \nu_i(x)$.

1.3.1. Open resp. closed subspaces associated with elements $a \in L$ are represented by *open* resp. *closed sublocales*

$$\mathfrak{o}(a) = \{a \to x \mid x \in L\} = \{x \in L \mid a \to x = x\} \quad \text{resp.} \quad \mathfrak{c}(a) = {\uparrow} a;$$

they are complemented with each other and one has

$$\mathfrak{o}(\bigvee a_i) = \bigvee \mathfrak{o}(a_i) \quad \text{and} \quad \mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b).$$

The associated nuclei are $\nu_{\mathfrak{o}(a)}(x) = a \to x$ and $\nu_{\mathfrak{c}(a)}(x) = a \vee x$. (See [23, 20].)

1.3.2. Open and closed sublocales reflect in general sublocales by the natural law

$$\mathfrak{o}(a) \cap S = \mathfrak{o}_S(\nu_S(a)) \quad \text{and} \quad \mathfrak{c}(a) \cap S = \mathfrak{c}_S(\nu_S(a))$$

(see [23], III.6.2).

Further rules we will use:

1.3.3. Lemma.
(1) *For each* $S \in \mathcal{S}\ell(L)$, $\mathfrak{o}(a) \cap \overline{S} \neq \mathsf{O}$ *iff* $\mathfrak{o}(a) \cap S \neq \mathsf{O}$.
(2) $\mathfrak{c}(b) \subseteq \mathfrak{o}(a)$ *iff* $a \vee b = 1$.
(3) $\mathfrak{o}(a) \cap \mathfrak{c}(b) \neq \mathsf{O}$ *iff* $a \not\leq b$.

Proof: (1) $\mathfrak{o}(a) \cap S = \mathsf{O}$ iff $S \subseteq \mathfrak{c}(a)$ iff $\overline{S} \subseteq \mathfrak{c}(a)$ iff $\mathfrak{o}(a) \cap \overline{S} = \mathsf{O}$.
(2) $\Rightarrow$: If $\mathfrak{c}(b) \subseteq \mathfrak{o}(a)$ then $L = \mathfrak{c}(b) \vee \mathfrak{o}(b) \leq \mathfrak{o}(a) \vee \mathfrak{o}(b) = \mathfrak{o}(a \vee b)$ and hence $a \vee b = 1$.
$\Leftarrow$: If $a \vee b = 1$ then $\mathfrak{o}(a) \vee \mathfrak{o}(b) = 1$ and $\mathfrak{c}(b) \subseteq \mathfrak{o}(a)$ by complementation.
(3) $\mathfrak{o}(a) \cap \mathfrak{c}(b) = \mathsf{O}$ iff $\mathfrak{o}(a) \subseteq \mathfrak{o}(b)$ iff $a \leq b$. ■

1.4. Sublocales and subframes. Each sublocale is a frame, but this concept should not be confused with that of a *subframe* where the embedding is a frame homomorphism. We will need to understand the following construction with subframes and sublocales. Suppose that we have a subframe $L' \subseteq L$ and a sublocale $S \subseteq L$ with the associated frame homomorphism $\nu_S \colon L \to S$. Then we have the frame homomorphism

$$L' \xrightarrow{j=\subseteq} L \xrightarrow{\nu_S} S$$

and hence an onto frame homomorphism

$$\mu = (x \to \nu_S(x)) \colon L' \to \nu_S[L'].$$

Now obviously $\nu_S(L)$ is a subframe of S. It is not precisely a sublocale of L': it is a subset of L but not necessarily a subset of L'; but it is almost that: if we set

$$S' = \{x \mid x = \bigvee\{y \mid \mu(x) = \mu(y)\}\}$$

then S' is a sublocale of L' isomorphic with the $\nu_S[L']$ by the isomorphism $x \mapsto \nu_S(x)$.

1.5. Fit and subfit. According to Isbell [13], a frame is *fit* if each closed sublocale is an intersection (meet) of open sublocales, that is, using 1.3.3,

$$\forall a \in L, \quad \mathfrak{c}(a) = \bigcap\{\mathfrak{o}(x) \mid \mathfrak{o}(x) \supseteq \mathfrak{c}(a)\} = \bigcap\{\mathfrak{o}(x) \mid a \vee x = 1\}$$

which can be expressed as

$$a \nleq b \quad \Rightarrow \quad \exists c,\ a \vee c = 1 \text{ and } c \to b \nleq b.$$

A frame is *subfit* if each open sublocale is a join of closed ones, that is,

$$\forall a \in L, \quad \mathfrak{o}(a) = \bigvee\{\mathfrak{c}(x) \mid a \vee x = 1\}.$$

This can be expressed by

$$a \nleq b \quad \Rightarrow \quad \exists c,\ a \vee c = 1 \text{ and } b \vee c \neq 1.$$

Subfitness is sometimes referred to as *conjunctivity* (Simmons [27]).

A sublocale of a fit locale is fit, while subfitness is not hereditary. In actual fact, fitness is hereditary subfitness. This is a standard fact, but we will present a short proof (first, because it is much shorter than what can be usually found in literature, and, second, because the same procedure will be use later in the biframe context).

Proposition. *If every sublocale of a locale L is subfit then L is fit.*

Proof: Suppose it is not. Then there is an $a \in L$ such that

$$\mathfrak{c}(a) \not\supseteq S = \bigcap\{\mathfrak{o}(x) \mid a \vee x = 1\}.$$

Thus, there is a $b \in S$ that is not in $\mathfrak{c}(a)$. We have $a \in S$ (since $(a \vee x) \to a = (a \to a) \wedge (x \to a) = x \to a$) and $b \in S$, $a \nleq b$. Suppose $a \vee c = 1$ for a $c \in S$. Since $c \in S$ and $a \vee c = 1$ (so that $\mathfrak{c}(a) \subseteq \mathfrak{o}(c)$ and hence $S \subseteq \mathfrak{o}(c)$) we have in particular $c \in \mathfrak{o}(c)$ and hence $c = c \to c = 1$. Thus, $b \vee c = 1$ and S is not subfit. ∎

2. Bilocales

2.1. Recall that a *biframe* ([2]) is a triple (L, L_1, L_2) of frames where L_1, L_2 are subframes of L and

$$\forall a \in L,\ a = \bigvee\{a_1 \wedge a_2 \mid a_i \in L_i, a_1 \wedge a_2 \leq a\}.$$

The frame L is usually called the *total part* of the biframe.

A *biframe homomorphism* $h\colon (L, L_1, L_2) \to (M, M_1, M_2)$ is a frame homomorphism $h\colon L \to M$ for which $h[L_i] \subseteq M_i$ $(i = 1, 2)$.

In the sequel, we use L_i, L_j to denote L_1 or L_2, always assuming that $i, j = 1, 2$, $i \neq j$.

2.2. Subbilocales. Let (L, L_1, L_2) be a biframe. A subbilocale of (L, L_1, L_2) is a (S, S_1, S_2) where S is a sublocale of L and $S_i = \nu_S[L_i]$ for $i = 1, 2$ (cf. 1.4).

2.2.1. Observation. *Each subbilocale of a biframe is a biframe.*

Proof: If $a \in S$ we have

$$\begin{aligned} a = \nu_S(a) &= \nu_S(\bigvee\{a_1 \wedge a_2 \mid a_i \in L_i, a_1 \wedge a_2 \leq a\}) \\ &= \bigvee\{\nu_S(a_1) \wedge \nu_S(a_2) \mid a_i \in L_i, a_1 \wedge a_2 \leq a\}, \end{aligned}$$

and $\nu_S(a_i) \in S_i$. ■

2.3. Regularity, fitness and subfitness in biframes. We will extend the definition of fitness and subfitness to biframes in analogy with the extension of regularity. In the next section we will see that it is not just a formal matter: the concepts will be seen to be equivalent with another important property.

Recall that a biframe (L, L_1, L_2) is said to be *regular* ([2]) if

$$\forall a \in L_i, \quad a = \bigvee\{c \in L_i \mid c \prec_i a\},$$

where $c \prec_i a$ means that there is a $b \in L_j$ $(j \neq i)$ such that $c \wedge b = 0$ and $a \vee b = 1$ (equivalently, $c \prec_i a$ iff $c^\bullet \vee a = 1$ where $c^\bullet$ is the biframe *pseudocomplement*

$$\bigvee\{y \in L_j \mid y \wedge c = 0\}$$

of c in L_j [26]).

2.3.1. Proposition. *A biframe is regular iff*

$$\forall a \in L_i, \forall x \in L, \quad a \nleq x \Rightarrow \exists b \in L_j \colon a \vee b = 1,\ b^\bullet \nleq x. \tag{Reg}$$

Proof: $\Rightarrow$: Let $a \nleq x$ with $a \in L_i$ and $x \in L$. Since $a = \bigvee\{c \in L_i \mid c \prec_i a\}$, there is a $c \in L_i$ such that $c \prec_i a$ and $c \nleq x$. Let $b = c^\bullet \in L_j$. Then $b \vee a = 1$ and since $b^\bullet = c^{\bullet\bullet} \geq c$, $b^\bullet \nleq x$.

$\Leftarrow$: By contradiction, if $a \nleq \bigvee\{c \in L_i \mid c \prec_i a\}$ then there is some $b \in L_j$ with $a \vee b = 1$ and

$$b^\bullet \nleq \bigvee\{c \in L_i \mid c \prec_i a\}.$$

But this is a contradiction since $b^\bullet \in L_i$ and $b^\bullet \prec_i a$ (indeed, $b^{\bullet\bullet} \vee a \geq b \vee a = 1$). ■

2.3.2. Recall 1.5. We define a biframe (L, L_1, L_2) to be *fit* if

$$\forall a \in L_i, \quad \mathfrak{c}(a) = \bigcap\{\mathfrak{o}(b) \mid b \in L_j, \mathfrak{o}(b) \supseteq \mathfrak{c}(a)\} = \bigcap\{\mathfrak{o}(b) \mid b \in L_j, a \vee b = 1\}.$$

Similarly, we say that a biframe (L, L_1, L_2) is *subfit* if

$$\forall a \in L_i, \quad \mathfrak{o}(a) = \bigvee\{\mathfrak{c}(b) \mid b \in L_j, \mathfrak{c}(b) \subseteq \mathfrak{o}(a)\} = \bigvee\{\mathfrak{c}(b) \mid b \in L_j, a \vee b = 1\}.$$

2.3.3. Using De Morgan law in the co-frame $\mathcal{S}\ell(L)$ we immediately obtain

Proposition. *Every fit biframe is subfit.* ■

2.4. Proposition. *A biframe is fit iff*

$$\forall a \in L_i, \forall x \in L, \quad a \nleq x \Rightarrow \exists b \in L_j \colon a \vee b = 1, b \to x \nleq x. \tag{Fit}$$

Proof: $\Rightarrow$: Let $a \nleq x$ with $a \in L_i$ and $x \in L$. We have

$$\mathfrak{c}(x) \nsubseteq \mathfrak{c}(a) = \uparrow a = \bigcap\{\mathfrak{o}(b) \mid b \in L_j, a \vee b = 1\}$$

and consequently there is a $b \in L_j$ such that $a \vee b = 1$ and $\mathfrak{c}(x) \nsubseteq \mathfrak{o}(b)$. Furthermore, if $b \to x = x$ for all $b \in L_j$ such that $a \vee b = 1$ then $a \leq x$.

$\Leftarrow$: Let $a \in L_i$. By contradiction, if

$$\mathfrak{c}(a) \subsetneq \bigcap\{\mathfrak{o}(b) \mid b \in L_j, a \vee b = 1\},$$

then there is an s in the intersection of such opens $\mathfrak{o}(b)$ with $s \notin \mathfrak{c}(a)$, that is, $s \ngeq a$. By hypothesis, there is a $c \in L_j$ such that $c \vee a = 1$ and $c \to s \nleq s$.

Since c is one of those b's, $s \in \mathfrak{o}(c)$, that is, $s = c \to x$ for some $x \in L$. Then we get a contradiction:

$$c \to s = c \to (c \to x) = c \to x = s. \quad \blacksquare$$

2.5. Proposition. *A biframe is subfit iff*

$$\forall a \in L_i, \forall x, y \in L, \quad a \nleq y \to x \Rightarrow \exists b \in L_j \colon a \vee b = 1, y \nleq b \vee x. \qquad \text{(Sfit)}$$

Proof: $\mathfrak{o}(a) = \bigvee\{\mathfrak{c}(b) \mid b \in L_j, a \vee b = 1\}$ is the same as $\mathfrak{o}(a) \subseteq \bigvee\{\mathfrak{c}(b) \mid b \in L_j, a \vee b = 1\}$. Set

$$S = \bigvee\{\mathfrak{c}(b) \mid b \in L_j, a \vee b = 1\}.$$

In the language of nuclei we have $\nu_{\mathfrak{o}(a)} \geq \nu_S$ and hence, by 1.3.1,

$$a \to x \geq \bigwedge\{a \vee x \mid b \in L_j, a \vee b = 1\}$$

for all x; thus, if $y \leq b \vee x$ for all $b \in L_j$ such that $a \vee b = 1$ we have $y \leq a \to x$, that is, $a \leq y \to x$. $\blacksquare$

2.5.1. In particular, if a biframe is subfit then the case $y = 1$ yields

$$\forall a \in L_i, \forall x \in L, \quad a \nleq x \Rightarrow \exists b \in L_j \colon a \vee b = 1 \neq x \vee b \qquad (*)$$

the standard formula for subfitness in frames. The reader may wonder what makes the difference, that is, why the $(*)$ fails (or at least seems to fail) to characterize subfitness also in the biframe context. If we have $a \nleq y \to x$, that is, $a \wedge y \nleq x$, the standard subfitness gives a $b \in L$ such that $a \vee b = 1 \neq b \vee x$ but such b is not necessarily in L_j.

The difference can be seen already in spaces. We have

2.5.2. Proposition. *Let $(X, \mathcal{O}_1 X, \mathcal{O}_2 X)$ be a bitopological space. Then, denoting by* Cl, Cl_1 *and* Cl_2, *respectively, the closures in $\mathcal{O}X = \mathcal{O}_1 X \vee \mathcal{O}_2 X$, $\mathcal{O}_1 X$ and $\mathcal{O}_2 X$, we have:*

(1) *$(X, \mathcal{O}_1 X, \mathcal{O}_2 X)$ satisfies $(*)$ iff for any $A \in \mathcal{O}_i X$ $(i = 1, 2)$ and any $x \in A$, there is a $y \in \mathrm{Cl}(\{x\})$ such that $\mathrm{Cl}_j(\{y\}) \subseteq A$ $(j \neq i)$.*
(2) *$(X, \mathcal{O}_1 X, \mathcal{O}_2 X)$ is subfit iff for any $A \in \mathcal{O}_i X$ $(i = 1, 2)$, $U \in \mathcal{O}X$ and any $x \in A \cap U$, there is a $y \in \mathrm{Cl}(\{x\}) \cap U$ such that $\mathrm{Cl}_j(\{y\}) \subseteq A$ $(j \neq i)$.*

Proof: (1) $\Rightarrow$: Let $A \in \mathcal{O}_i X$, $x \in A$ and $V = X \smallsetminus \mathrm{Cl}(\{x\})$. By the hypothesis, there is a $B \in \mathcal{O}_j X$ such that $A \cup B = X \neq V \cup B$. Any element in $X \smallsetminus (V \cup B)$ is the required y.

$\Leftarrow$: Let $A \in \mathcal{O}_i X$ and $V \in \mathcal{O} X$ with $A \not\subseteq V$, $x \in A \smallsetminus V$ and the corresponding y given by the hypothesis. Then $B = X \smallsetminus \mathrm{Cl}_j(\{y\}) \in \mathcal{O}_j X$ satisfies $A \cup B = X$ (since $\mathrm{Cl}_j(\{y\}) \subseteq A$) and $V \cup B \neq X$ (since $y \in \mathrm{Cl}(\{x\}) \subseteq X \smallsetminus V$).

(2) $\Rightarrow$: Let $A \in \mathcal{O}_i X$, $U \in \mathcal{O} X$ and $x \in A \cap U$. Take $V = X \smallsetminus \mathrm{Cl}(\{x\})$. By the hypothesis, there is a $B \in \mathcal{O}_j X$ such that $A \cup B = X$ and $U \not\subseteq B \cup V$. In particular, there is a $y \in U$ that is not in $B \cup V$. Clearly, this is the required y, since $\mathrm{Cl}_j(\{y\}) \subseteq X \smallsetminus B \subseteq A$.

$\Leftarrow$: Let $A \in \mathcal{O}_i X$ and $U, V \in \mathcal{O} X$ with $A \cap U \not\subseteq V$. Consider $x \in (A \cap U) \smallsetminus V$ and the corresponding y given by the hypothesis. Then $B = X \smallsetminus \mathrm{Cl}_j(\{y\}) \in \mathcal{O}_j X$ satisfies $A \cup B = X$ (since $\mathrm{Cl}_j(\{y\}) \subseteq A$) and $U \not\subseteq B \cup V$ (since $y \in \mathrm{Cl}(\{x\}) \subseteq X \smallsetminus V$). ■

2.5.3. It may be of interest to see a "Heyting" reason why (Fit)$\Rightarrow$(Sfit), without reference to the coframe $\mathcal{S}\ell(L)$:

Let $a \not\leq x \to y$. Then, by (Fit), there is a $b \in L_j$ satisfying $a \vee b = 1$ and $b \to (x \to y) \not\leq x \to y$. Then $b \vee x = b \vee y$ leads to a contradiction:

$$\begin{aligned} b \to (x \to y) &= (b \wedge x) \to y = ((b \wedge x) \to y) \wedge (y \to y) = \\ &= ((b \wedge x) \vee y) \to y = ((b \wedge y) \vee x) \to y = \\ &= ((b \wedge y) \to y) \wedge (x \to y) \leq x \to y. \end{aligned}$$

2.6.1. Proposition. *A subbilocale of a fit biframe is fit.*

Proof: Let $a = \nu_S(a) \in S_i$, $a \in L_i$. Then $\mathfrak{c}_S(a) = \mathfrak{c}(a) \cap S$ and $\mathfrak{c}(a) = \bigwedge\{\mathfrak{o}(x) \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(x),\ x \in L_j\}$ so that

$$\begin{aligned} c_S(a) &= S \cap \bigwedge\{\mathfrak{o}(x) \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(x),\ x \in L_j\} \\ &= \bigwedge\{\mathfrak{o}(x) \cap S \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(x),\ x \in L_j\}. \end{aligned}$$

By 1.3.1, $\mathfrak{o}(x) \cap S = \mathfrak{o}_S(\nu_S(x))$ and hence the statement follows. ■

2.6.2. Proposition. *A biframe is fit iff each of its subbilocales is subfit.*

Proof: It suffices to prove that if every subbilocale of a biframe (L, L_1, L_2) is subfit then (L, L_1, L_2) is fit. Suppose it is not. Then there is an i and an $a \in L_i$ such that

$$\mathfrak{c}(a) \subsetneq \bigcap\{\mathfrak{o}(x) \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(x),\ x \in L_j\}.$$

That is, there is a b in $\bigcap\{\mathfrak{o}(x) \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(x),\ x \in L_j\}$ that is not in $\mathfrak{c}(a)$. Set

$$S = \bigcap\{\mathfrak{o}(x) \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(x),\ x \in L_j\}.$$

We have $a \not\leq b$ in S and $a = \nu_S(a) \in S_i$. Suppose $a \vee c = 1$ for a $c \in S_j$. Since $c \in S$ and $a \vee c = 1$ (so that $\mathfrak{c}(a) \subseteq \mathfrak{o}(c)$ and hence $S \subseteq \mathfrak{o}(c)$) we have in particular $c \in \mathfrak{o}(c)$ and hence $c = c \rightarrow c = 1$. Thus, $b \vee c = 1$ and S is not subfit. ∎

3. Biframes and quasi-nearness

The cover approach (Tukey 1940) does not allow, without radical modification, a non-symmetric variant of the concept of nearness while there are no such obstacles when approaching the structures in the entourage way (Weil 1938). Thus, the reader may expect us to proceed right away to the latter. This will be discussed in the following section; first, however, we will approach the non-symmetry, via biframes and their paircovers. This is not an idle detour. It will be seen that even if one decides for the entourages, the biframe structure naturally emerges.

3.1. (Generalized) nearness. The standard cover nearness structure in the pointfree context ([3, 1, 4]), as compared with the nearness as defined originally in spaces by Herrlich (see [10, 11]) corresponds, rather, to the *regular nearness*. The pointfree structure corresponding to general nearness (or, rather, its general admissibility) was introduced in [12]. As in the regular case, a cover of a frame L is a subset $A \subseteq L$ such that $\bigvee A = 1$, a cover A refines a cover B, written $A \leq B$, if for every $a \in A$ there is a $b \in B$ such that $a \leq b$, covers A, B have a common refinement

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\},$$

and a nearness $\mathcal{N}$ is a filter in the preorder of refinement. The difference comes with the definition of admissibility: whereas in the standard (regular) case we assume that for each $a \in L$, $a = \bigvee\{x \mid \exists C \in \mathcal{N}, Cx \leq a\}$, in the generalized case one requires that

$$\mathfrak{o}(a) = \bigvee\{S \in \mathcal{S\ell}(L) \mid \exists C \in \mathcal{N}, CS \leq a\}. \tag{gadm}$$

Here,

$$Cx = \bigvee\{c \in C \mid c \wedge x \neq 0\} \quad \text{and} \quad CS = \bigvee\{c \in C \mid \mathfrak{o}(c) \cap S \neq \mathsf{O}\};$$

hence $Cx \leq a$ iff $C\mathfrak{o}(x) \subseteq \mathfrak{o}(a)$, that is, the former is the latter reduced to the open sublocales only. Now if we recall 1.3.3.1 we see that we can reduce (gadm) to

$$\mathfrak{o}(a) = \bigvee\{\mathfrak{c}(x) \mid \exists C \in \mathcal{N}, C\mathfrak{c}(x) \leq a\}$$

and hence each L admitting a nearness is subfit (recall 1.5). In fact, one has (see [12]) that L admits a (generalized) nearness iff it is subfit.

3.2. Paircovers. A subset $C \subseteq L_1 \times L_2$ is a *paircover* of a biframe (L, L_1, L_2) if

$$\widetilde{C} = \{c_1 \wedge c_2 \mid (c_1, c_2) \in C\}$$

is a cover of L (cf. [5, 6]). A paircover C is *strong* if, for any $(c_1, c_2) \in C$, $c_1 \wedge c_2 = 0$ implies $(c_1, c_2) = (0, 0)$.

A paircover C refines a paircover D, written $C \leq D$, if for every $(c_1, c_2) \in C$ there is a $(d_1, d_2) \in D$ such that $c_i \leq d_i$, $i = 1, 2$. Paircovers C, D have a common refinement

$$C \wedge D = \{(c_1 \wedge d_1, c_2 \wedge d_2) \mid (c_1, c_2) \in C, (d_1, d_2) \in D\}.$$

3.2.1. For any paircover C of (L, L_1, L_2) and any $x \in L$ define

$$C_1x = \bigvee\{c_1 \mid (c_1, c_2) \in C,\ c_2 \wedge x \neq 0\}, \quad C_2x = \bigvee\{c_2 \mid (c_1, c_2) \in C,\ x \wedge c_1 \neq 0\}$$

$$\text{and} \quad CD = \{(C_1d_1, C_2d_2) \mid (d_1, d_2) \in D\}.$$

3.2.2. Observations. *For every $x, y \in L$ and every paircovers C, D, we have:*

(1) $C \leq D \quad \& \quad x \leq y \Rightarrow C_ix \leq D_iy$ $(i = 1, 2)$.
(2) $C_i(D_ix) \leq (CD)_ix$ $(i = 1, 2)$.
(3) *If $C_ix \leq y$ then $x \prec y$, that is, $x^* \vee y = 1$ $(i = 1, 2)$. In case $x, y \in L_i$ then $C_ix \leq y$ implies $x \prec_i y$.*

More generally, for a sublocale S of L set

$$C_1S = \bigvee\{a \mid (a,b) \in C,\ \mathfrak{o}(b) \cap S \neq \mathsf{O}\} \quad \text{and}$$
$$C_2S = \bigvee\{b \mid (a,b) \in C,\ \mathfrak{o}(a) \cap S \neq \mathsf{O}\}.$$

3.2.3. Observations.

(1) $C_iS \le a,\ D_iT \le b \Rightarrow (C \wedge D)_i(S \cap T) \le a \wedge b$ $(i = 1, 2)$.
(2) *If* $C_iS \le a$ *then* $S \subseteq \mathfrak{o}(a)$ $(i = 1, 2)$.
(3) $C_i\mathfrak{o}(x) = C_ix$ $(i = 1, 2)$.
(4) $C_1\mathfrak{c}(x) = \bigvee\{a \mid (a,b) \in C, b \not\le x\}$ *and* $C_2\mathfrak{c}(x) = \bigvee\{b \mid (a,b) \in C, a \not\le x\}$.
(5) $C_iS = C_i\overline{S}$ $(i = 1, 2)$.

(For the last one recall 1.3.3.1.)

3.3. In a biframe (L, L_1, L_2) one has more: for a sublocale S of L let

$$\mathrm{cl}_i(S) = \bigcap\{\mathfrak{c}(a) \mid a \in L_i, S \subseteq \mathfrak{c}(a)\} = \mathfrak{c}(\bigvee\{a \in L_i \mid S \subseteq \mathfrak{c}(a)\}) \quad (i = 1, 2).$$

Of course, $S \le \overline{S} \le \mathrm{cl}_i(S)$.

3.3.1. Lemma.

(1) *Let* $a \in L_i$. *Then* $\mathfrak{o}(a) \cap S = \mathsf{O}$ *iff* $\mathfrak{o}(a) \cap \mathrm{cl}_i(S) = \mathsf{O}$.
(2) *For each paircover* C *and each sublocale* S *of* L, $C_iS = C_i\mathrm{cl}_j(S)$.

Proof: (1) $\mathfrak{o}(a) \cap S = \mathsf{O} \Leftrightarrow S \le \mathfrak{c}(a) \Leftrightarrow \mathrm{cl}_i(S) \le \mathfrak{c}(a) \Leftrightarrow \mathfrak{o}(a) \cap \mathrm{cl}_i(S) = \mathsf{O}$.
(2) This is an immediate consequence of (1):

$$\begin{aligned} C_1S &= \bigvee\{a \mid (a,b) \in C, \mathfrak{o}(b) \cap S \neq \mathsf{O}\} \\ &= \bigvee\{a \mid (a,b) \in C, \mathfrak{o}(b) \cap \mathrm{cl}_2(S) \neq \mathsf{O}\}. \quad \blacksquare \end{aligned}$$

3.4. It follows immediately from 3.3 that

$$\begin{aligned} \bigvee\{S \in \mathcal{S}\ell(L) \mid \exists C \in \mathcal{N}\colon C_iS \le a\} &= \bigvee\{\mathrm{cl}_j(S) \mid \exists C \in \mathcal{N}, C_i\mathrm{cl}_j(S) \le a\} \\ &= \bigvee\{\mathfrak{c}(b) \mid b \in L_j, \exists C \in \mathcal{N}, C_i\mathfrak{c}(b) \le a\} \end{aligned}$$

and we may introduce the *quasi-admissibility* of a system of paircovers $\mathcal{N}$ by requiring that

$$\forall a \in L_i, \quad \mathfrak{o}(a) = \bigvee\{S \in \mathcal{S}\ell(L) \mid \exists C \in \mathcal{N}, C_iS \le a\}.$$

3.5. A *quasi-nearness* on a biframe (L, L_1, L_2) is a non-void set $\mathcal{N}$ of paircovers such that

(N1) The family of strong paircovers of $\mathcal{N}$ is a filter-base for $\mathcal{N}$ with respect to $\wedge$ and $\leq$ defined above, and

(N2) $\mathcal{N}$ is quasi-admissible.

3.6. Proposition. *A biframe admits a quasi-nearness iff it is subfit.*

Proof: $\Rightarrow$ follows from 3.4.
$\Leftarrow$: We will use the subfitness condition from 2.3.2 to prove the formula in 3.4 for the system of all paircovers. Let $a \in L_1$. Consider a $b \in L_2$ such that $\mathfrak{c}(b) \subseteq \mathfrak{o}(a)$ (that is, such that $a \vee b = 1$). We will prove that $C_1\mathfrak{c}(b) \leq a$ for a suitable paircover C. Take

$$C = \{(a, 1), (1, b)\}.$$

Since $b \leq b$ we have $\mathfrak{o}(b) \cap \mathfrak{c}(b) = \mathsf{O}$ and hence in $C_1\mathfrak{c}(b)$ only $(a, 1)$ qualifies and $C_1\mathfrak{c}(b) \leq a$. ■

From 2.6.1 we now immediately obtain

3.6.1. Corollary. *A biframe (L, L_1, L_2) is fit iff each of its subbilocales admits a quasi-nearness.* ■

4. (Generalized) nearness: entourages

In this section we will discuss, at last, the general nearness based on entourages (modelling the "neighbourhoods of the diagonal"). As it was mentioned earlier, here there is no immediate preference of symmetry. But a biframe and paircover structure emerges anyway, and the reader will see that the other approach is natural, and may be even preferred for some purposes.

Recall that the product $L \oplus L$ of a locale L (i.e., the coproduct of L by itself in **Frm**) can be constructed as follows (see e.g. [23, 14]):

First take the Cartesian product $L \times L$ as a poset and the corresponding set of down-sets

$$\mathfrak{D}(L \times L) = \{U \subseteq L \times L \mid {\downarrow} U = U \neq \emptyset\}.$$

Call a $U \in \mathfrak{D}(L \times L)$ *saturated* if

(1) $\forall A \subseteq L,\, \forall b \in L,\, A \times \{b\} \subseteq U \Rightarrow (\bigvee A, b) \in U.$

(2) $\forall B \subseteq L$, $\forall a \in L$, $\{a\} \times B \subseteq U \Rightarrow (a, \bigvee B) \in U$.

(A and B can be void and hence, in particular, each saturated set contains the subset $\mathsf{n} = \{(a,0),(0,b) \mid a,b \in L\}$ and for each $(a,b) \in L \times L$,

$$a \oplus b = {\downarrow}(a,b) \cup \mathsf{n}$$

is saturated).

Then $L \oplus L$ is the frame of all saturated U in $\mathfrak{D}(L \times L)$.

An element E of the localic product $L \oplus L$ is an *entourage* of L whenever

$$\bigvee\{x \mid (x,x) \in E\} = 1 \quad ([16,\ 19]).$$

Let $E \in L \oplus L$ (or, more generally, a down-set of $L \times L$). For an $x \in L$ we write

$$E_1x = \bigvee\{a \mid (a,b) \in E, b \wedge x \neq 0\} \quad \text{and} \quad E_2x = \bigvee\{b \mid (a,b) \in E, a \wedge x \neq 0\}.$$

We have, among other, the following obvious

4.1. Observations. *For any $x, y \in L$, $E, F \in L \oplus L$ and $i, j = 1, 2$ ($i \neq j$),*
(1) *if $E \subseteq F$ and $x \leq y$ then $E_ix \leq F_iy$,*
(2) $E_i(F_ix) \leq (E \circ F)_ix$,
(3) $(E^{-1})_ix = E_jx$ *(where $E^{-1} = \{(y,x) \mid (x,y) \in E\}$),*
(4) $E_ix \wedge y = 0$ *iff* $x \wedge E_jy = 0$.

More generally, for a sublocale S of L, we set

$$E_1S = \bigvee\{a \mid (a,b) \in E, \mathfrak{o}(b) \cap S \neq \mathsf{O}\} \quad \text{and}$$
$$E_2S = \bigvee\{b \mid (a,b) \in E, \mathfrak{o}(a) \cap S \neq \mathsf{O}\}.$$

4.2. Proposition. *For any $E, F \in L \oplus L$ and $S, T \in \mathcal{S}\ell(L)$,*
(P1) *if $E \subseteq F$ and $S \leq T$ then $E_iS \leq F_iT$,*
(P2) $E_i(F_iS) \leq (E \circ F)_iS$,
(P3) $(E^{-1})_iS = E_jS$,
(P4) $E_i\mathfrak{o}(x) = E_ix$,
(P5) $E_1\mathfrak{c}(x) = \bigvee\{a \mid (a,b) \in E, b \not\leq x\}$, $E_2\mathfrak{c}(x) = \bigvee\{b \mid (a,b) \in E, a \not\leq x\}$,
(P6) $E_iS = E_i\overline{S}$,
(P7) $E_iS = E_i\mathrm{cl}_j(S)$,
(P8) *if E is an entourage and $E_iS \leq a$ then $S \leq \mathfrak{o}(a)$.*

Proof: (P1), (P2) and (P3) are obvious. (P4) is a consequence of the fact that $\mathfrak{o}(a) \cap \mathfrak{o}(x) = \mathsf{O}$ iff $a \wedge x = 0$, (P5) is a consequence of the fact that $\mathfrak{o}(a) \cap \mathfrak{c}(x) = \mathsf{O}$ iff $a \leq x$, while (P6) follows from the equivalence $\mathfrak{o}(a) \cap S = \mathsf{O} \Leftrightarrow \mathfrak{o}(a) \cap \overline{S} = \mathsf{O}$. (P7) follows similarly as in 3.3.1.

(P8): By (P6), it suffices to prove that $E_i\overline{S} \leq a \Rightarrow \overline{S} \leq \mathfrak{o}(a)$. Since E is an entourage,

$$\overline{S} = \overline{S} \cap \mathfrak{o}(\bigvee\{x \mid (x,x) \in E\}) = \overline{S} \cap \bigvee\{\mathfrak{o}(x) \mid (x,x) \in E\}.$$

We are in a co-frame, nevertheless the distribution law

$$\overline{S} \cap \bigvee\{\mathfrak{o}(x) \mid (x,x) \in E\} = \bigvee\{\overline{S} \cap \mathfrak{o}(x) \mid (x,x) \in E, \overline{S} \cap \mathfrak{o}(x) \neq \mathsf{O}\}$$

holds since $\overline{S}$ is complemented. Then it follows from $E_i\overline{S} \leq a$ that $x \leq a$ and thus $\bigvee\{\overline{S} \cap \mathfrak{o}(x) \mid (x,x) \in E, \overline{S} \cap \mathfrak{o}(x) \neq \mathsf{O}\} \leq \mathfrak{o}(a)$. ■

Whenever E is a symmetric entourage (i.e., $E^{-1} = E$) we denote the common element $E_1S = E_2S$ just by ES.

Now, for an $A \in \mathfrak{D}(L \times L)$ we define

$$\kappa_0(A) = \{(x, \bigvee S) \mid \{x\} \times S \subseteq A\} \cup \{(\bigvee S, y) \mid S \times \{y\} \subseteq A\}$$

and let $\kappa(A)$ denote the element $\bigcap\{E \in L \oplus L \mid E \supseteq A\}$ of $L \oplus L$. The following useful fact generalizes Lemma 3.1 of [18]:

4.3. Lemma *Let $A \in \mathfrak{D}(L \times L)$. For any sublocale S of L and $i = 1, 2$,*

$$\kappa(A)_iS = A_iS.$$

Proof: Consider a sublocale S of L and the non-empty set

$$\mathcal{E} = \{U \in \mathfrak{D}(L \times L) \mid A \subseteq U \subseteq \kappa(A),\ U_iS = A_iS\}.$$

(1) If $U \in \mathcal{E}$ then $\kappa_0(U) \in \mathcal{E}$. Indeed:

It suffices to check that $\kappa_0(U)_1S \leq U_1S$. Consider $(a,b) \in \kappa_0(U)$ with $\mathfrak{o}(b) \cap S \neq \mathsf{O}$). If $(x,y) = (x, \bigvee Y)$ for some Y such that $\{x\} \times Y \subseteq U$, then there is a non-zero $y' \in Y$ such that $\mathfrak{o}(y') \cap S \neq \mathsf{O}$ and $(x,y') \in U$, and therefore $x \leq U_1S$. Otherwise, if $(x,y) = (\bigvee X, y)$ for some X with $X \times \{y\} \subseteq U$, then, immediately, $x = \bigvee X \leq U_1S$.

(2) For any non-void $\mathcal{X} \subseteq \mathcal{E}$, $\bigcup \mathcal{X} \in \mathcal{E}$, since $\bigcup \mathcal{X}_iS = \bigvee_{U\in\mathcal{X}} U_iS$. Consequently, $T = \bigcup_{U\in\mathcal{E}} U$ belongs to $\mathcal{E}$, i.e., $\mathcal{E}$ has a largest element T.

(3) Then, by (1), $\kappa_0(T) \in \mathcal{E}$. Hence $T = \kappa_0(T)$, i.e., $T \in L \oplus L$. Finally, $\kappa(A) = T \in \mathcal{E}$ and therefore $\kappa(A)_i S = A_i S$. ■

4.3.1. This helps in computing the result of the operators $E_1 S$ and $E_2 S$ for a concrete E. For example, for the entourage

$$E^{ab} = (a \oplus 1) \vee (1 \oplus b)$$

where $a, b \in L$ satisfy $a \vee b = 1$, we have

$$\begin{aligned}(E^{ab})_1 \mathfrak{c}(b) &= [(a \oplus 1) \cup (1 \oplus b)]_1 \mathfrak{c}(b) \\ &= \bigvee \{x \mid (x, y) \in (a \oplus 1) \cup (1 \oplus b), y \not\leq b\} = a\end{aligned} \tag{$*$}$$

and similarly

$$(E^{ab})_2 \mathfrak{c}(a) = b. \tag{$**$}$$

4.4. A *quasi-nearness* on L is a nonvoid system $\mathcal{N}$ of entourages in L such that:

(N1) $E \in \mathcal{N} \quad \& \quad E \subseteq F \Rightarrow F \in \mathcal{N}$.
(N2) $E, F \in \mathcal{N} \Rightarrow E \cap F \in \mathcal{N}$.

Of course, if $\mathcal{N}$ is a quasi-nearness on L, then the filter $\mathcal{N}^{-1}$ consisting of the inverse entourages E^{-1} ($E \in \mathcal{N}$) is also a quasi-nearness on L.

4.5. Proposition. *Let $\mathcal{N}$ be a quasi-nearness on L. For each $i = 1, 2$,*

$$L_i(\mathcal{N}) = \Big\{a \in L \mid \mathfrak{o}(a) = \bigvee\{S \in \mathcal{S}\ell(L) \mid E_i S \leq a \textit{ for some } E \in \mathcal{N}\}\Big\}$$

is a subframe of L.

Proof: We prove it for L_1. Obviously $1 \in L_1$, and $0 \in L_1$ by (P8).
(1) L_1 is closed under binary meets: in fact, if $a, b \in L_1$ then

$$\begin{aligned}\mathfrak{o}(a \wedge b) &= \mathfrak{o}(a) \wedge \mathfrak{o}(b) \\ &= \bigvee\{S \cap T \mid S, T \in \mathcal{S}\ell(L), \exists E, F \in \mathcal{N}, E_1 S \leq a, F_1 T \leq b\} \\ &\leq \bigvee\{S \in \mathcal{S}\ell(L) \mid \exists E \in \mathcal{N}, E_1 S \leq a \wedge b\} \\ &\leq \mathfrak{o}(a \wedge b)\end{aligned}$$

where the last inequalities are consequence of (P1) and (P8) respectively.

(2) L_1 is closed under arbitrary joins: if $a_i \in L_1$ $(i \in I)$ then

$$\begin{aligned}\mathfrak{o}(\bigvee_{i\in I} a_i) &= \bigvee_{i\in I} \mathfrak{o}(a_i) = \bigvee_{i\in I} \bigvee \{S \in \mathcal{S}\ell(L) \mid \exists E_i \in \mathcal{N}, (E_i)_1 S \leq a_i\} \\ &= \bigvee \{S \in \mathcal{S}\ell(L) \mid \exists E \in \mathcal{N}\ \exists i \in I, E_1 S \leq a_i\} \\ &\leq \bigvee \{S \in \mathcal{S}\ell(L) \mid \exists E \in \mathcal{N}, E_1 S \leq \bigvee_{i\in I} a_i\} \\ &\leq \mathfrak{o}(\bigvee_{i\in I} a_i). \quad \blacksquare\end{aligned}$$

4.6. We say that the pair $(L, \mathcal{N})$ is a *quasi-nearness frame* whenever the quasi-nearness $\mathcal{N}$ is *quasi-admissible* on L, that is, whenever

(N3) the triple $(L, L_1(\mathcal{N}), L_2(\mathcal{N}))$ is a biframe.

We refer to $(L, L_1(\mathcal{N}), L_2(\mathcal{N}))$ as the biframe *induced* by $\mathcal{N}$ on L.

Note that this is an extension of the symmetric quasi-admissibility condition. Indeed, whenever $\mathcal{N}$ is symmetric, that is,

(N4) $E \in \mathcal{N} \Rightarrow E^{-1} \in \mathcal{N}$

then, evidently, $L_1(\mathcal{N}) = L_2(\mathcal{N})$ and therefore (N3) means precisely that $L = L_1(\mathcal{N}) = L_2(\mathcal{N})$.

On the other hand, for a general non-symmetric nearness $\mathcal{N}$, axiom (N3) means that any $x \in L$ is of the form

$$x = \bigvee_{i\in I} (x_i^1 \wedge x_i^2)$$

for some $x_i^1 \in L_1(\mathcal{N})$ and $x_i^2 \in L_2(\mathcal{N})$, where for any i,

$$\mathfrak{o}(x_i^1) = \bigvee \{\mathfrak{c}(b_1) \mid \exists E \in \mathcal{N}, E_1 \mathfrak{c}(b_1) \leq x_i^1\}$$

and

$$\mathfrak{o}(x_i^2) = \bigvee \{\mathfrak{c}(b_2) \mid \exists F \in \mathcal{N}, F_2 \mathfrak{c}(b_2) \leq x_i^2\}.$$

But by (P1) and (P3), if $E_1 \mathfrak{c}(b_1) \leq x_i^1$ and $F_2 \mathfrak{c}(b_2) \leq x_i^2$, then

$$(E \cap F^{-1}) \mathfrak{c}(b_1 \vee b_2) \leq x_i^1 \wedge x_i^2.$$

Consequently, if we denote by $\widehat{\mathcal{N}}$ the nearness generated by the quasi-nearnesses $\mathcal{N}$ and $\mathcal{N}^{-1}$, that is, the filter of entourages generated by $\{E \cap E^{-1} \mid E \in \mathcal{N}\}$, we may write

$$\begin{aligned}\mathfrak{o}(x) &= \bigvee_{i\in I}[\mathfrak{o}(x_i^1)\wedge\mathfrak{o}(x_i^2)]\\ &= \bigvee\{\mathfrak{c}(b_1)\wedge\mathfrak{c}(b_2)\mid \exists E,F\in\mathcal{N}, E_1\mathfrak{c}(b_1)\le x, F_2\mathfrak{c}(b_2)\le x\}\\ &\le \bigvee\{\mathfrak{c}(b)\mid \exists E\in\widehat{\mathcal{N}}, E\mathfrak{c}(b)\le x\}\le\mathfrak{o}(x).\end{aligned}$$

Hence,

$$\mathfrak{o}(x)=\bigvee\{S\in\mathcal{S}\ell(L)\mid \exists E\in\widehat{\mathcal{N}}, ES\le x\}$$

for every $x\in L$.

Remark. Note that if $\mathcal{N}$ is a quasi-admissible quasi-nearness on L, then its conjugate $\mathcal{N}^{-1}$ is also a quasi-admissible quasi-nearness on L.

Analogously with 3.6 we have

4.7. Proposition. *A frame L admits a quasi-nearness iff it is the total part of a subfit biframe.*

Proof: Let $(L,\mathcal{N})$ be a quasi-nearness frame with induced biframe

$$(L,L_1(\mathcal{N}),L_2(\mathcal{N})).$$

Then, for each $a\in L_i(\mathcal{N})$, using properties (P7) and (P8) we may obtain

$$\begin{aligned}\mathfrak{o}(a) &= \bigvee\{S\in\mathcal{S}\ell(L)\mid E_iS\le a \text{ for some } E\in\mathcal{N}\}\\ &\le \bigvee\{\mathrm{cl}_j(S)\mid E_i\mathrm{cl}_j(S)\le a \text{ for some } E\in\mathcal{N}\}\\ &= \bigvee\{\mathfrak{c}(b)\mid b\in L_j, E_i\mathfrak{c}(b)\le a \text{ for some } E\in\mathcal{N}\}\\ &\le \bigvee\{\mathfrak{c}(b)\mid b\in L_j, a\vee b=1\}\le\mathfrak{o}(a),\end{aligned}$$

which shows that $(L,L_1(\mathcal{N}),L_2(\mathcal{N}))$ is subfit.

Conversely, let (L,L_1,L_2) be a subfit frame and let $\mathcal{N}$ be the quasi-nearness on L generated by the subbasic family of entourages

$$\{E^{ab}\mid a\in L_1, b\in L_2, a\vee b=1\}$$

with induced subframes $L_1(\mathcal{N})$ and $L_2(\mathcal{N})$. By the subfitness of (L, L_1, L_2) and identities $(*)$ and $(**)$ in 4.3.1, we get immediately for each $a \in L_i$,

$$\begin{aligned}\mathfrak{o}(a) &= \bigvee\{\mathfrak{c}(b) \mid b \in L_j, a \vee b = 1\} \\ &\leq \bigvee\{S \in \mathcal{S}\ell(L) \mid \exists E \in \mathcal{N}, E_i S \leq a\} \leq \mathfrak{o}(a).\end{aligned}$$

This means that $L_i \subseteq L_i(\mathcal{N})$ for $i = 1, 2$. Hence $(L, L_1(\mathcal{N}), L_2(\mathcal{N}))$ is also a biframe and $(L, \mathcal{N})$ is a quasi-nearness frame. ∎

Remark 4.8. As we have proved elsewhere (see [16, 17, 18, 21, 22]), there is a Galois correspondence between the cover and the entourage structures, which yields an equivalence precisely in the quasi-uniform setting (the non-symmetric case) and the uniform one (the symmetric case): the refinement axiom is crucial for our proof of the isomorphism. In the generalized nearness setting, as it is shown in the present paper, there is still a striking parallel between the two approaches (in the sense that every result one gets on the former has a corresponding exact counterpart concerning the latter). It remains an open problem to decide wether they produce isomorphic categories; the answer in the positive would be a surprise, though.

References

[1] B. Banaschewski, *Completion in pointfree topology*, Lecture Notes in Math. and Applied Math., Vol. 2, University of Cape Town, 1996.

[2] B. Banaschewski, G. C. L. Brümmer and K. A. Hardie, Biframes and bispaces, *Quaest. Math.* 6 (1983) 13–25.

[3] B. Banaschewski and A. Pultr, Cauchy points of uniform and nearness frames, *Quaest. Math.* 19 (1996) 101–127.

[4] T. Dube, Strong nearness frames, in: *Festschrift* (*on the occasion of the 60th birthday of Guillaume Brümmer*) incorporating Proceedings Symposium on Categorical Topology (Univ. Cape Town 1994), University of Cape Town, 1999, pp. 103–112.

[5] J. Frith, *Structured frames*, Doctoral Dissertation, University of Cape Town, 1987.

[6] J. Frith and A. Schauerte, The Samuel compactification for quasi-uniform biframes, *Topology Appl.* 156 (2009) 2116–2122.

[7] J. Frith and A. Schauerte, Quasi-nearnesses on biframes and their completions, *Quaest. Math.* 33 (2010) 507–530.

[8] J. Frith and A. Schauerte, Quasi-nearness biframes: unique completions and related covering properties, *Quaest. Math.* 35 (2012) 35–56.

[9] T. E. Gantner and R. C. Steinlage, Characterizations of quasi-uniformities, *J. London Math. Soc.* 5 (1972) 48–52.

[10] H. Herrlich, A concept of nearness, *Gen. Topology Appl.* 5 (1974) 191–212.

[11] H. Herrlich, *Topologie II: Uniforme Räume*, Heldermann Verlag, Berlin, 1988.

[12] H. Herrlich and A. Pultr, Nearness, subfitness and sequential regularity, *Appl. Categ. Structures* 8 (2000) 67–80.
[13] J. R. Isbell, Atomless parts of spaces, *Math. Scand.* 31 (1972) 5–32.
[14] P. T. Johnstone, *Stone Spaces*, Cambridge Studies in Advanced Mathematics, Vol. 3, Cambridge University Press, Cambridge, 1982.
[15] K. Morita, On the simple extension of a space with respect to a uniformity I, *Proc. Japan Acad.* 27 (1951) 65–72.
[16] J. Picado, Weil uniformities for frames, *Comment. Math. Univ. Carolin.* 36 (1995) 357–370.
[17] J. Picado, Weil nearness spaces, *Port. Math.* 55 (1998) 233–254.
[18] J. Picado, Frame quasi-uniformities by entourages, in: *Festschrift* (*on the occasion of the 60th birthday of Guillaume Brümmer*), incorporating Proceedings Symposium on Categorical Topology (Univ. Cape Town 1994), University of Cape Town, 1999, pp. 161–175.
[19] J. Picado, Structured frames by Weil entourages, *Appl. Categ. Structures* 8 (2000) 351–366.
[20] J. Picado and A. Pultr, *Locales treated mostly in a covariant way*, Textos de Matemática, Vol. 41, University of Coimbra, 2008.
[21] J. Picado and A. Pultr, Cover quasi-uniformities in frames, *Topology Appl.* 158 (2011) 869–881.
[22] J. Picado and A. Pultr, On strong inclusions and asymmetric proximities in frames, *Order* 29 (2012) 513–531.
[23] J. Picado and A. Pultr, *Frames and Locales: topology without points*, Frontiers in Mathematics, Vol. 28, Springer, Basel, 2012.
[24] J. Picado and A. Pultr, Entourages, covers and localic groups, *Appl. Categ. Structures* 21 (2013) 49–66.
[25] A. Pultr, Pointless uniformities I, *Comment. Math. Univ. Carolin.* 25 (1984) 91–104.
[26] A. Schauerte, *Biframes*, Doctoral Dissertation, McMaster University, 1992.
[27] H. Simmons, The lattice theoretic part of topological separation properties, *Proc. Edinburgh Math. Soc. (2)* 21 (1978) 41–48.
[28] H. Simmons, Regularity, fitness, and the block structure of frames, *Appl. Categ. Structures* 14 (2006) 1–34.

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