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THE NUMBER OF STRINGS ON ESSENTIAL TANGLE DECOMPOSITIONS OF A KNOT CAN BE UNBOUNDED

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ABSTRACT: We show the existence of infinitely many knots that for arbitrarily high n have n-string essential tangle decompositions.

KEYWORDS: Knot, essential tangle decomposition, meridional essential surface.
AMS SUBJECT CLASSIFICATION (2010): 57M25, 57N10.

1. Introduction

A *n-string tangle* $(B, \mathcal{T})$ is a ball B together with collection of n disjoint arcs $\mathcal{T}$ properly embedded in B, for $n \in \mathbb{N}$. We say that $(B, \mathcal{T})$ is *essential*, if n is 1 and its arc is knotted*, or if n is bigger than 1 and there is no properly embedded disk in B disjoint from $\mathcal{T}$ and separating the components of $\mathcal{T}$ in B. Otherwise, we say that the tangle is *inessential*. (See Figure 1 for examples.)

Let K be a knot in S^3 and S a 2-sphere in general position with K. Each ball bounded by S in S^3 intersects K in the same number n of arcs. So, these balls together with the arcs of intersection with K are n-string tangles. In this case, we say that S defines a *n-string tangle decomposition* of K, and if both tangles are essential we say that the tangle decomposition of K defined by S is *essential*. A knot is composite if, and only if, it has a 1-string essential tangle decomposition. Note also that S defines an essential tangle decomposition for K if, and only if, the intersection of S with the exterior of K, $E(K)$†, is an essential surface in $E(K)$. (See Definition 1.)

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*An arc of $\mathcal{T}$ is *unknotted* if it co-bounds a disk embedded in B together with an arc in ∂B, otherwise it is said to be *knotted*.

†We denote the *exterior* of a knot K, that is $S^3 - int\, N(K)$ where $N(K)$ is a regular neighborhood of K, by $E(K)$.

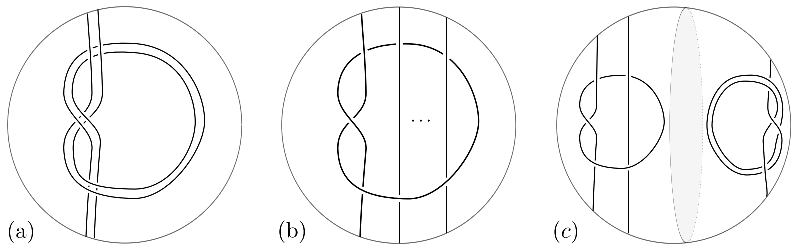

FIGURE 1: Examples of essential tangles, in (a) and (b), and an inessential tangle, in (c).

A tangle decomposition of a knot is natural and has been relevant for knot theory and its applications. The concept of "tangle" was first used in the work of Conway [3], where he defines and classifies (2-string) rational tangles and uses it as an instrument to list knots. The concept of essential tangle was first used in [8] where Kirby and Lickorish prove that any knot is concordant to a prime knot. They actually define *prime tangle*, that is an essential tangle with no local knots‡. Another example is the work of Lickorish in [9] where he proves for instance that if a knot has a 2-string prime tangle decomposition then the knot is prime. Tangles are also used in applied mathematics to study the DNA topology. The paper [2] by Buck surveys the subject concisely, and also explains how tangles are useful to the study of the topological properties of DNA, an application pioneered by Ernst and Sumners in [5].

This paper addresses the question if the number of strings on essential tangle decompositions of a fixed knot is bounded. There are results showing some evidence for this to be true. For instance, knots with no closed incompressible surfaces [4], tunnel number one knots [6] and free genus one knots [10] have no essential tangle decompositions. There also knots with an unique essential tangle decomposition [12]. Furthermore, in Proposition 2.1 of [11], Mizuma and Tsutsumi proved that for a given knot the number of strings in essential tangle decompositions, without parallel strings§, is bounded. The proof of this result allows a more general statement. That is, the number of strings that are not parallel to other strings in an essential tangle decomposition of a fixed knot is bounded. So, from this flow of results

‡A tangle $(B, \mathcal{T})$ has no local knots if any 2-sphere intersecting $\mathcal{T}$ transversely in two points bounds a ball in B meeting $\mathcal{T}$ in an unknotted arc.

§Two strings of a tangle in a ball B are parallel if there is an embedded disk in B co-bounded by these strings and two arcs in ∂B.

and intuition on essential tangle decompositions the following theorem and its corollary are surprising.

Theorem 1. *There are infinitely many prime knots with a n-string essential tangle decomposition for all $n \geq 2$.*

Corollary 1.1. *There are infinitely many knots with a n-string essential tangle decomposition for all $n \geq 1$.*

Essential surfaces are very important to the study of 3-manifolds topology. And as observed above, to each n-string essential tangle decomposition of a knot corresponds a meridional essential surface in the exterior of the knot, with $2n$ boundary components. Therefore, from the results in this paper there are knots with meridional planar essential surfaces in their exteriors for all possible numbers of boundary components. Furthermore, from Lemma 1.2 in [1], the double cover of S^3 along these knots contains genus g closed incompressible surfaces, meeting the fixed point set of the covering action in $2(g+1)$ points, and separating the double cover in irreducible and boundary irreducible components, for all $g \geq 1$.

The reference used for standard definitions and results of knot theory is Rolfsen's book [13], and throughout this paper we work in the piecewise linear category.
In Section 2, we show the existence of handlebody-knots (see Definition 2) with incompressible planar surfaces in their exteriors with b boundary components, for all $b \geq 2$. In Section 3, we use these handlebody knots to prove Theorem 1 and its corollary. The main techniques used are the standard innermost curve argument from 3-manifold topology. Along the paper, the number of connected components of a topological space X is denoted by $|X|$.

2. Meridional incompressible planar surfaces in handlebody-knots complements

To prove Theorem 1 we use the correspondence between n-string essential tangle decompositions of a knot and meridional planar essential surfaces in the knot exterior. So, we start by defining these surfaces.

Definition 1. Let H be a handlebody embedded in S^3.
A *planar surface* is obtained from a 2-sphere by cutting the interior of a finite number of disks.

A surface P properly embedded in $H^c = S^3 - intH$ is *meridional* if each boundary component of P bounds a disk in H.
An embedded disk D in H^c is a *compressing disk* for P if $D \cap P = \partial D$ and ∂D doesn't bound a disk in P. We say that P is *incompressible* if there is no compressing disk for P in H^c.
An embedded disk D in H^c is a *boundary compressing disk* for P if $\partial D \cap P = \alpha$, with α a connected arc not cutting a disk from P, and $\partial D - \alpha = \beta$ a connected arc in ∂H. We say that P is *boundary incompressible* if there is no boundary compressing disk for P in H^c.
The surface P is *essential* if it is incompressible and boundary incompressible.

In this section, we present handlebody-knots whose complements contain meridional incompressible planar surfaces with n boundary components for any $n \geq 2$. This embedding will later be used in the proof of Theorem 1. So, next we define a handlebody-knot.

Definition 2. A *handlebody-knot* of genus g in S^3 is an embedded handlebody of genus g in S^3. A spine γ of a handlebody-knot Γ is an embedded graph in S^3 with Γ as a regular neighborhood.

Let Γ be the genus two handlebody-knot 4_1 from the list of [7], with spine γ, as in Figure 2. Consider also a collection of knots C_i, for $i \in \mathbb{N}$, with distinct knot group, and C some other non-trivial knot. We work with γ as if defined by two vertices, two loops e_1, e_2, one for each vertex, and an edge e between the two vertices. (See Figure 2.)

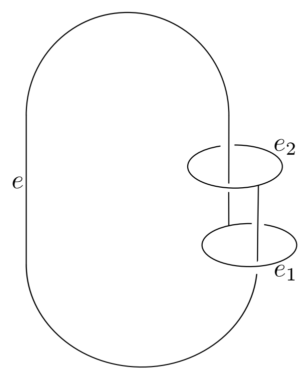

FIGURE 2: The spine γ of the handlebody-knot Γ.

Consider two disjoint closed arcs a_1 and a_2 in e, as in Figure 3(a). We proceed to a connected sum operation between γ and the knots considered

before along the arcs a_1 and a_2 with an usual connect sum operation. That is, we take a ball in S^3 intersecting γ in a_1, and a ball in S^3 intersecting C at a single arc. A connected sum operation is obtained by removing both balls and gluing their boundaries through a homeomorphism in a way that the boundary points of a_1 are mapped to the boundary points of the chosen arc in C_i. A similar operation can be obtained from the arc a_2 and C. From these operations we get the handlebody-knots as represented schematically in Figure 3(b), that we denote by Γ_i with a respective spine γ_i. In this figure we also have represented an embedded 2-sphere S_2 in S^3 that intersects γ at two points, p_1 and p_2, and denote the ball bounded by S_2 containing a single component of e by B_{21} and the other by B_{22}. Denote by l_1, resp. l_2, the component of $B_{22} \cap \gamma_i$ that contains e_1, resp. e_2 and note that l_j intersects S_2 at p_j, $j = 1, 2$.
From Van-Kampen's Theorem, and from the knots C_i having distinct knot group we have that the handlebody knots Γ_i are not ambient isotopic.

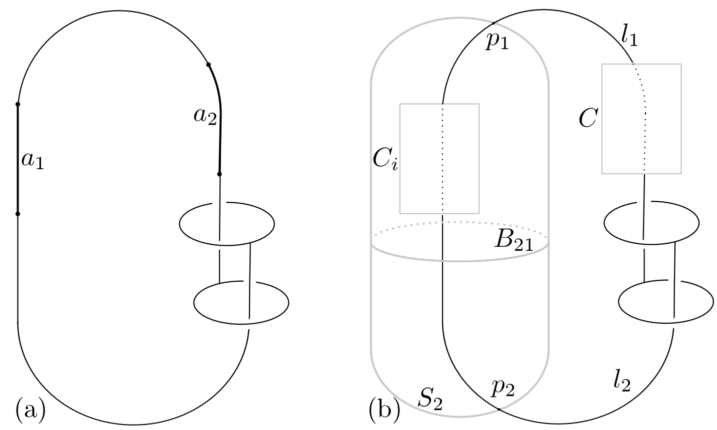

FIGURE 3: (a) The arcs a_1 and a_2 in γ; (b) The spines γ_i of the handlebody-knots Γ_i and the sphere S_2. Note that C_i and C label the pattern of the respective knots.

Both loops e_1 and e_2 co-bound an embedded annulus in B_{22}, parallel to the component of e in B_{22} each encircles, with interior disjoint from γ_i and intersecting S in the other boundary component, as it can easily be observed from Figure 3(b). Considering such an annulus with boundary l_1, say A_1, we proceed with an isotopy of γ_i through A_1 taking l_1 passing by S_2 and we obtain γ_i as in Figure 4(a). We refer to this isotopy as an *annulus isotopy* of γ_i. After this isotopy we denote S_2 by S_3, considering its relative position with Γ_i, and the respective balls it bounds by B_{31} and B_{32}. We assume that l_1 intersects S_3 at p_1. Note that all intersections of γ_i and S_3 are in the arc of e between p_1 and p_2. Again, we consider an embedded annulus A_2 in B_{31}, co-bounded by e_1, parallel to the component of e in B_{31} disjoint from e_1 and in the direction of the opposite end from the previous isotopy. By an annulus isotopy of γ_i through A_2 taking l_1 passing by S_3 we obtain γ_i as in Figure 4(b). After this isotopy we denote S_3 by S_4, and the respective balls it bounds by B_{41} and B_{42}. The ball B_{31} intersects γ_i in two parallel arcs, we still assume that $l_1 \cap S_4$ is p_1. Note again that all intersections of γ_i and S_3 are in the arc of e between p_1 and p_2.

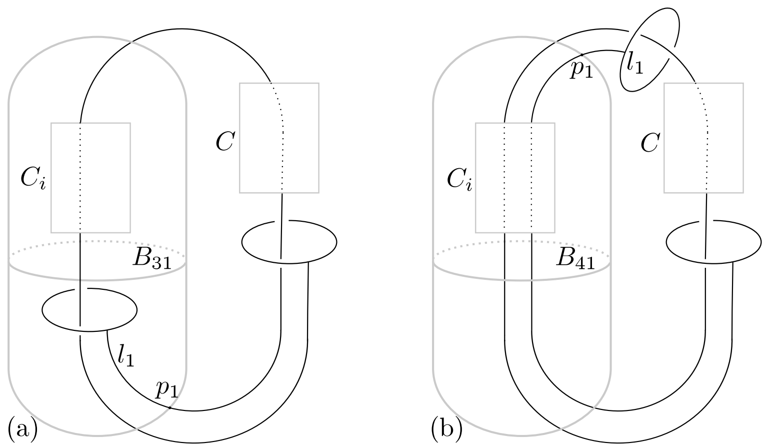

FIGURE 4: The spine γ_i after one, (a), and two, (b), annulus isotopies and the spheres S_3 and S_4.

For a canonical position, we isotope e_1 along the component of $e \cap B_{42}$, disjoint from e_1 and e_2, encircling l_2. (See Figure 5(a).) We can now continue the previous process. Consider again an annulus A_4 in B_{42}, co-bounded by e_1, parallel to the components of $e \cap B_{42}$ other than l_1. By an annulus isotopy of γ_i through A_4 taking l_1 passing by S_4 we obtain γ_i as in Figure 5(b). After this isotopy we denote S_4 by S_5, and the respective balls it bounds by B_{51} and B_{52}. Again, l_1 intersects S_5 at p_1, and all intersections of S_5 with γ_i are in the arc of e between p_1 and p_2. For the next step proceed with an annulus isotopy along annulus A_5 in B_{42} co-bounded by e_1, parallel to components of $e \cap B_{42}$ disjoint from e_1, in the direction of the opposite end from the previous isotopy.

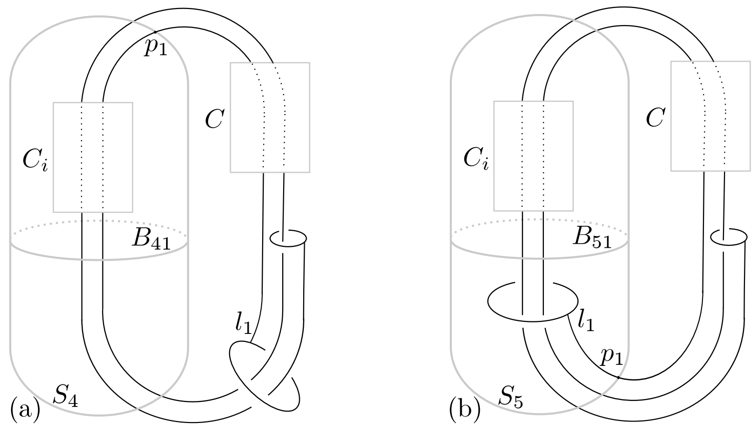

FIGURE 5: The spine γ_i of Figure 4(b) in (a) and after another annulus isotopy in (b), and the spheres S_4 and S_5.

After $2(k-1)$, $k=1,2,\ldots$, annulus isotopies as the ones explained above we get γ_i as in Figure 6(a). From S_2 we obtain S_{2k}, and the respective balls it bounds by $B_{(2k)1}$ and $B_{(2k)2}$. The ball $B_{(2k)1}$ intersects γ_i in k parallel arcs with the pattern of C_i, and the ball $B_{(2k)2}$ intersects γ_i in $k-2$ parallel arcs with the pattern of C, another arc with the pattern of C encircled by l_2, and l_1 that encircles all these other components.
After $2k-1$, $k=1,2,\ldots$, annulus isotopies we obtain γ_i as in Figure 6(b). From S_2 we obtain S_{2k+1}, and the respective balls it bounds by $B_{(2k+1)1}$ and $B_{(2k+1)2}$. The ball $B_{(2k+1)1}$ intersects γ_i in n parallel arcs with the pattern of C_i and l_1 encircling these arcs, and the ball $B_{(2k+1)2}$ intersects γ_i in $k-1$ parallel arcs with the pattern of C, together with another arc with the pattern of C and l_2 which encircles this arc.
Note after each isotopy we assume that l_j intersects S_n, $n=2,3,\ldots$, in p_j and that all points of $S_n\cap\gamma_i$ are in the arc between p_1 and p_2 in e.

Let Q_n, for $n=2,3,\ldots$, be the intersection of S_n with the complement of Γ_i in S^3 (or, of γ_i in S^3; we use the same notation in both cases).

Lemma 1. *The surface Q_n is incompressible.*

Proof: 1. Suppose n is even. Then S_n is as in Figure 6(a).
(a) In this case, the ball B_{n1} intersects γ_i in a collection of $k=\frac{n}{2}$ parallel knotted arcs. Then $(B_{n1}, B_{n1}\cap\gamma_i)$ is an essential tangle. In fact, suppose there is a compressing disk D for Q_n in B_{n1}. Then D separates the arcs $B_{n1}\cap\gamma_i$ in two collections. Let s_1 and s_2 be two arcs in the opposites sides separated by D in B_{n1}. As s_1 and s_2 are parallel there is a disk E with

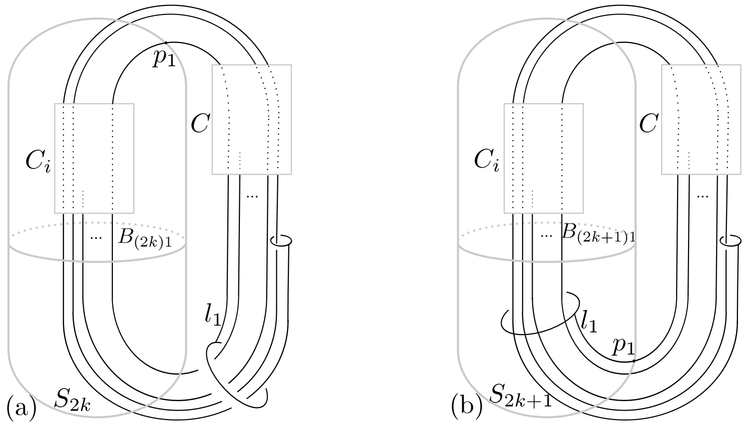

FIGURE 6: The spine γ_i after an even number, in (a), and an odd number, in (b), of annulus isotopies and the corresponding spheres S_{2k} and S_{2k+1}, $k \in \mathbb{N}$.

boundary $s_1 \cup s_2$ and two arcs in Q_n, α_1 and α_2, each with one end in s_1 and the other in s_2. Consider D and E in general position and suppose that $|D \cap E|$ is minimal. If D intersects E in simple closed curves or in arcs with both ends in α_1 or both in α_2, by an innermost arc type of argument we can reduce we can reduce $|D \cap E|$, which is a contradiction. Therefore, all arcs of $D \cap E$ have one end in α_1 and the other end in α_2. Hence, both s_1 and s_2 are parallel to outermost arcs of $D \cap E$, which implies that s_1 and s_2 are parallel to S_n. This a contradiction because the arcs s_1 and s_2 are knotted by construction.
(b) If $n \leq 4$ then the ball B_{n2} intersects γ_i in l_1, l_2, and when $n = 4$ also in an arc encircled by both l_1 and l_2. In this case if there is a compressing disk for Q_n in B_{n2} it separates the components l_1 or l_2 from the other components. This implies that e_1 or e_2 bound a disk in the complement of γ_i, which is a contradiction with Γ_i being a knotted handlebody-knot. Otherwise, suppose that $n > 4$. Thus, B_{n2} intersects γ_i in $\frac{n}{2} - 2$ parallel arcs with the pattern of C, another arc with the pattern of C encircled by l_2, and the component l_1 that encircles the $\frac{n}{2} - 2$ parallel arcs. With exception to l_1 and l_2, all other arcs are parallel. Thus, if a compressing disk for Q_n in B_{n2} separates these arcs, following an argument as in 1(a) we have a contradiction with these arcs being knotted. Therefore, a compressing disk for Q_n in B_{n2} separates a single component l_1 or l_2 from all the other components, or it separates both components l_1 and l_2 from the other parallel arcs. As e_1 bounds a disk disjoint from l_2, in both cases e_1 bounds a disk in the complement of γ_i,

which is a contradiction with Γ_i being a knotted handlebody-knot.

2. Suppose now that n is odd. Then S_n is as in Figure 6(b).
(a) The ball B_{n1} intersects γ_i in a collection of $\frac{n-1}{2}$ parallel arcs and l_1 which encircles these arcs. If there is a compressing disk D of Q_n in B_{n1} separating the parallel arcs, following an argument as in 1(a) we have a contradiction with these arcs being knotted. If D separates the component l_1 from the other components, following an argument as in 1(b) we have a contradiction with Γ_i being a knotted handlebody-knot.
(b) If $n = 3$ the ball B_{n2} intersects γ_i in an arc with pattern C and l_2 which encircles the arc. If there is a compressing disk for Q_n in B_{n2} in this case, then it separates the component l_2 from the other one. From the same argument used in 1(b) we have a contradiction with Γ_i being a knotted handlebody-knot. If $n > 3$ then the ball B_{n2} intersects γ_i in $\frac{n-1}{2}$ parallel arcs, and l_2 which encircles one of the previous arcs. Without considering l_2, if a compressing disk for Q_n in B_{n2} separates the parallel arcs then following an argument as in 1(a) we have a contradiction with the arcs being knotted. Then, if Q_n contains a compressing disk in B_{n2} it isolates the component l_2 from the other components, and following the argument as in 1(b) we have a contradiction with Γ_i being a knotted handlebody-knot. ∎

The surface Q_n is boundary compressible in the complement of Γ_i, as there are boundary compressing disks over the regular neighborhoods of l_1 and l_2. However, our construction of the handlebody knots Γ_i could have been made in a way that the surfaces Q_n are compressible and boundary compressible in their complements. For that purpose, we could do a connect sum of γ_i with two knots along two arcs in e_1 and e_2. After this operation, there won't be boundary compressing disks of Q_n over the regular neighborhoods of l_1 and l_2. And as these are the only possible boundary compressing disks, because all other components $\gamma_i - \gamma_i \cap S_n$ correspond to knotted arcs in their respective balls, the surfaces Q_n are also boundary incompressible in the complement of the handlebody knots after these connected sums. But for the purpose of this paper, we will use the the handlebody-knots Γ_i.

3. Knots with essential tangle decompositions with arbitrarily high number of strings

In this section, we use the handlebody-knots Γ_i to construct infinitely many examples of knots with essential tangle decompositions for all numbers of strings.

Let K_1 and K_2 be torus knots in the boundary of the solid tori T_1 and T_2. We connect sum K_1 and K_2 in a way that T_1 and T_2 are glued along a disk D, and denote by H the resulting genus two handlebody and $K = K_1 \# K_2$. Note that the knot K is parallel to the boundary of H. In Figure 7 we have the examples of the connected sum of two trefoils that we will use as reference for the remainder of the paper.

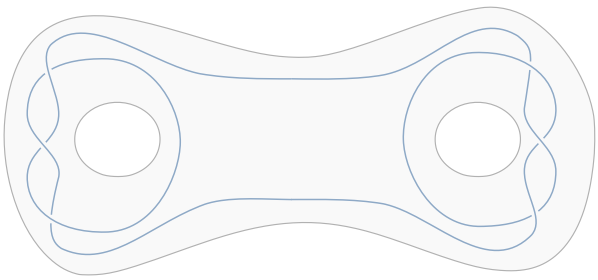

FIGURE 7: The handlebody H with the connected sum of two trefoil knots.

Consider disks D_1 and D_2 parallel to D in H, such that the cylinder $C_{1,2}$ cut by $D_1 \cup D_2$ from H intersects K in two parallel arcs, each with one end in D_1 and the other in D_2. We also keep denoting by T_1 and T_2 the solid torus cut from H by D_1 and D_2.
Let s be a spine of H that intersects $C_{1,2}$ in a single arc. We denote by d_i the point $D_i \cap s$, and by t_i the intersection of s with T_i, for $i = 1, 2$. See Figure 8.

We now embed the knot K in Γ_i as follows. Consider an embedding h_i of H in S^3 taking H homeomorphically to Γ_i, such that $h_i(s) = \gamma_i$, $h_i(d_j) = p_j$, $h_i(t_j)$ and also that $h_i(T_j) = L_j$, for $j = 1, 2$.

Proof of Theorem 1: Denote by K_i the knots $h_i(K)$, $i \in \mathbb{N}$, for a fixed knot K as above. As ∂H is essential in the complement of K in H, we have that $\partial\Gamma_i$ is essential in the complement of K_i. Hence, as the complement of

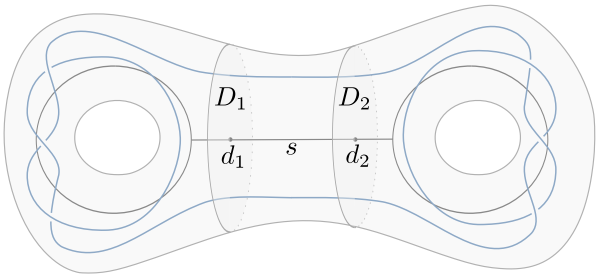

FIGURE 8: The handlebody H an the spine s with the connected sum of two trefoil knots.

the handlebody-knots Γ_i have non-isomorphic fundamental groups, the knot groups of the knots K_i, $i \in \mathbb{N}$, are also non-isomorphic.

We will prove that the spheres S_n, $n \geq 2$, define a n-string essential tangle decomposition for the knots K_i, and that these knots are prime. Consequently, we have infinitely many prime knots with n-string essential tangle decompositions for all $n \geq 2$, as in the statement of Theorem 1.

We start by proving that S_n defines an n-string essential tangle decomposition of K_i. Let $E(K_i)$ be the exterior of K_i in S^3, that is $S^3 - intN(K_i)$, and let P_n be the intersection of S_n with $E(K_i)$, for a fixed n. To prove that S_n defines an essential tangle decomposition for K_i, we need to prove that P_n is essential in $E(K_i)$, *i.e.* that P_n is incompressible and boundary incompressible.

First, we observe that P_n is boundary incompressible. In fact, as the strings of $K \cap B_{ni}$ in B_{ni}, $i = 1, 2$, are knotted, there is no boundary compressing disk for P_n in $E(K_i)$.

Now we prove that P_n is incompressible in $E(K_i)$. Let D_j, $j = 1, \ldots, n$, be the disks of intersection between Γ_i and S_n with $D_1 = L_1 \cap S_n$ and $D_n = L_2 \cap S_n$. Denote by $C_{j,j+1}$ the cylinder cut by $D_j \cup D_{j+1}$ from Γ_i. Note that $C_{j,j+1} \cap K$ is a collection of two arcs parallel to $\partial C_{j,j+1} - D_j \cup D_{j+1}$, each with one end in D_j and the other in D_{j+1}.
Suppose that P_n is compressible in $E(K_i)$ with D a compressing disk in general position with Γ_i. Thus, D is properly embedded in B_{n1} or B_{n2}.
Suppose that D intersects Γ_i and that $|D \cap \Gamma_i|$ is minimal.
In particular, assume that D intersects some cylinder $C_{j,j+1}$. If $D \cap \cup_{j=1}^{n-1} C_{j,j+1}$

contains a simple closed curve or an arc with both ends in the same disk of $\Gamma_i \cap S_n$, by considering an outermost one between such curves and arcs, and cutting and pasting along the disk it bounds or co-bounds, which contradicts the minimality of $|D \cap \Gamma_i|$. Thus, $D \cap \cup_{j=1}^{n-1} C_{j,j+1}$ is a collection of arcs with ends in distinct disks of $\Gamma_i \cap S_n$. Consider an outermost arc of $D \cap \cup_{j=1}^{n-1} C_{j,j+1}$ in D, say a, and, without loss of generality, suppose it belongs to $C_{j,j+1}$. The arc a is parallel to a string of the tangle defined by S_n that is in $C_{j,j+1}$, which contradicts the fact that all string of the tangle decomposition of K_i defined by S_n are knotted. Consequently, we can assume that $D \cap \cup_{j=1}^{n-1} C_{j,j+1}$ is empty.
Assume that D intersects Γ_i at L_1 or L_2, and also that D is general position with l_1 and l_2. If D is disjoint from $l_1 \cup l_2$ then we have a contradiction with Lemma 1. Then, D intersects $l_1 \cup l_2$ and we can assume that the intersection of D with L_1 and L_2 is a collection of disks, that we suppose to be minimal. Denote by s_j the string component, of the tangle decomposition of K_i defined by S_n, in L_j, $j = 1, 2$. Note that s_j is parallel to $\partial L_j - (L_j \cap S_n)$. So, we push s_j slightly off L_j and consider a disk O_j, in the complement of Γ_j, with boundary s_j, an arc α_j in L_j and two arcs δ_{1j}, δ_{2j} in S_n.

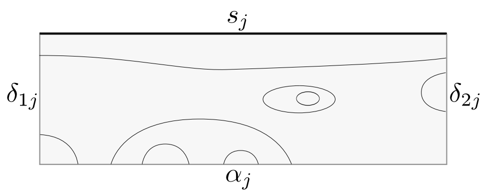

FIGURE 9: The string s_j, the corresponding disk O_j and arcs of intersection with D.

As s_j is essential in L_j, *i. e.* there are no essential disks in L_j disjoint from s_j, we have that D intersects O_j. Assuming D in general position with the disks O_j, consider the intersection of D with O_j and suppose $|D \cap O_j|$ to be minimal. The disk D intersects O_j in arcs with ends in δ_{1j}, δ_{2j} or α_j, or in simple closed curves. See Figure 9 for representations of these possible arcs in O_j. If D intersects O_j in simple closed curves, or in arcs with both ends in α_j, both in δ_{1j} or both in δ_{2j}, by considering an outermost component of $D \cap O_j$ between these in O_j and by cutting and pasting along the disk bounded or co-bounded by it in O_j, we get a contradiction with the minimality of $|D \cap O_j|$. Suppose that D intersects O_j in arcs with both ends in α_j.

Consider an outermost of such arcs in O_j, say a. Note that the ends of a are in two disks of $D \cap L_j$. If we proceed with an isotopy of D along the disk cut by a in O_j we can reduce the number of disks of $D \cap L_j$, which contradicts the minimality of $|D \cap (L_1 \cup L_2)|$. Then, D intersects O_j in arcs with one end in α_j and the other in δ_{1j} or δ_{2j}, or arcs with one end in δ_{1j} and the other in δ_{2j}.
If D intersects O_j in arcs with one end in δ_{1j} or δ_{2j}, and the other in α_j, consider an outermost arc between these in O_j, say a again, and the outermost disk it bounds in O_j. We proceed with an isotopy of D along this disk into and through L_j. After this isotopy we reduce the number of disks of $D \cap L_j$, which contradicts the minimality of $|D \cap (L_1 \cap L_2)|$. So, as D is disjoint from s_j, at this point all components of $D \cap O_j$ are arcs with one end in δ_{1j} and the other in δ_{2j}. We have that s_j is parallel to these arcs. By considering an outermost of such arcs in D this implies that s_j is unknotted, which is a contradiction because s_j is a knotted string with pattern a torus knot.
Therefore, D is disjoint from Γ_i, and consequently we have a contradiction with Lemma 1. So, we have that P_n is essential in the complement of K_i, which ends the proof that S_n defines an n-string essential tangle decomposition of K_i.

Now we prove that the knots K_i are prime. From Theorem 1 of [1], if a knot has a 2-string prime tangle decomposition, that is the tangles are essential and with no local knots, the knot is prime. We have that the knot K_i has a 2-string essential tangle decomposition defined by S_2. So, to prove that it is prime, we just need to show that the tangle decomposition defined by S_2 has no local knots. The ball B_{21} intersects K_i in two parallel arcs. Hence, if there is a 2-sphere intersecting only one of the arcs at a single component, this component has to be unknotted. The ball B_{22} intersects γ_i in l_1 and l_2, then it intersects K_i at two strings each with the pattern of a torus knot. Note that, even though the pattern of the knot C is in l_2, it doesn't affect the topological type of the string in L_2. Suppose, the tangle in B_{22} contains a local knot. That is, there is a ball Q intersecting only one of the strings, and at a knotted component. As the torus knots are prime, this knotted component is the all pattern of the string. Therefore, as the strings in B_{22} are parallel to the boundary of L_1 and L_2, we have Q contains either e_1 or e_2. But then, either e_1 or e_2 bound a disk in the complement of γ_i, which implies, as observed in 1(b) from the proof of Lemma 1, that the handlebody-knots Γ_i are unknotted. Consequently, the tangle decomposition defined by S_2 contains no local knots and the knots K_i are prime. ∎

Corollary 1.1 is now an immediate consequence.

Proof of Corollary 1.1: In Theorem 1 we proved that the spheres S_n, $n \geq 2$, define a n-string essential tangle decomposition for the knots K_i. Hence, considering the knots K_i connected sum with some other knot, we have infinitely many knots with n-string essential tangle decompositions for all $n \in \mathbb{N}$, as in the statement of this corollary. ∎

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